An Engineer's approach to the Riemann Hypothesis and why it is true

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Abstract: I am an Engineer very passionate with Prime numbers. They are the atoms of mathematics and mathematics is needed to make sense of the real world. Finding the Prime number structure and eventually being able to crack their code is the ultimate goal in what is called Number Theory. From the evolution of species to cryptography, Nature finds help in Prime numbers.

One of the most important advance in the study of Prime numbers was the paper by Bernhard Riemann in November 1859 called "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (On the number of primes less than a given quantity).

In this paper, Riemann gave a formula for the number of primes less than x in terms the integral of 1/log(x) and the roots (zeros) of the zeta function, defined by:

(1) $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$

The Zeta function, ζ(z), is a function of a complex variable z that analytically continues the Dirichlet series.

Riemann also formulated a conjecture about the location of these zeros, which fall into two classes: the "trivial zeros" -2, -4, -6, etc., and those whose real part lies between 0 and 1. Riemann's conjecture Riemann hypothesis [RH] was formulated as this:

[RH] The real part of every non-trivial zero z* of the Riemann Zeta function is 1/2.

Thus, if the hypothesis is correct, all the non-trivial zeros lie on the critical line consisting of the complex numbers 1/2 + iß, where ß is a real number and i is the imaginary unit.

In this paper, we will analyze the Riemann Zeta function and provide an analytical/geometrical proof of the Riemann Hypothesis. The proof will be based on the fact that if we decompose the ζ(z) in a difference of two functions, both functions need to be equal when ζ(z)=0, so their distance to the origin or modulus must be equal and we will prove that this can only happen when Re(z)=1/2 for certain values of Im(z).

We will also prove that all non-trivial zeros of ζ(z) in the form z=1/2+iß have all ß related by an algebraic expression. They are all connected and not independent.

Finally, we will show that as a consequence of this connection of all ß, the harmonic function Hn can be expressed as a function of each ß zero of ζ(z) with infinite representations.

We will use mathematical and computational methods available for engineers.

A. Nomenclature and conventions

- $\zeta(z)=\lim_{n\to\infty}\sum_{k=1}^{n} k^{-z}$ is the Zeta function of Riemann in the complex plane
- We will name z* any non-trivial solution of the Zeta function verifying that ζ(z*)=0+i0. We always mean by default a non-trivial zero of ζ(z) whenever we mention a zero of ζ(z).
- R(n) is the n^{th} zero of the Riemann function in the critical line x=1/2 in *C*
- α=Re(z) is the real part of z
- ß=Im(z) is the imaginary part of z
- If z=α+iß, we define Modulus(z)= $|z|^2 = \alpha^2+ß^2$
- For notation simplification, all modulus of complex functions in this paper, such as $|\zeta(z)|^2$, $|x(z)|^2$ and $|y(z)|^2$, that will be represented in the form of infinite series when $n \to \infty$, must be understood as functions in R over the variables α, ß, n.

B. The function $\zeta(s)$ in the Real line

As defined earlier:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This series converges for s>1 to the following values:

s	ζ(s)	Known ζ(s) representations over π
2	1.6449179	$\pi^2/6$
3	1.2020569	
4	1.0823232	$\pi^4/90$
5	1.0369278	
6	1.0173431	$\pi^6/945$
7	1.0083493	
8	1.0040774	$\pi^8/9450$
9	1.0020084	
10	1.0009946	$\pi^{10}/93555$

Table 1. Values of ζ(s)

What happens with the odd values of s? Do they have also a representation in the form ?

$$\zeta(s) = \frac{\pi^s}{k}$$

To solve this question, let's analyze the behavior of the rate of growth of the ζ(s) function. It is easy to calculate that:

$$\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)+1}\right)^{1/s} = 1$$

And:

$$\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)-1}\right)^{1/s} = 2$$

Based on this expression, we can say that for s sufficiently large, we can represent $\zeta(s)$ as a multiple of π^s:

$$\zeta(s) = \frac{\pi^s}{K_s} \qquad \text{with } K_s = (2^s - 1) * \frac{\pi^s}{2^s}$$

with a very good approximation given by:

$$K_s^* = int\left((2^s - 1) * \frac{\pi^s}{2^s}\right) - 1 \qquad \text{where int(k) is the integer part of k.}$$

The error between the K_s^* calculated and K_s actual is very small for s>4.

Some calculated values of K_s^* calculated and K_s actual:

s	Calculated	Actual
2	6.0	6.0
3	26.0	25.8
4	90.0	90.0
5	295.0	295.1
6	945.0	945.0
7	2,995.0	2,995.3
8	9,450.0	9,450.0
9	29,749.0	29,749.4
10	93,555.0	93,555.0
11	294,059.0	294,058.7
12	924,042.0	924,041.8
13	2,903,321.0	2,903,321.0
14	9,121,613.0	9,121,612.5
15	28,657,270.0	28,657,269.4
16	90,030,846.0	90,030,845.0

Table 2. Values of K_s^* calculated and K_s actual

(2) A consequence of this is that we can use the following approximation for increasing values of s:

$$\zeta(s)=\frac{2^s}{2^{s-1}}$$

The error of using this approximation is less than 1% for s>5.

(3) The $\zeta(s)$ has multiple representations linking it to other Number Theory concepts. We will give a representative list to place the $\zeta(s)$ in a wider perspective:

(3.1) $\zeta(s)$ and the Bernoulli Numbers:

$$K_s \quad \zeta(s) = B_s \pi^s$$

Where K_s as defined in the previous table and B_s the Bernoulli numbers

(3.2) Integral representation of $\zeta(s)$ as a function of the Gamma function:

$$\zeta(x) \equiv \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} \, du,$$

(4) We can see that $\zeta(s)$ diverges for s=1 as $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(5) An interesting relationship involving $\zeta(1)$ delivers the Euler-Mascheroni constant:

$$\zeta(1) - \lim_{n\to\infty} \left(\frac{1}{k}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln(k) = \gamma = 0.57721566490153 \ldots$$

(6) $\zeta(s)$ is intimately related to the distribution of prime numbers. This was first reveal by Euler through the identity:

$$\zeta(\text{s}) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime}^{\infty} \frac{1}{1-\frac{1}{p^s}}$$

The infinite sum over all natural numbers equal the infinite product over all primes. This is an amazing revelation and provides a look into the distribution of primes.

(7) We could ask ourselves what would change in Euler's expression if we did the infinite sum also on the prime numbers instead of all the natural numbers. As the number of primes is a fraction of the naturals, this infinite sum should be smaller.

The most interesting finding is the way this partial sum over primes evolves as s increases.

Let's call:

$$\zeta^p(s) = \sum_{p\ prime}^{\infty} \frac{1}{p^s}$$

Let $\theta(\text{s}) = \zeta^n(\text{s}) \,/\, \zeta^p(\text{s})$ with $\zeta^n(\text{s})$ the regular zeta function as a sum over n naturals and $\zeta^p(\text{s})$ the zeta function as a sum over the first n primes, then:

$$\lim_{s\to\infty} \frac{\theta(\text{s})}{\theta(\text{s}-1)} = 2$$

The error $\frac{\theta(\text{s})}{2*\theta(\text{s}-1)} - 1$ is less than 1% for s>9

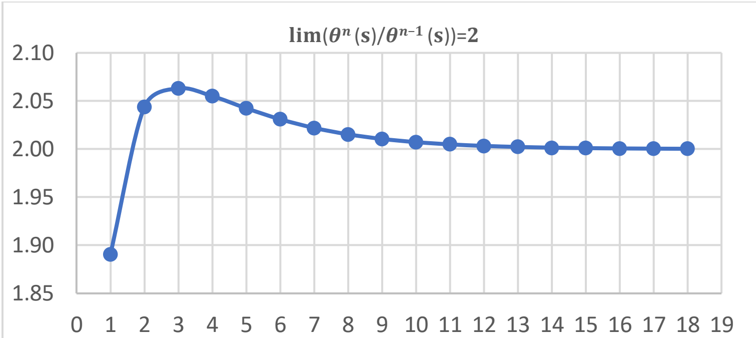

Figure 1.

