Generalizations of Schwarzschild and (Anti) de Sitter Metrics in Clifford Spaces *

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Abstract

After a very brief introduction to generalized gravity in Clifford spaces (C-spaces), generalized metric solutions to the C-space gravitational field equations are found, and inspired from the (Anti) de Sitter metric solutions to Einstein's field equations with a cosmological constant in ordinary spacetimes. C-space analogs of static spherically symmetric metrics solutions are constructed. Concluding remarks are devoted to a thorough discussion about Areal metrics, Kawaguchi-Finsler Geometry, Strings, and plausible novel physical implications of C-space Relativity theory.

Keywords : Extended Relativity in Clifford Spaces; Gravity; Strings, Area metrics, Kawaguchi-Finsler Geometry.

1 Geometry of Clifford-spaces

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [5]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing.

*Dedicated to the loving memory of Don Guillermo Zambrano Lozano, a maverick and visionary entrepreneur from Monterrey, Mexico.

Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [5]. The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate).

These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background. C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\rho}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. When $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions it can be decomposed as

$$\mathbf{X} \; = \; s\, \mathbf{1} + x^\mu \, \gamma_\mu + x^{\mu\nu} \, \gamma_\mu \wedge \gamma_\nu + x^{\mu\nu\rho} \, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + x^{\mu\nu\rho\tau} \, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\tau \quad (1.1)$$

where we have omitted *combinatorial* numerical factors for convenience in the expansion of eq-(1.1). To avoid introducing powers of a length parameter L (like the Planck scale L_p), in order to match physical units in the expansion of the polyvector $\mathbf{X}$ in eq-(1.1), we can set it to unity to simplify matters.

The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval

$$(d\Sigma)^2 \; = \; (ds)^2 \; + \; dx_\mu \; dx^\mu \; + \; dx_{\mu\nu} \; dx^{\mu\nu} \; + \; \ldots \quad (1.2)$$

that is *invariant* under $Cl(1,3)$ transformations and which are the Clifford-algebraic extensions of the $SO(1,3)$ Lorentz transformations [5]. One should emphasize that $d\Sigma$ is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu} dx^\mu dx^\nu = dx_\mu dx^\mu$. Generalized Lorentz transformations (poly-rotations) in flat C-spaces were discussed in [5]. An extensive analysis of the C-space generalized Lorentz transformations and their physical implications can be found in [2].

Given $\mathbf{X} = X^A \gamma_A$, where A is a polyvector-valued index and γ_A span over all the generators of the Clifford algebra, the quadratic form is defined as

$$< \mathbf{X}^\dagger \, \mathbf{X} > \; = \; X_A \, X^A \; = \; s^2 + X_\mu \, X^\mu + X_{\mu_1\mu_2} \, X^{\mu_1\mu_2} + ... \, X_{\mu_1\mu_2.....\mu_D} \, X^{\mu_1\mu_2....\mu_D} \quad (1.3)$$

where $\mathbf{X}^\dagger$ denotes the reversal operation obtained by reversing the order of the gamma generators in the wedge products. The symbol $< \; \gamma_A \; \gamma_B \; >$ denotes taking the *scalar* part in the Clifford geometric product of $\gamma_A \; \gamma_B$. It is the analog of taking the trace of a product of matrices.

In curved C-space [5], [6] one introduces the $\mathbf{X}$-dependent basis generators γ_M, γ^M defined in terms of the beins E^A_M, inverse beins E^M_A and the flat tangent space generators γ_A, γ^A as follows $\gamma_M = E^A_M(\mathbf{X})\gamma_A, \gamma^M = E^M_A(\mathbf{X})\gamma^A$. The curved C- space metric expression $g_{MN} = E^A_M E^B_N \eta_{AB}$ also agrees with taking the scalar part of the Clifford geometric product $< \gamma_M \ \gamma_N >= g_{MN}$.

From now one we shall denote the curved C-space basis generators γ_M, γ^M by E_M, E^M, and the flat tangent space generators γ_A, γ^A by E_A, E^A. The indices $A, B, C, ...$ from the beginning of the alphabet represent the tangent space indices, while those from the middle of the alphabet $L, M, N, ...$ represent the base world indices. The covariant derivative of $E^A_M(\mathbf{X}), E^M_A(\mathbf{X})$ involves the generalized connection and spin connection and are defined as

$$\nabla_K E^A_M \ = \ \partial_K E^A_M \ - \ \Gamma^L_{KM} \ E^A_L \ + \ \omega^A_{KB} \ E^B_M \tag{1.4}$$

$$\nabla_K E^M_A \ = \ \partial_K E^M_A \ + \ \Gamma^M_{KL} \ E^L_A \ - \ \omega_{KA}{}^B \ E^M_B \tag{1.5}$$

If the nonmetricity is zero then $\nabla_K E^A_M = 0, \ \nabla_K E^M_A = 0$ in eqs-(1.4,1.5).

