Theory of exact trigonometric periodic solutions to quadratic Liénard type equations

J. Akande[1],D.K.K. Adjaï[1], L.H. Koudahoun[1], Y.J.F. Kpomahou[2], M.D. Monsia[1*].

1. Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01. BP.526, Cotonou, BENIN(*E.mail: monsiadelphin@yahoo.fr).
2. Department of Industrial and Technical Sciences, ENSET-Lokossa, University of Abomey, Abomey, BENIN.

Abstract

The problem of finding exact trigonometric periodic solutions to non-linear differential equations is still an open mathematical research field. In this paper it is shown that the Painlevé-Gambier XVIII equation and its inverted version may exhibit exact trigonometric periodic solutions as well as other quadratic Liénard type equations but with amplitude-dependent frequency. Other inverted Painlevé-Gambier equations are also shown to admit exact periodic solutions.

Keywords: Liénard equation, Painlevé-Gambier equation, periodic solution, generalized Sundman transformation.

1- Introduction

In mathematics analytical properties of trigonometric functions are well known so that in practice many engineering applications are performed on the basis of the linear harmonic oscillator equation. However in mathematical modeling Liénard equations are widely used for the description of mechanical and electrical systems, for example. In particular the quadratic Liénard type differential equation has become an attractive research subject in mathematical physics since it exhibits position-dependent mass oscillator features having permitted the development of many engineering applications not only from the classical mechanics viewpoint but also from the quantum mechanics standpoint [1]. It appears then reasonable to be interested to the theory of exact harmonic periodic behavior to quadratic Liénard type nonlinear differential equations. As a nonlinear differential equation, the quadratic Liénardtype differential equation could not be in general solved in terms of exact solution. If the problem of determining approximate harmonic periodic behavior has been more or less solved in the theory of dissipative nonlinear differential equations, the problem

* Corresponding author. E-mail : monsiadelphin@yahoo.fr

of exact periodic solutions is less so. A fortiori, the identification of trigonometric functions as a class of exact periodic solutions to Liénard type dissipative non-linear differential equations remains a problem not yet fully or even partially solved. However, finding exact solutions to non-linear differential equations is a vital question for modern engineering applications which require a high reliability in the mathematical representation of physical systems [2]. Consequently, to highlight such a class of dissipative nonlinear differential equations of the Liénard type whose exact solutions are trigonometric functions is undoubtedly a fact of high scientific significance. In this way, Mustafa [3] used a generalized Sundman transformation for detecting a class of quadratic Liénard type differential equations which incorporates as special cases some physically important nonlinear oscillator equations like the Mathews-Lakshmanan oscillator equation and the position-dependent mass Morse type oscillator equation. The Mathews-Lakshmanan equation was for over four decades considered as the unique non-linear oscillator equation having an exact trigonometric solution but with amplitude-dependent frequency. The problem in this paper is to show that other quadratic Liénard type dissipative non-linear differential equations may exhibit exact harmonic periodic solutions with amplitude-dependent frequency. The question is then more precisely: Do the inverted Painlevé-Gambier XVIII equation and other quadratic Liénard type equations exhibit exact harmonic periodic behavior but with amplitude-dependent frequency? This work predicts such quadratic Liénard type equations which exhibit trigonometric functions as exact periodic solutions. This prediction may allow a best understanding of oscillators and dynamical systems which may be represented by these equations but also a creation of new physical systems. To demonstrate the prediction, the required generalized Sundman transformation is first recalled (section 2) and applied to the general second order linear equation for establishing [4] the general class of mixed Liénard type differential equations (section 3) from which, secondly, the general class of quadratic Liénard type equations [4] is deduced with illustrative examples (section 4) and finally the trigonometric functions are highlighted as exact periodic solutions to the Painlevé-Gambier XVIII equation and other quadratic Liénard type equations (section 5) and a conclusion for the work is addressed.