We know that $\lim_{s\to\infty} \frac{\zeta^n(s)}{\zeta^n(s-1)} = 1$ so we can conclude that:

$$\lim_{s\to\infty} \frac{\zeta^p(s)}{\zeta^p(s-1)} = \frac{1}{2}$$

The sum over all primes of $\zeta^p(s)$ decreases by 50% for each unit increment of s. The analysis of the infinite sums and products over naturals and primes will probably reveal other interesting relationships.

C. The ζ(z) in the Complex plane

As defined above, the zeta function ζ(z) with z=α+iß a complex number is defined for Re(z)>1. However, ζ(z) has a unique analytic continuation to the entire complex plane, excluding the point z=1. This analytic continuation let us work with ζ(z) in the entire complex plane.

(8) $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$

The following identity by Euler still holds:

(9) $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p\ prime}^{\infty} \frac{1}{1-\frac{1}{p^z}}$

The Riemann zeta function can also be defined in the complex plane by the contour integral

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \oint_{\gamma} \frac{u^{z-1}}{e^{-u}-1}\, du$$ for all $z \neq 1$

As stated before, there are two types of zeros in the complex plane:

Trivial zeros	z^T= -2k	k∈ *N*
Non-Trivial zeros	z*= ½+iß	Riemann Hypothesis

D. Decomposition of ζ(z) in the Complex plane into two functions such that ζ(z)=x(z) – y(z)

The goal of this decomposition will help prove the Riemann Hypothesis. The basic idea is that if ζ(z)=x(z) – y(z) and ζ(z)=0 then we can check if x(z) = y(z) and for what values of z=z* that identity is true. Eventually we can prove that for non-trivial zeros of ζ(z) the value of Re(z) must be ½.

D.1. The τ -transformation in R

To define x(z) and y(z) we will define a transformation given by:

(10) $\tau(f(n)) = \sum_1^n f(k) - \int^n f(x)\,dx$ with $\int^n f(x) = \int \; f(x)\,dx$ at x=n

With f(x) continuous and $\lim_{x\to\infty} f(x)= 0$ for x>0

If we apply this transformation in R to f(x)=1/x $(x > 0 \in R)$

(11) $\tau(\mathrm{f(n)}) = \sum_{k=1}^n (1/k - \ln(n))$

This is a known expression we showed before with a known limit for $n \to \infty$:

(12) $\lim_{n\to\infty} \tau(\mathrm{f(n)}) = \gamma = 0.57721566490153\ldots$

Which is the Euler-Mascheroni constant.

If we apply τ-transformation to f(x) = $x^{-\alpha}$ $\alpha > 1$ $(x > 0 \in R)$

(13) $\tau(f(n)) = \sum_{k=1}^n 1/k^\alpha$- $\int^n x^{-\alpha}\ dx$

(14) $\tau(f(n)) = \sum_{k=1}^n 1/k^\alpha$- $n^{1-\alpha}/(1-\alpha)$

We can calculate values of $\tau\left(f(n \to \infty)\right)$:

α	$\tau(f(n \to \infty)\,\vert\alpha)$	
2	1.6449….	= ζ(2)
3	1.2020…	= ζ(3)
4	1.0823…	= ζ(4)
6	1.0173…	= ζ(6)

Table 3. Values of $\tau(\boldsymbol{f}(\boldsymbol{x}))$ *for f(x) =* $x^{-\alpha}$

We have added to the table the fact that these values are the values of the ζ(α) for α∈ R. We can conclude that $\lim_{n\to\infty} \tau(\boldsymbol{f}(\boldsymbol{n}))$ = ζ(α) Where *ζ(α)* is the Zeta function of Riemann in R defined for α>1.

D.2. The τ −transformation in C

Let's now apply the same methodology to calculate the $\boldsymbol{\tau}$ transformation of the function: f(x) = x^{-z} for z = α+iß a complex number and α≥0.

Applying τ-transformation to f(z) we obtain:

(15) $\tau\big(f(n,z)\big) = \sum_{k=1}^{n} k^{-z}$ - $\int^{n} x^{-z}$ dx

Applying the exponential expression to the power of k∈ R to a complex number z∈ C:

(16) $k^{-z} \ = k^{\alpha}$ [cos(ß $*$ ln(k)) + i (sin(ß $*$ ln(k))]

And:

(17) $\int^{n} x^{-z}\, dx = \frac{1}{(1-\alpha)-iß} n^{(1-\alpha)-ßi}$

Or:

(18) $\int^{n} x^{-z}\, dx = [n^{(1-\alpha)}$ [cos(ß $*$ $ln(n) - i\, sin$(ß $*$ $ln(n)$]] $*$ $\frac{[(1-\alpha)+iß]}{[(1-\alpha)^2+ß^2]}$

We can now express the real and imaginary components of (16) as:

(19) Re($\tau\big(f(n,z)\big)$) = $\sum_{k=1}^{n} k^{-\alpha}$ (cos(ß $*$ ln(k))-
$- n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]}$ [(1-α)*cos(ß*ln(n))+ß* sin(ß*ln(n))]

(20) Im($\tau\big(f(n,z)\big)$) = $\sum_{k=1}^{n} k^{-\alpha}$ (sin(ß $*$ ln(k))+
$+ n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))]

We can calculate the following table:

(21)

z= α+iß	$\tau\big(f(z)\big)$	ζ(z)
(2,0)	1.644934 + i*0	ζ(2,0)
(3,0)	1.202057 + i*0	ζ(3,0)
(1, 1)	0.582096 + i* 0.9269	ζ(1,1)
(1/2, 14.134725…)	0 + i*0	Zero of the ζ function

Table 4. Values of $\tau\big(f(z)\big)$ for f(x) = x^{-z}

We can see that if z =*α+iβ* ∈ C with α>0, *then* $\lim_{n\to\infty} \tau\big(f(z)\big)$= ζ($z$) when Re(z)= α≥0

D.3. Definition of x(z) and y(z) such that ζ(z) = x(z) – y(z) for z∈ C *and* Re(z)= α≥0

From (19) and (20) let's define:

(22) x1(z) = $\sum_{k=1}^{n} k^{-\alpha}$ $(\cos(ß * \ln(k))$

(23) x2(z) = [$n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [(1-α)*cos(ß*ln(n))+ß*sin(ß*ln(n))]]

(24) y1(z) = $\sum_{k=1}^{n} k^{-\alpha}$ $(\sin(ß * \ln(k))$

(25) y2(z) = - $n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))]

And let's call:

(26) x(z) = x1(z) +i* y1(z)

(27) y(z) = x2(z) +i* y2(z)

In general, we can now express that any solution in Z of ζ(z) as:

(28) ζ(z) = [x1(z)-x2(z)] + i * [y1(z)-y2(z)]

Or

(29) ζ(z) = x(z) - y(z)

Where:

(30) x(z) = $\sum_{k=1}^{n} k^{-\alpha}$ $(\cos(ß * \ln(k))$ + i* $\sum_{k=1}^{n} k^{-\alpha}$ $(\sin(ß * \ln(k))$