The coefficients (functions) $W_{LM}{}^N$ associated to the Clifford geometric product are defined by

$$E_A \ E_B \ = \ W_{AB}{}^C \ E_C, \ \ given \ \ E_L = E^A_L \ E_A, \ E_M \ = \ E^A_M \ E_A \ \Rightarrow$$

$$E_L \ E_M \ = \ W_{LM}{}^N \ E_N \Rightarrow W_{LM}{}^N \ = \ E^A_L \ E^B_M \ E^N_C \ W_{AB}{}^C \tag{1.6}$$

the Clifford algebra structure functions $f_{LM}{}^N, d_{LM}{}^N$ are defined by

$$[E_A, \ E_B] \ = \ f_{AB}{}^C \ E_C, \ \ [E_L, \ E_M] \ = \ f_{LM}{}^N \ E_N \Rightarrow f_{LM}{}^N \ = \ E^A_L \ E^B_M \ E^N_C \ f_{AB}{}^C \tag{1.7}$$

$$\{E_A, \ E_B\} \ = \ d_{AB}{}^C \ E_C, \ \ \{E_L, \ E_M\} \ = \ d_{LM}{}^N \ E_N \Rightarrow d_{LM}{}^N \ = \ E^A_L \ E^B_M \ E^N_C \ d_{AB}{}^C \tag{1.8}$$

For simplicity we shall set the nonmetricity Q^L_{MN} to zero. The *torsionless* Levi-Civita connection is given by [1]

$$^{(lc)}\Gamma^L{}_{MN} \ = \ \{^L_{MN}\} \ + \ \frac{1}{2} \ g^{LK} \ (\ f_{MKN} \ + \ f_{NKM} \ + \ f_{MNK} \ ) \tag{1.9}$$

where

$$\{^L_{MN}\} \ = \ \frac{1}{2} \ g^{LK} \ (\ \partial_N \ g_{KM} \ + \partial_M \ g_{KN} \ - \partial_K \ g_{MN} \ ) \tag{1.10}$$

and f_{MKN} are the Clifford algebra structure functions (coefficients). We should notice that the Levi-Civita (LC) connection in eq-(1.10) has a symmetric $^{(lc)}\Gamma^L{}_{(MN)}$ and antisymmetric $^{(lc)}\Gamma^L{}_{[MN]}$ piece. The symmetric piece is given by the first three terms in (1.9), while the antisymmetric piece is given by the last term in (1.9).

The Torsion is defined by

$$\mathbf{T} \ = \ \nabla_{\mathbf{X}} \ \mathbf{Y} \ - \ \nabla_{\mathbf{Y}} \ \mathbf{X} \ - \ [\mathbf{X}, \mathbf{Y}] \tag{1.11}$$

so that by inspection one can see that the Levi-Civita (LC) connection (1.9) is torsionless

$$^{(lc)}T_{MN}^{\ \ \ L} \ \equiv \ ^{(lc)}\Gamma^{L}_{\ \ MN} \ - \ ^{(lc)}\Gamma^{L}_{\ \ NM} - f_{MN}^{\ \ \ L} \ = \ 0 \tag{1.12}$$

The last term $-f_{MN}^{\ \ \ L}$ in the expression for the torsion (2.17) originates from the non-vanishing $[\mathbf{X}, \mathbf{Y}] \neq 0$ contribution and resulting from the fact that $[E_M, E_N] = f_{MN}^{\ \ \ L} \ E_L \neq 0$.

The curvature is defined as

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}) \ \mathbf{Z} \ = \ [\nabla_{\mathbf{X}}, \ \nabla_{\mathbf{Y}}] \ \mathbf{Z} \ - \ \nabla_{[\mathbf{X}, \mathbf{Y}]} \ \mathbf{Z} \tag{1.13}$$

such that the explicit curvature components are given by

$$\mathbf{R}_{MNJ}^{\ \ \ \ \ K} \ = \ \partial_M \ \Gamma_{NJ}^{\ \ \ K} \ - \ \partial_N \ \Gamma_{MJ}^{\ \ \ K} \ - \ \Gamma_{MJ}^{\ \ \ L} \ \Gamma_{NL}^{\ \ \ K} \ + \ \Gamma_{NJ}^{\ \ \ L} \ \Gamma_{ML}^{\ \ \ K} \ - \ f_{MN}^{\ \ \ L} \ \Gamma_{LJ}^{\ \ \ K} \tag{1.14}$$

In [1] it is shown explicitly that the curvature (1.14) transforms homogeneously under coordinate transformations $X^M \to \tilde{X}^M(X^N)$ despite that the connection Γ^K_{MJ} transforms inhomogeneously.

