Absorption of angular momentum of a plane electromagnetic wave

Radi I. Khrapko[1]
Moscow Aviation Institute - Volokolamskoe shosse 4, 125993 Moscow, Russia

It is demonstrated that dielectric or magnetic, which absorbs a circularly polarized plane electromagnetic wave, absorbs the angular momentum, which is contained in the wave according to the canonical spin tensor of electrodynamics. Lorentz transformations are used for energy, momentum, and angular momentum flux density because a moving absorber is considered. The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum.
Key Words: classical spin, circular polarization, spin tensor
PACS 75.10.Hk

1. Introduction. A symmetric absorber

The reflection of light from a moving boundary of two mediums is essentially exhaustively investigated in a famous paper [1]. Nevertheless, it seems interesting to demonstrate the implementation of the conservation laws with respect to energy and angular momentum fluxes within concrete situations.

The reflection of light from a moving mirror is considered in article [2]. But a mirror does not absorb energy and angular momentum (It absorbs only momentum). The absorption and reflection of light from dielectric at rest is considered in article [3]. In the present paper, we consider interaction of light with *moving* dielectric and magnetic, which is characterized by $\varepsilon = \mu$ (This is a "*symmetric absorber*"). Such a medium does not require a reflected wave; it is simplifies formulas.

So, let a plane monochromatic circularly polarized electromagnetic wave

$$\breve{\mathbf{E}} = E(\mathbf{x} + i\mathbf{y})\exp(ikz - i\omega t)\ \text{[V/m]}, \quad \breve{\mathbf{H}} = -i\varepsilon_0 c\breve{\mathbf{E}}\ \text{[A/m]}, \quad ck = \omega \tag{1.1}$$

impinges normally on a flat x,y-surface of the absorber, which is characterized by complex permittivity and permeability $\breve{\varepsilon} = \breve{\mu}$ (we mark complex numbers and vectors by *breve*) and moves along z axis with velocity v.

As is known, wave (1.1) contains volume density of mass-energy u, flux density of mass-energy (the Poynting vector) Π, volume density of momentum G, and flux density of momentum (pressure) $\mathscr{P}$, according to formulas

$$u = \frac{\varepsilon_0 E^2}{c^2}\left[\frac{\text{kg}}{\text{m}^3}\right], \quad \Pi = G = \frac{\varepsilon_0 E^2}{c}\left[\frac{\text{kg}}{\text{m}^2 s}\right], \quad \mathscr{P} = \varepsilon_0 E^2\left[\frac{\text{kg}}{\text{m}s^2} = \frac{\text{N}}{\text{m}^2}\right], \tag{1.2}$$

but because of Doppler Effect [4 § 48], our wave has lesser frequency and, according to [1], has lesser amplitude *relative to the moving absorber*

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}, \qquad E' = E\sqrt{\frac{1-\beta}{1+\beta}}. \tag{1.3}$$

($\beta = v/c$). So, relative to the absorber, the impinging wave is expressed by the formulas

$$\breve{\mathbf{E}}' = E'(\mathbf{x} + i\mathbf{y})\exp(ik'z - i\omega' t), \quad \breve{\mathbf{H}}' = -i\varepsilon_0 c\breve{\mathbf{E}}', \quad ck' = \omega' \tag{1.4}$$

Accordingly, the Poynting vector and the momentum flux density prove to be lesser relative to the moving surface

$$\Pi' = \frac{\varepsilon_0 E'^2}{c} = \frac{\varepsilon_0 E^2}{c}\frac{1-\beta}{1+\beta}, \quad \mathscr{P}' = \varepsilon_0 E'^2 = \varepsilon_0 E^2\frac{1-\beta}{1+\beta}. \tag{1.5}$$

[1] Email: khrapko_ri@hotmail.com, http://khrapkori.wmsite.ru

2. The Lorentz transformations

However, from the viewpoint of an observer at rest, these latter quantities, i.e. densities of energy and momentum fluxes through the surface, have other values. These values must be found by the Lorentz transformations for coordinates of a 4-point and for components of 4-momentum

$$t = \frac{t' + vz'/c^2}{\sqrt{1-\beta^2}}, \quad z = \frac{z' + vt'}{\sqrt{1-\beta^2}}, \quad m = \frac{m' + vp'/c^2}{\sqrt{1-\beta^2}}, \quad p = \frac{p' + vm'}{\sqrt{1-\beta^2}}. \tag{2.1}$$