2- Sundman transformation

In this section the generalized Sundman transformation required to demonstrate the preceding prediction is recalled. Such a transformation is a non-local transformation which applies to map in general any second order non-linear ordinary differential equation into a second order linear ordinary differential equation [4] to find closed-form solutions. However, given a general second order linear ordinary differential equation, the generalized Sundman transformation, conversely, may be applied for the investigation of problem of detecting general classes of second order non-linear ordinary differential equation exactly integrable [4]. This formalism has been applied in [4-5] to deduce mainly a class of quadratic Liénard type dissipative differentialequations having trigonometric functions as exact periodic solutions. Consider now the general second order linear differential equation is of the form

$$y'' + b y' + a^2 y = 0 \quad (1)$$

where prime means differentiation with respect to τ, a and b are arbitrary parameters and the more generalized Sundman transformation [4]

$$y(\tau) = F(t,x), \quad d\tau = G(t,x)dt, \quad G(t,x)\frac{\partial F(t,x)}{\partial x} \neq 0$$

such that

$$F(t,x) = \int g(x)^l\, dx, \quad G(t,x) = \exp(\gamma\, \varphi(x)) \quad (2)$$

where the exponents l and γ are arbitrary parameters, $g(x) \neq 0$, and $\varphi(x)$ are arbitrary functions of x. So, the application of (2) to (1) may give the desired general class of mixed Liénard type differential equations [4]

3- General class of mixed Liénard type equations

By application of the generalized Sundman transformation (2), the general second order linear ordinary differential equation (1) may be mapped onto the general class of mixed Liénard type equations [4]

$$\ddot{x} + \left(l\,\frac{g'(x)}{g(x)} - \gamma\, \varphi'(x) \right)\dot{x}^2 + b\,\dot{x}\exp(\gamma\, \varphi(x)) + a^2\, \frac{\exp(2\gamma\, \varphi(x))\int g(x)^l\, dx}{g(x)^l} = 0 \quad (3)$$

By application of $l = \gamma$, $\varphi(x) = \ln(g(x))$, the equation (3) reduces to

$$\ddot{x} + b\,\dot{x}\, g(x)^l + a^2\, \frac{g(x)^{2l}\int g(x)^l\, dx}{g(x)^l} = 0 \quad (4)$$

where $\ln$ designates the natural logarithm.

For $l=1$, $b=1$, and $a^2=\frac{2}{9}$, Eq. (4) reduces to the Musielak equation [6]

$$\ddot{x}+g(x)\,\dot{x}+\frac{2}{9}g(x)\int g(x)dx=0 \qquad (5)$$

The choice $\varphi(x)=\ln\left(f(x)\right)$gives as equation

$$\ddot{x}+\left(l\,\frac{g'(x)}{g(x)}-\gamma\,\frac{f'(x)}{f(x)}\right)\dot{x}^2+b\,\dot{x}\,f^{\gamma}(x)+a^2\,\frac{f(x)^{2\gamma}\int g(x)^l\,dx}{g(x)^l}=0 \qquad (6)$$

For $l=\gamma=1$, $g(x)=f(x)=u'(x)$ $,b=\frac{3}{2}$, and $a^2=\frac{1}{2}$, (6) becomes the physically interesting equation studied in [5]

$$\ddot{x}+\frac{3}{2}\dot{x}u'(x)+\frac{1}{2}u'(x)u(x)=0 \qquad (7)$$

where $u(x)$ is an arbitrary function such that by making $u(x)=kx^2$, (7) reduces to the well known modified Emden equation

$$\ddot{x}+3k\,x\dot{x}+k^2x^3=0 \qquad (8)$$

which is widely studied in the literature. However this equation may directly be obtained from (6) by putting $l=\gamma=1$, $g(x)=f(x)=x$, $b=3k$ and $a=k$.

Now, from (3), one may deduce the general class of quadratic Liénard type differential equations from which the class of quadratic Liénard type equations having trigonometric functions as exact periodic solutions is highlighted[4-5].

4- General class of quadratic Liénard type equations

4-1 Derivation of the equation

The parametric choice $b=0$ in (3) yields

$$\ddot{x}+\left[l\,\frac{g'(x)}{g(x)}-\gamma\,\varphi'(x)\right]\dot{x}^2+\frac{a^2e^{2\gamma\varphi(x)}\int g(x)^l\,dx}{g(x)^l}=0 \qquad (9)$$

The equation (9) represents the required general class quadratic Liénard type dissipative differential equations. One may notice that a judicious parametric choice as well as a judicious selection of function $g(x)$ and $\varphi(x)$ may lead to physically interesting non-linear oscillator equations. An interesting case of (9) may be, for $\varphi(x)=\ln\left(f(x)\right)$, obtained as

$$\ddot{x}+\left(l\frac{g'(x)}{g(x)}-\gamma\frac{f'(x)}{f(x)}\right)\dot{x}^2+\frac{a^2 f(x)^{2\gamma}\int g(x)^l\,dx}{g(x)^l}=0 \qquad (10)$$

So with that some examples may be given to illustrate moreover the high mathematical significance of the work developed in this paper.

