Spin transferred to a mirror reflecting light

Radi I. Khrapko[1]
Moscow Aviation Institute - Volokolamskoe shosse 4, 125993 Moscow, Russia

We consider the incidence of a plane circularly polarized electromagnetic wave on a mirror at an angle φ and its reflection from it. We have calculated the transfer of a momentum and a spin to the mirror and, accordingly, the pressure and density of a mechanical torque on the mirror. The given calculations show that spin occurs to be the same natural property of the plane electromagnetic wave, as momentum. Recognizing the existence of photons with momentum, energy and spin in a plane electromagnetic wave, it is strange to deny the presence of spin in such a wave, as is done in modern electrodynamics.

Key Words: classical spin; circular polarization; electrodynamics torque
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1. Introduction

It was suggested as early as 1899 by Sadowsky [1] and as 1909 by Poynting [2] that any circularly polarized light has angular momentum *density*. That is the angular momentum is present in any point of the light.

According to the Lagrange formalism, this angular momentum density is *spin* density. The spin of electromagnetic waves is described by a spin tensor [3 -5].

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\alpha)}, \qquad (1.1)$$

where $\mathcal{L}$ is a Lagrangian and A^λ is the magnetic vector potential of the electromagnetic field. So, any infinitesimal 3-volume dV_ν contains spin

$$dS^{\lambda\mu} = \Upsilon^{\lambda\mu\nu}dV_\nu. \qquad (1.2)$$

Against this, an oppinion is spread that an electromagnetic plane wave of arbitrary extent has no angular momentum.

> **Heitler W:** "A plane wave travelling in z-direction and with infinite extension in the xy-directions can have no angular momentum about the z-axis, because Π is in the z-direction and $(\mathbf{r}\times\Pi)_z = 0$ " [6]

Here Π is the Poynting vector.

> **Simmonds J. W., Guttmann M. J.:** "The electric and magnetic fields can have a nonzero z-component only within the skin region of this wave. Having z-components within this region implies the possibility of a nonzero z-component of angular momentum within this region. So, the skin region (of a beam) is the only in which the z-component of angular momentum does not vanish" [7, p. 227]
>
> **Allen L., Padgett M. J.:** "For a plane wave there is no (radial intensity) gradient and the spin density is zero" [8]

On the other hand, according to [9], this oppinion is a mistake. The Poynting's and Sadowsky's concept is true: spin angular momentum is present in any point of a circularly polarized electromagnetic beam and, accordingly, torque acts on any point of an absorber of such a beam.

In this paper, we confirm the Poynting's and Sadowsky's concept by our calculation.

Since 1905, when Einstein explained the photoelectric effect, it has become clear that an electromagnetic wave consists of photons. Photons have energy, momentum and spin (internal angular momentum), and if the wave is circularly polarized, spins of all the photons are directed in the same direction that is parallel to that of the momentum. Therefore, use is made of such notions

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[1] Email: khrapko_ri@hotmail.com, http://khrapkori.wmsite.ru

as volume density and flux density of momentum, energy, and spin as well as number of photons in an electromagnetic wave. Densities of the energy and momentum are quantitatively described by the Maxwell energy–momentum tensor. The density of a spin should be described by the spin tensor, and the number density of photons is obtained by dividing the energy density of a wave by the energy of a single photon, or by dividing the density of a spin wave by the spin of a single photon (with circular polarization).

In our previous papers, we have examined the implementation of the law of conservation of energy, momentum and spin angular momentum for normal incidence of an electromagnetic wave on a moving mirror [10], on the surface of a fixed insulator [11], on the surface of a moving symmetric absorber [12].

In this paper, we consider the incidence of a plane circularly polarized electromagnetic wave on a fixed mirror at an angle of incidence, φ. In this situation, the electromagnetic energy is not transferred to the mirror, and the momentum and the spin of the photons change their direction upon reflection. As a result, the mirror receives a doubled normal component of the wave momentum in the form of pressure and a doubled tangential component of the spin in the form of a distributed torque. The fact is that in the process of reflection, the wave helicity is reversed, i.e., the mutual orientation of the momentum and spin changes into the opposite one (see Fig. 1).