By inserting the torsionless connection expression in eq-(1.9) of the form $\Gamma^L_{MN} = \{^L_{MN}\} + f^L_{MN}$..... terms, and after using the covariantly constancy condition on the curved C-space Clifford algebra structure functions $\nabla_M f_{JKL} = 0$ [1], one can decompose the symmetrized part of the Ricci tensor as $\mathbf{R}_{(MJ)} \sim R_{MJ} + f_M^{KL} f_{KLJ} + f_J^{KL} f_{KLM}$, and the Ricci scalar as $\mathbf{R} \sim R + f^{JKL} f_{JKL}$. $R_{MJ} = R_{JM}, R$ are the Ricci tensor and Ricci scalar analogs in C-space associated with the symmetric Christoffel connection $\{^L_{MN}\} = \{^L_{NM}\}$.

The physical significance of this curvature decomposition is that these extra terms involving the curved C-space Clifford algebra structure functions can be interpreted as an $effective$ stress energy tensor which can *mimic* the effects of "dark" matter/energy. To see how the cosmological constant Λ emerges, it is straightforward to infer that the contraction $f^{JKL} \ f_{JKL}$ involving the Clifford-algebra structure functions in curved C-space turns out to be *equal* to $f^{ABC} \ f_{ABC} \sim \Lambda_1 =$ constant, when f^{ABC}, f_{ABC} are the tangent space Clifford algebra structure *constants*. This finding is just a consequence of the definitions of f^{JKL} and f_{JKL} in terms of the beins E^A_J , and inverse beins E^J_A obeying $E^J_A \ E^A_M = \delta^J_M, ...$

Therefore, to sum up, when the torsion is set to zero, the generalized vacuum field equations in C-space were shown to be given by [1]

$$\mathbf{R}_{MN}(\{\}) \ - \frac{1}{2} \ \mathbf{g}_{MN} \ \mathbf{R}(\{\}) \ + \ \Lambda \ \mathbf{g}_{MN} \ = \ 0 \tag{1.15}$$

where the cosmological constant Λ stems from the contractions involving the quadratic terms of the form $f^{JKL} f_{JKL}$, and the curvature terms are defined in terms of the symmetric Christoffel like connection coefficients $\{^L_{MN}\} = \{^L_{NM}\}$.

Having presented this brief introduction of C-gravity in the next section we shall explore metric solutions to the C-space gravitational field equations (1.15).

2 Generalized Metrics in C-spaces

2.1 (Anti) de Sitter Metrics in C-Spaces

The d-dim Anti de Sitter space AdS_d can be parametrized in terms of stereographic coordinates by embedding the d-dim hyperboloid (whose throat radius is $L/2$) in a $d+1$-dim pseudo-Euclidean flat space $R^{d-1,2}$ of signature $(-,+,+,\cdots,+,-)$ as follows

$$y^\mu = \frac{x^\mu}{(1 - x_\mu x^\mu / L^2)}, \quad \mu = 0,1,2,\cdots,d-1 \tag{2.1}$$

$$y^{d+1} = \frac{L}{2} \frac{(1 + x_\mu x^\mu / L^2)}{(1 - x_\mu x^\mu / L^2)}, \quad x_\mu x^\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + \ldots + (x^{d-1})^2 \tag{2.2}$$

one can infer from eqs-(2.1,2.2) that

$$-(y^{d+1})^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 + \ldots + (y^{d-1})^2 = -(\frac{L}{2})^2 \tag{2.3}$$

The d-dim de Sitter space dS_d can be parametrized by the stereographic coordinates by embedding the d-dim hyperboloid (whose throat radius is $L/2$) into a $d+1$-dim pseudo-Euclidean flat space $R^{d,1}$ of signature $(-,+,+,\cdots,+,+)$ as follows

$$y^\mu = \frac{x^\mu}{(1 + x_\mu x^\mu / L^2)}, \quad \mu = 0,1,2,\cdots,d-1 \tag{2.4}$$

$$y^{d+1} = \frac{L}{2} \frac{(1 - x_\mu x^\mu / L^2)}{(1 + x_\mu x^\mu / L^2)}, \quad x_\mu x^\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + \ldots + (x^{d-1})^2 \tag{2.5}$$

obeying

$$(y^{d+1})^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 + \ldots + (y^{d-1})^2 = (\frac{L}{2})^2 \tag{2.6}$$