4-2Examples

Illustrative examples related to (10) are given in this section.

4-2-1 $l=\frac{1}{2}$, $\gamma=1$

The parametric choice $l=\frac{1}{2}$ and $\gamma=1$ in (10) yields

$$\ddot{x}+\frac{1}{2}\left(\frac{g'(x)}{g(x)}-2\frac{f'(x)}{f(x)}\right)\dot{x}^2+\frac{a^2 f(x)^2\left(g(x)\right)^{\frac{1}{2}}\int\left(g(x)\right)^{\frac{1}{2}}dx}{g(x)}=0 \qquad (11)$$

This case corresponds to the class of quadratic Liénard type differential equations constructed in [3]. Indeed, the substitution of $l=\frac{1}{2}$, $\gamma=1$, and $\varphi(x)=\ln\left(f(x)\right)$, into the non-local transformation (2), yields the generalized Sundman transformation introduced in [3] to build the generalized position-dependent mass Euler-Lagrange equation (Eq.(12) of [3]) identical to (11). The equation (11) is shown in [3] to include as special cases several interesting position-dependent mass non-linear oscillator equations like the celebrated Mathews-Lakshmanan equations, the quadratic Morse type equation, etc. It is therefore no longer necessary to again carry out these calculations to prove the usefulness of the generalized Sundman transformation (2) used in this paper. Let us consider, nevertheless, other interesting illustrative examples in addition to those mentioned in [3].

4-2-2 Other cases $\gamma\neq 0$, and $l\neq 0$

Making now $f(x)=x^2$, and $g(x)=x$, (10) becomes

$$\ddot{x}+\left(l-2\gamma\right)\frac{\dot{x}^2}{x}+\frac{a^2}{l+1}x^{4\gamma+1}=0 \qquad (12)$$

The use of non-local transformation (2) leads to

$$y(\tau)=\frac{1}{l+1}x^{l+1},\qquad d\tau=x^{2\gamma}dt,\quad l\neq -1 \quad (13)$$

that is

$$x(t)=\left[(l+1)A_0\right]^{\frac{1}{l+1}}\sin^{\frac{1}{l+1}}\left(a\phi(t)+\alpha\right) \qquad (14)$$

where the function $\tau=\phi(t)$ satisfies

$$\left[(l+1)A_0\right]^{\frac{2\gamma}{l+1}}\left(t+K\right)=\int\frac{d\phi(t)}{\sin^{\frac{2\gamma}{l+1}}\left(a\phi(t)+\alpha\right)} \qquad (15)$$

with K an integration constant and

$$y(\tau)=A_0\sin\left(a\tau+\alpha\right) \qquad (16)$$

is the solution to (1)where $b=0$.

4-2-2-1 $\gamma\neq 0$, $l=-\frac{2}{3}$

The equation (12) reduces to

$$\ddot{x}-\left(\frac{2}{3}+2\gamma\right)\frac{\dot{x}^2}{x}+3a^2x^{4\gamma+1}=0 \quad (17)$$

such that (15) becomes

$$\left(\frac{A_0}{3}\right)^{6\gamma}\left(t+K\right)=\int\frac{d\phi(t)}{\sin^{6\gamma}\left(a\phi(t)+\alpha\right)} \qquad (18)$$

In this perspective the solution $x(t)$ reads

$$x(t)=\frac{A_0^3}{9}\sin^3\left[a\phi(t)+\alpha\right] \qquad (19)$$

where $\phi(t)$ verifies (18). For an integer γ or $\gamma=\frac{1}{6}$, (18) may be easily computed and the solution $x(t)$ may be expressed in terms of elementary functions.

