Periodic solutions for nonlinear oscillations in Elastic Structures via Energy Balance Method

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Abstract

A mathematical model describing the nonlinear oscillations in elastic structures is proposed. The Energy Balance Method (EBM) is applied to solve the generalized nonlinear Duffing equation obtained in absence of excitation. The numerical results show an excellent agreement with the periodic solutions obtained through the Energy Balance Method. Finally the effects of different parameters on the system behavior are studied.

Keywords: Generalized Duffing equation, periodic solution, Energy Balance Method, Numerical simulation.

Introduction

A best understanding of dynamical behavior of engineering structures requires a mathematical modeling of these structures. This is a primary stage for various scientific applications in mechanical engineering, civil engineering, artificial intelligence, aircraft industry, motor-car industry, etc. Several nonlinear mathematical models of different complexities have been proposed in literature. Most of these models represent a generalization of existing nonlinear oscillators such as Mathieu, Rayleigh, Helmholtz-Duffing, Van-der-pol and Duffing equations [1]. These equations are widely used in many areas of physics and engineering applications [2, 3]. Among these oscillators, Duffing equation is most used for modeling many practical engineering systems and various physical phenomena [4, 5]. In mathematical modeling of the elastic structure depicted on Figure 1, most of investigators take into account only the linear elasticity of the system. But, it is well-known that the components of engineering systems show in general a nonlinear behavior under an exciting force [6]. Therefore, a good modeling of elastic structures must then take into consideration the nonlinear elastic behavior of structure components. Thus the resulting differential equations become more complex to be resolved. Due to this fact, obtaining an exact analytical solution becomes very difficult [7-10] or even sometimes impossible. Therefore, several approximate methods have been proposed for solving the nonlinear differential equations. The perturbation methods [5, 11, 12] are commonly used in the study of vibrations modeling and control. These techniques are based on the existence of small parameters, the so-called perturbation quantity. Nevertheless other approaches have also been developed recently in

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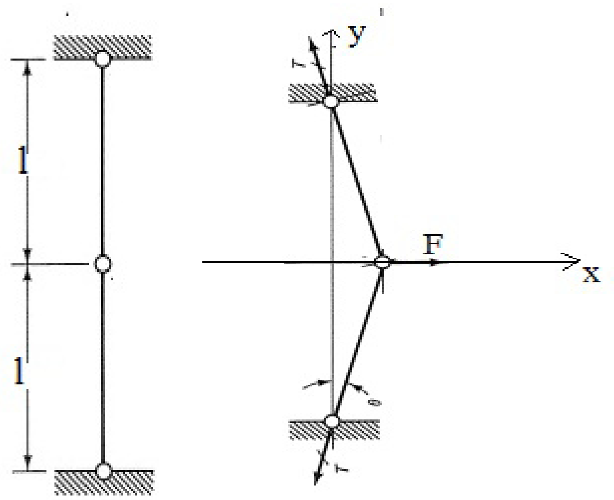
literature for determining an approximate analytical solution of nonlinear differential equations without existing the perturbation quantity. Variational iteration method [13- 16], He's frequency-amplitude formulation [17-20], energy balance method [21], harmonic balance method [10, 22, 24], Hamiltonian approach [3,7 , 13, 25, 27], and coupled Homotopy-variational formulation [27-29] are some examples. Among all these techniques, it is well-known that the energy balance method is widely used in many practical problems to solve the nonlinear equations [27, 30, 35]. The objective of this paper consists of taking into account the nonlinear elastic behavior for analyzing the periodic solutions for specific values of parameters of mechanical structure represented on Figure1 by applying energy balance method. In this way, we perform first, the mathematical model of the problem (section 2), afterward we carry out the system analysis via the Energy Balance Method (section 3) and the results are discussed (section 4) and finally some conclusions of the work are given in the last section.

2 Mathematical modeling

2.1- Formulation of the problem

We consider a simple system constituted of a mass m, attached in the middle of a uniform string of cross-section A and total length 2ℓ, as shown in Figure 1(a). When an excitation is applied to the mass, the system takes the elastic configuration shown in Figure 1(b) [36].



(a) (b)

Figure 1- The physical model of the elastic structure

Now we purpose to determine the governing equation of motion of the system represented in Figure 1.

2.2 Equation of motion

When the mass moves in the x direction (Figure 1-b), the application of the fundamental relation of the dynamics yields:

$$m\ddot{x} + 2T\sin\theta = F(t) \quad (1)$$

where F and T represent the excitation and the tension in the elastic structure, respectively.