(31) y(z)= [($n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [(1-α)*cos(ß*ln(n))+ß*sin(ß*ln(n))])
+ i ($n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))])]

Let's observe that if z*= α+iß as a non-trivial zero of ζ(z), then:

(32) x1(z) = x2(z) and y1(z) = y2(z)

Re(ζ(z*))= 0 and Im(ζ(z*))= 0 imply:

(33) $\sum_{k=1}^{n} k^{-\alpha}$ $(\cos(ß * \ln(k))$= $n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [(1-α)*cos(ß*ln(n))+ß*sin(ß*ln(n))]

(34) $\sum_{k=1}^{n} k^{-\alpha}$ $(\sin(ß * \ln(k))$= - $n^{(1-\alpha)}$ $\frac{1}{[(1-\alpha)^2+ß^2]}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))]

Let's represent graphically the following four wave functions x1(z), -x2(z), y1(z), -y2(z) when $n \to \infty$ for different values of z=α+iß :

z= ½ + i*5

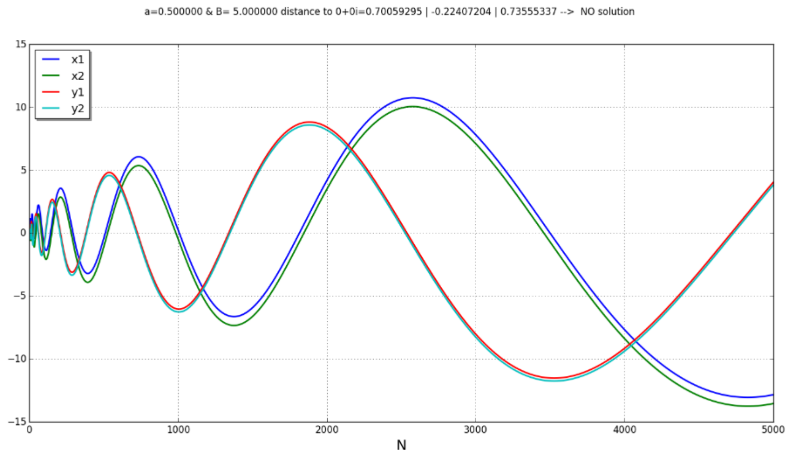

Figure 2

z=1+ i*2

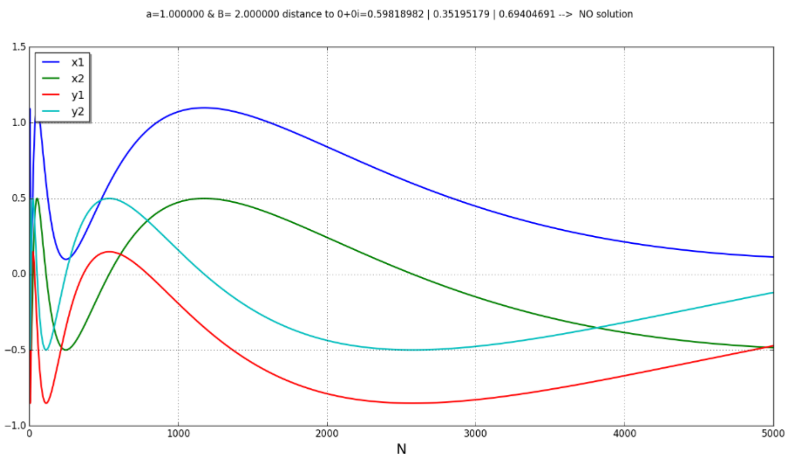

Figure 3

z= ½ + 14.1347251417346 *i

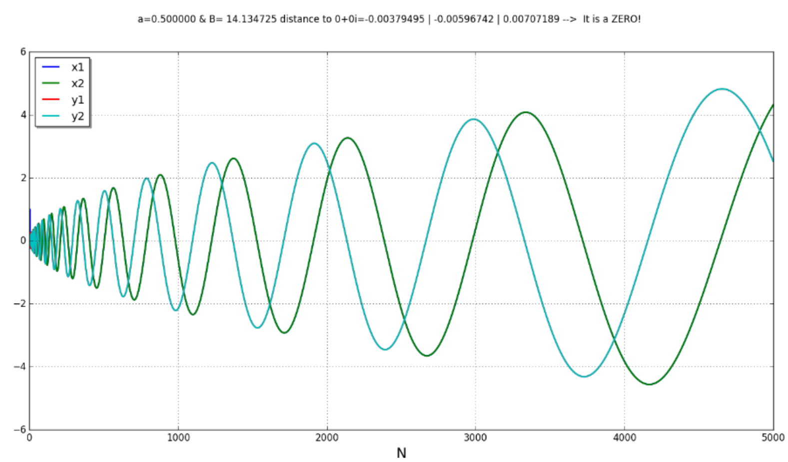

Figure 4

The observations that can be drawn from these charts for multiple iterations over α,ß are :

- The graphs are symmetrical with respect of a certain horizontal axis if α<1
- The graphs, as wave functions, evolve around the x axis if α=1/2 as all four partial functions [24-27] are of the type f(x)=f($\sqrt{n}$):

  f(n) = A$\sqrt{n}$ $(B * \cos(\ln(Cn) \pm D * \sin(\ln(Cn))$ with A, B, C, D =constant

- If z* is a known non-trivial zero of ζ(z), such as R(1)=1/2+i*14.134725… the 4 graphs collapse into 2, as in figure 4. We just checked the obvious evidence that for z* non-trivial zeros of ζ(z) the following obvious equalities happen:
-
    - x1(z)= - x2(z) and
    - y1(z) = - y2(z)

Let's calculate and plot the distance of ζ(z) to the origin given by its modulus defined for z=a+ib as:

(35) $| z |^2 = x^2 + b^2$

The modulus is the distance of z to the origin.

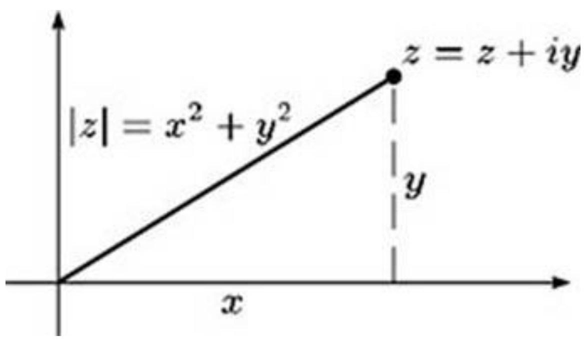

Figure 5. Modulus of a complex number

The modulus of ζ(z) will be given by:

(36) $| \zeta(z) |^2 = [x1(z)) - x2(z)]^2 + [(y1(z) - y2(z)]^2$

This modulus must be zero when z=z* a zero of ζ(z).

Let's see this fact in two different graphs. We will add the line $| \zeta(z)|^2$ to the previous graphs for x1(z), - x2(z), y1(z), - y2(z) for different values of z.