Conversely, for a non-integer γ, (18) will be evaluated in terms of special functions, and the argument $a\phi(t)+\alpha$, becomes a complicated function of t. So, for example, $\gamma=\frac{1}{6}$, (18) gives

$$\frac{A_0}{3}(t+K)=\frac{1}{a}\ln\left(tg\,\frac{a\phi(t)+\alpha}{2}\right)(20)$$

that is [8]

$$\exp\left(\frac{aA_0}{3}(t+K)\right)=tg\,\frac{a\phi(t)+\alpha}{2} \qquad (21)$$

and the solution $x(t)$ may be written as

$$x(t)=\frac{A_0^3}{9}\sin^3\left[2tg^{-1}\left(\exp\left(\frac{aA_0}{3}(t+K)\right)\right)\right] \qquad (22)$$

4-2-2-2 $\gamma=-\frac{3}{2}$**,** $l=-2$

The parametric choice $\gamma=-\frac{3}{2}$, and $l=-2$, according to (12), leads to the equation

$$\ddot{x}+\frac{\dot{x}^2}{x}-\frac{a^2}{x^5}=0 \qquad (23)$$

After (13) the solution $x(t)$ takes the form

$$x(t)=-\frac{1}{A_0\sin(a\phi(t)+\alpha)} \qquad (24)$$

where $\phi(t)$ satisfies

$$-A_0^3(t+K)=\int\frac{d\phi(t)}{\sin^3(a\phi(t)+\alpha)} \qquad (25)$$

that is [8]

$$-\frac{\cos(a\phi(t)+\alpha)}{2\,a\sin^2(a\phi(t)+\alpha)}+\frac{1}{2a}\ln\left[tg\left(\frac{a\phi(t)+\alpha}{2}\right)\right]=-A_0^3(t+K) \qquad (26)$$

The equation (23) admits a position-dependent mass dynamics so that the mass $m(x)=m_0x^2$ and the potential energy $V(x)=\frac{m_0a^2}{2}\frac{1}{x^2}$, where m_0 is an arbitrary constant. Such a potential is the so-called singular inverse square potential and has been widely studied in the quantum mechanics. However it is for the first time the existence of such a potential has been mathematically established. This highlights the physical importance of (23). Let us consider now other interesting classes of quadratic Liénard type differential equations obtained for $l=0$ or $\gamma=0$.

5- $\gamma=0$, and $l=1$

In this situation (9) reduces to, for $\gamma=0$

$$\ddot{x}+l\frac{g'(x)}{g(x)}\dot{x}^2+\frac{a^2\int(g(x))^l dx}{(g(x))^l}=0 \quad (27)$$

By application of $g(x)=h'(x)$, (27) becomes

$$\ddot{x}+l\frac{h''(x)}{h'(x)}\dot{x}^2+\frac{a^2\int(h'(x))^l dx}{(h'(x))^l}=0 \quad (28)$$

So, for $l=1$, (28) gives

$$\ddot{x}+\frac{h''(x)}{h'(x)}\dot{x}^2+\frac{\omega_0^2 h(x)}{h'(x)}=0 \quad (29)$$

where $a^2=\omega_0^2$. In [9] it has been analyzed that (29) admits the eight parameter Lie point symmetry $sl(3R)$ algebra and exhibits isochronous property whereas in [10], (29) was studied from quantum viewpoint. Let us take now into account the case $l=0$, which appears to be of high interest since it highlights the trigonometric functions as a class of exact periodic solutions to quadratic Liénard type equations.

6- Class of quadratic Liénard type equations having trigonometric functions as exact periodic solutions

For $l=0$, or $g(x)=1$, (9) reduces to [5]

$$\ddot{x}-\gamma\varphi'(x)\dot{x}^2+a^2 x e^{2\gamma\varphi(x)}=0 \quad (30)$$

According to the generalized Sundman transformation (2), and (16) the solution to (30) may read

$$x(t)=A_0\sin(a\phi(t)+\alpha) \quad (31)$$

where $\tau=\phi(t)$ obeys

$$d\phi(t) = \exp(\gamma\varphi(x))dt \tag{32}$$

that is

$$\exp(-\gamma\varphi(x))d\phi(t) = dt \tag{33}$$

The above shows clearly that the class of equations (30) is of high interest since it exhibits trigonometric functions with well known analytical properties and engineering applications as exact periodic solutions but with amplitude-dependent frequency characterizing nonlinear oscillator equations. So the problem of finding $x(t)$ reduces to solve equation (33) once the function $\varphi(x)$ and the parameter γ are defined. That being so, some illustrative examples are studied in this paragraph and in the section devoted to Painlevé-Gambier equations [11]. An interesting case is to consider $\varphi(x) = \ln(f(x))$ such that equation (30) becomes

$$\ddot{x} - \gamma \frac{f'(x)}{f(x)} \dot{x}^2 + a^2 x (f(x))^{2\gamma} = 0 \tag{34}$$

