Absorption of angular momentum of a plane electromagnetic wave

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It is demonstrated that dielectric or magnetic, which absorbs a circularly polarized plane electromagnetic wave, absorbs the angular momentum, which is contained in the wave according to the canonical spin tensor of electrodynamics. Lorentz transformations are used for energy, momentum, and angular momentum flux density because a moving absorber is considered. The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum.
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1. Introduction

It was suggested as early as 1899 by Sadowsky [1] and as 1909 by Poynting [2] that any circularly polarized light contains angular momentum *density*. That is the angular momentum is present in any point of the light.

According to the Lagrange formalism, this angular momentum density is *spin* density. The spin of electromagnetic waves is described by a spin tensor [3 -5].

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial\mathscr{L}}{\partial(\partial_\nu A_\alpha)}, \qquad (1.1)$$

where $\mathscr{L}$ is a Lagrangian and A^λ is the magnetic vector potential of the electromagnetic field. So, any infinitesimal 3-volume dV_ν contains spin

$$dS^{\lambda\mu} = \Upsilon^{\lambda\mu\nu}dV_\nu. \qquad (1.2)$$

Against this, an oppinion is spread that an electromagnetic plane wave of infinit extension has no angular momentum.

> **Heitler W:** "A plane wave travelling in z-direction and with infinite extension in the xy-directions can have no angular momentum about the z-axis, because Π is in the z-direction and $(\mathbf{r}\times\Pi)_z = 0$ " [6]

Here Π is the Poynting vector.

> **Simmonds J. W., Guttmann M. J.:** "The electric and magnetic fields can have a nonzero z-component only within the skin region of this wave. Having z-components within this region implies the possibility of a nonzero z-component of angular momentum within this region. So, the skin region (of a beam) is the only in which the z-component of angular momentum does not vanish" [7, p. 227]
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> **Allen L., Padgett M. J.:** "For a plane wave there is no (radial intensity) gradient and the spin density is zero" [8]

On the other hand, according to [9], this oppinion is a mistake. The Poynting's and Sadowsky's concept seems to be true: (spin) angular momentum is present in any point of a circularly polarized electromagnetic beam and, accordingly, torque acts on any point of an absorber of such a beam.

In this paper, we confirm the Poynting's and Sadowsky's concept by a new calculation.

Since 1905, when Einstein explained the photoelectric effect, it has become clear that an electromagnetic wave consists of photons. Photons have energy, momentum and spin (internal angular momentum), and if the wave is circularly polarized, spins of all the photons are directed in the same direction that is parallel to that of the momentum. Therefore, use is made of such notions as volume density and flux density of momentum, energy, and spin as well as number of photons in an electromagnetic wave. Densities of the energy and momentum are quantitatively described by the

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Maxwell energy-momentum tensor. The density of the spin should be described by the spin tensor (1.1). The number density of photons is obtained by dividing the energy density of a wave by the energy of a single photon $\hbar\omega$, or by dividing the spin density by the spin of a single photon $\hbar$ (if polarization is circular).

In a famous paper [10], the reflection of light from a moving boundary of two mediums is essentially exhaustively investigated. In addition, in our previous papers, we have examined the implementation of the law of conservation of energy, momentum and spin angular momentum for incidence of a plane electromagnetic wave on a mirror [11,12,13], and on the surface of a fixed dielectric [14]. In this paper, we consider the incidence of such a wave on the surface of a moving "symmetric absorber".

2. A symmetric absorber

We call "symmetric absorber" a medium, which is dielectric and magnetic with $\varepsilon = \mu$. Such a medium does not require a reflected wave; it simplifies formulas.