The (Anti) de Sitter metric in stereographic coordinates become respectively

$$(d\tau)^2_{AdS} = \frac{(dx_\mu)\ (dx^\mu)}{(1 - x_\mu x^\mu / L^2)^2}, \quad (d\tau)^2_{dS} = \frac{(dx_\mu)\ (dx^\mu)}{(1 + x_\mu x^\mu / L^2)^2} \tag{2.7}$$

namely, the metric is conformally flat. It is well known (to the experts) that the scalar curvature of the d-dim Lorentzian spacetime corresponding to the conformally flat metric $g = e^{2\phi}\eta_{\mu\nu} = \Omega^2\eta_{\mu\nu}$, and written in terms of inertial coordinates, is given by the expression

$$R(g) \;=\; \Omega^{-2}\; [\; -2\;(d-1)\;(\partial_\mu\partial^\mu ln\;\Omega)\;-\;(d-2)\;(d-1)\;(\partial_\mu ln\;\Omega)\;(\partial^\mu ln\;\Omega)\;] \tag{2.8}$$

hence, given the conformal factors displayed above and plugging their values into eq-(2.8) one ends up, respectively, with

$$R_{AdS} \;=\; -\;\frac{d\;(d-1)}{(L/2)^2},\;\; R_{dS} \;=\; \frac{d\;(d-1)}{(L/2)^2} \tag{2.8}$$

Given this preamble we are going to exploit the conformally flat nature of (Anti) de Sitter spaces and show that the generalization of the d-dim Anti de Sitter space AdS_d metric to C-spaces is given

$$(d\Sigma)^2 \;=\; \frac{(dX_M)\;(dX^M)}{(1-X_MX^M/L^2)^2} \tag{2.9}$$

the C-space conformal factor is

$$\Omega^2(X_M) \;=\; \frac{1}{(1-X_MX^M/L^2)^2} \tag{2.10}$$

the infinitesimal displacement squared is

$$(dX_M)\;(dX^M) \;=\; (L_P)^2\;(ds)^2\;+\;(dx_\mu)\;(dx^\mu)\;+\;(L_P)^{-2}\;(dx_{\mu\nu})\;(dx^{\mu\nu})\;+$$
$$(L_P)^{-4}\;(dx_{\mu\nu\rho})\;(dx^{\mu\nu\rho})\;+\;\ldots \tag{2.11}$$

The norm squared is

$$X_MX^M \;=\; (L_P)^2\;s^2\;+\;x_\mu x^\mu\;+\;(L_P)^{-2}\;x_{\mu\nu}\;x^{\mu\nu}\;+\;(L_P)^{-4}\;x_{\mu\nu\rho}\;x^{\mu\nu\rho}\;+\;\ldots \tag{2.12}$$

The Clifford scalar s is chosen to be dimensionless. We choose X_MX^M to have units of $(length)^2$ and for this reason suitable powers of the Planck scale L_P must appear in eqs-(2.11, 2.12).

The bivectors, trivectors, ... infinitesimal displacements containing the temporal direction will appear with a negative sign due to the chosen Lorentzian signature

$$(dx_\mu)\;(dx^\mu) \;=\; -\;(dx^0)^2\;+\;(dx^1)^2\;+\;(dx^2)^2\;+\;\ldots\;+\;(dx^{d-1})^2 \tag{2.13a}$$

$$(dx_{\mu\nu})\;(dx^{\mu\nu}) \;=\; -\,(dx^{01})^2-(dx^{02})^2-(dx^{03})^2-\;\ldots\;+(dx^{12})^2+(dx^{13})^2+\ldots \tag{2.13b}$$

$$(dx_{\mu\nu\rho})\,(dx^{\mu\nu\rho}) \;=\; -\,(dx^{012})^2 - (dx^{013})^2 - (dx^{014})^2 - \;\ldots\; + (dx^{123})^2 + (dx^{124})^2 + \;\ldots \qquad (2.13c)$$

etc. There is an ambiguity in choosing the sign in the Clifford scalar part $(ds)^2$ of eq-(2.11). We choose the + sign so the overall signature of the 2^d-dimensional C-space is *split* into an equal number of positive/negative signs.