We can observe in Figure 8 graphically in the figure that, if z* is a known non-trivial zero value of the ζ(z) function then obviously $|\zeta (z^*)|^2$=0 when $n \to \infty$

z= ½ + i*5 $|\zeta(z)|^2 > 0.7005\ldots$ for n>5000 and growing as $n \to \infty$

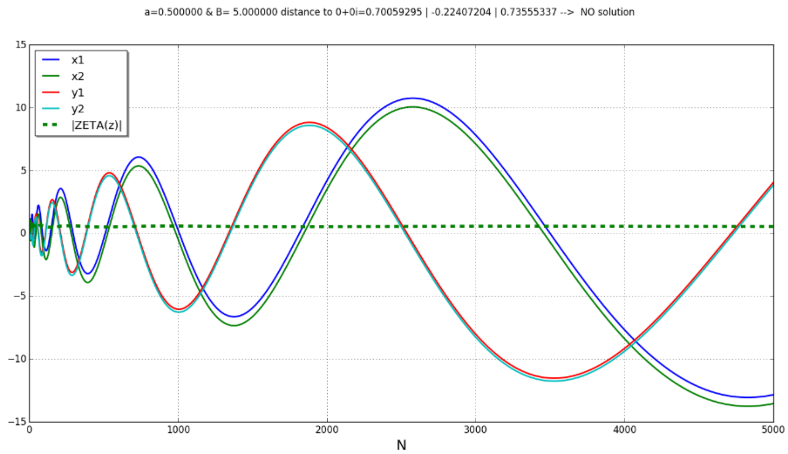

Figure 6

z= 1 + i*2 $|\zeta(z)|^2 > 0.5981\ldots$ for n>5000 $n \to \infty$

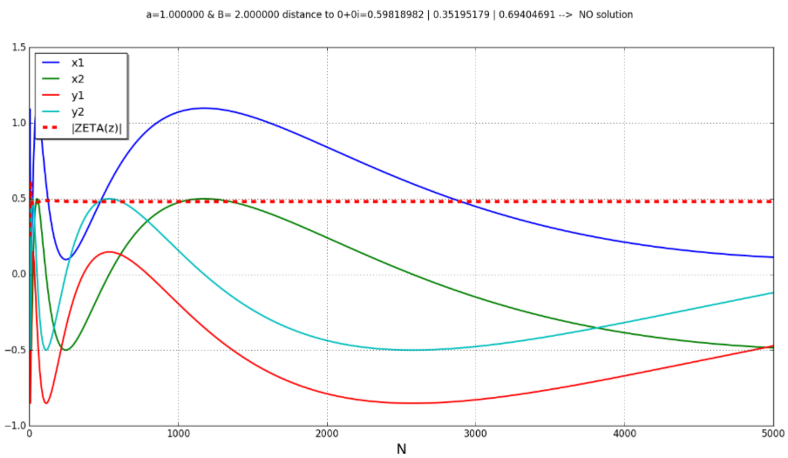

Figure 7

z= ½ + 14.1347251417346 *i $|\zeta(z)|^2 <$ infinitesimal for n>5000 and $\to 0$

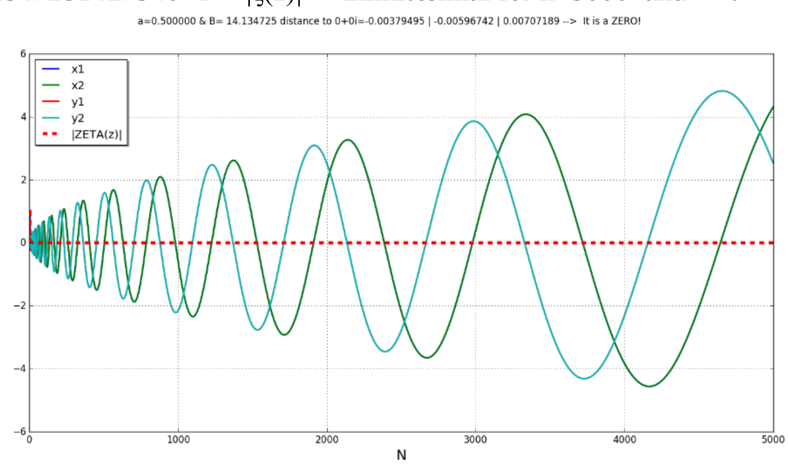

Figure 8

E. The function y(z)

E.1. Definition of y(z)

From (23), (25), (27) we can express y(z):

(37) $$y(z)= [(n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))]) + i\, (n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [ß*\cos(ß*\ln(n))-(1-\alpha)*\sin(ß*\ln(n))])^{]}$$

the value of $|y(z)|^2$ is therefore:

(38) $$|y(z)|^2= [(n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))])^2 + (n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [ß*\cos(ß*\ln(n))-(1-\alpha)*\sin(ß*\ln(n))])^{2]}$$

Simplifying, we obtain a very important formula for |y(z)|:

(39) $$|y(z)|^2 = n^{2(1-\alpha)} * \frac{1}{[ß^2+(1-\alpha)^2]}$$

$|y(z)|^2$ is a polynomial function as we can see in the chart for different values of α:

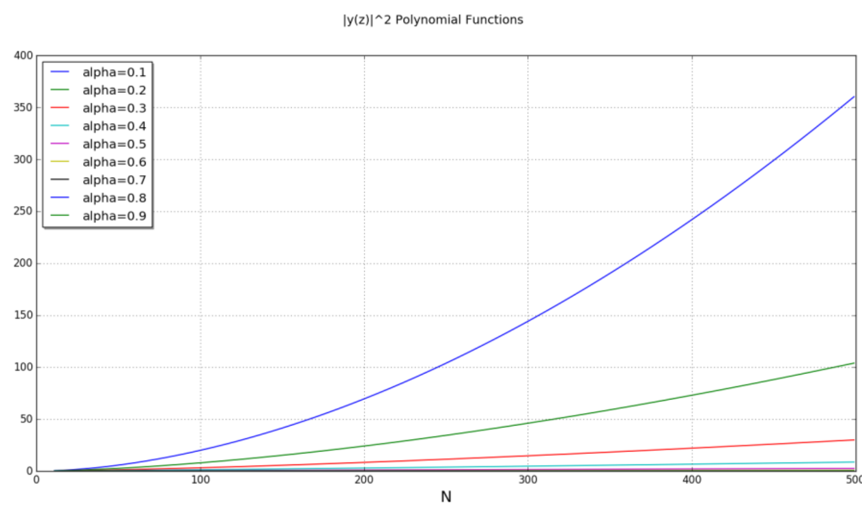

Figure 9. Polynomial representations of $|y(z)|^2$

E.2. Lemma: |y(z)| is a straight line for only for α=1/2

The slope for any $|y(z)|^2$ with respect to n is given by:

(40) $$\text{slope}(|y(z)|^2) = d(|y(z)|)/dn$$

Which equals to:

(41) $$\text{slope}(|y(z)|^2)= 2(1-\alpha)\, n^{1-2\alpha} * \frac{1}{[ß^2+(1-\alpha)^2]}$$

|y(z)| can only be a line when the slope is constant, which can only happen if and only if (1-2α)=0 =>

(42) $\alpha=1/2$

E.3. Conclusion: On the function |y(z)| for Re(z)=1/2 and $n \to \infty$:

We have calculated that for all $z \in Z$:

- ⇨ the slope $|y(z)|^2$ is constant if and only if α=1/2
- ⇨ and slope $|y(z)|^2 = \frac{1}{[ß^2+1/4]}$ for α=1/2 with z=α+iß

F. Analysis of the function x(z)

F.1. Definition of x(z)

We have already formulated x(z) in the complex plane as (22), (24), (26):

(43) $x(z) = \sum_{k=1}^{n} k^{-\alpha} (\cos(ß * \ln(k)) + i* \sum_{k=1}^{n} k^{-\alpha} (\sin(ß * \ln(k))$

The modulus of x(z) will be calculated by:

(44) $|x(z)|^2 = (\sum k^{-\alpha} \cos(ß \ln(n))^2 + (\sum k^{-\alpha} \sin(ß \ln(n))^2$

