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Shapes with the Minimum Moment of Inertia

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Abstract
A recent paper by one of the authors presented a new formula found experimentally for the shape of the free surface of stable vortices. In this paper we show that the given shape is the concave spinning top with the minimum moment of inertia. A new variation method is used to find this shape, since no existing method seems to cope with the Lagrangian involving $r(z)^4$, inequality constraints and the solution with r(0) = infinity. In the process we find spinning tops with the minimum moment of inertia under various constraints.

Keywords: Variations Calculus, Optimisation, Minimisation, Moment of Inertia, Inequality Constraints

1. Introduction

It has been shown experimentally (Germuska 2016) that the shape of the free surface of water and air vortices is described by the function (1.1) called Vir.

$$r(z) = a\left(\frac{a}{z}\right)^{\alpha} \qquad \text{where} \qquad \alpha \succ 0 \qquad a \succ 0 \tag{1.1}$$

The exponent α determines the shape and the parameter a the size. Vir function is a generalisation of the function for the irrotational vortex in the ideal fluid where the exponent is fixed at $\alpha = ½$ (Kundu 2008). We use the calculus of variations to show that the solid of rotation generated by Vir function (1.1) is the concave spinning top with the minimum moment of inertia, i.e. vortices have a concave shape with least resistance to spinning. More details about concave tops are in section 2.

In physics calculus of variations is used to solve a wide variety of problems, including finding the trajectory $f(t)$ of a particle that is exposed to some forces. In this case $f(t)$ gives the minimum value to the integral of the difference between the kinetic and potential energy (Landau & Lifshitz 1976). Function $f(t)$ is found by solving Euler-Lagrange differential equations (Arfken 1968; Jeffreys & Jeffreys 1980). More details about the principles of variation are in section 3. The details of the method used here are in sections 4 to 9. The rest of this introduction gives a survey of the existing variation methods and the reasons why none of them is suitable to find the solution we seek.

A little over 100 years ago Ritz and Galerkin showed that some differential equations can be solved approximately by expressing function f(t) as a series $\Sigma c_n \phi_n(t)$ and then finding the optimal coefficients c_n (Akheizer 1962; Kantorovich 1964; Courant & Hilbert 1989). With the introduction of computers this method has became widespread. For the expression in the integral with the 1st and 2nd powers of $f(t)$ and possibly also of $f\,'(t)$ the method results in N linear equations of N coefficients c_n. If the base functions ϕ_n are orthogonal then these equations have a sparse matrix with regular patterns, for which it is possible to find an algebraic solution (Chen & Hsiao 1975; Hwang & Shih 1983; Chang & Wang 1983; Hwang & Shih 1985; Horng & Chou 1986; Razzaghi & Razzaghi 1988; Paraskevopoulos et al. 1990; Razzaghi & Marzban 2000; Hsiao 2004).

However, using 3rd and 4th powers of $f(t)$ results in nonlinear equations involving terms $c_m c_n$ and $c_k c_m c_n$ respectively. Since there is no general formula for the cubic or higher power equations, algebraic solutions are extremely rare.

The equations may be nonlinear and hence difficult to solve also for other reasons, e.g. $\sqrt{f(t)}$ or $1/f(t)$ in the integral. Many methods have been developed to solve some of these problems, for example: VIM (He 2007), perturbation (Yildirim & Ozis 2007), linearization (Maleki & Mashali-Firouzi 2010), ADM (Wazwaz 2011), SAM (Jafari 2014), discretization (Maleki & Hashim 2014) and fractional variation (Atanacković et al. 2014).

Finding the shape of the spinning top we seek is more difficult than it may seem at first sight and none of the methods mentioned above seems suitable for this purpose. One of the difficulties is that the solution function (1.1) has an infinity at $z = 0$ and thus some integrals used in variation become infinite. Another difficulty is the requirement that the spinning top must be concave, which is an inequality constraint, but with differential equations one can not use inequalities, only equalities. Yet another difficulty arises because the expression in the integral (Lagrangian) involves the fourth power $r(z)^4$ which results in nonlinear equations, while most methods can solve only problems that involve the second power $r(z)^2$ at the most, thus resulting in linear equations. The variation method used in this paper overcomes these difficulties as outlined below.

To avoid the problem of the infinity at zero we introduce the condition that $r(z) = R$ for $0 < z < \zeta$, where R and ζ are constants. We may choose ζ as small and R as large as we like, within reason considering the accuracy of computation. We also demand that $r(z)$ has no discontinuity at $z = \zeta$.

The problem with the inequality constraint is overcome by converting the inequality involved into an equality using the integral of the absolute value of the function involved, as shown below, which is then treated as any other equality constraint, using the Lagrange multiplier λ.

$$\ddot{r}(z) \geq 0 \quad \text{for} \quad z = [\zeta, 1] \qquad \text{becomes} \qquad \int_{\zeta}^{1} \ddot{r}(z)dz = \int_{\zeta}^{1} |\,\ddot{r}(z)\,|\,dz$$

The problem with the fourth power $r(z)^4$ is resolved by writing $r(z) = r_c(z) + r_d(z)$, with the large "known" coefficients c_n and the small "unknown" coefficients d_n. We then use the binomial expansion of $r(z)^k = [r_c(z) + r_d(z)]^k$ to decompose integrals of $r(z)^k$ and their partial derivatives by d_k into powers of $r_d(z)$, see sections 4 and 5 respectively. The 1st power $r_d(z)^1$ involves only terms d_k and the 2nd power $r_d(z)^2$ only $d_k d_m$. The minimizing equations involve the partial derivatives of the integrals by d_k and the known coefficients c_n which are constants. Thus if we ignore the 3rd and the higher powers $r_d(z)^k$ we obtain approximate linear equations for the coefficients d_n.

$$r(z) = \sum_{n=0}^{N} (c_n + d_n)\phi_n(z) = r_c(z) + r_d(z) \qquad r_c(z) = \sum_{n=0}^{N} c_n \phi_n(z) \qquad r_d(z) = \sum_{n=0}^{N} d_n \phi_n(z)$$

Using the ignored 3rd and the higher powers $r_d(z)^k$ we compute the error in each linear equation involved, then using iteration we obtain a more and more accurate solution. To do so the initial values of the coefficients c_0 and c_1 are guessed and the rest are set to zero. The known coefficients c_n are the input to an iteration cycle and the unknown coefficients d_n are the output. For the next cycle we use $c_n = c_n + d_n$. The coefficients d_n converge to zero and c_n converge to the solution. The iteration process outlined above is described in detail in section 7.

The solution is presented in three stages, at each stage more constraints are added and different shapes with the minimum moment of inertia I are found. The 1st stage finds the spherical spinning top ($Ix = Iy = Iz$) with the minimum moment of inertia of given mass and height. The 2nd adds the condition that the spinning top has a Saturn-like disk of radius R and thickness ζ. The 3rd adds three more constraints, the profile $r(z)$ must be continuous, descending and convex.

2. Concave spinning top

In everyday English the meaning of "convex" and "concave" is clear and easily demonstrated using optical lenses as examples. In functional analysis a function $r(z)$ is convex if its second derivative $r''(z) \geq 0$. If $r(z)$ is convex then $-r(z)$ is concave and vice versa. A more general and more visual definition is based on Jensen inequality, i.e. $r(z)$ is convex if a straight line drawn between any two of its points is above $r(z)$ or at most coincides with it. In geometry a planar or a three dimensional object is convex if a straight line joining any two boundary points never goes outside the object, for example an ellipse. For more details see (Boyd & Vandenberghe 2004). The opposite object, where a straight line joining any two boundary points never goes inside the object doesn't exist. Hence there is no generally accepted equivalent definition of a concave object.