On substituting the following expression of the tension T when the system takes the elastic configuration

$$T = T_0 + A\sigma \tag{2}$$

where T_0 denotes the initial tension when the system is at rest, A the cross-section and σ the stress in the system, into the equation (1), we obtain

$$m\ddot{x} + 2(T_0 + A\sigma)\sin\theta = F(t) \tag{3}$$

Now, on supposing that the stress in the system can be written as follows

$$\sigma = E\varepsilon^n \tag{4}$$

where E means the elastic module, ε represents the normal strain in the x direction and n designates the strain hardening exponent, the equation (3) becomes

$$m\ddot{x} + 2x\left[\frac{T_0 + AE\left(\sqrt{1+\frac{x^2}{l^2}} - 1\right)^n}{l\sqrt{1+\frac{x^2}{l^2}}}\right] = F(t) \tag{5}$$

with $\varepsilon = \sqrt{1+\frac{x^2}{l^2}} - 1$

and

$$\sin\theta = \frac{x}{l\sqrt{1+\frac{x^2}{l^2}}}$$

Using the Taylor-series expansion of the expressions ε and $\sin\theta$ to the third order for small x, the equation (5) can be written as

$$\ddot{x} + \frac{2T_0}{ml}x + \frac{2AE}{2^n ml^{2n+1}}x^{2n+1} - \frac{T_0}{ml^3}x^3 - \frac{AE}{2^n ml^{2n+3}}x^{2n+3} = \frac{F(t)}{m} \tag{6}$$

which can be rewritten in nondimensional form as

$$u'' + u - \frac{1}{2}u^3 + \frac{\alpha}{2^n}\left(1 - \frac{1}{2}u^2\right)u^{2n+1} = f(\tau) \tag{7}$$

where $\tau = \omega_0 t$, is nondimensional time, $\omega_0 = \sqrt{\frac{2T_0}{ml}}$, $u = \frac{x}{l}$, $f = \frac{F}{ml\omega_0^{\;2}}$, $\alpha = \frac{AE}{T_0}$, and the symbol ' denotes the differentiation with respect to nondimensional time τ.

So, note that the obtained equation (7) represents a generalized Duffing oscillator equation subjected to forcing excitation. At present, we determine in the section 3 the approximate analytical solution of this equation under unforced oscillations conditions.

3 System analysis

3.1 Unforced equation

In this subsection, we analyze the nonlinear oscillations of the elastic system in the absence of the external excitation force, that is to say, when $f(\tau)=0$. In this condition, the equation (7) becomes

$$u''+u-\frac{1}{2}u^3+\frac{\alpha}{2^n}\left(1-\frac{1}{2}u^2\right)u^{2n+1}=0 \quad (8)$$

It is necessary to notice that for different values of strain hardening exponent n one can obtain various behaviors of the Duffing oscillator equation. It is again necessary to signal that for $n=1$ and $n=2$, the exact and approximate analytical solutions of the corresponding oscillator equation can be determined [32, 33, 37].

Now, it is interesting to solve the generalized Duffing nonlinear oscillator equation (8) by means of the Energy Balance Method (EBM) (subsection 3.2)

3.2 Application of the Energy Balance Method (EBM)

In the Energy Balance Method (EBM) the nonlinear oscillator equation (8) can be put in the form [8, 38]

$$u''+g(u)=0 \quad (9)$$

under following initial conditions

$$\begin{aligned} u(0)&=A \\ u'(0)&=0 \end{aligned} \quad (10)$$

Taking into account the equations (8) and (9) we obtain

$$g(u)=u-\frac{1}{2}u^3+\frac{\alpha}{2^n}\left(1-\frac{1}{2}u^2\right)u^{2n+1}$$

which will be used to determine variation and Hamiltonian formulations of the equation (8).