So, let a plane monochromatic circularly polarized electromagnetic wave

$$\breve{\mathbf{E}} = E(\mathbf{x} + i\mathbf{y})\exp(ikz - i\omega t) \ \text{[V/m]}, \quad \breve{\mathbf{H}} = -i\varepsilon_0 c\breve{\mathbf{E}} \ \text{[A/m]}, \quad ck = \omega \tag{2.1}$$

impinges normally on a flat x,y-surface of the absorber, which is characterized by complex permittivity and permeability $\breve{\varepsilon} = \breve{\mu}$ (we mark complex numbers and vectors by *breve*) and moves along z axis with speed v.

As is known, wave (2.1) contains volume density of mass-energy u, flux density of mass-energy (the Poynting vector) Π, volume density of momentum G, and flux density of momentum (pressure) $\mathscr{P}$, according to formulas

$$u = \frac{\varepsilon_0 E^2}{c^2}\left[\frac{\text{kg}}{\text{m}^3}\right], \ \Pi = G = \frac{\varepsilon_0 E^2}{c}\left[\frac{\text{kg}}{\text{m}^2 s}\right], \ \mathscr{P} = \varepsilon_0 E^2 \left[\frac{\text{kg}}{\text{m}s^2} = \frac{\text{N}}{\text{m}^2}\right], \tag{2.2}$$

but because of Doppler Effect [15 § 48], our wave has lesser frequency and, according to [10], has lesser amplitude *relative to the moving absorber*

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}, \qquad E' = E\sqrt{\frac{1-\beta}{1+\beta}}. \tag{2.3}$$

($\beta = v/c$). So, relative to the absorber, the impinging wave is expressed by the formulas

$$\breve{\mathbf{E}}' = E'(\mathbf{x} + i\mathbf{y})\exp(ik'z - i\omega' t), \quad \breve{\mathbf{H}}' = -i\varepsilon_0 c\breve{\mathbf{E}}', \quad ck' = \omega' \tag{2.4}$$

Accordingly, the Poynting vector and the momentum flux density prove to be lesser relative to the moving surface

$$\Pi' = \frac{\varepsilon_0 E'^2}{c} = \frac{\varepsilon_0 E^2}{c}\frac{1-\beta}{1+\beta}, \quad \mathscr{P}' = \varepsilon_0 E'^2 = \varepsilon_0 E^2 \frac{1-\beta}{1+\beta}. \tag{2.5}$$

3. The Lorentz transformations

However, from the viewpoint of an observer at rest, these latter quantities, i.e. mass-energy flux densities and momentum fluxes density through the surface, have other values. These values must be found by the Lorentz transformations for coordinates of a 4-point and for components of 4-momentum

$$t = \frac{t' + vz'/c^2}{\sqrt{1-\beta^2}}, \quad z = \frac{z' + vt'}{\sqrt{1-\beta^2}}, \quad m = \frac{m' + vp'/c^2}{\sqrt{1-\beta^2}}, \quad p = \frac{p' + vm'}{\sqrt{1-\beta^2}}. \tag{3.1}$$

We denote these flux densities by $\Pi_0, \mathscr{P}_0$. Taking into account that densities satisfy the equations,

$$\Pi_0 = m/at, \ \mathscr{P}_0 = p/at, \ \Pi' = m'/at', \ \mathscr{P}' = p'/at', \tag{3.2}$$

where a is a not being transformed area, and substituting values (3.1), when $z' = 0$, into expression (3.2), we get Lorentz transformations for the flux densities

$$\Pi_0 = \Pi' + v\mathscr{P}'/c^2, \ \mathscr{P}_0 = \mathscr{P}' + \Pi' v. \tag{3.3}$$

So, from the viewpoint of an observer at rest, the flux density of mass-energy, which enters into the absorber, equals

$$\Pi_0 = \Pi' + \frac{v\mathscr{P}'}{c^2} = \frac{\varepsilon_0 E^2}{c}\frac{1-\beta}{1+\beta} + \frac{v}{c^2}\varepsilon_0 E^2 \frac{1-\beta}{1+\beta} = \frac{\varepsilon_0 E^2}{c}(1-\beta) \tag{3.4}$$