An inflated cone is a convex object and using the lens analogy it seems reasonable to describe a deflated cone as a concave object. In this paper by a "concave spinning top" we will understand a solid of rotation that can be generated by a continuous curve consisting of a positive constant and a positive convex function, as shown in figure 1. Figure 2 shows the same spinning top, we can tell that it is concave because rotated by 90 degrees it can be generated as described.

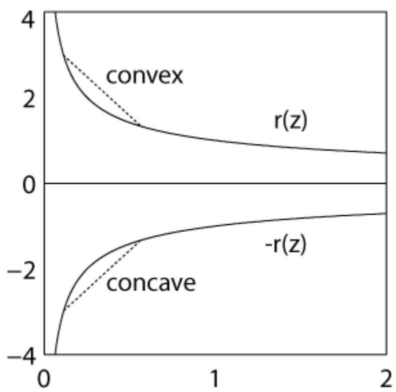

FIGURE 1. Concave spinning top

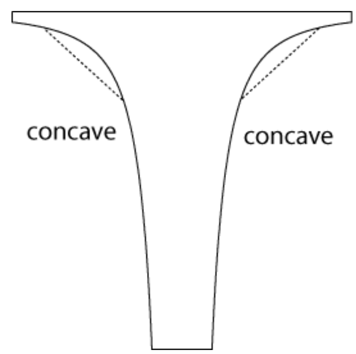

FIGURE 2. Concave spinning top

A symmetric concave spinning top is symmetrical around its centre of mass, as shown in figure 3. Formulas for the moments of inertia are much simpler for symmetrical spinning tops, hence it is this shape generated by Vir function (1.1) for which we prove that it has the minimum moment of inertia of all concave spinning tops. However, the same applies also to the shape in figures 1 and 2.

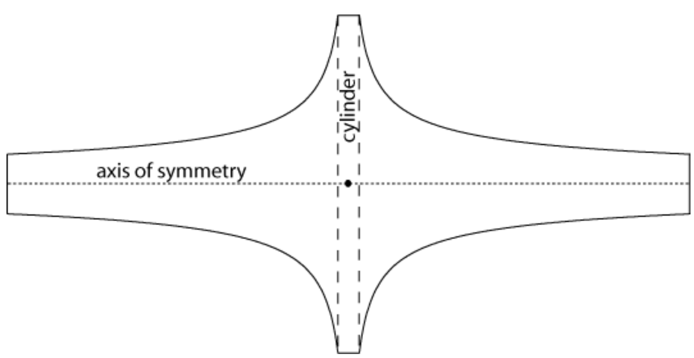

FIGURE 3. Symmetric concave spinning top

3. Euler-Lagrange Equations

Consider the problem of finding the function $f(t)$ which gives a stationary value, i.e. minimum, maximum or an inflection point, of the functional $J(f)$

$$J(f) = \int_a^b F[t, f(t), \dot{f}(t)]dt \tag{3.1}$$

The necessary condition for this is that the function F should satisfy the Euler equation (Arfken 1968; Jeffreys & Jeffreys 1980)

$$\frac{\partial F}{\partial f} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{f}}\right) = 0 \tag{3.2}$$

Let us approximate function $f(t)$ by a series of N chosen functions $\phi_n(t)$ with coefficients c_n

$$f(t) = \sum_{n=0}^{N} c_n \phi_n(t) \tag{3.3}$$

The functional J becomes dependent only on the vector **c**, i.e. the coefficients c_n

$$J(\mathbf{c}) = \int_a^b F[t, \sum_{n=0}^{N} c_n \phi_n(t), \sum_{n=0}^{N} c_n \dot{\phi}_n(t)]dt \tag{3.4}$$

and the condition (3.2) becomes

$$J'_n = \frac{\partial J(\mathbf{c})}{\partial c_n} = 0 \tag{3.5}$$

Assume that we impose a constraint on the function $f(t)$ in the form of an integral

$$\int_a^b G[t, f(t), \dot{f}(t)]dt = \text{constant} \tag{3.6}$$

Using the Lagrange multiplier λ equation (3.1) becomes (Arfken 1968)

$$J(f) = \int_a^b F[t, f(t), \dot{f}(t)] + \lambda \cdot G[t, f(t), \dot{f}(t)]dt \tag{3.7}$$

Treating the integrand as a new function L, usually called the Lagrangian (Landau & Lifshitz 1976), we write $J(f)$ as

$$J(f) = \int_a^b L[t, f(t), \dot{f}(t), \lambda]dt \tag{3.8}$$

The Euler-Lagrange equations that include the constraints are then solved simultaneously to find the solution function $f(t)$, i.e. N unknown c_n and K unknown λ_k using N+K equations.

4. Integrals of the function we seek

A solid of revolution that rotates about axis z is given by function $r(z)$. The maximum z can be set to 1, because it amounts to the choice of the unit of measurement. Since $r(z) = r(-z)$ it is sufficient to consider $r(z)$ on the interval [0,1]. The coefficients in the series $r(z)$ are split into two parts c_n and d_n. Thus we can use the binomial expansion of $[r_c(z) + r_d(z)]^k$ and express it in the powers of **d**.

$$r(z) = \sum_{n=0}^{N} (c_n + d_n)\phi_n(z) = r_c(z) + r_d(z) \qquad (4.1)$$

$$r_c(z) = \sum_{n=0}^{N} c_n\phi_n(z) \qquad r_d(z) = \sum_{n=0}^{N} d_n\phi_n(z) \qquad (4.2)$$

Any real power of the function $r(z)$ can be expanded into natural powers of the function $r_d(z)$. Then the 1st and 2nd powers of the series $r_d(z)$ can be expressed as scalar products of the vector **d**, while the higher powers become scalar functions of tensors $d_i d_i \ldots d_m$ of the corresponding rank.

Integral of $r^2(z)$

$$A = \int_0^1 r(z)^2 dz = \int_0^1 [r_c(z) + r_d(z)]^2 dz = \int_0^1 [r_c(z) + \sum_{n=0}^{N} d_n\phi_n(z)]^2 dz \qquad (4.3a)$$

$$A0 = \int_0^1 r_c(z)^2 dz \qquad \mathbf{A1}_n = 2\int_0^1 r_c(z)\cdot\phi_n(z)dz \qquad \mathbf{A2}_{n,m} = \int_0^1 \phi_n(z)\cdot\phi_m(z)dz \qquad (4.3b)$$

$$A = A0 + \mathbf{A1}\cdot\mathbf{d} + \mathbf{d}^T\cdot\mathbf{A2}\cdot\mathbf{d} \qquad (4.3c)$$

where $A0$ is a scalar, **A1** is a vector and **A2** is a symmetric tensor of rank two, i.e. a matrix.

Integral of $r^4(z)$

$$B = \int_0^1 r(z)^4 dz = \int_0^1 [r_c(z) + r_d(z)]^4 dz = \int_0^1 [r_c(z) + \sum_{n=0}^{N} d_n\phi_n(z)]^4 dz \qquad (4.4a)$$

$$B0 = \int_0^1 r_c(z)^4 dz \qquad \mathbf{B1}_n = 4\int_0^1 r_c(z)^3\cdot\phi_n(z)dz \qquad \mathbf{B2}_{n,m} = 6\int_0^1 r_c(z)^2\cdot\phi_n(z)\cdot\phi_m(z)dz \qquad (4.4b)$$

$$B3 = \tfrac{4}{3}\int_0^1 r_c(z)\cdot r_d(z)^3 dz \qquad B4 = \tfrac{1}{4}\int_0^1 r_d(z)^4 dz \qquad (4.4c)$$

$$B = B0 + \mathbf{B1}\cdot\mathbf{d} + \mathbf{d}^T\cdot\mathbf{B2}\cdot\mathbf{d} + 3\cdot B3(d_i d_j d_k) + 4\cdot B4(d_i d_j d_k d_m) \qquad (4.4d)$$

where $B3$ is a scalar function of the tensor $d_i d_j d_k$ and $B4$ is a scalar function of the tensor $d_i d_j d_k d_m$.