Because the C-space corresponding to the Clifford algebra $Cl(d-1,1)$ is 2^d-dimensional one can show, after some straightforward and lengthy algebra is performed in the defining expressions for the connection and curvature in eqs-(1.9, 1.10, 1.14) , that the generalization of the Anti de Sitter space scalar curvature to the 2^d-dimensional C-space, and evaluated for the symmetric Christoffel connection, is

$$\mathbf{R}(\{\}) \;=\; \Omega^{-2}\;\Big[\; -\,2\;(2^d-1)\;(\partial_M\partial^M ln\;\Omega)\;\Big]\;\; -$$

$$\Omega^{-2}\;\Big[\;(2^d-2)\;(2^d-1)\;(\partial_M ln\;\Omega)\;(\partial^M ln\;\Omega)\;\Big] \qquad (2.14)$$

where the expression for the C-space conformal factor $\Omega(X_M)$ is given by eq-(2.10). Hence, one arrives finally at

$$\mathbf{R} \;=\; -\;\frac{2^d\;(2^d-1)}{(L/2)^2} \qquad (2.15)$$

The generalization of the de Sitter space scalar curvature to the 2^d-dimensional C-space is derived from the C-space metric

$$(d\Sigma)^2 \;=\; \frac{(dX_M)\;(dX^M)}{(1+X_M X^M/L^2)^2} \qquad (2.16)$$

leading to the (positive) value

$$\mathbf{R} \;=\; \frac{2^d\;(2^d-1)}{(L/2)^2} \qquad (2.17)$$

Concluding this subsection, the generalized vacuum field equations in C-space [1] displayed in eq-(1.15) are obeyed when the values for Λ associated with the C-space version of (Anti) de Sitter spacetimes are respectively given by

$$\Lambda \;=\; -\;\frac{(2^d-1)\;(2^d-2)}{2(L/2)^2},\;\;\Lambda \;=\; \frac{(2^d-1)\;(2^d-2)}{2(L/2)^2} \qquad (2.18)$$

These results are consistent with a throat radius $\rho = L/2$ of the underlying (Anti) de Sitter spacetimes. The generalized Ricci tensors are respectively given by

$$\mathbf{R}_{MN} \;=\; -\;\frac{(2^d-1)}{(L/2)^2}\;\mathbf{g}_{MN},\;\;\mathbf{R}_{MN} \;=\; \frac{(2^d-1)}{(L/2)^2}\;\mathbf{g}_{MN} \qquad (2.19)$$

The embedding of the 2^d-dimensional C-space "hyperboloid" into an *abstract* space of 2^d+1 dimensions in the Anti de Sitter version of C-space can be attained by writing

$$Y^M = \frac{X^M}{(1 - X_M X^M/L^2)}, \quad M = 1,2,\cdots,2^d \tag{2.20a}$$

$$Y^{M+1} = \frac{L}{2}\,\frac{(1 + X_M X^M/L^2)}{(1 - X_M X^M/L^2)}, \tag{2.20b}$$

whereas for the de Sitter version one has

$$Y^M = \frac{X^M}{(1 + X_M X^M/L^2)}, \quad M = 1,2,\cdots,2^d \tag{2.21a}$$

$$Y^{M+1} = \frac{L}{2}\,\frac{(1 - X_M X^M/L^2)}{(1 + X_M X^M/L^2)} \tag{2.21b}$$

and leading to the a generalization of eqs-(2.1-2.6). Note that $2^d + 1 \neq 2^{d+1}$, unless $d = 0$, however the abstract space of 2^d+1 dimensions is associated to the dimensions of the direct sum of the Clifford algebras $Cl(d-1,1) \oplus Cl(0)$.

2.2 A different family of C-space metrics

Another C-space metric associated with the generalization of the d-dim Anti de Sitter space AdS_d to C-spaces is given by a "diagonal sum" of the Clifford scalar, vector, bivector, trivector, ... contributions

$$(d\Sigma)^2 = (L_P)^2\,\frac{(ds)^2}{(1-s^2)^2} + \frac{(dx_\mu)\,(dx^\mu)}{(1 - x_\mu x^\mu/L^2)^2} + (L_P)^{-2}\,\frac{(dx_{\mu\nu})\,(dx^{\mu\nu})}{(1 - x_{\mu\nu}x^{\mu\nu}/L^4)^2} + (L_P)^{-4}\,\frac{(dx_{\mu\nu\rho})\,(dx^{\mu\nu\rho})}{(1 - x_{\mu\nu\rho}x^{\mu\nu\rho}/L^6)^2} + \ldots \tag{2.22}$$

The above C-space metric is *not* the same as

$$(d\Sigma)^2 = \frac{(dX_M)\,(dX^M)}{(1 - X_M X^M/L^2)^2} \tag{2.23}$$

and for this reason the metric (2.20) does *not* obey the field equations (1.15).