The square of an infinite series will need some algebraic manipulation. We will simplify this calculation using the following expressions:

(45) $(\sum_{n=1}^{N} a_n)(\sum_{n=1}^{N} b_n) = \sum_{n=1}^{N} a_n b_n + \sum_{n=1}^{N} \sum_{m \neq n}^{N} a_n * b_n$

(46) cos(ßln(k))*cos(ßln(j)) + sin(ßln(k))*sin(ßln(j)) = cos (ßln(k)-ßln(k)) $= cos(ß\,(\ln\left(\frac{k}{j}\right)))$

to obtain a workable expression for |x(z)|:

(47) $|x(z)|^2 = \sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j \neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)$

F.2. Lemma: $|x(n)|^2$ converges when $n \to \infty$ and α>1 to $|\zeta(2\alpha,ß)|^2$

This Lemma provides the limit of $|x(z)|^2$ outside the critical strip [0,1]

(48) $\lim_{n\to\infty} |x(z)|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j \neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)$

(49) $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j \neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right) = 0$ for α>1

And we obtain the known expression:

(50) $$\lim_{n\to\infty} |x(z)|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} = |\zeta(2\alpha,ß)|^2 \qquad \text{for } \alpha>1$$

As we can see in some examples in the following table where z=α+iß:

α	ß	$\lim_{n\to\infty} \|x(z)\|^2$	ζ(2α,ß)
1.0	7	1.074711506185445	1.074756
1.0	10	1.4413521753699579	1.441430
2.5	7	1.0093487944300192	1.009349
2.5	10	1.0507402208589398	1.050740

Table 5

And also in the following figure:

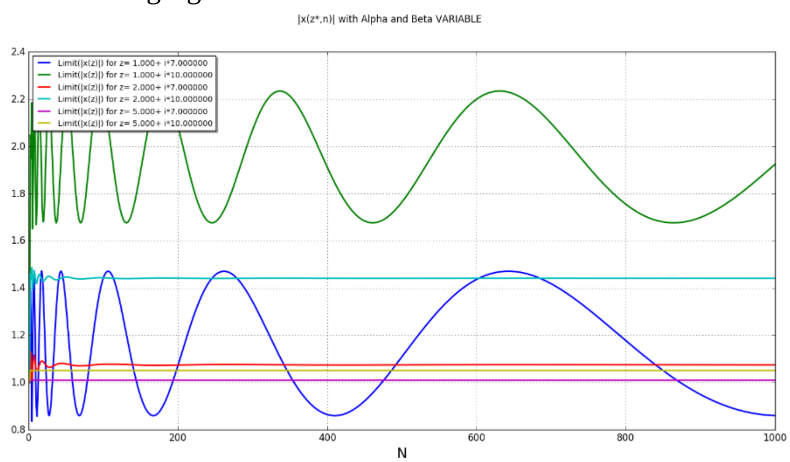

Figure 10. $|x(n)|^2$ converges when $n \to \infty$ and α>1

The graphs for α=1 do not converge while all other graphs for α>1 they all converge to a ζ(2α,ß). We will use this observation to prove later that there are no zero values of ζ(z) for z with Re(z)=α>1.

F.3. Lemma: $|x(n)|^2$ diverges when $n \to \infty$ for α<1

This Lemma provides the limit of $|x(z)|^2$ inside the critical strip. The function actually diverges to ∞ when $\boldsymbol{n} \to \infty$ for α<1 because:

(51) $$|\cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)| > -1$$

(52) $$|\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha}| \quad > \sum_{k=1}^{n} k^{-2\alpha}$$

And

(53) $$|\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha}| \quad \text{diverges for } \alpha<1$$

Now let's evaluate if the function $|x(z)|^2$admits a polynomial representation inside the critical strip [0,1]

F.4. Lemma: $|x(z)|^2$ does not collapse to any polynomial function f(n)=C*n^t for t>1

Let's f(n) = C n^t where C and t are constants in N, with C>0 and t>0

If |x(|z|) = C n^t then:

(54) $\lim_{n\to\infty} |x(z)|^2/n^t = C$

But:

(55) $\lim_{n\to\infty} |x(z)|^2 / n^t =$

$$\frac{1}{n^t} * \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} + \frac{1}{n^t} * \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)$$

And:

(56) $\frac{1}{n^t} * \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} = 0 \quad for\ t > 1$

(57) $\frac{1}{n^t} * \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) \quad = \ 0\ for\ t > 1$

So C must be 0 which is an absurd.

Let's see this graphically for f(n) = n^2

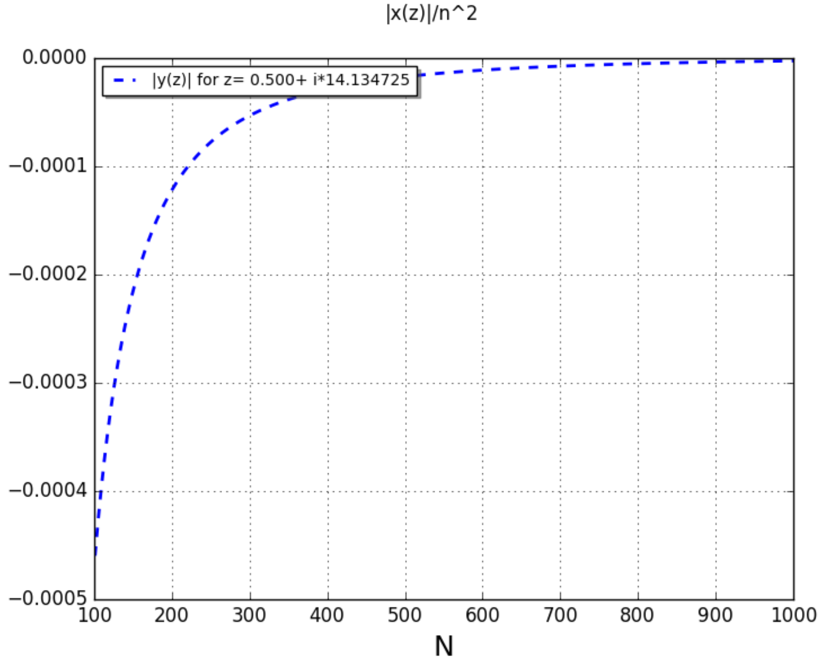

Figure 11

F.5. Lemma: $|x(z)|^2$ collapses to a straight-line f(n)=C*n if Re(z)=1/2

The proposition says that the following limit exists only for Re(z) = 1/2

(58) $\lim_{n\to\infty} (|x(z)|^2/n) \quad = S$

And we know the expression:

(59) $\lim_{n\to\infty} (|x(z)|^2/n) \ = \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) \ )$

F.5.1. For α>1/2 , we can see that $\lim_{n\to\infty} (|x(z)|^2/n) \quad = 0$:

(60) $\lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha}) = 0$ because 2α>1 and the series is convergent

(61) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) \ ) < \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} (k^{-\alpha} * j^{-\alpha})$

$< \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha})$

So:

(62) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) \ ) = 0$

F.5.2. For α<1/2, we can see that $\lim_{n\to\infty} (|x(z)|^2/n) \quad = \infty$ as:

(63) $\lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha}) < \lim_{n\to\infty} \frac{1}{n} \left(n * \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$

And:

(64) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) \ ) > \lim_{n\to\infty} (\frac{1}{n} * n^2 * \frac{1}{n^{2\alpha}}) = \infty$

Where we replace the summations by the number of elements in the matrix (n x n) times the smallest value in each row (1/n) and (2-n-2α)>0 when α<1/2

F.5.3. Let's calculate the limit for α=1/2.