Integral of *r*(*z*)^2*z*^{2}

$$C = \int_0^1 r(z)^2 \cdot z^2 dz = \int_0^1 [r_c(z) + r_d(z)]^2 \cdot z^2 dz = \int_0^1 [r_c(z) + \sum_{n=0}^{N} d_n \phi_n(z)]^2 \cdot z^2 dz \tag{4.5a}$$

$$C0 = \int_0^1 r_c(z)^2 \cdot z^2 dz \qquad \mathbf{C1}_n = 2\int_0^1 r_c(z) \cdot z^2 \cdot \phi_n(z) dz \qquad \mathbf{C2}_{n,m} = \int_0^1 z^2 \cdot \phi_n(z) \cdot \phi_m(z) dz \tag{4.5b}$$

$$C = C0 + \mathbf{C1} \cdot \mathbf{d} + \mathbf{d}^T \cdot \mathbf{C2} \cdot \mathbf{d} \tag{4.5c}$$

Integral of the 1st derivative of *r*(*z*)

This integral is the difference between the start and end of the curve $r(z)$, i.e. $[r(1) - r(\zeta)]$.

$$X = \int_\zeta^1 [\dot{r}_c(z) + \dot{r}_d(z)] dz = [r_c(1) - r_c(\zeta)] + [r_d(1) - r_d(\zeta)] = r(1) - r(\zeta) \tag{4.6a}$$

$$X = \int_\zeta^1 [\sum_{n=0}^{N} c_n \dot{\phi}_n(z) + \sum_{n=0}^{N} d_n \dot{\phi}_n(z)] dz = \sum_{n=0}^{N} (c_n + d_n)(\phi_n(1) - \phi_n(\zeta)) \tag{4.6b}$$

$$X1_n = \phi_n(1) - \phi_n(\zeta) \qquad\qquad X0 = \mathbf{X1} \cdot \mathbf{c} \tag{4.6c}$$

$$X = X0 + \mathbf{X1} \cdot \mathbf{d} \tag{4.6d}$$

Integral of the 2nd derivative of *r*(*z*)

This integral is the difference between the 1st derivatives of $r(z)$ at $z = 1$ and $z = \zeta$.

$$Y = \int_\zeta^1 [\ddot{r}_c(z) + \ddot{r}_d(z)] dz = [\dot{r}_c(1) - \dot{r}_c(\zeta)] + [\dot{r}_d(1) - \dot{r}_d(\zeta)] = \dot{r}(1) - \dot{r}(\zeta) \tag{4.7a}$$

$$Y = \int_\zeta^1 [\sum_{n=0}^{N} c_n \ddot{\phi}_n(z) + \sum_{n=0}^{N} d_n \ddot{\phi}_n(z)] dz = \sum_{n=0}^{N} (c_n + d_n)(\dot{\phi}_n(1) - \dot{\phi}_n(\zeta)) \tag{4.7b}$$

$$Y1_n = \dot{\phi}_n(1) - \dot{\phi}_n(\zeta) \qquad\qquad Y0 = \mathbf{Y1} \cdot \mathbf{c} \tag{4.7c}$$

$$Y = Y0 + \mathbf{Y1} \cdot \mathbf{d} \tag{4.7d}$$

Using the integrals shown above one can construct similar integrals, e.g. integral of $r(z)^3$, $r(z)^n$. Using Taylor expansion any integral can be decomposed into natural powers of coefficients d_n.

5. Partial derivatives of the integrals by coefficients *d*

To construct the Euler-Lagrange minimizing equations we need the partial derivatives by the coefficients d_k. They are obtained simply from the expanded integrals, as shown below.

Derivative of integral *A*

$$A'_k = \frac{\partial A}{\partial d_k} = \frac{\partial}{\partial d_k}[A0 + \mathbf{A1} \cdot \mathbf{d} + \mathbf{d}^T \cdot \mathbf{A2} \cdot \mathbf{d}] = (\mathbf{A1} + 2 \cdot \mathbf{A2} \cdot \mathbf{d})_k \tag{5.1}$$

Derivative of integral *B*

$$B'_k = \frac{\partial B}{\partial d_k} = \frac{\partial}{\partial d_k}[B0 + \mathbf{B1} \cdot \mathbf{d} + \mathbf{d}^T \cdot \mathbf{B2} \cdot \mathbf{d} + 3 \cdot B3 + 4 \cdot B4] \tag{5.2a}$$

$$B3'_k = \frac{\partial B3}{\partial d_k} = \frac{\partial}{\partial d_k} \tfrac{4}{3}\int_0^1 r_c(z) \cdot r_d(z)^3 dz = 4\int_0^1 r_c(z) \cdot r_d(z)^2 \cdot \phi_k(z) dz \tag{5.2b}$$

$$B4'_k = \frac{\partial B4}{\partial d_k} = \frac{\partial}{\partial d_k} \tfrac{1}{4}\int_0^1 r_d(z)^4 dz = \int_0^1 r_d(z)^3 \cdot \phi_k(z) dz \tag{5.2c}$$

$$B'_k = \frac{\partial B}{\partial d_k} = (\mathbf{B1} + 2 \cdot \mathbf{B2} \cdot \mathbf{d} + 3 \cdot \mathbf{B3'} + 4 \cdot \mathbf{B4'})_k \tag{5.2d}$$

Derivative of integral *C*

$$C'_k = \frac{\partial C}{\partial d_k} = \frac{\partial}{\partial d_k}[C0 + \mathbf{C1} \cdot \mathbf{d} + \mathbf{d}^T \cdot \mathbf{C2} \cdot \mathbf{d}] = (\mathbf{C1} + 2 \cdot \mathbf{C2} \cdot \mathbf{d})_k \tag{5.3}$$

Derivative of integral *X*

$$X'_k = \frac{\partial X}{\partial d_k} = \frac{\partial}{\partial d_k}[X0 + \mathbf{X1} \cdot \mathbf{d}] = X1_k \tag{5.4}$$

Derivative of integral *Y*

$$Y'_k = \frac{\partial Y}{\partial d_k} = \frac{\partial}{\partial d_k}[Y0 + \mathbf{Y1} \cdot \mathbf{d}] = Y1_k \tag{5.5}$$

Observe the pattern of the constants 1, 2, 3, 4 in 1.**B1**, 2.**B2**, 3.***B3'***, 4.***B4'*** of the expression **B'** and similar but shorter patterns of **A'**, **C'**, ***X'*** and **Y'**.

6. Stages in finding the solution

The solution we seek is the shape of the concave spinning top with the minimum moment of inertia. The moment of inertia about a given axis is defined as $\Sigma m_i r^2_i$ and we will find that when expressed in terms of the mass density the squared radius r^2 turns into r^4. The moment of inertia has a fundamental role in the mechanics of rotating bodies, as fundamental as the mass for non-rotating bodies. We can see this by considering spherical spinning tops, for which the moments of inertia about all three axes (*x, y, z*) are equal. If we substitute the moment of inertia *I* for the mass *m* and the angular velocity ω for the linear velocity *v* then the expressions for the energy *E* and the momentum *M* become the same as for non-rotating bodies (Landau & Lifshitz 1976).

$$E = \tfrac{1}{2} m v^2 \qquad E = \tfrac{1}{2} I \omega^2 \qquad\qquad M = mv \qquad M = I\omega \tag{6.1}$$

Hence finding the shape of bodies with the minimum moments of inertia under various constraints is of fundamental interest in physics and engineering (Diaz, Herrera & Martinez 2006) while the methods used to find the shapes is of interest in mathematics (Freitas, Laugesen & Liddell 2007). In the rest of this paper we present the solution to our problem in three stages described below, at each stage more constraints are added to the problem.