Thus referring to [8], the variational principle for nonlinear oscillation equation (8) can be written as:

$$J(u)=\int_0^{\tau}\left(-\frac{1}{2}u'^2+\frac{1}{2}u^2-\frac{1}{8}u^4+\frac{\alpha}{2^n(2n+2)}u^{2n+2}-\frac{\alpha}{2^{n+1}(2n+4)}u^{2n+4}\right)d\tau \quad (11)$$

The Hamiltonian of the equation (8) can be obtained in the form:

$$\begin{aligned} H(u)&=\frac{1}{2}u'^2+\frac{1}{2}u^2-\frac{1}{8}u^4\,\frac{\alpha}{2^n(2n+2)}u^{2n+2}-\frac{\alpha}{2^{n+1}(2n+4)}u^{2n+4} \\ &=\frac{1}{2}A^2-\frac{1}{8}A^4\,\frac{\alpha}{2^n(2n+2)}A^{2n+2}-\frac{\alpha}{2^{n+1}(2n+4)}A^{2n+4} \end{aligned} \quad (12)$$

where A denotes the initial amplitude. Choosing the following trial function

$$u(\tau)=A\cos\omega\tau \quad (13)$$

we obtain the following residual equation

$$R(\tau)=\frac{\omega^2}{2}\sin^2\omega\tau+\frac{A^2}{2}\cos^2\omega\tau-\frac{A^4}{8}\cos^4\omega\tau+\frac{\alpha A^{2n+2}}{2^n(2n+2)}\cos^{2n+2}\omega\tau-\frac{\alpha A^{2n+4}}{2^{n+1}(2n+4)}\cos^{2n+4}\omega\tau$$
$$-\left(\frac{A^2}{2}-\frac{A^4}{8}+\frac{\alpha A^{2n+2}}{2^n(2n+2)}-\frac{\alpha A^{2n+4}}{2^{n+1}(2n+4)}\right)=0 \tag{14}$$

If we collocate at $\omega\tau=\frac{\pi}{4}$, we obtain after some algebraic operations the following result:

$$\omega=\sqrt{1-\frac{3A^2}{8}-\frac{\alpha A^{2n}}{2^{2n}}\left[\frac{1-2^{n+1}}{n+1}+\frac{A^2\left(2^{n+1}-\frac{1}{2}\right)}{2n+4}\right]} \tag{15}$$

The period can be written in the form:

$$T=\frac{2\pi}{\sqrt{1-\frac{3A^2}{8}-\frac{\alpha A^{2n}}{2^{2n}}\left[\frac{1-2^{n+1}}{n+1}+\frac{A^2\left(2^{n+1}-\frac{1}{2}\right)}{2n+4}\right]}} \tag{16}$$

Substituting the equation (15}) into the equation (13), we obtain the following approximate solution:

$$u(\tau)=A\cos\left(\sqrt{1-\frac{3A^2}{8}-\frac{\alpha A^{2n}}{2^{2n}}\left[\frac{1-2^{2n+1}}{n+1}+\frac{A^2\left(2^{n+1}+\frac{1}{2}\right)}{2n+4}\right]}\,\tau\right)$$

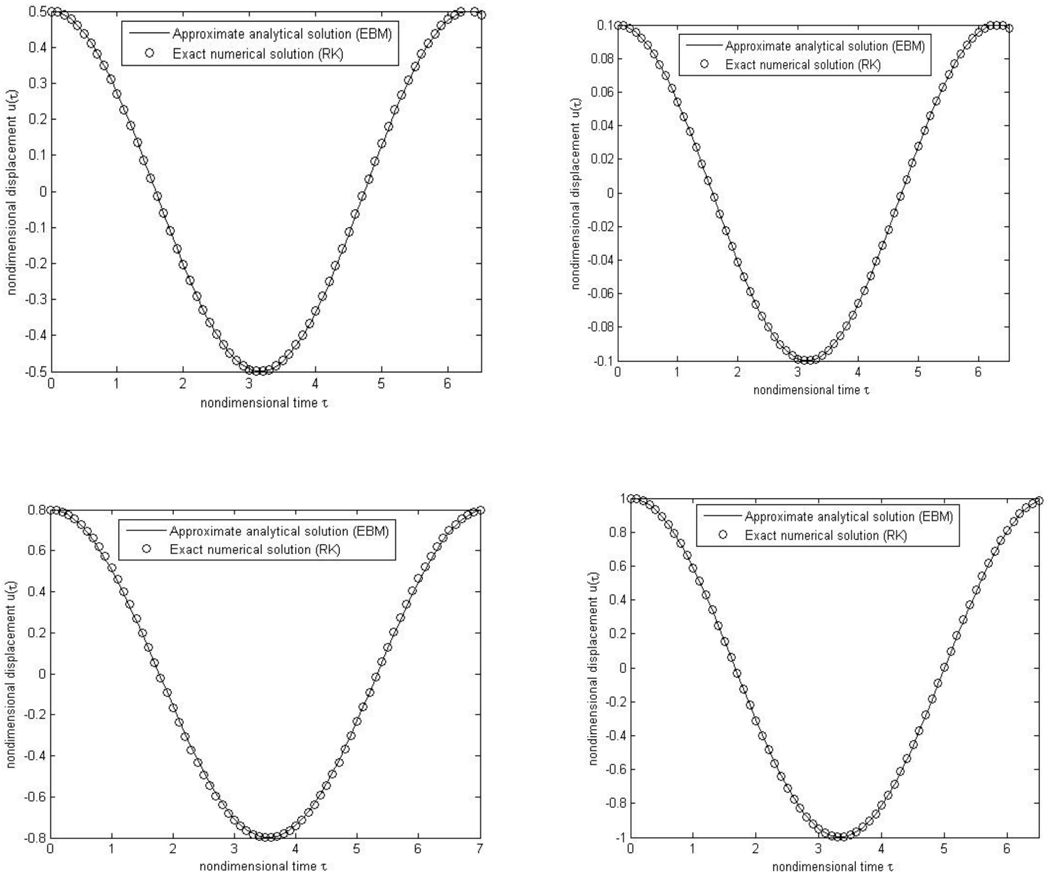
Therefore, with various n, different behaviors of solution of the equation (8) can be determined.