Before calculating this limit, let's see graphically that the limit actually exists for certain z=z* (in the graph ß=R(1)=14.134725…) with α=1/2:

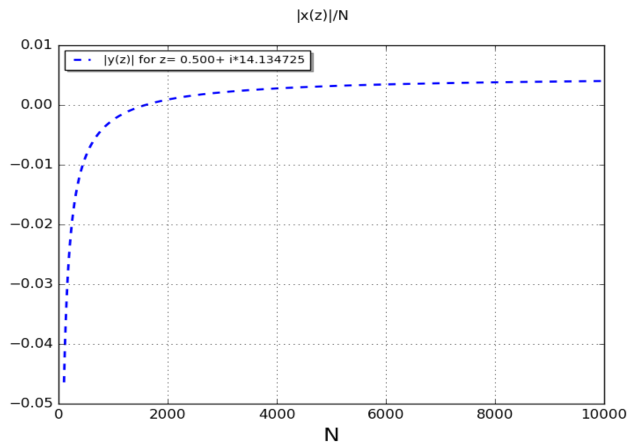

Figure 12
The chart shows that for z=1/2+i 14.134725
the limit |x(z)|=0.0044999 for n → ∞*

When α=1/2, we can express ($|x(z)|^2/n$) as:

$$\lim_{n\to\infty}(|x(z)|^2/n) = \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}k^{-1}+\sum_{k=1}^{n}\sum_{j\neq k}^{n}k^{-1/2}*j^{-1/2}*\cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)\right)=$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}k^{-1}\right)+\lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n}k^{-1/2}*j^{-1/2}*\cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)\right)=$$

$$= 0+\lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n}k^{-1/2}*j^{-1/2}*\cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)\right)=$$

$$= \lim_{n\to\infty}\frac{2n}{n}\left(\sum_{j=1}^{n-1}n^{-1/2}*j^{-1/2}*\cos\left(ß\left(\ln\left(\frac{n}{j}\right)\right)\right)\right)=$$

$$= \lim_{n\to\infty}2\left(n^{-\frac{1}{2}}\sum_{j=1}^{n-1}*j^{-\frac{1}{2}}*\cos\left(ß\left(\ln\left(\frac{n}{j}\right)\right)\right)\right)=$$

Using the integral approximation of the infinite series

$$= 2*\lim_{n\to\infty}\frac{2*\sqrt{n}*\cos\left(ß*ln\left(\frac{n}{n}\right)\right)-2*ß*\sin(ß*\ln(\frac{n}{n})}{4*ß^2+1}*n^{-\frac{1}{2}}$$

$$= 2*\frac{2*\sqrt{n}}{4*ß^2+1}n^{-\frac{1}{2}} = 2*\frac{2}{4*ß^2+1} = \frac{1}{ß^2+1/4}$$

So, if $\lim_{n\to\infty}(|x(z)|^2/n)$ exists will be equal to:

(65) $$\lim_{n\to\infty}(|x(z)|/n) = \frac{1}{ß^2+1/4} \qquad \text{if } z=1/2+iß$$

F.6. Lemma: The slope of $|x(z)|^2$/n is only constant at α=1/2 for certain values of ß

Let's define the function P(z,n) in R such that:

(66) $$P(z,n)=|(x(z)|^2/n - 1/n\sum_{k=1}^{n}k^{-1}=\frac{1}{n}\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n}k^{-\frac{1}{2}}*j^{-\frac{1}{2}}*\cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)\right)$$

P(z,n) and $|(x(z)|^2/n$ have the same limits when $n -> \infty$.

(67) $$\lim_{n\to\infty} P(z,n) = \lim_{n\to\infty} \frac{|(x(z)|^2}{n}$$

And for this function to have a limit, the function must be monotonously increasing or decreasing, therefore it can only have one zero if any.

Let's represent P(z,n) for different values of z=1/2+iß so we can see that P(z,n) can have one or multiple zeros.

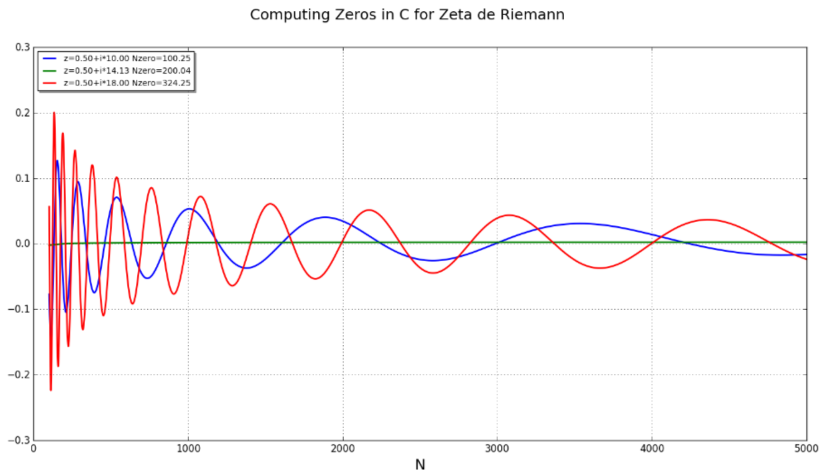

Figure 13. P(z,n)

P(z,n) is a wave function. To have one zero, P(z,n) must collapse to a polynomial line and cross the axis y=0 only once. We know that from (67) and (65) that Re(z) must be ½ for this to happen. This is an easy algorithm to program and the calculation gives the following zeros for P(z,n) for z=1/2+iß with ß>[0,∞)

n such that P(z, n)=0 once
200.1
442.2
625.8
926.0
1085.0
1413.0
1674.6
1877.5
2304.8
2477.7
2806.2
3186.5
3522.4
3700.8

Table 6. Showing only the first 14 Zeros of P(z,n)

And the values of $\lim_{n\to\infty} P(z*, n)$ are :

n such that P(z, n)=0 once	Limit (P(z*,n))
200.1	0.0049975
442.2	0.0022614
625.8	0.0015980
926.0	0.0010799
1085.0	0.0009217
1413.0	0.0007077
1674.6	0.0005972
1877.5	0.0005326
2304.8	0.0004339
2477.7	0.0004036
2806.2	0.0003564
3186.5	0.0003138
3522.4	0.0002839
3700.8	0.0002702

Table 7. Limit of P(z,n)*

From (67) we know that these limits are also the slopes of the straight lines $|x(z)|^2/n$ when z=z*, and from (65) that the limit is equal to $\frac{1}{ß^2+1/4}$

So we can calculate the ß* that fit each of those limits (n approximated to 1 decimal point):

n such that P(z, n)=0 once	Limit (P(z*,n))	ß fromP(z,n)
200.1	0.0049975	14.13683
442.2	0.0022614	21.02261
625.8	0.0015980	25.01100
926.0	0.0010799	30.42614
1085.0	0.0009217	32.93554
1413.0	0.0007077	37.58657
1674.6	0.0005972	40.91882
1877.5	0.0005326	43.32724
2304.8	0.0004339	48.00573
2477.7	0.0004036	49.77399
2806.2	0.0003564	52.97122
3186.5	0.0003138	56.44688
3522.4	0.0002839	59.34770
3700.8	0.0002702	60.83215

Table 8. with P(z,n)=0 and ß calculated to the 1st decimal place

We can state from (67) that these ß calculated from the zeros of P(z,n) must be the zeros of the Riemann Zeta function as $|(y(z)|^2=|(x(z)|^2$ at these z* and $|\zeta(z)|^2$ is zero from the definition of ζ(z)=x(z)-y(z).