4. Filling of the space with mass

Flux density Π_0 (3.4) is lesser than flux density Π (2.2), which is brought by the incident wave. The difference between the mass fluxes (2.2) and (3.4) is spent on filling of the space, which become free of the moving absorber. This filling requires mass flux density, which we denote $\tilde{\Pi}$,

$$\tilde{\Pi} = uv = \frac{\varepsilon_0 E^2}{c^2} v = \frac{\varepsilon_0 E^2}{c}\beta . \tag{4.1}$$

As a result, we have an elementary equality

$$\Pi = \tilde{\Pi} + \Pi_0 = \frac{\varepsilon_0 E^2}{c} . \tag{4.2}$$

But it is desirable to demonstrate the mechanism of the absorption of mass flux density Π' (2.5) in the symmetric absorber

5. Absorption of energy and angular momentum

According to (2.4), the wave propagated in the absorber is expressed by the formulas

$$\breve{\mathbf{E}}' = E'(\mathbf{x} + i\mathbf{y})\exp(ik'\breve{k}z - i\omega' t') , \quad \breve{\mathbf{H}}' = -i\varepsilon_0 c\breve{\mathbf{E}}' , \quad ck' = \omega' \quad \breve{k} = \sqrt{\breve{\varepsilon}\breve{\mu}} = \breve{\varepsilon} = \breve{\mu} = k_1 + ik_2 \tag{5.1}$$

A mechanism of the absorption in dielectric was explained by Feynman [16] very good. According to the explanation, the rotating electric field $\breve{\mathbf{E}}' = \breve{E}'(\mathbf{x} + i\mathbf{y})\exp(-i\omega' t)$ exerts a torque $\tau = \breve{\mathbf{d}} \times \breve{\mathbf{E}}'$ on rotating dipole moments of molecules $\mathbf{d}$ of the polarized dielectric and makes a work. The power volume density of this work is

$$w_e = \left|\breve{\mathbf{P}}_e \times \breve{\mathbf{E}}'\right| \omega' \ \ [\mathrm{J/m^3 s}], \quad \breve{\mathbf{P}}_e = (\breve{\varepsilon} - 1)\varepsilon_0 \breve{\mathbf{E}}' , \tag{5.2}$$

$\breve{\mathbf{P}}_e$ is the polarization vector, and $\breve{\mathbf{P}}_e \times \breve{\mathbf{E}}'$ [J/m^3] is a *torque volume density*. The calculation gives

$$w_e = \frac{\omega'}{2}\Re\{\breve{P}_{ex}\overline{E}'_y - \breve{P}_{ey}\overline{E}'_x\} = \frac{\omega'\varepsilon_0}{2}\Re\{(\breve{\varepsilon}-1)(\breve{E}'_x\overline{E}'_y - \breve{E}'_y\overline{E}'_x)\} = \frac{\omega'\varepsilon_0}{2}\exp(-2k'k_2 z)\Re\{(\breve{\varepsilon}-1)(-i-i)\}E'^2$$
$$= \omega'\varepsilon_0 \exp(-2k'k_2 z)\Im(\breve{\varepsilon}-1)E'^2 = \omega'\varepsilon_0\exp(-2k'k_2 z)k_2 E'^2 . \tag{5.3}$$

It is naturally that the rotating magnetic field of electromagnetic wave (5.1) makes the same work over rotating magnetic dipoles in our absorber.

$$w_m = \left|\breve{\mathbf{P}}_m \times \breve{\mathbf{H}}'\right|\mu_0\,\omega' \ \ [\mathrm{J/m^3 s}], \quad \breve{\mathbf{P}}_m = (\breve{\mu} - 1)\mathbf{H}' , \tag{5.4}$$

$$w_m = \omega'\Re\{\breve{P}_{mx}\overline{H}'_y - \breve{P}_{my}\overline{H}'_x\}\mu_0 / 2 = \omega'\mu_0\Re\{(\breve{\mu}-1)(\breve{H}'_x\overline{H}'_y - \breve{H}'_y\overline{H}'_x)\}/2 . \tag{5.5}$$