4 Numerical results and discussion

In this section, the accuracy of the Energy Balance Method (EBM) is tested in the firstt time by comparing the approximate analytical solution with exact numerical solution obtained by Matlab (subsection 4.1). Secondly the effects of different parameters on the response have been analyzed. At last the variations of frequency with amplitude A, strain hardening exponent n and α have been performed, respectively.

4.1 Comparison between approximate analytical result and exact numerical solution

This subsection compares the approximate analytical solution represented in solid line with the exact numerical solution plotted in circles for different values of the parameters of the system. Thus, we observe in Figure 2(a)-(d), that the approximate analytical solution obtained by Energy Balance Method is in excellent agreement with the exact numerical solution obtained by Matlab ode solver 45. The mean square error (mse) values that justify this agreement are 1.9719e-009 for Figure 2 (a), 1.9336e-005 for Figure 2 (b), 3.3917e-005 for Figure 2 (c) and 7.8925e-005 for Figure 2 (d).



4.2 Effects of the parameters on the system response

Here, the effects of the nondimensional tension α and strain hardening exponent n are carried out in the phase plane by keeping the initial amplitude constant. Thus the Figure 3 represents the effect of the hardening parameter n on the system response for six different values of n. The values of other parameters are kept constants. This Figure 3 shows elliptical trajectories of horizontal focal axis.

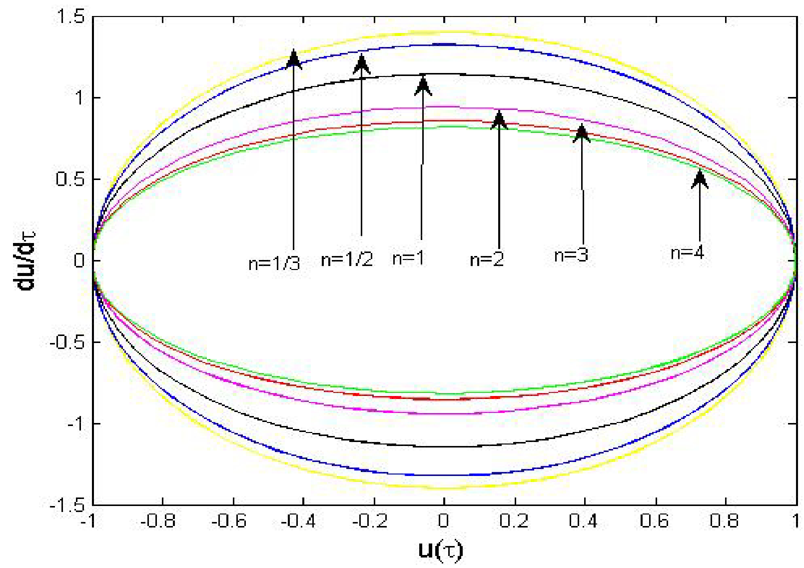


Figure 3- Phase portrait showing the effects of the strain hardening exponent n for $A=1$ and $\alpha=3$

In Figure 4, we also note that the trajectories of the system are ellipses of horizontal focal axis.