We denote these values of the flux densities by $\Pi_0, \mathscr{P}_0$. Taking into account that densities satisfy the equations,

$$\Pi_0 = m/at\,, \ \mathscr{P}_0 = p/at\,, \ \Pi' = m'/at'\,, \ \mathscr{P}' = p'/at'\,, \tag{2.2}$$

where a is a not being transformed area, and substituting values (2.1), when $z' = 0$, into expression (2.2), we get Lorentz transformations for the densities

$$\Pi_0 = \Pi' + v\mathscr{P}'/c^2\,, \ \ \mathscr{P}_0 = \mathscr{P}' + \Pi' v\,. \tag{2.3}$$

So, from the viewpoint of an observer at rest near the absorber surface, the flux density of mass-energy, which enters into the absorber, equals

$$\Pi_0 = \Pi' + \frac{v\mathscr{P}'}{c^2} = \frac{\varepsilon_0 E^2}{c}\frac{1-\beta}{1+\beta} + \frac{v}{c^2}\varepsilon_0 E^2 \frac{1-\beta}{1+\beta} = \frac{\varepsilon_0 E^2}{c}(1-\beta) \tag{2.4}$$

3. Filling of the space with mass

Flux density Π_0 (2.4) is lesser than mass flux density Π (1.2), which is brought with the incident wave. The difference between the mass flux (1.2) and (2.4) is spent on filling of the space, which become free of the moving absorber. This filling requires mass flux density, which we denote $\tilde{\Pi}$,

$$\tilde{\Pi} = uv = \frac{\varepsilon_0 E^2}{c^2} v = \frac{\varepsilon_0 E^2}{c}\beta\,. \tag{3.1}$$

As a result, we have an elementary equality

$$\Pi = \tilde{\Pi} + \Pi_0 = \frac{\varepsilon_0 E^2}{c}. \tag{3.2}$$

But it is desirable to demonstrate the mechanism of the absorption of mass flux density Π' (1.5) in the symmetric absorber

4. Absorption of energy and angular momentum

Accordingly to (1.4), the wave propagated in the absorber is expressed by the formulas

$$\breve{\mathbf{E}}' = E'(\mathbf{x} + i\mathbf{y})\exp(ik'\breve{k}z - i\omega' t')\,, \quad \breve{\mathbf{H}}' = -i\varepsilon_0 c\breve{\mathbf{E}}'\,, \quad ck' = \omega' \quad \breve{k} = \sqrt{\breve{\varepsilon}\breve{\mu}} = \breve{\varepsilon} = \breve{\mu} = k_1 + ik_2 \tag{4.1}$$

A mechanism of the absorption in dielectric was explained by Feynman [5] very good. Accordingly to the explanation, the rotating electric field $\breve{\mathbf{E}}' = \breve{E}'(\mathbf{x} + i\mathbf{y})\exp(-i\omega' t)$ exerts a moment of force $\tau = \breve{\mathbf{p}} \times \breve{\mathbf{E}}'$ on rotating dipole moments of molecules of the polarized dielectric and makes a work. The power volume density of this work is

$$w_e = \left|\breve{\mathbf{P}}_e \times \breve{\mathbf{E}}'\right|\omega' \ \ [\mathrm{J/m^3s}], \quad \breve{\mathbf{P}}_e = (\breve{\varepsilon} - 1)\varepsilon_0 \breve{\mathbf{E}}'\,, \tag{4.2}$$

$\breve{\mathbf{P}}_e$ is the polarization vector. The calculation gives

$$w_e = \frac{\omega'}{2}\Re\{\breve{P}_{ex}\overline{E}'_y - \breve{P}_{ey}\overline{E}'_x\} = \frac{\omega'\varepsilon_0}{2}\Re\{(\breve{\varepsilon}-1)(\breve{E}'_x\overline{E}'_y - \breve{E}'_y\overline{E}'_x)\} = \frac{\omega'\varepsilon_0}{2}\exp(-2k'k_2 z)\Re\{(\breve{\varepsilon}-1)(-i-i)\}E'^2$$
$$= \omega'\varepsilon_0 \exp(-2k'k_2 z)\Im(\breve{\varepsilon}-1)E'^2 = \omega'\varepsilon_0 \exp(-2k'k_2 z)k_2 E'^2\,. \tag{4.3}$$

It is naturally that the rotating magnetic field of electromagnetic wave (4.1) makes the same work over rotating magnetic dipoles in our absorber.