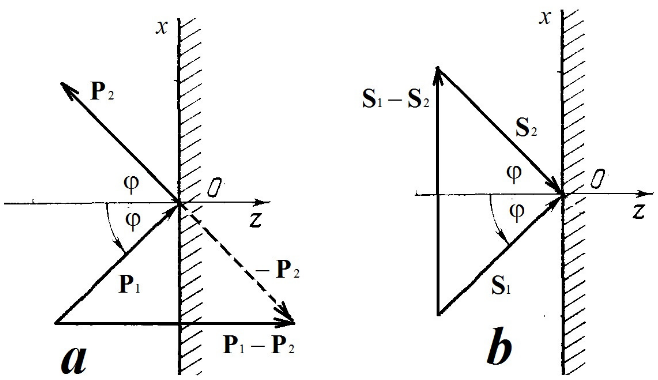

Figure 1. (***a***) Momentum of incident and reflected photons and the momentum gained by the mirror, and (***b***) spin of incident and reflected photons and the spin gained by the mirror.

We will calculate the flux densities of the momentum and spin in incident and reflected waves, and make sure that the change in the momentum and the spin of the reflected wave correspond to the pressure and density of the torque experienced by the mirror. The conclusions of this paper have been published elsewhere [13].

2. Electromagnetic waves in question

To write the expression for a wave incident at an angle φ, we will make use of the expression for a right-hand circularly polarized electromagnetic wave incident normally on the xy-surface in the coordinates x', y', z':

$$E_1^{x'} = \cos(z'-t), \quad E_1^{y'} = -\sin(z'-t), \quad B_1^{x'} = \sin(z'-t), \quad B_1^{y'} = \cos(z'-t) \tag{2.1}$$

(for simplicity we put $\omega = k = c = \varepsilon_0 = \mu_0 = 1$). The coordinate transformations

$$x' = x\cos\varphi - z\sin\varphi, \quad z' = x\sin\varphi + z\cos\varphi, \quad y' = y \tag{2.2}$$

give expressions

$$E_1^x = \cos\varphi\cos(x\sin\varphi + z\cos\varphi - t), \quad B_1^x = \cos\varphi\sin(x\sin\varphi + z\cos\varphi - t), \tag{2.3}$$

$$E_1^y = -\sin(x\sin\varphi + z\cos\varphi - t), \quad B_1^y = \cos(x\sin\varphi + z\cos\varphi - t), \tag{2.4}$$

$$E_1^z = -\sin\varphi\cos(x\sin\varphi + z\cos\varphi - t), \quad B_1^z = -\sin\varphi\sin(x\sin\varphi + z\cos\varphi - t). \tag{2.5}$$

for the right-hand circularly polarized wave incident at an angle φ.

To write the expression for a wave reflected at an angle #, we will make use of the expression for a left-hand circularly polarized electromagnetic wave originating along the normal from the xy-surface in the coordinates x', y', z':

$$E_2^{x'} = -\cos(z'+t), \quad E_2^{y'} = -\sin(z'+t), \quad B_2^{x'} = -\sin(z'+t), \quad B_2^{y'} = \cos(z'+t) . \tag{2.6}$$

The coordinate transformations

$$x' = x\cos\varphi + z\sin\varphi, \quad z' = -x\sin\varphi + z\cos\varphi, \quad y' = y \tag{2.7}$$

give expressions

$$E_2^x = -\cos\varphi\cos(-x\sin\varphi + z\cos\varphi + t), \quad B_2^x = -\cos\varphi\sin(-x\sin\varphi + z\cos\varphi + t) , \tag{2.8}$$

$$E_2^y = -\sin(-x\sin\varphi + z\cos\varphi + t), \quad B_2^y = \cos(-x\sin\varphi + z\cos\varphi + t) , \tag{2.9}$$

$$E_2^z = -\sin\varphi\cos(-x\sin\varphi + z\cos\varphi + t), \quad B_2^z = -\sin\varphi\sin(-x\sin\varphi + z\cos\varphi + t) . \tag{2.10}$$

for the wave reflected at an angle φ.

One can easily see that the boundary conditions

$$\left[E_1^x + E_2^x\right]_{z=0} = \left[E_1^y + E_2^y\right]_{z=0} = \left[B_1^z + B_2^z\right]_{z=0} = 0 . \tag{2.11}$$

are fulfilled on the surface of the mirror (an ideal conductor).