and (33) yields

$$dt = \frac{d\phi(t)}{(f(x))^{\gamma}} \tag{35}$$

6-1 $\gamma = 2$**, and** $\varphi(x) = \frac{1}{2}\ln(1+\mu x)$

For $\gamma = 2$,and $\varphi(x) = \frac{1}{2}\ln(1+\mu x)$, the equation (34) takes the form

$$\ddot{x} - \frac{\mu}{1+\mu x} \dot{x}^2 + a^2 x (1+\mu x)^2 = 0 \tag{36}$$

The equation (35) gives in this perspective

$$(t+K) = \int \frac{d\phi(t)}{1+\mu A_0 \sin(a\phi(t)+\alpha)} \tag{37}$$

that is [8]

$$a\phi(t) + \alpha = 2tg^{-1}\left[\sqrt{1-\mu A_0^2}\, tg\left(\frac{a(t+K)\sqrt{1-\mu A_0^2}}{2}\right) - \mu A_0\right] \tag{38}$$

so that the solution

$$x(t)=A_0\sin\left[2tg^{-1}\left[\sqrt{1-\mu A_0^2}\,tg\left(\frac{a(t+K)\sqrt{1-\mu A_0^2}}{2}\right)-\mu A_0\right]\right] \tag{39}$$

6-2 $\gamma=-1$**, and** $f(x)=\dfrac{1}{\sqrt{1+\mu x^2}}$

By applying $\gamma=-1$,and $f(x)=\dfrac{1}{\sqrt{1+\mu x^2}}$, (34) gives as equation

$$\ddot{x}-\frac{\mu x}{1+\mu x^2}\dot{x}^2+a^2x\left(1+\mu x^2\right)=0 \tag{40}$$

The equation (40) admits then as solution

$$x(t)=B_0\cos\left(a\phi(t)-\beta\right) \tag{41}$$

where B_0 and β are arbitrary parameters

such that

$$\left(t+K\right)=\int\frac{d\phi(t)}{\sqrt{1+\mu B_0\cos^2\left(a\phi(t)-\beta\right)}} \tag{42}$$

The evaluation of the integral of the right hand side $J=\int\dfrac{d\phi(t)}{\sqrt{1+\mu B_0\cos^2(a\phi(t)-\beta)}}$ depends on the value of μB_0^2. J may be computed as

$$J=\int\frac{d\phi(t)}{\sqrt{1+d\cos^2\left(a\phi(t)\right)+e\cos\left(a\phi(t)\right)\sin\left(a\phi(t)\right)+f\sin^2\left(a\phi(t)\right)}} \tag{43}$$

where $d=\mu B_0^2b^2,\quad f=\mu B_0^2c^2,\quad e=2bc\mu B_0^2,\quad b=\cos\beta\ and\ c=\sin\beta$.

By setting

$A=1+d,\ B=2e,\ C=2-2d+4f,\ D=-2e,\ E=1+d,\ and\quad u=tg\left(\dfrac{a\phi}{2}\right)$, the above integral becomes [8]

$$J=\frac{2}{a}\int\frac{du}{\sqrt{A+Bu+Cu^2-Du^3+Eu^4}} \tag{44}$$

which may be reduced to the elliptic integral of the first kind for some values of parameters $A, B, C,\ D$ and E. On the other hand, it is worth to note that for $\gamma = 1$, and the same function $f(x)$, (34) reduces to the equation of motion of a particle moving on a rotating parabola

$$\ddot{x} + \frac{\mu x}{1+\mu x^2}\dot{x}^2 + \frac{a^2 x}{1+\mu x^2} = 0 \tag{45}$$

which admits according to the present theory an exact trigonometric periodic solution but with amplitude-dependent frequency. However the study of this equation will be performed in a subsequent work. It is now interesting to show that the proposed theory of nonlinear differential equations may be also used to solve exactly some well knownPainlevé-Gambier equations [11]

7- Inverted Painlevé-Gambier equations

In this paragraph, solutions to some inverted Painlevé-Gambier equations are expressed as mentioned in the above.