Substituting value (5.1) for the magnetic field into (5.5), we see that the work of the magnetic field is equal to the work of the electric field

$$w_m = \omega'\varepsilon_0\Re\{(\breve{\varepsilon}-1)(\breve{E}'_x\overline{E}'_y - \breve{E}'_y\overline{E}'_x)\}/2 = w_e . \tag{5.6}$$

The energy flux density, which comes on the surface of the absorber from the waves, can be obtained by an integration of the total power volume density, $w = w_e + w_m = 2w_e$, over z

$$\int_0^\infty 2w_e dz = 2\omega'\varepsilon_0\int_0^\infty \exp(-2k'k_2 z)k_2 E'^2 dz = \frac{\omega'\varepsilon_0}{k'}E'^2 = \varepsilon_0 c E'^2 = \Pi' c^2 \left[\frac{\mathrm{J}}{\mathrm{m}^2 s}\right] . \tag{5.7}$$

So, the total energy flux density coincides with (2.5) $\Pi' c^2$.

But we must recognize that the torque volume density[2] $\tau_{\sim} = \breve{\mathbf{P}}_e \times \breve{\mathbf{E}}' + \breve{\mathbf{P}}_m \times \breve{\mathbf{H}}'\mu_0$, which supply with energy inside the absorber, in the same time, is a volume density of *angular momentum flux*, which comes into the absorber. The torque volume density $\tau_{\sim}$ produces specific mechanical stresses in

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[2] We mark pseudo densities by index *tilda*. The torque volume density $\tau_{\sim}$ is a pseudo *density*, as opposed to the torque τ.

the dielectric [9]. And, as the volume density of angular momentum flux, the torque volume density requires angular momentum flux density, which comes on the absorber surface from the wave. We get this angular momentum flux density by integrating the torque volume density $\tau_{\sim}$ over z.

$$\Upsilon' = \int_0^\infty \left| \breve{\mathbf{P}}_e \times \breve{\mathbf{E}}' + \breve{\mathbf{P}}_m \times \breve{\mathbf{H}}'\mu_0 \right| dz = \frac{1}{\omega'} \int_0^\infty (w_e + w_m) dz = \frac{\Pi' c^2}{\omega'} = \frac{\varepsilon_0 c}{\omega'} E'^2 \quad \left[\frac{\mathrm{J}}{\mathrm{m}^2}\right]. \tag{5.8}$$

Using formulas (2.3), we can express this angular momentum flux density in terms of the incident wave (2.1)

$$\Upsilon' = \frac{\varepsilon_0 c}{\omega'} E'^2 = \frac{\varepsilon_0 c}{\omega} E^2 \sqrt{\frac{1-\beta}{1+\beta}}, \tag{5.9}$$

And in order to transform it to the laboratory at rest, we must take into account that an angular momentum flux density satisfies equalities

$$\Upsilon_0 = J / at, \quad \Upsilon' = J' / at',, \tag{5.10}$$

where a is a not being transformed area, and $J = J'$ is a not being transformed angular momentum relative the axis z. Taking into account (3.1), equations (5.10) yield the angular momentum flux density entered into the absorber from the viewpoint of an observer at rest:

$$\Upsilon_0 = \Upsilon' t' / t = \frac{\varepsilon_0 c}{\omega} E^2 \sqrt{\frac{1-\beta}{1+\beta}} \sqrt{1-\beta^2} = \frac{\varepsilon_0 c}{\omega} E^2 (1-\beta). \tag{5.11}$$

Results of this Section concerning the absorbtion of energy and angular momentum in dielectric were first published in paper [17].