First stage

In the first stage we find the spinning top with the minimum moment of inertia that has given mass and height. To compare like with like the ratio of the moments of inertia about the axes x, y, z must be fixed. For simplicity we choose a spherical spinning top. Examples are a sphere, cube and any symmetric round body such as a cylinder or two cones, where the ratio of height to radius is just right. Thus we have three constraints, that the top is spherical, has given mass and $r(z)$ is non-negative. It also has one boundary condition, namely the maximum value of $z = 1$. We find that the generating function of rotation $r(z)$ is an elliptical curve and the body is a truncated spheroid.

This problem has an algebraic solution and we use it to show that our method works correctly.

Second stage

In the second stage we add the boundary condition that the spinning top has a Saturn-like disk of given radius *R* and thickness ζ. This results in a truncated spheroid again, but this time with a discontinuity at $z = \zeta$.

Due to the discontinuity we don't seek an algebraic solution which is much more difficult to find.

Third stage

The third stage adds three constraints, that the profile $r(z)$ is continuous, descending and convex.

Function $r(z)$ is convex if its second derivative $r''(z) \geq 0$ (Troutman 1983; Boyd & Vandenberghe 2004). A more general and more visual definition is based on Jensen inequality, i.e. $r(z)$ is convex if a straight line drawn between any two of its points is above $r(z)$ or at most coincides with it. Thus a wavy curve is not convex, because each valley is convex, but each crest is concave.

Because convexity and descendancy are inequality constraints there is no algebraic solution and the iterative solution becomes much more difficult. We find that the function $r(z)$ that gives the minimum moment of inertia under the specified constraints has a singularity at zero, i.e. $r(0) = \infty$.

7. First stage

In this stage we seek the shape of the spinning top with the minimum moment of inertia that has given mass and height from z = -1 to 1. A planar disk has the smallest moment of inertia of all planar shapes with the same area. Hence in three dimensional space the body with the minimum moment of inertia is a solid of rotation, with $Ix = Iy$. We solve the problem algebraically and then by iteration to show that the iterative method works correctly.

For a profile curve $r(z)$ the expressions for mass m, the moments of inertia Iz, Ix and the condition that the top is spherical, i.e. $Iz = Ix$, are

$$I_z = \rho \cdot \pi \int_0^1 r(z)^4 dz \qquad I_x = \rho \cdot \pi \int_0^1 \tfrac{1}{2} \cdot r(z)^4 + 2 \cdot r(z)^2 \cdot z^2 dz \tag{7.1}$$

$$m = 2 \cdot \rho \cdot \pi \int_0^1 r(z)^2 dz \qquad I_z - I_x = \rho \cdot \pi \int_0^1 \tfrac{1}{2} \cdot r(z)^4 - 2 \cdot r(z)^2 \cdot z^2 dz = 0 \tag{7.2}$$

For simplicity we drop the constant π and the mass density ρ from the formulas above, which amounts to changing the units of measurement. With this simplification the functional $J(r)$ is

$$J(r) = \int_0^1 [r(z)^4 + \lambda_1 (\tfrac{1}{2} \cdot r(z)^4 - 2 \cdot r(z)^2 \cdot z^2) + \lambda_2 \cdot r(z)^2] dz \tag{7.3}$$

In the absence of the derivative $r'(z)$ by z in the Lagrangian the Euler-Lagrange equations (3.2) are

$$\frac{\partial L}{\partial r} = [4 \cdot r(z)^3 + \lambda_1 (2 \cdot r(z)^3 - 4 \cdot r(z) \cdot z^2) + \lambda_2 \cdot 2 \cdot r(z)] = 0 \tag{7.4}$$

Carrying out the partial derivatives above gives us the algebraic equation

$$r(z) \cdot [2 \cdot r(z)^2 + \lambda_1 (r(z)^2 - 2 \cdot z^2) + \lambda_2] = 0 \tag{7.5}$$

The mass constraint cannot be satisfied if $r(z) = 0$ for all z and hence the solution is

$$r^2 (2 + \lambda_1) - 2\lambda_1 \cdot z^2 = -\lambda_2 \tag{7.6}$$

The solution $r(z)$ is an ellipse if $-2 < \lambda_1 < 0$ and $\lambda_2 < 0$ else a hyperbola if $\lambda_1 < -2$ and $\lambda_2 < 0$. The values of λ_1 and λ_2 are obtained by substituting $r(z)$ from (7.6) into the constraints (7.2).

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \lambda_1 \cdot \left(\frac{2}{3} - \frac{m}{2} \right) - m \tag{7.7}$$

$$a = \frac{2}{5} - \frac{1}{2} \left(\frac{2}{3} - \frac{m}{2} \right)^2 \qquad b = \frac{8}{5} - \left(\frac{2}{3} - \frac{m}{2} \right) \cdot \left(\frac{4}{3} - m \right) \qquad c = m \cdot \left(\frac{4}{3} - \frac{m}{2} \right) \tag{7.8}$$

To obtain the solution by iteration we start with the Lagrangian $L(r, z)$ of the integral $J(r)$ in (7.3)

$$L(z,r) = r^4 + \lambda_1(\tfrac{1}{2} \cdot r^4 - 2 \cdot r^2 z^2) + \lambda_2 \cdot r^2 \tag{7.9}$$

and use it as a template to express the integral J in terms of the integrals A, B, C

$$J(A,B,C) = B + \lambda_1(\tfrac{1}{2} \cdot B - 2 \cdot C) + \lambda_2 \cdot A \tag{7.10}$$

The partial derivatives of J by vector **d**, which is vector **J’**, are obtained in the same way

$$\mathbf{J}'(\mathbf{A}',\mathbf{B}',\mathbf{C}') = \mathbf{B}' + \lambda_1(\tfrac{1}{2} \cdot \mathbf{B}' - 2 \cdot \mathbf{C}') + \lambda_2 \cdot \mathbf{A}' \tag{7.11}$$

and so are the constituents of **J’** that correspond to different powers of the components d_n

$$\mathbf{L1} = 1 \cdot \mathbf{L}(\mathbf{A1},\mathbf{B1},\mathbf{C1}) = 1 \cdot [\mathbf{B1} + \lambda_1(\tfrac{1}{2} \cdot \mathbf{B1} - 2 \cdot \mathbf{C1}) + \lambda_2 \mathbf{A1}] \tag{7.12a}$$

$$\mathbf{L2} = 2 \cdot \mathbf{L}(\mathbf{A2},\mathbf{B2},\mathbf{C2}) = 2 \cdot [\mathbf{B2} + \lambda_1(\tfrac{1}{2} \cdot \mathbf{B2} - 2 \cdot \mathbf{C2}) + \lambda_2 \mathbf{A2}] \tag{7.12b}$$

$$\mathbf{L3} = 3 \cdot \mathbf{L}(\mathbf{A3}',\mathbf{B3}',\mathbf{C3}') = 3 \cdot [\mathbf{B3}' + \lambda_1 \tfrac{1}{2} \cdot \mathbf{B3}'] \tag{7.12c}$$

$$\mathbf{L4} = 4 \cdot \mathbf{L}(\mathbf{A4}',\mathbf{B4}',\mathbf{C4}') = 4 \cdot [\mathbf{B4}' + \lambda_1 \tfrac{1}{2} \cdot \mathbf{B4}'] \tag{7.12d}$$