7-1 Inverted Painlevé-Gambier XVIII equation

The Painlevé-Gambier XVIII equation may read [11]

$$\ddot{x} - \frac{1}{2}\frac{\dot{x}^2}{x} - 4x^2 = 0 \tag{46}$$

so its inverted version becomes

$$\ddot{x} - \frac{1}{2}\frac{\dot{x}^2}{x} + 4x^2 = 0 \tag{47}$$

which only differs from equation (46) by a sign. The equation (47) belongs to the class of quadratic Liénard type nonlinear differential equation represented by (30) under the considerations that $\gamma = \frac{1}{4}, \varphi(x) = \ln(x^2)$ and $a^2 = 4$. As a result the solution to (47) may be expressed following equation (31) as

$$x(t) = B_0 \cos(a\phi(t) - \beta) \tag{48}$$

such that $\phi(t)$ verifies

$$dt = B_0^{-\frac{1}{2}} \cos^{-\frac{1}{2}}(a\phi(t) - \beta)d\phi(t) \tag{49}$$

The integration of the right hand side of equation (49) may be evaluated as

$$J = B_0^{-\frac{1}{2}} \int \frac{d\phi(t)}{\sqrt{\cos(a\phi(t) - \beta)}} \tag{50}$$

that is [8]

$$J = \frac{2}{a\sqrt{B_0}} \int \frac{d\psi}{\sqrt{-1 + 2\cos^2\psi}} \tag{51}$$

where $a\phi = 2\psi + \beta$. Using the identity $\cos^2\psi = 1 - \sin^2\psi$, the integral J becomes

$$J = \frac{2}{a\sqrt{B_0}} \int \frac{d\psi}{\sqrt{1 - 2\sin^2\psi}} \tag{52}$$

that is

$$J = \frac{1}{a}\sqrt{\frac{2}{B_0}} F\left(\delta, \frac{\sqrt{2}}{2}\right) \tag{53}$$

where $\delta = \arcsin\left(\sqrt{2}\sin\psi\right)$, and $F(\theta, k)$ is the elliptic integral of the first kind. So, one may find $a\phi(t)$ such that ψ is given by

$$\cos\left[\sin^{-1}(\sqrt{2}\sin\psi)\right] = cn\left(\frac{a\sqrt{2B_0}}{2}t, \frac{\sqrt{2}}{2}\right) \tag{54}$$

that is, making $a = 2$, in (54), one may recover the explicit solution (48), where $cn(z, k)$ is the Jacobi elliptic function. The solution (48) would allow in principle, to compute the exact solution of the initial Painlevé-Gambier XVIII equation as a trigonometric function by replacing the parameter a by ia, where i is the purely imaginary number. In other words, the solution (48) gives the exact solution to the Painlevé-Gambier XVIII equation (46) replacing (54) by

$$\cos\left[\sin^{-1}(\sqrt{2}\sin\psi)\right] = cn\left(\frac{ia\sqrt{2B_0}}{2}t, \frac{\sqrt{2}}{2}\right) \tag{55}$$

where $a = 2$, knowing the identity [8]

$$cn\left(\frac{ia\sqrt{2B_0}}{2}t, \frac{\sqrt{2}}{2}\right) = \frac{1}{cn\left(\frac{a\sqrt{2B_0}}{2}t, \frac{\sqrt{2}}{2}\right)} \tag{56}$$

7-2 Inverted Painlevé- Gambier XXXII equation

The Painlevé-Gambier XXXII equation, after [11] is written as

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}+\frac{1}{2x}=0 \tag{57}$$

The inverted version may then be written in the form

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}-\frac{2a^2}{x}=0 \tag{58}$$

where $a^2=\frac{1}{4}$. The equation (58) may be obtained by substituting $\gamma=-\frac{1}{2}$

and $l=-\frac{3}{2}$ in (12). In this perspective the solution takes the form following (14)

$$x(t)=\frac{4}{A_0^2\sin^2\left(a\phi(t)+\alpha\right)} \tag{59}$$

where $\phi(t)$ obeys

$$\frac{A_0^2}{4}\left(t+K\right)=\int\frac{d\phi(t)}{\sin^2\left(a\phi(t)+\alpha\right)} \tag{60}$$

or

$$\frac{A_0^2}{4}\left(t+K\right)=-\frac{1}{a}\cot\left(a\phi(t)+\alpha\right) \tag{61}$$

Therefore the preceding solution $x(t)$ becomes

$$x(t)=\frac{4}{A_0^2\sin^2\left[\cot^{-1}\left(-\frac{aA_0^2}{4}t+\frac{aA_0^2}{4}K\right)\right]} \tag{62}$$

Here also, the preceding remark on the solution to the Painlevé-Gambier XVIII equation is valuable for the equation (57) to find its exact periodic solution.