The above "diagonal sum" version in the de Sitter case is

$$(d\Sigma)^2 = (L_P)^2\,\frac{(ds)^2}{(1+s^2)^2} + \frac{(dx_\mu)\,(dx^\mu)}{(1 + x_\mu x^\mu/L^2)^2} + (L_P)^{-2}\,\frac{(dx_{\mu\nu})\,(dx^{\mu\nu})}{(1 + x_{\mu\nu}x^{\mu\nu}/L^4)^2} + (L_P)^{-6}\,\frac{(dx_{\mu\nu\rho})\,(dx^{\mu\nu\rho})}{(1 + x_{\mu\nu\rho}x^{\mu\nu\rho}/L^6)^2} + \ldots \tag{2.24}$$

The above C-space metric does *not* solve the field equations (1.15) and does not have the form

$$(d\Sigma)^2 \ = \ \frac{(dX_M)\ (dX^M)}{(1+X_M X^M/L^2)^2} \tag{2.25}$$

2.3 Analog of Static Spherically Symmetric Metrics in C-spaces

To search for a generalization of static spherically symmetric metrics in C-spaces, let us focus on the Clifford algebra $Cl(3,1)$ associated with a four-dim Lorentzian spacetime and which is $2^4 = 16$ dimensional. The C-space metric defining the infinitesimal interval $(d\Sigma)^2$ has a *split* signature $(8,8)$ [5]. Let us examine what would be the analog of a "spherically" symmetric metric in C-space. The analog of the "spatial radial distance" squared in the 16-dim C-space is

$$|X|^2 \ = \ (L_P)^2\ s^2 \ + \ (x^1)^2 \ + \ (x^2)^2 \ + \ (x^3)^2 \ + \\ (L_P)^{-2} \ \big(\ (x^{12})^2 \ + \ (x^{13})^2 \ + \ (x^{23})^2\ \big) \ + (L_P)^{-4}\ (x^{123})^2 \tag{2.26}$$

from which one can infer that

$$d|X| \ = \ |X|^{-1} \ \big[\ (L_P)^2\ sds \ + \ x^1 dx^1 \ + \ x^2 dx^2 \ + \ x^3 dx^3\ \big] \ + \\ |X|^{-1} \ \big[\ (L_P)^{-2} \ \big(\ x^{12}dx^{12} \ + \ x^{13}dx^{13} \ + \ x^{23}dx^{23}\ \big) \ + (L_P)^{-4}\ x^{123}dx^{123}\ \big] \tag{2.27}$$

where $|X|$ is the square root of eq-(2.26).

The analog of the "temporal radial distance" squared in the 16-dim C-space is

$$|T|^2 \ = \ (x^0)^2 \ + (L_P)^{-2} \ \big(\ (x^{01})^2 \ + \ (x^{02})^2 \ + \ (x^{03})^2\ \big) \ + \\ (L_P)^{-4} \ \big(\ (x^{012})^2 \ + \ (x^{013})^2 \ + \ (x^{023})^2\ \big) \ + (L_P)^{-6}\ (x^{0123})^2 \tag{2.28}$$

from which one can infer the expression for the infinitesimal temporal displacement

$$d|T| \ = \ |T|^{-1} \ \big[\ x^0 dx^0 \ + (L_P)^{-2} \ \big(\ x^{01}dx^{01} \ + \ x^{02}dx^{02} \ + \ x^{03}dx^{03}\ \big)\ \big] \ + \\ |T|^{-1} \ \big[(L_P)^{-4} \ \big(\ x^{012}dx^{012} \ + \ x^{013}dx^{013} \ + \ x^{023}dx^{023}\ \big) \ + (L_P)^{-6}\ x^{0123}dx^{0123}\ \big] \tag{2.29}$$

where $|T|$ is the square root of eq-(2.28).