The partial derivatives **J’** by the components of vector **d** can now be expressed as

$$\mathbf{J}' = \mathbf{L1} + \mathbf{L2} \cdot \mathbf{d} + \mathbf{L3}(d_i d_j) + \mathbf{L4}(d_i d_j d_k) \tag{7.13}$$

The solution to the variational problem given by the equations (3.5) can now be written as

$$\mathbf{L1} + \mathbf{L2} \cdot \mathbf{d} + \mathbf{L3}(d_i d_j) + \mathbf{L4}(d_i d_j d_k) = 0 \tag{7.14}$$

In general we cannot always solve this equation algebraically because vectors **L3** and **L4** are functions of the products $d_j d_j$ and $d_i d_j d_k$ respectively. However, on the assumption that the components of vector **d** are much smaller than those of vector **c**, then the length of vectors **L3**, **L4** is much less than of vectors **L1** and **L2.d**. Thus we can neglect vectors **L3**, **L4** and get a set of linear equations for an approximate value of vector **d** and its (scalar) error ΔL

$$\mathbf{L2} \cdot \mathbf{d} \cong \text{-}\mathbf{L1} \qquad \Delta L = \sqrt{(\mathbf{L3} + \mathbf{L4})^T \cdot (\mathbf{L3} + \mathbf{L4})} \tag{7.15}$$

If the function $r(z)$ is a polynomial then all integrals involved have an algebraic formula. However, except for the polynomials of very low order the integral formulas are hundreds of characters long, not feasible to work out manually. Instead we use computer symbolic mathematics to obtain them.

All matrices that make up matrix **L2** are symmetrical since $\phi_m\ \phi_n = \phi_n\ \phi_m$ and hence **L2** is also symmetrical. Thus **L2** can always be diagonalised and the linear equations above always have a unique non-trivial solution **d**, for any values of λ_1 and λ_2, provided that neither **L1** nor **L2** is null. Symmetrical matrices are a subset of Hermetian matrices used in quantum mechanics for which very precise and fast numerical methods exist. It is also easy to check the precision of the solution since $\mathbf{L2}.\mathbf{L2}^{-1}$ should be the identity matrix, thus there is no point trying to find $\mathbf{L2}^{-1}$ algebraically.

Now we describe the iteration process. We choose the initial values for the coefficients c_0 and c_1 to give a suitable initial function $r(z)$. This is used to compute the initial integrals A, B, C and their partial derivatives **A'**, **B'**, **C'**. We also choose a suitable value for the initial range of λs and the number of discrete values of λ in this range. For the 1st and all subsequent iterations an approximate vector **d** is computed for all possible pairs (λ_1, λ_2) using the linear equations. The overall error ε, i.e. the solution precision, for each vector **d** is computed using the formulas below.

$$\varepsilon = \sqrt{\Delta L^2 + dI^2 + dm^2} \tag{7.16}$$

where
$$dI = \frac{\frac{1}{2} \cdot B - 2 \cdot C}{B} \qquad dm = \frac{2A - m}{m} \tag{7.17}$$

In the problem considered here, there is also a tacit constraint, namely that we must have $r(z) \geq 0$. We convert this inequality constraint into an equality constraint using the integral of $|r(z)|$ and reject all vectors **d** that do not satisfy it.

$$\int_0^1 r(z)dz = \int_0^1 |\, r(z)\,|\, dz \tag{7.18}$$

In general, the integrals involving an absolute value do not have an algebraic solution. This is because to obtain the solution involves finding the roots of the integrand. If the integrand is a polynomial of degree N we know that there are N complex roots, but we cannot find them algebraically if $N \geq 3$. The numerical evaluation of integrals is inaccurate if the derivative $r'(z)$ has high absolute values. The derivatives of $|r(z)|$ and of $r(z)$ have the same absolute value for all z. Hence, if the integral of $r(z)$ evaluated numerically differs by some Δ from the same integral evaluated symbolically then the integral of $|r(z)|$ also has the accuracy of the order of Δ.

After each successful iteration step the range of λ is reduced by a chosen constant factor in such a way that the pair (λ_1, λ_2) with the minimum ε is in the centre of the range. The vector **d** corresponding to the minimum ε is used to update the vector **c** = **c** + **d**. The reduction factor of the λ range and the number of the discrete values of λ remain the same. Thus the accuracy of (λ_1, λ_2) and hence of the solution function $r(z)$ improves as the iteration goes on. The maximum Δ for all integrals of $r(z)$, $r'(z)$, $r''(z)$ evaluated during the iteration is monitored. Taking into account all three stages the maximum $\Delta = 10^{-13}$, hence the iteration is stopped when the solution accuracy ε exceeds 10^{-12}.

Iterating on λ values we change the N-dimensional problem of finding potentially a large number of coefficients c_n to an L-dimensional problem of finding the λ values, which in this case is two. Using a programming language like MATLAB the base functions may be defined as strings so that they can easily be changed. For this problem we use very simple non-orthogonal base functions:

$$\phi_n(z) = z^n \tag{7.19}$$

The solution precision, is of the order of 10^{-13} for any length of the series ϕ from $N = 4$ to 10. For $N > 10$ the solution is affected by computational precision and rounding errors. The series $r(z)$ converges point-wise to the ellipse that was found by the exact solution. The maximum error for the series with $N = 4$ is 10^{-2} and for $N = 10$ it is 10^{-4}. This shows that the method works correctly.

8. Second stage

In this stage we demand that the body from the first stage must have a Saturn-like ring with given radius R and thickness ζ.

$$r(z) = R \qquad \text{for} \qquad z = [0, \zeta) \tag{8.1}$$

The ring is effectively a constraint on the shape of $r(z)$, but we do not use a Lagrange multiplier λ. Instead we split the integration over two intervals $[0, \zeta]$ and $[\zeta, 1]$. The contribution from the 1st interval is always the same and independent of vectors **c**, **d**. Thus this contribution needs to be taken into account only for the constraints $Iz = Ix$ and mass = m (7.17), i.e. the errors dI and dm.

$$dI = \frac{\frac{1}{2}\cdot B - 2\cdot C}{B} \qquad\qquad dm = \frac{2A - m}{m} \tag{8.2}$$

For this problem we use the same base functions ϕ_n as in the first stage (7.19)

$$\phi_n(z) = z^n \tag{8.3}$$

The Lagrangians **L1** to **L4** have the same form as in the 1st stage (7.12). Figure 4 shows the graph of $r(z)$ and figure 5 the surface map of $1/\varepsilon$ where ε is the solution precision for the pairs (λ_1, λ_2).

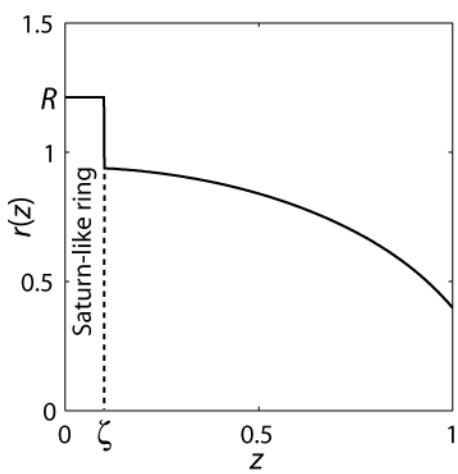

FIGURE 4. Radius $r(z)$ Graph

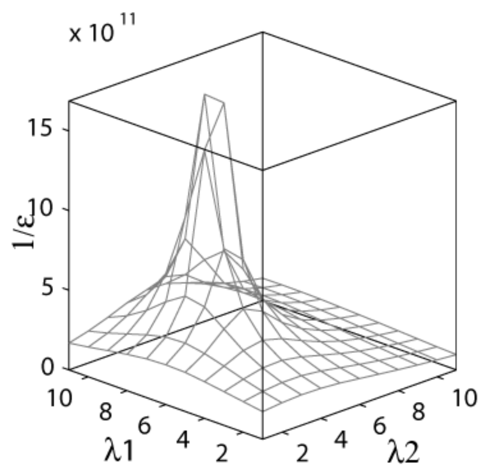

FIGURE 5. Surface Map $1/\varepsilon$ over (λ_1, λ_2)

We use the inverted error $1/\varepsilon$ because it results in a sharp peak which is easy to see, while the error ε results in a valley with a flat bottom and an unclear minimum that is difficult to locate visually.