6. Calculation of the angular momentum flux density, which is contained in the electromagnetic wave

By the account that angular momentum (5.11) is absorbed under every square meter of the absorber surface per second, one can concludes that the angular momentum is brought to the surface by the wave (2.1). To calculate this bringing angular momentum flux, it is natural to use the electrodynamics canonical spin tensor [3,4]

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda} F^{\mu]\nu}, \tag{6.1}$$

here $F^{\mu\nu}$ is the electromagnetic field tensor, and A^λ is the magnetic vector potential.

Angular momentum flux density, which is directed along z-axis to xy surface, is given by the component

$$\Upsilon^{xyz} = -2A^{[x} F^{y]z} = A_x H_x + A_y H_y \ [\mathrm{J/m^2}]. \tag{6.2}$$

Note that the lowering of the spatial index of the vector potential is related to the change of the sign in the view of the metric signature $(+---)$. Since $A_k = -\int E_k dt = -iE_k / \omega$ for a monochromatic field, densities (6.1), (6.2) can be expressed through the electromagnetic field:

$$\Upsilon^{xyz} = (-iE_x H_x - iE_y H_y) / \omega. \tag{6.3}$$

In our case, in addition to (2.2), we have spin flux density

$$\Upsilon = \langle \Upsilon^{xyz} \rangle = \Re\{-iE_x \overline{H}_x - iE_y \overline{H}_y\} / 2\omega = \frac{\varepsilon_0 c}{\omega} E^2 = \frac{\Pi c^2}{\omega}. \tag{6.4}$$

for the incident wave (2.1). This quantity, (6.4), is larger than the angular momentum flux density Υ_0 (5.11), which enters into the absorber. The difference between the angular momentum fluxes (6.4) and (5.11) is spent on filling of the space vacated by the absorber, moving at a speed of v. This filling requires angular momentum flux density, which we denote $\tilde{\Upsilon}$. Angular momentum volume density is given by the component

$$\Upsilon^{xyt} = -2A^{[x} F^{y]t} = -A_x D_y + A_y D_x = (iE_x D_y - iE_y D_x) / \omega \tag{6.5}$$

of the spin tensor (6.1). By the time averaging, we get

$$\langle \Upsilon^{xyt} \rangle = \Re\{(iE_x \overline{D}_y - iE_y \overline{D}_x) / 2\omega = \varepsilon_0 E^2 / \omega \ [\mathrm{Js/m^3}]. \tag{6.6}$$

So, the filling of the space requires

$$\tilde{\Upsilon} = <\Upsilon^{xyt}> v = \frac{\varepsilon_0 E^2}{\omega} v = \frac{\varepsilon_0 c E^2}{\omega} \beta . \qquad (6.7)$$

As a result, we have an elementary equality similary to (4.2)

$$\Upsilon = \tilde{\Upsilon} + \Upsilon_0 = \frac{\varepsilon_0 c E^2}{\omega} . \qquad (6.8)$$

7. Conclusion

The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum. Recognizing the existence of photons with momentum, energy and spin in a plane electromagnetic wave, it is strange to deny the existence of spin in such a wave, as is done in modern electrodynamics.

I am eternally grateful to Professor Robert Romer, having courageously published my question: "Does a plane wave really not carry spin?" [18].