Hence, an ansatz for the analog of a "static spherically symmetric" metric in the 16-dim C-space of split signature $(8, 8)$ is

$$(d\Sigma)^2 \ = \ -f(|X|)\,(d|T|)^2 - |T|^2\,(d\chi_7)^2 + h(|X|)\,(d|X|)^2 + |X|^2\,(d\Omega_7)^2 \quad (2.30)$$

where $|X|^2 (d\Omega_7)^2$ is the C-space metric analog a 7-dim sphere determined by the spatial directions, and $|T|^2 (d\chi_7)^2$ is the C-space metric analog of a 7-dim sphere determined by the temporal directions. Ω_7, χ_7 are the respective solid angles of the 7-dim spheres. All the other terms in (2.30) are defined by eqs-(2.26-2.29). The real-valued functions $f(|X|), h(|X|)$ in (2.30) are determined by solving the very *complicated* C-space field equations (1.15). The flat C-space limit is attained when $f(|X|) = h(|X|) = 1$.

The $4D$ (Anti) de Sitter-Schwarzschild metric in natural units $\hbar = c = G = 1$

$$(d\tau)^2 \ = \ -\,(1 - \frac{2M}{r} - \frac{\lambda}{3}\,r^2)\,(dt)^2 + (1 - \frac{2M}{r} - \frac{\lambda}{3}\,r^2)^{-1}\,(dr)^2 + r^2\,(d\Omega_2)^2 \quad (2.31)$$

is a solution of Einstein's field equations in $4D$ with a cosmological constant ($\lambda < 0$ in the AdS case). This metric is just a *slice* of 16-dim C-space of split signature $(8, 8)$ gived by eq-(2.30). Guided by this metric (2.31) one could attempt to find the real-valued functions $f(|X|), h(|X|)$ in (2.30) which solve the C-space field equations (1.15).

We finalize this section by discussing the very restricted class of C-space metrics ($\mathbf{g}_{MN} = \mathbf{g}_{NM}$) that can be *decomposed* into products of ordinary metrics in spacetime. Firstly, one needs to have a C-space metric whose components have the *same* grade like

$$g_{\mathbf{00}}, \quad g_{\mu\nu}, \quad g_{\mu_1\mu_2\ \nu_1\nu_2}, \quad \ldots, \quad g_{\mu_1\mu_2\ldots\mu_D\ \nu_1\nu_2\ldots\nu_D} \qquad (2.32)$$

and which can be decomposed as

$$g_{[\mu_1\mu_2]\ [\nu_1\nu_2]}(x^\mu) \ = \ g_{\mu_1\nu_1}(x^\mu)\ g_{\mu_2\nu_2}(x^\mu) \ - \ g_{\mu_2\nu_1}(x^\mu)\ g_{\mu_1\nu_2}(x^\mu)$$

$$g_{[\mu_1\mu_2\ldots\mu_k]\ [\nu_1\nu_2\ldots\nu_k]}(x^\mu) \ = \ det\ G_{\mu_i\nu_j} \ = \ \epsilon^{j_1 j_2\ldots j_k}\ g_{\mu_1\nu_{j_1}}\ g_{\mu_2\nu_{j_2}}\ \ldots\ g_{\mu_2\nu_{j_k}}, \qquad (2.33)$$

The determinant of $G_{\mu_i\nu_j}$ can be written as

$$det \begin{pmatrix} g_{\mu_1\nu_1}(x^\mu) & \ldots & \ldots\ g_{\mu_1\nu_k}(x^\mu) \\ g_{\mu_2\nu_1}(x^\mu) & \ldots & \ldots\ g_{\mu_2\nu_k}(x^\mu) \\ ------------- & & --------------- \\ g_{\mu_k\nu_1}(x^\mu) & \ldots & \ldots\ g_{\mu_k\nu_k}(x^\mu) \end{pmatrix}, \qquad (2.34)$$

The metric component $g_{\mathbf{00}}$ involving the Clifford scalar "directions" $X_{\mathbf{0}} = s$ of the Clifford polyvectors in C-space must also be included. $X_{\mathbf{0}} = s$ must not be confused with the temporal coordinate x_0. $g_{\mathbf{00}}$ behaves like a Clifford scalar under coordinate transformations in C-space. The other component $g_{[\mu_1\mu_2\ldots\mu_D]\ [\nu_1\nu_2\ldots\nu_D]}$ involves the pseudo-scalar "direction". The latter scalar and pseudo-scalar components of the C-space metric might bear some connection to the dilaton and axion fields in Cosmology and particle physics. In the most general case the C-space metric *does not factorize* into antisymmetrized sums of products of ordinary metrics . We presented above examples of metrics in C-space which cannot be decomposed into antisymmetrized sums of products of ordinary metrics.