The fact that these are equal to the zeros of ζ(z) can be seen in the following table:

n such that P(z, n)=0 once	Limit (P(z*,n))	ß fromP(z,n)	Known zero of Zeta
200.1	0.0049975	14.13683	14.134725142
442.2	0.0022614	21.02261	21.022039639
625.8	0.0015980	25.01100	25.010857580
926.0	0.0010799	30.42614	30.424876126
1085.0	0.0009217	32.93554	32.935061588
1413.0	0.0007077	37.58657	37.586178159
1674.6	0.0005972	40.91882	40.918719012
1877.5	0.0005326	43.32724	43.327073281
2304.8	0.0004339	48.00573	48.005150881
2477.7	0.0004036	49.77399	49.773832478
2806.2	0.0003564	52.97122	52.970321478
3186.5	0.0003138	56.44688	56.446247697
3522.4	0.0002839	59.34770	59.347044003
3700.8	0.0002702	60.83215	60.831778525

Table 9. Comparing ß calculated with known zeros of ζ(z)

As an observation, it is very interesting to see that P(z*,n) has the following special properties for all z* zeros of ζ(z). If S= slope of $|x(z^*)|^2$

(68) P(z*, n*) = 0 when n*= 1/S

(69) $\lim_{n\to\infty} \mathrm{P}(z^*, \mathrm{n}) = \mathrm{S}$

Graphically:

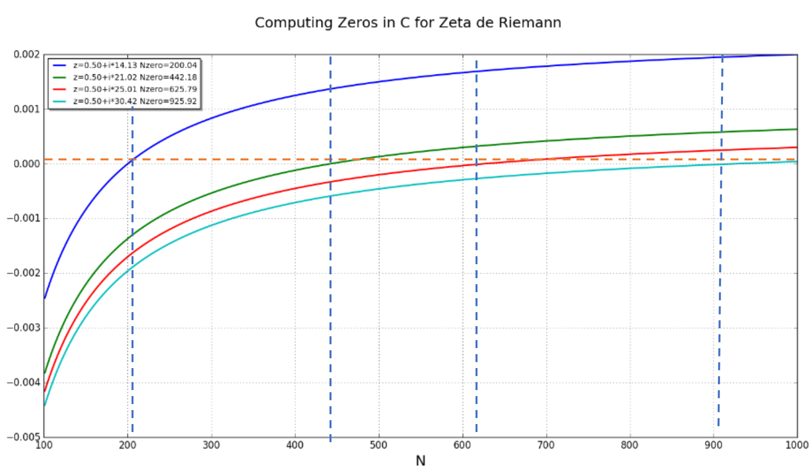

*Figure 14. P(z,n) for z=z**

F.7. Corollary: A linearization of the Harmonic series using zeros of ζ(z).

The precedent formulations describe also a way to approximate the Harmonic function to a straight line with slope $\frac{1}{[ß^2+(1-\alpha)^2]}$ where α=1/2 and ß=R(n):

(70) $$H_n = \frac{n}{[ß^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos(ß\,(\ln\left(\frac{k}{j}\right)) \qquad \text{when } n \to \infty$$

We can see this graphically for $ß_1$=14.134725… with O(n) given by:

$$O(n) = \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos(ß\,(\ln\left(\frac{k}{j}\right))$$

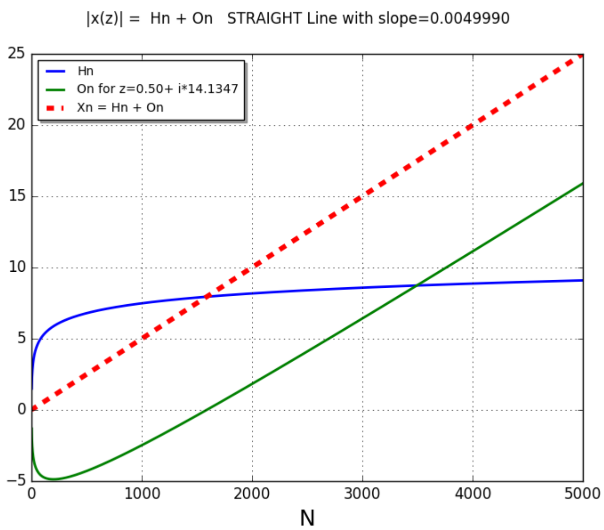

Figure 15. H_n *and* $|x(z)|^2$

F.8. Corollary: All ß zeros of ζ(z) are related algebraically.

The fact that the same H_n can be expressed in an infinite number of ways as a function of ß for every ß imaginary part of a non-trivial solution of ζ(z), provides an algorithm to calculate all non-trivial zeros from any known zero through the expression. If $ß_1$ and $ß_2$ are imaginary part of a non-trivial solution of ζ(z), then:

(71) $$\frac{n}{[ß_2^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos(ß_2\,(\ln\left(\frac{k}{j}\right)) = \frac{n}{[ß_1^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos(ß_1\,(\ln\left(\frac{k}{j}\right))$$

when $n \to \infty$, where the size of n will determine the degree of accuracy of the solution.

F.9. Corollary: There are no zeros of ζ(z) when Re(z)>1

From Figure 10:

- we proved that |x(z)| converges to a given value for α=Re(z)>1, which means that

- |x(z)| tends to a horizontal line with slope =0 as $n \rightarrow \infty$ when α>1 .
- We know that all zeros of ζ(z) must make |x(z)| a straight line with slope $\frac{1}{[ß^2+(1-\alpha)^2]}$.
- Therefore, this contradiction proves that there can't be any zeros of the ζ(z) function for α>1

F.10. Corollary: $|x(z)|^2$ is very sensitive to slight variations of ß with α=1/2

Let's review the sensitivity of this solution for several z* with α=1/2 and different values of ß around some R(n).

a) z= ½ + i*ß   with ß around R(1)=14.134725142

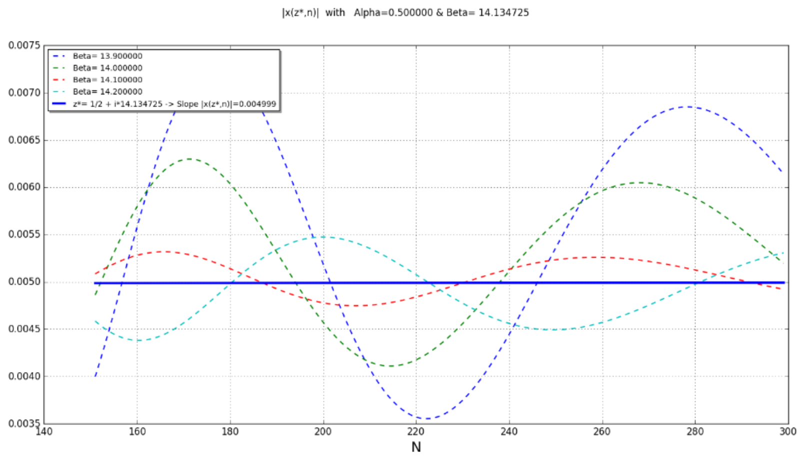

Figure 16.