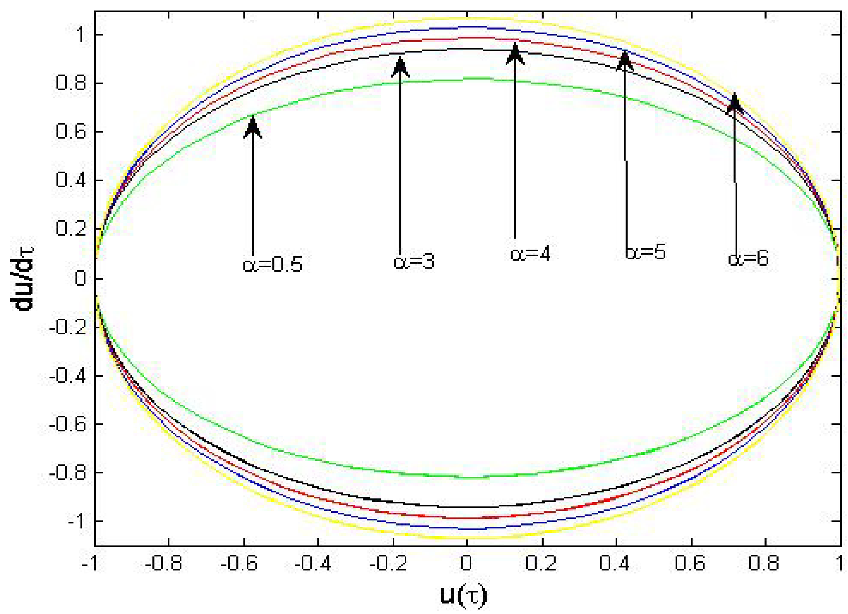


Figure 4- Phase portrait exhibiting the effects of the tension α for $A=1$ and $n=2$

Therefore we can note that α and n have opposite effects on the system response. This observation is confirmed by the Figure 5 and Figure 6

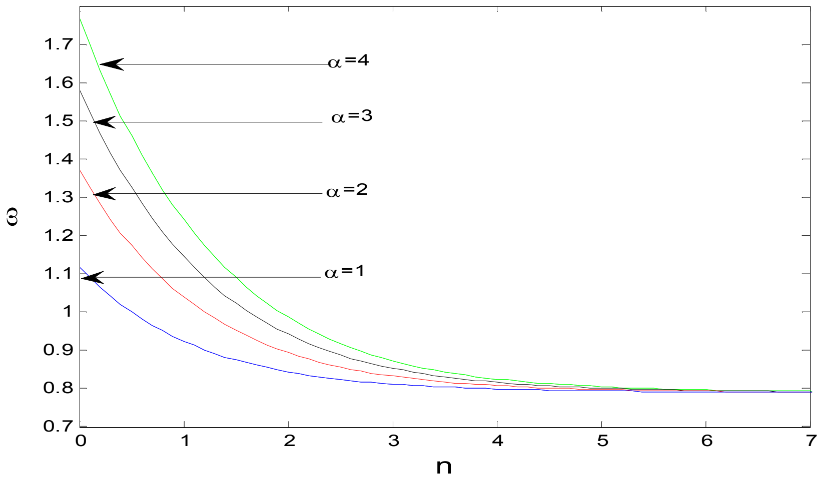


Figure 5- Frequency variation with strain hardening exponent (n) for $\alpha = 1, 2, 3, 4$

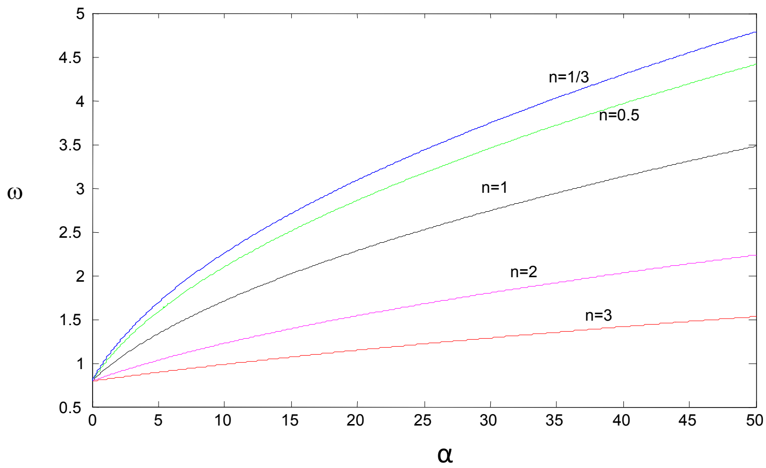


Figure 6- Frequency variation with α for $n = \frac{1}{3}, 0.5, 1, 2, 3$

4.3 Amplitude variation with angular frequency

The variation of the angular frequency with respect to amplitude is performed in Figure 7 and Figure 8 for different values of n and α, respectively. In Figure 7 we note that for positive values of n smaller than 1, the frequency increases with amplitude to attain fast its maximal value, and then decreases slowly to reach its minimal value. On the other hand, for higher values of n, the frequency decreases rapidly to attain its minimal value.

