New Expansions in Series for Tangent and Secant Functions

By Edigles Guedes

Abstract. In this paper, the author proved new expansions in series for tangent and secant functions.

1. Introduction

In this paper, I demonstrated the new expansion in series for the tangent function

$$\pi\tan(\pi x)=x\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{4}{n+2x}+\frac{4}{n-2x}-\frac{1}{n+x}-\frac{1}{n-x}\right)$$

and secant function

$$\pi^2\sec^2(\pi x)=\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{4}{n+2x}+\frac{4}{n-2x}-\frac{1}{n+x}-\frac{1}{n-x}\right.$$
$$\left.+\frac{8x}{(n-2x)^2}-\frac{8x}{(n+2x)^2}+\frac{x}{(n+x)^2}-\frac{x}{(n-x)^2}\right],$$

provided that $x\in\mathbb{R}$.

2. Preliminaries

Lemma 1. *For* $x\in\mathbb{C}-\{-1,-2,-3,\ldots\}$, *then*

$$\cos(\pi x)=\prod_{n=1}^{\infty}\frac{n^2-4x^2}{n^2-x^2},$$

where $\cos(x)$ *denotes the cosine function.*

Proof. I know the double-angle formula for sine function [1]

$$2\cos(\pi x)\sin(\pi x)=\sin(2\pi x)\Rightarrow\cos(\pi x)=\frac{1}{2}\cdot\frac{\sin(2\pi x)}{\sin(\pi x)}. \tag{1}$$

On the other hand, I know that [2]

$$\frac{\sin((x-y)\pi)}{(x-y)\pi}=\binom{x}{y}\cdot\binom{y}{x}\Rightarrow\sin((x-y)\pi)=(x-y)\pi\cdot\binom{x}{y}\cdot\binom{y}{x}. \tag{2}$$

Using (2) into (1), and putting $x=4x$ and $y=2x$, $x=2x$ and $y=x$, respectively, I encounter

$$\cos(\pi x)=\frac{1}{2}\cdot\frac{2x\pi\cdot\binom{4x}{2x}\cdot\binom{2x}{4x}}{x\pi\cdot\binom{2x}{x}\cdot\binom{x}{2x}}$$
$$\Rightarrow\cos(\pi x)=\frac{\binom{4x}{2x}\cdot\binom{2x}{4x}}{\binom{2x}{x}\cdot\binom{x}{2x}}=\frac{\Gamma(1-x)\Gamma(1+x)}{\Gamma(1-2x)\Gamma(1+2x)}. \tag{3}$$

The Euler's infinite product representation for gamma function [3] is given by

$$\Gamma(1+x)=\prod_{n=1}^{\infty}\left(\frac{1}{1+\frac{x}{n}}\right)\left(1+\frac{1}{n}\right)^{x}, \tag{4}$$

provided that $x \in \mathbb{C} - \{-1, -2, -3, ...\}$.

From (3) and (4), I conclude that

$$\cos(\pi x) = \prod_{n=1}^{\infty} \frac{n^2 - 4x^2}{n^2 - x^2},$$

which is the desired result. □

3. Theorem and Corollary

Theorem 2. *For $x \in \mathbb{R}$, then*

$$\pi \tan(\pi x) = x \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{4}{n+2x} + \frac{4}{n-2x} - \frac{1}{n+x} - \frac{1}{n-x} \right),$$

where $\tan(x)$ *denotes the tangent function.*

Proof. The logarithmic differentiation of the Lemma 1, give me

$$\begin{aligned} -\pi \tan(\pi x) &= -\sum_{n=1}^{\infty} \frac{6n^2 x}{n^4 - 5n^2x^2 + 4x^4} \\ \Rightarrow \frac{\pi \tan(\pi x)}{6x} &= \sum_{n=1}^{\infty} \frac{n^2}{(n+x)(n-x)(n+2x)(n-2x)} \\ \Rightarrow \frac{\pi \tan(\pi x)}{x} &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{4}{n+2x} + \frac{4}{n-2x} - \frac{1}{n+x} - \frac{1}{n-x} \right), \end{aligned}$$

which is the desired result. □

Corollary 3. *For $x \in \mathbb{R}$, then*

$$\begin{aligned} \pi^2 \sec^2(\pi x) = \sum_{n=1}^{\infty} \frac{1}{n} \Bigg[& \frac{4}{n+2x} + \frac{4}{n-2x} - \frac{1}{n+x} - \frac{1}{n-x} \\ & + \frac{8x}{(n-2x)^2} - \frac{8x}{(n+2x)^2} + \frac{x}{(n+x)^2} - \frac{x}{(n-x)^2} \Bigg], \end{aligned}$$

where $\sec(x)$ *is the secant function.*

Proof. The differentiation of the previous Theorem, get me

$$\frac{\pi^2 \sec^2(\pi x)}{x} - \frac{\pi \tan(\pi x)}{x^2} = \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{8}{(n-2x)^2} - \frac{8}{(n+2x)^2} + \frac{1}{(n+x)^2} - \frac{1}{(n-x)^2} \right]. \tag{5}$$

From Theorem 2 and (5), I obtain

$$\begin{aligned} & \frac{\pi^2 \sec^2(\pi x)}{x} - \sum_{n=1}^{\infty} \frac{1}{nx} \left(\frac{4}{n+2x} + \frac{4}{n-2x} - \frac{1}{n+x} - \frac{1}{n-x} \right) = \\ & = \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{8}{(n-2x)^2} - \frac{8}{(n+2x)^2} + \frac{1}{(n+x)^2} - \frac{1}{(n-x)^2} \right] \\ & \Rightarrow \frac{\pi^2 \sec^2(\pi x)}{x} = \sum_{n=1}^{\infty} \frac{1}{n} \Bigg\{ \frac{1}{x} \left(\frac{4}{n+2x} + \frac{4}{n-2x} - \frac{1}{n+x} - \frac{1}{n-x} \right) \\ & \quad + \left[\frac{8}{(n-2x)^2} - \frac{8}{(n+2x)^2} + \frac{1}{(n+x)^2} - \frac{1}{(n-x)^2} \right] \Bigg\}, \end{aligned}$$

wich is the desired result. □

References

[1] en.wikipedia.org/wiki/List_of_trigonometric_identities, avaliable in December 14, 2016.

[2] en.wikipedia.org/wiki/Binomial_coefficient, avaliable in December 14, 2016.

[3] en.wikipedia.org/wiki/Gamma_function, avaliable in December 14, 2016.