3 Concluding Remarks on Areal Geometry and Strings

C-space metrics are an extension of *areal* metrics of the form $(d\tau)^2 = \frac{1}{4}h_{ijkl}(dx^i \wedge dx^j) \otimes (dx^k \wedge dx^l)$ which were studied long ago by Cartan. An areal metric generalization of the usual metric to Finsler geometry was developed by [9]. Such a generalized notion of area, and more generally the volume of m-dimensional submanifolds embedded in an n-dimensional space, have been considered under the terminology of "areal geometry" [14]. In these considerations, the metric and connection in general depend not only on $\mathbf{x}$ but also on the derivatives of $\mathbf{x}$ with respect to world-volume coordinates. Applications of the Kawaguchi Lagrangian formulation to string theory and p-branes can be found in [10]. The classification of area metrics and the construction of vacuum field equations were analyzed in [8]. Another family of equations for area metrics that reduce to the vacuum Einstein's equations in very special cases were studied in [7]. Static spherical symmetric solutions were found for the generalized Einstein equation in vacuum, including the Schwarzschild solution as a special case.

The Nambu-Goto action corresponding to the bosonic string is defined in terms of its worldsheet area. Motivated by the possibility that string theory admits backgrounds where the notion of length is not well defined but a definition of area is, propelled the authors [7] to study space-time geometries based on the generalization of length metrics to area metrics. In analogy with Riemannian geometry, they defined the analogues of connections, curvatures and Einstein tensor.

In Einstein's theory of gravity, the Bianchi identity provides a hint on how to define Einstein's equation such that the conservation of energy-momentum tensor is guaranteed. The situation is different for the gravitational theory of area metrics [7]. The conservation of energy-momentum is a result of the invariance of the theory under general coordinate transformations. In the theory of area metrics, the gauge symmetry is still merely general coordinate transformations but the number of degrees of freedom of the areal metric, connection and curvature are much larger than in the case of ordinary metrics. Therefore, the authors [7] argued that one should not try to define the generalized Einstein equation from the generalized Bianchi identity as one did in Einstein's theory.

However, a key difference that gravity in C-spaces has is that one has full diffeomorphism invariance under the polyvector-valued coordinate changes

$X_M \to X'_M$, thus the generalized energy-momentum polytensor in C-space is conserved and consistent with the generalized C-space Bianchi identities, in the absence of torsion and nonmetricity, and which in turn, allows us to write down the generalization of Einstein equations in C-spaces [1]. A discussion of matter fields in C-spaces can be found in [5].

Another problem with the formulation of gravity of area metrics is that it does not seem to admit an action principle due to the fact that the tensor $\mathbf{R}_{ij}^{kl}{}_{mn}$ does *not* admit the definition of scalar curvature through the contraction of indices, if the only additional tensor available is the area metric [7]. A possibility is that the action principle for the area metric theory is available only in certain dimensions when the volume form can be used to do the trick to appropriately be able to contract indices [7]. Fortunately, in C-spaces this problem does *not* arise since all polyvector-valued indices are contracted with the C-space metric $\mathbf{g}_{MN} = \mathbf{g}_{NM}$, its inverse $\mathbf{g}^{MN} = \mathbf{g}^{NM}$, and δ_N^M which in general have polyvector valued indices M, N of the same and different grades : $\mathbf{g}_{[\mu_1\mu_2\cdots\mu_i]\ [\nu_1\nu_2\cdots\nu_j]}$, for example.

To finalize, we should point out that when the C-space metric components are of the same grade, and *admit* a decomposition as shown in eq-(2.34), it is plausible to have in the putative quantum gravitational theory cases where the expectation values of the areal metrics are not zero $< \hat{g}_{\mu\nu}\hat{g}_{\rho\sigma} > \neq 0$, despite that the expectation of the metric is $< \hat{g}_{\mu\nu} >= 0$ (Topological QFT's are characterized by physical correlations independent of the metric). This could be a very natural explanation as to why quantum gravitational effects could be essentially "stringy". If on average $< \hat{g}_{\mu\nu} >= 0$, one does not observe lengths but areas instead. Quantum gravitational effects are intrinsically relevant at the Planck-scale (there are quantum gravitational phenomena which have cosmological manifestations at larger scales due to inflation, and/or compounding effects). Since the Planck scale L_P is an essential ingredient in the construction of the extended relativity in C-spaces [5], and Quantum Gravity, this suggests that C-space geometry is a natural arena to be explored. For this reason, we believe that more novel physical phenomena could be unraveled behind C-space gravity than we previously thought.

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