7-3 Inverted Painlevé-Gambier XXI equation

After [11] the Painlevé-Gambier XXI equation reads

$$\ddot{x} - \frac{3}{4}\frac{\dot{x}^2}{x} + 1 = 0 \quad (63)$$

In this regard, the inverted equation may be written as

$$\ddot{x} - \frac{3}{4}\frac{\dot{x}^2}{x} - 1 = 0 \quad (64)$$

which may be obtained from equation (12) by setting $\gamma = -\frac{1}{4}$, $l = -\frac{5}{4}$, and $a^2 = \frac{1}{4}$.Thus the solution $x(t)$ becomes

$$x(t) = \frac{256}{A_0^4 \sin^4\left(a\phi(t) + \alpha\right)} \quad (65)$$

where $\phi(t)$ satisfies

$$\frac{A_0^2}{16}\left(t + K\right) = \int \frac{d\phi(t)}{\sin^2\left(a\phi(t) + \alpha\right)} \quad (66)$$

So, equation (65) may be written in the form

$$x(t) = \frac{256}{A_0^4 \sin^4\left[\cot^{-1}\left(-\frac{aA_0^2}{16}\left(t + K\right)\right)\right]} \quad (67)$$

The preceding remark is also valuable for the Painlevé-Gambier XXI equation to compute its exact periodic solution. Now, taking into account these illustrative examples a conclusion may be addressed for the work.

Concluding remarks

If the problem of determining approximate trigonometric periodic solutions to non-linear differential equations has been more or less solved, the problem of expressing exact periodic solutions as a trigonometric function to non-linear dissipative differential equations is yet an active mathematical research field. In such a situation the Liénard type non-linear dissipative equations are subject of an intensive study from mathematical viewpoint as well as physical standpoint. Different linearizing transformations with different complexities have been used to construct exact periodic solutions to Liénard nonlinear differential equations. In particular the generalized Sundman transformation has been widely used to establish exact solutions to diverse types of Liénard differential equations. Conversely such a transformation may also be used to detect diverse types of classes ofLiénard differential equations having exact analytical solutions. In this perspective, a generalized Sundman transformation is used in this work to highlight a general class of quadratic Liénard type non-linear differential equations whose exact periodic solutions are trigonometric functions. By doing so, it appears that the non-linear differential equation theory proposed may be used to exactly solve a number of physically important mixed and quadratic Liénard type equations as well as to generate new generalized non-linear differential equations of Liénard type for the mathematical modeling and creation of new physical systems characterized in particular by a harmonic potential and position-dependent mass.

References

[1]BijanBagchi, Supratim Das, Samiran Ghosh, SwarupPoria, Nonlinear dynamics of a position-dependent mass driven Duffing-type oscillator,arXiv: 1212.3159 v2 (2013).

[2]Y.J.F. Kpomahou, M.D. Monsia, Hardening nonlinearity effects on forced vibration of viscoelastic.

dynamical systems to external step perturbation field, Int. J. Adv.Appl.Math. and Mech.3(2)(2015)(16-32).

[3]Omar Mustafa, Position-dependent mass Lagrangians: nonlocal transformations, Euler-Lagrange invariance and exact solvability,arXiv: 1411.4405v3 (2005).

[4]M.D. Monsia, J. Akande, D. K. K. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou, A class of position-dependent mass Liénard differential equations via a general nonlocal transformation, viXra: 1608.0226v1 (2016).

[5]M.D. Monsia, J. Akande, D. K. K. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou, Exact analytical periodic solutions with sinusoidal form to a class of position-dependent mass Liénard-Type oscillator equations, viXra:1608.0368v1 (2016).

[6]Z.E. Musielak, Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients, J. Phys. A: Math. Theor. 41 (2008) 055205 (17pp).

[7]Jose F. Carinena, Manuel F. Ranada, Mariano Santander, Two important examples of nonlinear oscillators, arXiv: 0505028v1(2005).

[8]I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Seventh edition, 2007.

[9]Ajey K. Tiwari, S.N.Pandey, M. Senthilvelan, Classification of Lie point symmetries for quadratic Liénard type equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$, Journal of Mathematcal Physics 54, 053506 (19pp) (2013).

[10]G. Gubbbiotti, M.C.Nucci, Quantization of quadratic Liénard-type equations by preserving Noether symmetries, arXiv: 1406.0192v1 (2014).

[11] E.L. Ince Ordinary differential equations, Dover, New York. 1956.