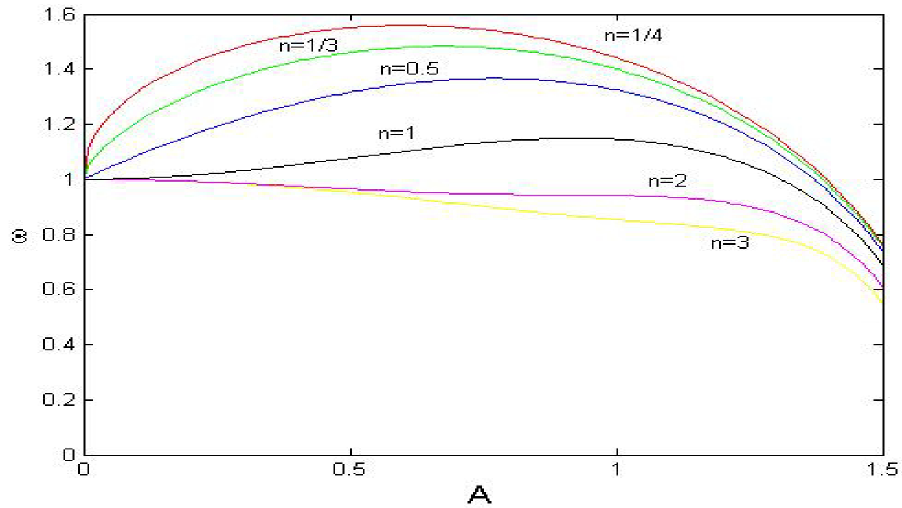


Figure 7- Effects of the strain hardening exponent n on frequency

The Figure 8 shows that the angular frequency decreases rapidly with amplitude when α decreases.

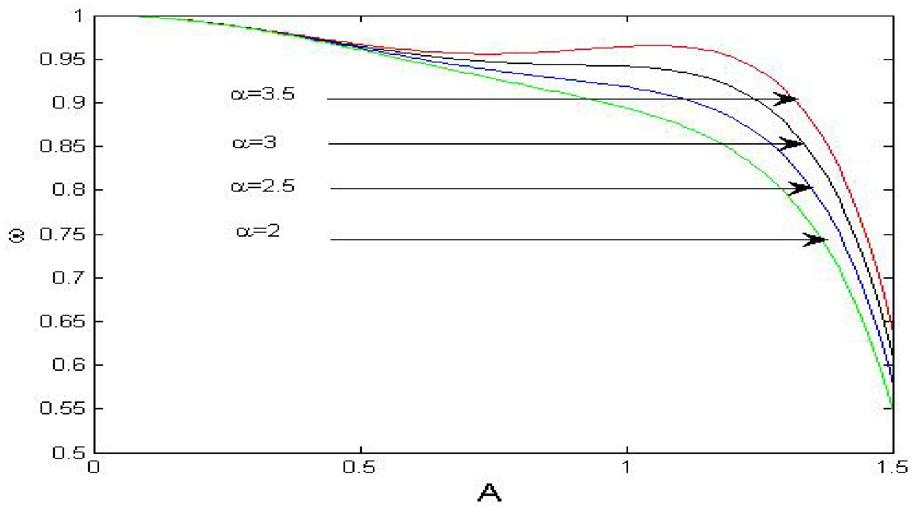


Figure 8- Effects of the stiffness α on frequency

Conclusion

A generalized Duffing oscillator equation was developed in this study for analyzing the unforced nonlinear oscillations in elastic structures. The Energy Balance Method (EBM) was used for determining the approximate analytical solution of the governing nonlinear equation. The accuracy of the approximate analytical solution obtained was verified by comparing with exact numerical solution given by Matlab ode solver 45. As result, an excellent agreement was obtained between the approximate analytical solution and exact numerical solution. The effects of the strain hardening exponent n, the tension α and amplitude are carried out.

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