The body with the minimum moment of inertia is again a spheroid, but with the specified Saturn-like ring that causes a discontinuity at $z = \zeta$.

For series of length $N = 7$ the peak height $1/\varepsilon = 1.7\text{e}+12$, to reach this precision takes 18 iterations.

9. Third stage

In this stage we again demand that the body must have a Saturn-like ring with given radius R and thickness ζ. However, this time the reason for using the ring is that it provides a mechanism for finding a solution if $r(z)$ tends to infinity at $z = 0$. We can choose R as large as we like and ζ as small as we like, within reason considering the accuracy of computation.

We also add three constraints, namely that the profile $r(z)$ is continuous, descending and convex. Thus, the function $r(z)$ must be continuous on the entire interval

$$r(z) = continuous \qquad \text{on} \qquad z = [0, 1] \tag{9.1}$$

The continuity is satisfied by using base functions ϕ_n that are zero at $z = \zeta$ except for ϕ_0

$$\phi_n(z) = (z-\zeta)^n \qquad \text{with} \qquad \phi_0(z) = 1 \tag{9.2}$$

and by fixing **L1**, **L2** so that we always get $d_0 = 0$, hence $c_0 = R$ and $r(\zeta)$ is always exactly R

$$L1_0 = 0 \qquad \text{and} \qquad L2_{0,n} = 0 \qquad L2_{n,0} = 0 \qquad L2_{0,0} = 1 \tag{9.3}$$

We also ask that $r(z)$ is convex beyond the Saturn ring

$$r(z) = convex \qquad \text{on} \qquad z = [\zeta, 1] \tag{9.4}$$

A function is convex if its 2nd derivative is non-negative, hence we have the inequality constraint

$$\ddot{r}(z) \geq 0 \qquad \text{on} \qquad z = [\zeta, 1] \tag{9.5}$$

A solution to the Euler-Lagrange equation is a stationary point, i.e. minimum, maximum or a point of inflection. There is nothing in the constraints above that requests the minimum. We found in the 1st stage that the solution with the minimum moment of inertia is descending. Thus to direct the optimisation towards the minimum solution we use the constraint that $r(z)$ must be descending.

$$\dot{r}(z) \leq 0 \qquad \text{on} \qquad z = [\zeta, 1] \tag{9.6}$$

Convexity and descendancy are inequalities that we convert to equalities using the integrals of absolute values. We evaluate the integrals with and without the absolute value numerically and check that they are sufficiently accurate as described in stage one.

$$\int_\zeta^1 \dot{r}(z)dz = -\int_\zeta^1 |\,\dot{r}(z)\,|\,dz = -\|\,X\,\| \qquad \int_\zeta^1 \ddot{r}(z)dz = \int_\zeta^1 |\,\ddot{r}(z)\,|\,dz = \|\,Y\,\| \tag{9.7}$$

In the formulas (9.7) above we used the notation $\|\ \|$ as a shorthand for the integrals of an absolute value. We will use these constraints with λ_3 and λ_4 as we used λ_1 for mass = m constraint in (7.2), (7.3). The difference from the mass constraint is that the required value m of the mass does not change, while the values of the integrals of the absolute value change for every curve $r(z)$.

With the additional constraints the integral J (7.10) and the overall solution error ε (7.16) become

$$J(A,B,C,X,Y) = B + \lambda_1(\tfrac{1}{2}\cdot B - 2\cdot C) + \lambda_2\cdot A + \lambda_3\cdot X + \lambda_4\cdot Y \tag{9.8}$$

$$\varepsilon = \sqrt{\Delta L^2 + dI^2 + dm^2 + dx^2 + dy^2} \tag{9.9}$$

$$dx = \frac{\| X \| + X}{\| X \|} \qquad dy = \frac{\| Y \| - Y}{\| Y \|} \tag{9.10}$$

The Lagrangians **L2**, **L3**, **L4** retain the same form as in the 1st stage (7.12) while **L1** becomes

$$\mathbf{L1} = \mathbf{L}(\mathbf{A1},\mathbf{B1},\mathbf{C1},\mathbf{X1},\mathbf{Y1}) = [\mathbf{B1} + \lambda_1(\tfrac{1}{2}\cdot\mathbf{B1} - 2\cdot\mathbf{C1}) + \lambda_2\cdot\mathbf{A1} + \lambda_3\cdot\mathbf{X1} + \lambda_4\cdot\mathbf{Y1}] \tag{9.11}$$

Series $r(z)$ of four terms gives the dotted convex arc in figure 6, while the eventual solution is the solid arc. It satisfies the constraints mass = m and $Iz = Ix$ poorly compared to the previous stages. Figures 7, 8 show almost flat $1/\varepsilon$ maps with the maximum height 40 instead of sharp peaks 10^{+12}.

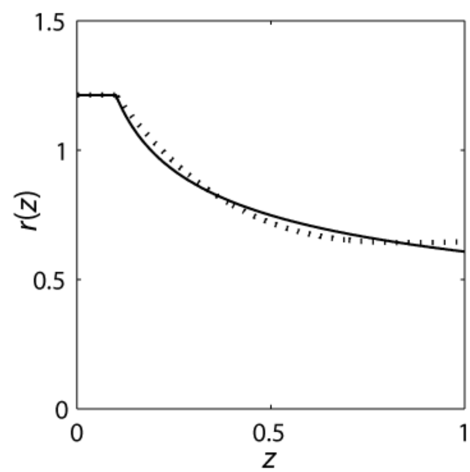

FIGURE 6. Radius $r(z)$ Graph

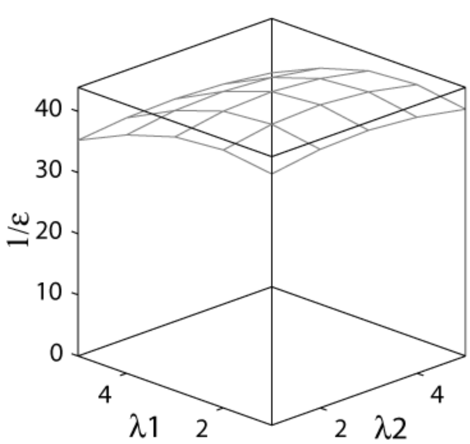

FIGURE 7. Surface Map $1/\varepsilon$ over (λ_1, λ_2)

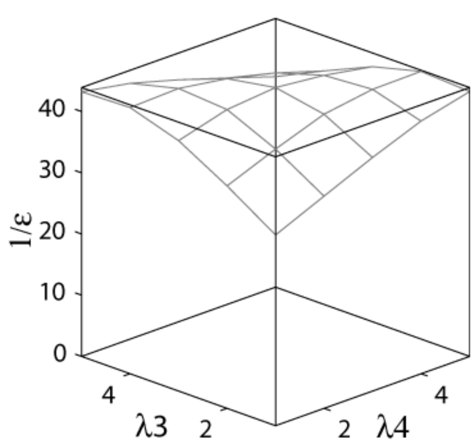

FIGURE 8. Surface Map $1/\varepsilon$ over (λ_3, λ_4)