References

1. Sadowsky A. *Acta et Comm. Imp. Universitatis Jurievensis* 7, No. 1-3 (1899)
2. Poynting J. H., "The wave motion of a revolving shaft, and a suggestion as to the angular momentum in a beam of circularly polarised light". *Proc. R. Soc. Lond. A* 82, 560-567 (1909)
3. Corson E M *Introduction to tensors, spinors, and reativistic wave-equation* NY, Hafner, 1953 p.71
4. Soper D. E., *Classical Field Theory* (N.Y.: Dover, 2008), p. 114
5. Barut A. O. *Electrodynamics and Classical Theory of Particles and Fields* (Macmillan, New York, 1964), p. 102
6. Heitler W *The Quantum Theory of Radiation* (Oxford: Clarendon, 1954) p. 401.
7. Simmonds J. W., M. J. Guttmann, *States, Waves and Photons* (Addison-Wesley, Reading, MA, 1970)
8. Allen L., Padgett M. J. “Response to Question #79. Does a plane wave carry spin angular momentum?” Am. J. Phys. 70, 567 (2002).
9. Khrapko R.I. "Mechanical stresses produced by a light beam" *J. Modern Optics*, 55, 1487-1500 (2008)
10. Bolotovskii B. M, Stolyarov S, N “Reflection of light from a moving mirror and related problems” *Sov. Phys. Usp*. **32** 813–827 (1989) http://ufn.ru/en/articles/1989/9/e/
11. Khrapko R.I. "Reflection of light from a moving mirror" http://khrapkori.wmsite.ru/ftpgetfile.php?id=153&module=files
12. Khrapko R.I. "Spin transferred to a mirror reflecting light" http://viXra.org/abs/1702.0032
13. Khrapko R.I. "Spin transferred to a mirror reflecting light" (2005) http://www.mai.ru/science/trudy/published.php?ID=34126 ,
14. Khrapko R.I. "The use of the electrodynamics canonical spin tensor" http://khrapkori.wmsite.ru/ftpgetfile.php?id=154&module=files
15. Landau L. D., Lifshitz E. M. *The Classical Theory of Fields* (Pergamon, N. Y. 1975)
16. Feynman R. P., R. B. Leighton, M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, London, 1965) Vol. 3, p. 17–10.
17. Khrapko R.I. "Absorbtion of light in a dielectric " http://www.mai.ru/science/trudy/published.php?ID=34422 (2003) in Russian,: http://khrapkori.wmsite.ru/ftpgetfile.php?id=116&module=files in Ehglish
18. Khrapko R I "Does plane wave not carry a spin?" *Amer. J. Phys*. 69, 405 (2001) http://khrapkori.wmsite.ru/ftpgetfile.php?id=10&module=files

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We have reviewed your submission (our manuscript #29287) and determined that it is not appropriate for publication. David P. Jackson, Daniel V. Schroeder

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MB-1524. This paper has now been considered by the editorial office. Whilst I have no cause at this point to doubt the correctness of your work, I am of the opinion that it does not meet the standards required for publication in Optics Communications. I refer particularly here to our policy that "Manuscripts should offer clear evidence of novelty and significance" and that "small technical advances ... fall outside the journal scope". You work concerns matters that have been extensively discussed in the literature. These observations do not seem to have significant consequence in the wider field of research, so could reasonably considered a small technical advance. In view of this, I regret to say that I cannot accept this paper for publication in Optics Communications.
Martin Booth

Journal of Optics

Absorption the angular momentum of a circularly polarized electromagnetic wave is well known effect open by Sadovskii in 1889. The angular momentum of a classical electromagnetic plane wave of arbitrary extent is predicted to be, on theoretical grounds, exactly zero. However, finite sections of circularly polarized plane waves are found experimentally to carry angular momentum and it is known that the contribution to the angular momentum arises from the edges of the beam. A mathematical model that gives a quantitative account of this effect and resolves the paradox was done by A. M. Stewart in 2005, see "Angular momentum of the electromagnetic field: the plane wave paradox resolved" European Journal of Physics, Volume 26, Number 4 (2005), Published 6 May 2005. Therefore, I cannot see sufficient novelty in this paper to warrant consideration in JOPT.
Nikolay Zheludev

Author's reply

A. M. Stewart's papers are mistaken. Stewart's mistaks were exposed in: R. Khrapko "Mechanical stresses produced by a light beam" *J. Modern Optics*, **55**, 1487-1500 (2008)
http://khrapkori.wmsite.ru/ftpgetfile.php?id=9&module=files (2533 dowloads)

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