Slope= 0.004499… $= \frac{1}{[ß^2+1/4]}$

And the slope is not constant for any other α+iß for small variations of ß=R(1)

b) z= ½ + i**ß   with ß around R(7)=37.586178159

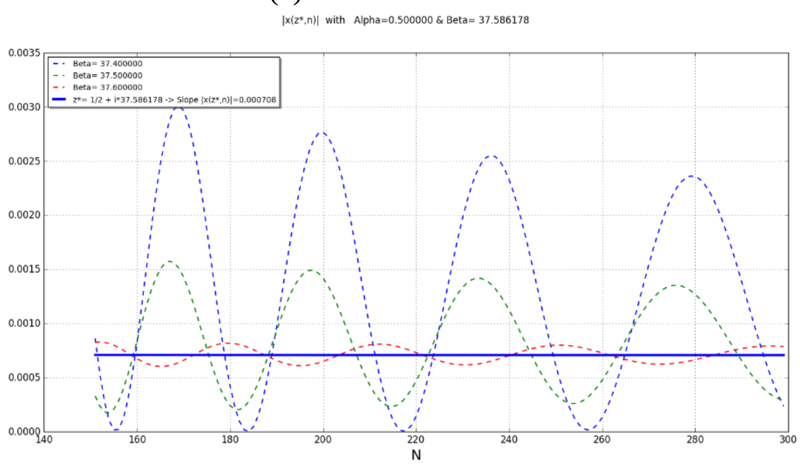

Figure 17.

Slope= 0.0007077… $= \frac{1}{[ß^2+1/4]}$

And the slope is not constant for any other α+iß for small variations of ß=R(7).

F.11. Corollary: $|x(z)|^2/n$ very sensitive to slight variations of α with ß=R(n)

Let's review the sensitivity of this solution for several z* with ß=R(n) and different values of α around α=1/2.

a) ß = R(1)   α + i*14.134725142   with α =0.3, 0.4, 0.5, 0.6

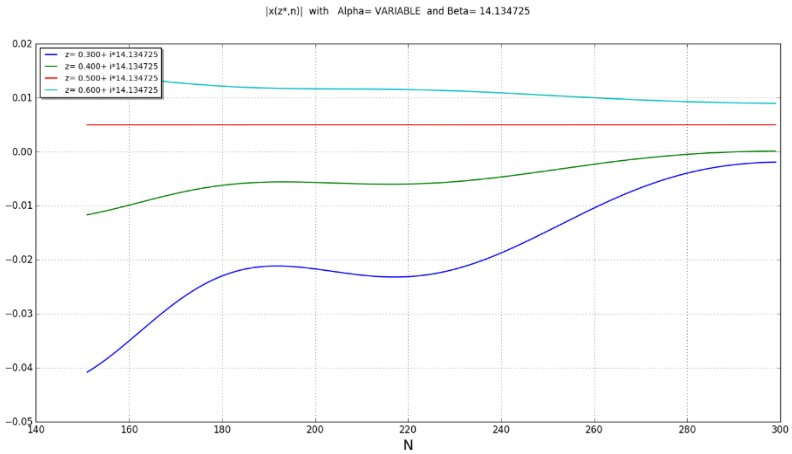

Figure 18

We can observe that for Im(z)=R(1) the slope of |x(z)| is not constant for any small variations around Re(z) = 1/2

b) ß = R(14)   z=α + i*60.831778525   with α =0.3, 0.4, 0.5, 0.6

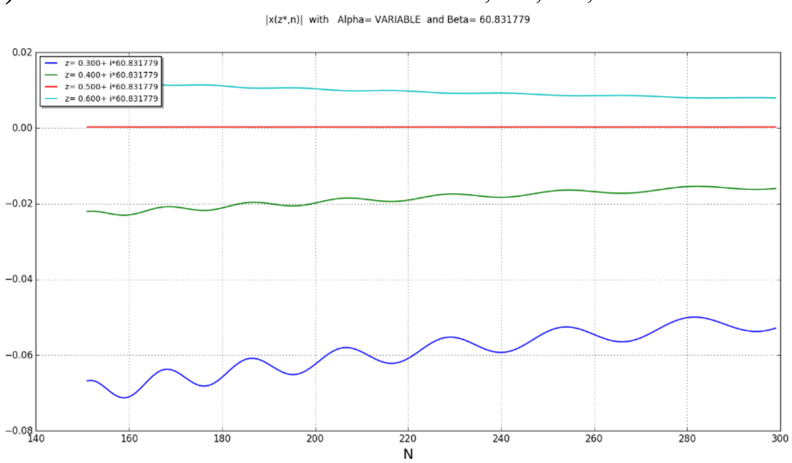

Figure 19

We can observe that for Im(z)=R(14) the slope of |x(z)| is not constant for any small variations around Re(z) = ½

F.12. Conclusion: On the function $|x(z)|^2$ for z=z*=α+iß:

For all z=z* such that $\zeta(z^*) = 0$, we have proved that:

- ⇨ The wave $|x(z)|^2$ collapses to a straight line when for certain values of z=1/2+iß*
- ⇨ The only values that makes $|x(z)|^2$ collapse into a line are given by:
  a. α=1/2
  b. ß* such that if S=$\frac{1}{[ß^{*2}+1/4]}$ then for n=1/S

$$\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(ß^* \left(\ln\left(\frac{k}{j}\right)\right)\right)\right) = 0$$

- ⇨ $|x(z^*)|^2$ is a straight line with slope = $\frac{1}{[ß^{*2}+1/4]}$ for z=1/2+iß*

G. Theorem. All non-trivial zeros of $\zeta(z^*)$ have Re(z*)=1/2

- ➢ We defined x(z) and y(z) as functions on *C* such that $\zeta(z) = x(z) - y(z)$
- ➢ We proved that $|x(z)|^2$ is a wave function that has only one polynomial representation in the form of a straight line if and only if Re(z)= ½ and for certain values of Im(z)=ß* that we calculated
- ➢ $|x(z^*)|^2 = \frac{n}{[ß^{*2}+1/4]}$ when $n \to \infty$ for z*=1/2+iß*
- ➢ We showed that $|y(z)|^2$ is always a polynomial line of any degree.
- ➢ We proved that $|y(z)|^2$ is only straight line if and only if Re(z)= ½
- ➢ $|y(z)|^2 = \frac{n}{[ß^2+1/4]}$ when $n \to \infty$ for all z=1/2+iß
- ➢ If z=z* is a zero of ζ(z) and $n \to \infty$ ➔
- ➢ $\zeta(z^*) = 0 + i0$ ➔
- ➢ $|\,\zeta(z\,)|^2$ must be 0 ➔
- ➢ All z* non-trivial solution of ζ(z) must have $|x(z^*)|^2 = |y(z^*)|^2$ when $n \to \infty$
- ➢ We proved that of all possible representations of $|x(z^*)|^2$ and $|y(z^*)|^2$ the only one in common for both functions is a representation as a straight line when Re(z)=1/2
- ➢ Therefore, all z* non-trivial solution of ζ(z) must have Re(z*) = ½ and we can also state that any zero of ζ(z) with z=α+iß meet these two conditions:

(condition 1) α=1/2

(condition 2) If $S=\frac{1}{[ß^2+1/4]}$ then for n=1/S ->

$$\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)\right) = 0$$

This is a finite sum, with n∈ $[1,\frac{1}{S}]$.

And we can calculate any zero $ß_2$ knowing any $ß_1$ with the following expression:

$$\frac{n}{[ß_2^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos(ß_2\left(\ln\left(\frac{k}{j}\right)\right) = \frac{n}{[ß_1^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos(ß_1\left(\ln\left(\frac{k}{j}\right)\right)$$

when $n \to \infty$, where the size of n will determine the degree of accuracy of the solution.

Q.E.D.

H. <u>REFERENCES</u>

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