3. Momentum flux density transferred to the mirror

To calculate the momentum flux density, i.e. the pressure $\mathscr{P}$, on the mirror, it is natural to use the component of the Maxwell stress tensor [14, (33.3)]

$$\mathscr{P} = T^{zz} = [-(E^z)^2 + (E^x)^2 + (E^y)^2 - (B^z)^2 + (B^x)^2 + (B^y)^2]/2 . \tag{3.1}$$

The incident and reflected waves do not interfere with each other. This can be verified by determining that the energy–momentum tensor of the total field is equal to the sum of the energy–momentum tensors of the incident and reflected waves. Therefore, we can calculate the pressure by substituting expressions for the incident wave (2.3) – (2.5) into formula (3.1) and double the result. Thus, we obtain

$$\mathscr{P} = 2\cos^2\varphi . \tag{3.2}$$

On the other hand, this result can be regarded as the action of the Lorentz force on charges and currents induced in the mirror. Indeed, we express the force $d\mathscr{F}$ acting on an infinitesimal area of the mirror surface da_z through the divergence of the component of the energy–momentum tensor (see Fig. 2)

$$d\mathscr{F} = \mathscr{P}da = T^{zz}da_z = -\oint_{\partial dV} T^{zi}da_i = -\partial_i T^{zi}dV . \tag{3.3}$$

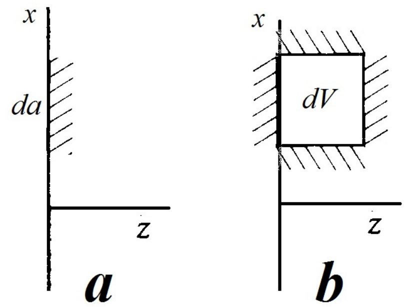

Figure 2. (***a***) Area da on the mirror and (***b***) area da forming a closed surface which is the boundary of the mirror material volume dV.

Here we assume integration over the boundary of volume dV, which is obtained by closing the area da inside the mirror material with changing the external orientation to the opposite one. Since [14, (33.7)] $\partial_\mu T^{\lambda\mu} = -j^\mu F^{\lambda\nu} g_{\mu\nu}$, the divergence of the tensor component is expressed in terms of the Lorentz force density

$$-\partial_i T^{zi} = \partial_t T^{zt} + j^t F^{zt} - j^x F^{zx} - j^y F^{zy} = \rho E^z + j^y B_x - j^x B_y = \rho E^z + [\mathbf{jB}]_z . \tag{3.4}$$

The momentum density T^{zt} in the direction of the mirror, as well as the Poynting vector, are zero. Therefore, we obtain

$$d\mathscr{F} = \mathscr{P} da = (\rho E^z + [\mathbf{jB}]_z) dV . \quad (3.5)$$

4. Spin tensor

To describe the spin density and spin flux density in electromagnetic waves, use is typically made of the canonical spin tensor, $\underset{c}{\Upsilon}^{\lambda\mu\nu}$, obtained by using the Lagrangian formalism from the canonical Lagrangian $\underset{c}{\mathrm{L}} = -F_{\mu\nu}F^{\mu\nu}/4$ [3 - 5]:

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta_{\alpha}^{\mu]}\frac{\partial \underset{c}{\mathrm{L}}}{\partial(\partial_{\nu}A_{\alpha})} = -2A^{[\lambda}F^{\mu]\nu}, \quad \underset{c}{\Upsilon}^{ijt} = -2A^{[i}F^{j]t} = -2A^{[i}E^{j]} = \mathbf{E}\times\mathbf{A} . \quad (4.1)$$

For example, Soper [4, p. 115] writes, referring to the component $\underset{c}{\Upsilon}^{xyt}$:

> "To describe a circularly polarized plane wave traveling in the z-direction, we can choose a potential
>
> $$A^x = a\cos[\omega(z-t)],\ A^y = -a\sin[\omega(z-t)] .$$
>
> The corresponding electric field is $E^k = -\partial_t A^k$. Thus the spin density carried by this wave is
>
> $$\mathbf{s} = a^2\omega\hat{\mathbf{z}} ,$$
>
> where $\hat{\mathbf{z}}$ is a unit vector pointing in the z-direction."