In figure 6 the function r(z) looks convex and descending, but in fact all four constraints are more or less equally unsatisfied, because the cubic polynomials are not flexible enough. On the other hand if we use seven terms in the series we get the reverse situation. The polynomials are too flexible and convexity with descendancy visibly dominate the errors, the function r(z) is too wavy. Virtually all series, e.g. Fourier, Chebyshev, etc. produce wavy curves. Thus to satisfy the convexity constraint completely is extremely difficult. To solve this problem we multiply the convexity error *dy* by a weight function *w* that decreases as the iteration proceeds and thus relaxes the convexity constraint.

$$dy = \frac{\| Y \| - Y}{\| Y \|} \cdot w \qquad w(iteration) = descending \qquad w(1) = 1 \quad w(\infty) = 0 \qquad (9.12)$$

When the weight function *w* decreases more rapidly we get a more wavy curve r(z). Thus, still using the cubic polynomials, the function *r*(*z*) in figure 9 intersects the solid curve only 3 times (including the starting point) while in figure 10 there are 4 intersections. The surface maps $1/\varepsilon$ in figure 11 and figure 12 refer to figure 10, but the maps for figure 9 are very similar.

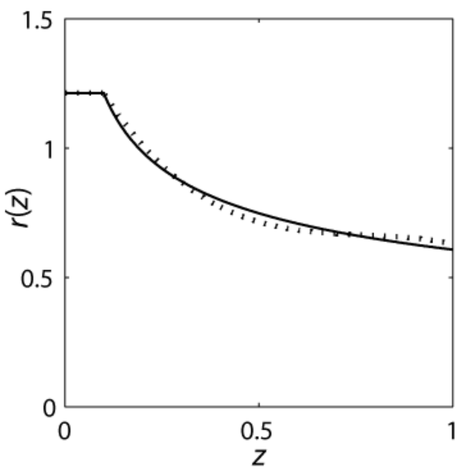

FIGURE 9. Radius *r*(*z*) Graph

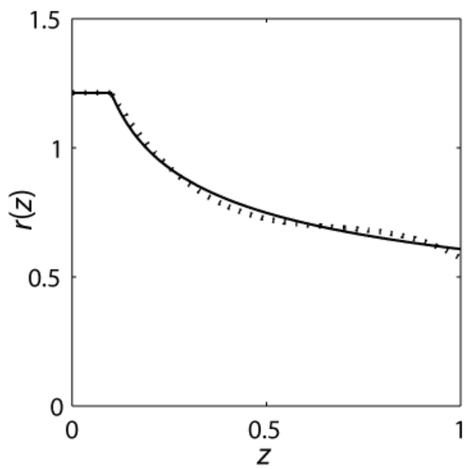

FIGURE 10. Radius *r*(*z*) Graph

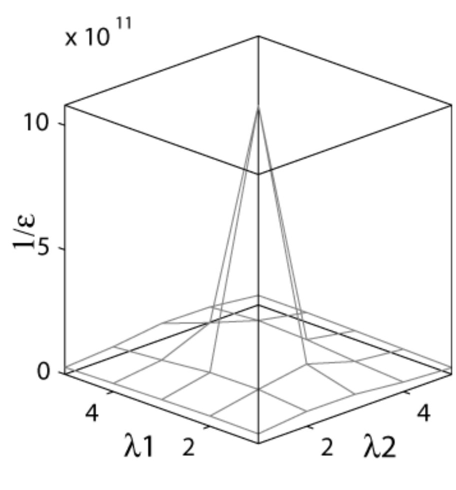

FIGURE 11. Surface Map $1/\varepsilon$ over (λ_1, λ_2)

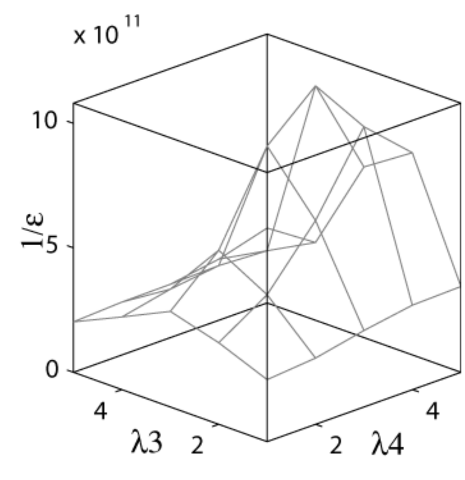

FIGURE 12. Surface Map $1/\varepsilon$ over (λ_3, λ_4)

Both solutions converge and after 29 iterations the peak of $1/\varepsilon$ attains the accuracy of 10^{+12}.

If we increase the number of terms in the series $r(z)$ from four to five we obtain the solution shown in figure 13 below as the dotted curve, which is visually almost identical to the solid curve.

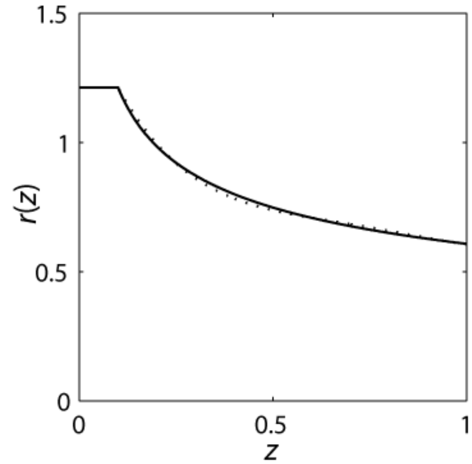

FIGURE 13. Radius $r(z)$ Graph

As the iteration proceeds the weight w decreases and the function $r(z)$ converges to the solid curve. The iteration ends on the 29th iteration when the overall error ε exceeds the accuracy of 10^{-12}. The surface maps of $1/\varepsilon$ on the grid (λ_1, λ_2) and (λ_3, λ_4) are shown in figures 14 and 15 below.

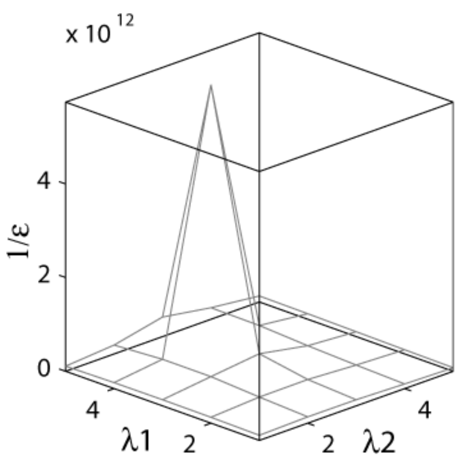

FIGURE 14. Surface Map $1/\varepsilon$ over (λ_1, λ_2)

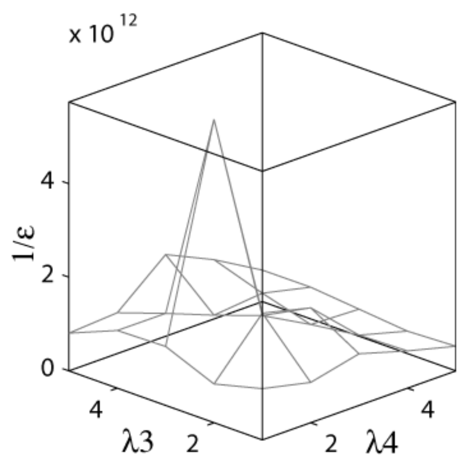

FIGURE 15. Surface Map $1/\varepsilon$ over (λ_3, λ_4)

The moment of inertia for the wavy dotted curve $r(z)$ is 0.39% greater than that for the solid curve, which is strictly convex, strictly descending, infinite at the origin and we call it *Vir*.

$$Vir(z) = a\left(\frac{a}{z}\right)^{\alpha} \quad \text{where} \quad a \succ 0 \quad \alpha \succ 0 \quad \text{and} \quad Vir(0) = \infty \tag{9.13}$$

Vir(z) function is the solution to which all approximations converge. It generates the concave spherical spinning_top with the minimum moment of inertia. *Vir* function was found by one of the authors as a conjecture based on physical considerations that are beyond the scope of this article.