$$w_m = \left|\breve{\mathbf{P}}_m \times \breve{\mathbf{H}}'\right|\mu_0\,\omega' \ \ [\mathrm{kg/m^3s}], \quad \breve{\mathbf{P}}_m = (\breve{\mu} - 1)\mathbf{H}'\,, \tag{4.4}$$

$$w_m = \omega'\Re\{\breve{P}_{mx}\overline{H}'_y - \breve{P}_{my}\overline{H}'_x\}\mu_0/2 = \omega'\mu_0\Re\{(\breve{\mu}-1)(\breve{H}'_x\overline{H}'_y - \breve{H}'_y\overline{H}'_x)\}/2\,. \tag{4.5}$$

Substituting value (4.1) for the magnetic field into (4.5), we see that the work of the magnetic field is equal to the work of the electric field

$$w_m = \omega'\varepsilon_0 \Re\{(\breve{\varepsilon}-1)(\breve{E}'_x \overline{E}'_y - \breve{E}'_y \overline{E}'_x)\}/2 = w_e \,. \qquad (4.6)$$

The energy flux density, which comes on the surface of the absorber from the waves, can be obtained by an integration of the total power volume density, $w = w_e + w_m = 2w_e$, over z

$$\Pi' c^2 = \int_0^\infty 2w_e dz = 2\omega'\varepsilon_0 \int_0^\infty \exp(-2k'k_2 z)k_2 E'^2 dz = \frac{\omega'\varepsilon_0}{k'} E'^2 = \varepsilon_0 c E'^2 \left[\frac{\mathrm{J}}{\mathrm{m}^2 s}\right]. \qquad (4.7)$$

So, the total energy flux density coincides with (1.5)

But we must recognize that the volume density of moment of force[2] $\tau_{\sim} = \breve{\mathbf{P}}_e \times \breve{\mathbf{E}}' + \breve{\mathbf{P}}_m \times \breve{\mathbf{H}}'\mu_0$, which supply with energy inside the absorber, in the same time, is a volume flux density of *angular momentum*, which comes into the absorber. The volume density of moment of force $\breve{\mathbf{P}}_e \times \breve{\mathbf{E}}' + \breve{\mathbf{P}}_m \times \breve{\mathbf{H}}'\mu$ produces specific mechanical stresses in the dielectric [6]. And, as a volume flux density of angular momentum, it requires angular momentum flux density, which comes on the absorber surface from the waves. We get this angular momentum flux density by integrating over z.

$$\Upsilon' = \int_0^\infty \left|\breve{\mathbf{P}}_e \times \breve{\mathbf{E}}' + \breve{\mathbf{P}}_m \times \breve{\mathbf{H}}'\mu_0\right| dz = \frac{1}{\omega'}\int_0^\infty (w_e + w_m) dz = \frac{\Pi' c^2}{\omega'} = \frac{\varepsilon_0 c}{\omega'} E'^2 \;\left[\frac{\mathrm{J\,s}}{\mathrm{m}^2 s}\right]. \qquad (4.8)$$

Using formulas (1.3), we can express this angular momentum flux density in terms of the incident wave (1.1)

$$\Upsilon' = \frac{\varepsilon_0 c}{\omega'} E'^2 = \frac{\varepsilon_0 c}{\omega} E^2 \sqrt{\frac{1-\beta}{1+\beta}}, \qquad (4.9)$$

And in order to transform it to the laboratory at rest, we must take into account that an angular momentum flux density satisfies equalities

$$\Upsilon_0 = J/at, \quad \Upsilon' = J'/at', , \qquad (4.10)$$

where a is a not being transformed area, and $J = J'$ is a not being transformed angular momentum relative the axis z. Taking into account (2.1), equations (4.11) yield the angular momentum flux density entered into the absorber from the viewpoint of an observer at rest:

$$\Upsilon_0 = \Upsilon' t'/t = \frac{\varepsilon_0 c}{\omega} E^2 \sqrt{\frac{1-\beta}{1+\beta}}\sqrt{1-\beta^2} = \frac{\varepsilon_0 c}{\omega} E^2 (1-\beta). \qquad (4.11)$$

Results of this Section concerning the absorbtion of energy and angular momentum in dielectric were first published in paper [7].

5. Calculation of the angular momentum flux density, which is contained in the electromagnetic wave

By the account that angular momentum (4.11) is absorbed under every square meter of the absorber surface per second, one can concludes that the angular momentum is brought to the surface by the wave (1.1). To calculate this bringing angular momentum flux, it is natural to use the electrodynamics canonical spin tensor [8,9]

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}, \qquad (5.1)$$

here $F^{\mu\nu}$ is the electromagnetic field tensor, and A^{λ} is the magnetic vector potential.