We have successfully used the canonical spin tensor elsewhere [8–10]. However, for this paper, it is very important that the canonical spin tensor *incorrectly* describes the spin flux in the directions that do not coincide with the wave propagation direction. This was pointed out in paper [9]. We wrote:

> "The canonical spin tensor contradicts experiments. For example, consider a circularly polarized plane wave,
>
> $$E^x = \cos(z-t),\ E^y = -\sin(z-t),\ B^x = \sin(z-t),\ B^y = \cos(z-t),$$
>
> A calculation of components of the canonical spin tensor yields
>
> $$\underset{c}{\Upsilon}^{zxy} = A^x B_x = \sin^2(z-t), \quad \underset{c}{\Upsilon}^{yzx} = A^y B_y = \cos^2(z-t) .$$
>
> This result is not adequate because it means that there are spin fluxes in the directions, which are transverse to the direction of the wave propagation."

More adequate, than the canonical spin tensor, is the spin tensor derived by the Lagrangian formalism from the Lagrangian of the massless vector field $\mathscr{L} = -\partial_{\mu}A^{\nu}\partial^{\mu}A_{\nu}/2$ [5]

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}\delta_{\alpha}^{\mu]}\frac{\partial\mathscr{L}}{\partial(\partial_{\nu}A_{\alpha})} = A^{\lambda}\partial^{\nu}A^{\mu} - A^{\mu}\partial^{\nu}A^{\lambda} . \quad (4.2)$$

Hereafter we will use this spin tensor.

5. Spin angular momentum flux density transferred to the mirror

In accordance with Fig. 1b, the S^{yz} component of the spin is transferred to the mirror. The flux density of this spin component on the mirror is given by the component

$$\Upsilon^{yzz} = A^y\partial^z A^z - A^z\partial^z A^y \quad (5.1)$$

of the spin tensor, and, in the absence of interference, it is possible to calculate this component only for the incident wave and to double it. From the formula $\mathbf{A} = -\int \mathbf{E}dt$ we obtain the magnetic vector potentials in the incident wave:

$$A_1^y = \cos(x\sin\varphi + z\cos\varphi - t), \quad A_1^z = -\sin\varphi\sin(x\sin\varphi + z\cos\varphi - t) . \quad (5.2)$$

Thus, given that $\partial^z = -\partial_z$ due to the metrics signature $(+---)$, the spin flux density on the mirror is equal to the expression

$$\Upsilon^{yzz} = 2(A_1^y \partial^z A_1^z - A_1^z \partial^z A_1^y) = \sin(2\varphi) \quad (5.3)$$

On the other hand, this quantity is the result of the distributed torque on the currents induced in the mirror. Indeed, we can express the torque $d\tau^{yz}$ acting on the area da_z of the mirror surface through the divergence of the spin tensor component (see Fig. 2)

$$d\tau^{yz} = \Upsilon^{yzz} da_z = -\oint_{\partial dV} \Upsilon^{yzi} da_i = -\partial_i \Upsilon^{yzi} dV \,. \quad (5.4)$$

Here we assume integration over the boundary of volume dV, which is obtained by closing the area da_z inside the mirror material with changing the external orientation to the opposite one. Since

$$-\partial_\nu \Upsilon^{\lambda\mu\nu} = -2\partial_\nu (A^{[\lambda} \partial^{|\nu|} A^{\mu]}) = 2 j^{[\lambda} A^{\mu]} \,, \quad (5.5)$$

and since the electromagnetic spin does not accumulate in the mirror, $\partial_t \Upsilon^{yzt} = 0$, the divergence of the spin tensor component is expressed in terms of the *torque density* $[\mathbf{jA}]$, which is an analogue of the Lorentz force density

$$-\partial_i \Upsilon^{yzt} = 2 j^{[y} A^{z]} = [\mathbf{jA}]_x \,, \quad d\tau^{yz} = [\mathbf{jA}]_x dV \,. \quad (5.6)$$

6. Conclusions

The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum. Recognizing the existence of photons with momentum, energy and spin in a plane electromagnetic wave, it is strange to deny the presence of spin in such a wave, as is done in modern electrodynamics.

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Note

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