10. Summary and conclusions

Using the calculus of variations we found that vortices have the shape of concave spinning tops with the minimum moment of inertia, called Vir. The term Vir to refers to the function and the 3D shape.

$$Vir(z) = a\left(\frac{a}{z}\right)^{\alpha} \qquad \text{where} \qquad a \succ 0 \qquad \alpha \succ 0 \qquad \text{note} \qquad Vir(0) = \infty$$

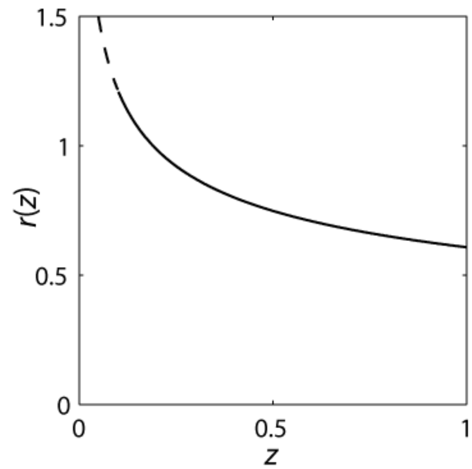

FIGURE 16. Radius $r(z)$ of *Vir*

The laws of Newtonian Mechanics, Special Relativity, General Relativity and Quantum Mechanics can be derived from the Principle of Least Action (Landau & Lifshitz 1965; Landau & Lifshitz 1975; Landau & Lifshitz 1976; Troutman 1983). Action is the integral of the difference between the kinetic and potential energy. Variation is used to find the function that minimises the action.

In Optics we cannot use "Action" because there is no concept of the potential energy of light. Instead Fermat derived the laws for the refraction and reflection of light from the Principle of Least Time using the calculus of variations (Meyer-Arendt 1972; Courant & Robbins 1978; Troutman 1983).

In a similar way to the examples above, we derived here a law of Fluid Mechanics for the shape of vortices by using the Principle of Least Moment of Inertia, i.e. Least Resistance to Spinning.

Vir function may be of interest also to other branches of physics, because in spite of having infinity at zero it generates a solid of rotation with a finite mass and moment of inertia if $\alpha < ¼$. Vir with the mass and height of a sphere can have the radius as large as we like and in free space behave as a sphere, i.e. it can be rotated with equal effort about any axis through its centre of mass.

To the best of our knowledge the variation method used here is new and may be suitable for solving problems involving non-linear equations, inequality constraints and an infinity in the solution.

The computer programme for finding the solution has two versions, with symbolic and numeric integration. Both produce the results accurate to 10^{-12} but the numerical version is 100 times faster.

It will be shown shortly in the paper entitled "Vir Theory of Elementary Particles" that elementary particles have the shape of twin vortices with the common spin axis connected at the large ends.

References

Akheizer N I 1962 *The Calculus of Variations* (Blaisdell Publishing Company) p.143

Arfken G 1968 *Mathematical Methods for Physicists* (Academic Press) p.617, 637

Atanacković T.M. et al. 2014 Expansion formula for fractional derivatives in variational problems *J. Math. Anal. Appl.* **409** 911

Boyd S & Vandenberghe L 2004 *Convex Optimization* (CUP) p.23-25, 67, 71, 77-78

Chang R & Wang M L 1983 Shifted Legendre Series Direct Method for Variational Problems *J. Optim. Theory Applic.* **39** 299

Chen C F & Hsiao C H 1975 A Walsh Series Direct Method for Solving Variational Problems *J. of The Franklin Inst.* **300** 265

Courant R & Hilbert D 1989 *Methods of Mathematical Physics* (John Wiley & Sons) p.175

Courant R & Robbins H 1978 *What is Mathematics?* (Oxford University Press) p.379-397

Diaz R A, Herrera W J, Martinez R 2006 Moments of inertia for solids of revolution and variational methods *Eur. J. of Phys.* **27** 183

Freitas P, Laugesen R S, Liddell G F 2007 On convex surfaces with minimal moment of inertia *J. Math. Phys.* **48** 122902

Germuska M J 2016 The Shape of Water and Air Vortices *viXra:1612.0211 [Classical Physics]*

He J-H 2007 Variational iteration method - Some recent results and new interpretations, *J. of Comput. and Appl. Math.* **207** 3

Horng I R & Chou J H & 1986 Aplication of Shifted Chebyshev Series to the Optimal Control of Scaled Systems *Int. J. Syst. Sci.* **17** 679

Hwang C & Shih Y P 1983 Laguerre Series Direct Method for Variational Problems *J. of Optim. Theory and Appl.* **39** 143

Hwang C & Shih Y P 1985 Optimal Control of Delay Syst. via Block Pulse Functions *J. of Optim. Theory and Appl.* **45** 101

Hsiao C 2004 Haar wavelet direct method for solving variational problems *Math. and Comput. in Simul.* **64** 569

Jafari H 2014 A comparison between the variational iteration method and the successive approximations method, *Appl. Math. Letters* **32** 1

Jeffreys H & Jeffreys B 1980 *Methods of Mathematical Physics* (CUP) p.314

Kantorovich L W 1964 *Approximate Methods of Higher Analysis* (P. Nordhoff) p.258

Landau L D and Lifshitz E M 1965 *Quantum Mechanics* (Pergamon Press) p.20, 25

Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* (Pergamon Press) p.24

Landau L D and Lifshitz E M 1976 *Mechanics* (Pergamon Press) p.2,3,8,100,106,131

Maleki M & Mashali-Firouzi M 2010 A numerical solution of problems in calculus of variation using direct method and nonclassical parameterization *J. Comp. Appl. Math.* **234** 1364

Maleki M & Hashim I 2014 Adaptive pseudospectral methods for solving constrained linear and nonlinear time-delay optimal control problems *J. of the Franklin Inst.* **351** 811

Malitsky Y V & Semenov V V 2015 A hybrid method without extrapolation step for solving variational inequality problems *J. of Glob. Optim.* **61** 193

Meyer-Ardent J R 1972 *Introduction to Classical and modern Optics* (Prentince-Hall) p.14

Paraskevopoulos P N et al. 1990 The operation matrix of integration for Bessel functions *J. of the Franklin Inst. Eng. and Appl. Math.* **327** 329

Razzaghi M & Razzaghi M 1988 Fourier Series Direct Method for Variational Problems *Int. J. Control* **48** 887

Razzaghi M & Marzban H 2000 Direct method for Variational Problems via Hybrid of Block-Pulse and Chebyshev Functions *Math. Probl. in Eng.* **6** 85

Troutman J 1983 *Variational Calculus with Elementary Convexity* (Springer Verlag) p.5-6, 230-232, 266-267

Wazwaz A M 2011 The variational iteration method for solving systems of equations of Emden–Fowler type *Int. J. of Comput. Math.* **88** 3406

Yildirim A & Ozis T 2007 Solutions of singular IVPs of Lane–Emden type by homotopy perturbation method *Phys. Lett.* **A 369** 70.