Angular momentum flux density, which is directed along z-axis to xy surface, is given by the component

$$\Upsilon^{xyz} = -2A^{[x}F^{y]z} = A_x H_x + A_y H_y \;\; [\mathrm{J/m^2}]. \qquad (5.2)$$

---

[2] We mark pseudo densities by index *tilda*. The volume density of moment of force $\tau_{\sim}$ .is a pseudo density, as opposed to the moment of force τ..

Note that the lowering of the spatial index of the vector potential is related to the change of the sign in the view of the metric signature $(+---)$. Since $A_k = -\int E_k dt = -iE_k / \omega$ for a monochromatic field, densities (5.1), (5.2) can be expressed through the electromagnetic field:

$$\Upsilon^{xyz} = (-iE_x H_x - iE_y H_y)/\omega. \quad (5.3)$$

In our case, similarly to (1.4), we have

$$\Upsilon = <\Upsilon^{xyz}> = \Re\{-iE_x \overline{H}_x - iE_y \overline{H}_y\}/2\omega = \frac{\varepsilon_0 c}{\omega} E^2. \quad (5.4)$$

for the incident wave (1.1). This quantity, (5.4), is larger than angular momentum flux density (4.12), which enters into the absorber. The difference between the angular momentum fluxes (5.4) and (4.12) is spent on filling of the space, which become free of the moving absorber. This filling requires angular momentum flux density, which we denote $\tilde{\Upsilon}$. Angular momentum volume density is given by the component

$$\Upsilon^{xyt} = -2A^{[x}F^{y]t} = -A_x D_y + A_y D_x = (iE_x D_y - iE_y D_x)/\omega \quad (5.5)$$

of the spin tensor (5.1). By the time averaging, we get

$$<\Upsilon^{xyt}> = \Re\{(iE_x \overline{D}_y - iE_y \overline{D}_x)/2\omega = \varepsilon_0 E^2/\omega \ \text{[Js/m}^3\text{]}. \quad (5.6)$$

So, the filling of the space requires

$$\tilde{\Upsilon} = <\Upsilon^{xyt}> v = \frac{\varepsilon_0 E^2}{\omega} v = \frac{\varepsilon_0 c E^2}{\omega}\beta. \quad (5.7)$$

As a result, we have an elementary equality

$$\Upsilon = \tilde{\Upsilon} + \Upsilon_0 = \frac{\varepsilon_0 c E^2}{\omega}. \quad (5.8)$$

6. Conclusion

The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum. Recognizing the existence of photons with momentum, energy and spin in a plane electromagnetic wave, it is strange to deny the existence of spin in such a wave, as is done in modern electrodynamics.

I am eternally grateful to Professor Robert Romer, having courageously published my question: "Does a plane wave really not carry spin?” [10]

References

1. Bolotovskii B. M, Stolyarov S, N “Reflection of light from a moving mirror and related problems” *Sov. Phys. Usp.* **32** 813–827 (1989) http://ufn.ru/en/articles/1989/9/e/
2. Khrapko R.I. "Reflection of light from a moving mirror" http://khrapkori.wmsite.ru/ftpgetfile.php?id=153&module=files , http://viXra.org/abs/1611.0167
3. Khrapko R.I. "The use of the electrodynamics canonical spin tensor" http://viXra.org/abs/1612.0232 "Использование канонического тензора спина" http://khrapkori.wmsite.ru/ftpgetfile.php?id=154&module=files
4. Landau L. D., Lifshitz E. M. *The Classical Theory of Fields* (Pergamon, N. Y. 1975)
5. Feynman R. P., R. B. Leighton, M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, London, 1965) Vol. 3, p. 17–10.
6. Khrapko R.I. Mechanical stresses produced by a light beam *J. Modern Optics*, 55, 1487-1500 (2008) http://khrapkori.wmsite.ru/ftpgetfile.php?id=9&module=files
7. Khrapko R.I. "Absorbtion of light in a dielectric " http://www.mai.ru/science/trudy/published.php?ID=34422 (2003) in Russian,: http://khrapkori.wmsite.ru/ftpgetfile.php?id=116&module=files in Ehglish
8. Corson E M *Introduction to tensors, spinors, and reativistic wave-equation* NY, Hafner, 1953 p.71
9. Soper D. E., *Classical Field Theory* (N.Y.: Dover, 2008), p. 114
10. Khrapko R I "Does plane wave not carry a spin?" *Amer. J. Phys*. 69, 405 (2001) http://khrapkori.wmsite.ru/ftpgetfile.php?id=10&module=files