A General type of Liénard Second Order Differential Equation: Classical and Quantum Mechanical Study

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We generate a general model of Liénard type of second order differential equation and study its classical solution. We also generate Hamiltonian from the differential equation and study its stable eigenvalues.

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I. Introduction

Liénard equations are widely used in many branches of science and engineering to model various types of phenomena like oscillations in mechanical and electrical systems. Particularly, more than fifty years, there has been a continued interest among different authors for paying attention of Liénard type differential equation [1,2]:

$$\frac{d^2x}{dt^2} + f(x)\left(\frac{dx}{dt}\right)^2 + xg(x) = 0 \tag{1}$$

where $f(x)$ and $g(x)$ are functions of x since it admits position-dependant mass dynamics useful for several applications of quantum physics. These types of second order differential equation are interesting for physicists provided one generates suitable Hamiltonian. For all possible values of $f(x)$ and $g(x)$, it may not be possible to generate Hamiltonian having stable eigenvalues. Secondly a classical model solution can also be obtained using He's approximation [3-6] by using procedure given below

$$\frac{d^2x}{dt^2} + f(x)\left(\frac{dx}{dt}\right)^2 + xg(x) = R(t) \tag{2}$$

Let us consider now two different values of x as

$$x_1 = A\cos\omega_1 t \tag{3}$$

and

$$x_2 = A\cos\omega_2 t \tag{4}$$

then

$$\omega^2 = \omega_2^2 = \frac{R_2(0)\omega_1^2 - R_1(0)\omega_2^2}{R_2(0) - R_1(0)} \tag{5}$$

here $\omega_1 = 1$. In this paper, we address the above differential equation by selecting a general type of values on $f(x)$ and $g(x)$, and generate suitable Hamiltonian and study its stable eigenvalues.

II. General type of Differential Equation and Solution

Here we consider a general type of differential equation as

$$\frac{d^2x}{dt^2} + \frac{N\lambda x^{N-1}}{2(1+\lambda x^N)}\left(\frac{dx}{dt}\right)^2 + \omega_0^2 \frac{Kx^{K-1}}{2(1+\lambda x^N)} = 0 \tag{6}$$

where N, K =2, 4, 6,…… In this equation one has to fix the value of K and vary N or vice versa. Let us consider that the general solution of this differential equation be

$$x = A\cos\omega t \tag{7}$$

then

$$\omega = \omega_0\sqrt{\frac{KA^{K-2}}{2(1+\lambda A^N)}} \tag{8}$$

III. Hamiltonian generation

In order to generate Hamiltonians we multiply the differential equation by $\dot{x}$ and rewrite it as

$$\frac{d\left[\frac{\dot{x}^2(1+\lambda x^N) + \omega_0^2 x^K}{2}\right]}{dt} = 0 \tag{9}$$

Let the bracket term be denoted as H where

$$H = \frac{1}{2}\left[\dot{x}^2(1+\lambda x^N) + \omega_0^2 x^K\right] \tag{10}$$

Now define momentum p as

$$p = \frac{\partial H}{\partial \dot{x}} \tag{11}$$

Hence one can write H as

$$H = \frac{1}{2}\left[\frac{p^2}{(1+\lambda x^N)} + \omega_0^2 x^K\right] \tag{12}$$

One can interpret this Hamiltonian as a model in which mass varies with distance [3].

IV. Eigenvalues of Generated Hamiltonian

Here we solve the eigenvalue relation

$$H|\psi\rangle = E|\psi\rangle \tag{13a}$$

using matrix diagonalization method [7]. In which

$$|\psi\rangle = \sum_m A_m |m\rangle \tag{13b}$$

Here $|m\rangle$ satisfy the exact eigenvalue relation

$$H_0|m\rangle = \left(p^2 + x^2\right)|m\rangle = (2m+1)|m\rangle \tag{14}$$

Now using the above formalism, we get the following recursion relation satisfied by A_m as

$$\sum_{k=2,4,6,.....} P_m A_{m-k} + S_m A_m + R_m A_{m+k} = 0 \tag{15}$$

where

$$P_m = \langle m|H|m-k\rangle \tag{16a}$$

$$R_m = \langle m|H|m+k\rangle \tag{16b}$$

$$S_m = \langle m|H|m\rangle - E \tag{16c}$$

In fact one will notice that the above Hamiltonian is not invariant under exchange of momentum p and $\frac{1}{(1+\lambda x^N)}$. Hence following the literature [8] we write the invariant Hamiltonian as

$$H = \frac{1}{2}\left[p\frac{1}{(1+\lambda x^N)}p + \omega_0^2 x^K\right] \tag{17}$$

and reflect the first four states eigenvalues in table-1.

Table -1: First four eigenvalues of Hamiltonians with $\omega_0 = 1,\ \lambda = 1$.

Hamiltonian	Value of n	Eigenvalue
$H = \frac{1}{2}\left[p\frac{1}{(1+x^2)}p + x^2\right]$	0	0.355 026 280
	1	1.226 397 537
	2	1.846 999 994
	3	2.445 481 398
$H = \frac{1}{2}\left[p\frac{1}{(1+x^4)}p + x^2\right]$	0	0.338 179 394
	1	1.199 312 190
	2	1.770 479 342
	3	2.154 962 590
$H = \frac{1}{2}\left[p\frac{1}{(1+x^6)}p + x^2\right]$	0	0.320 091 281
	1	1.169 152 075
	2	1.662 103 201
	3	1.897 043 406
$H = \frac{1}{2}\left[p\frac{1}{(1+x^2)}p + x^4\right]$	0	0.342 163 615
	1	1.447 762 223
	2	2.733 381 643
	3	3.824 351 590
$H = \frac{1}{2}\left[p\frac{1}{(1+x^4)}p + x^4\right]$	0	0.326 786 311
	1	1.447 762 223
	2	2.733 381 643
	3	3.824 351 590
$H = \frac{1}{2}\left[p\frac{1}{(1+x^6)}p + x^4\right]$	0	0.306 713 747
	1	1.392 267 754
	2	2.676 140 588
	3	3.519 276 808
$H = \frac{1}{2}\left[p\frac{1}{(1+x^2)}p + x^6\right]$	0	0.354 476 360
	1	1.652 542 050
	2	3.294 555 429
	3	5.270 061 821
$H = \frac{1}{2}\left[p\frac{1}{(1+x^4)}p + x^6\right]$	0	0.341 508 635
	1	1.617 435 142
	2	3.393 428 656
	3	5.181 678 146

V. Phase portrait in the (p, x) plane

Classical Phase trajectories of the system (Eqn. (12)) are represented in the figures (Figure 1 to 8) for different parametric choices. The quantum mechanical Phase trajectories of the system (Eqn. (12)) are also represented in the figures (Figure 9 to 16) for different parametric choices. Further we plot $|\psi|^2$ of first four eigenstates of Hamiltonian given in eqn. (17) (Fig. 17 - 25).

VI. Conclusion

In this paper we have generated a general form of differential equation which can be termed as Liénard type. Further, we find classical solution and quantum eigenvalues of the generated system. We hope interested reader can follow the present approach and generate many similar type of Hamiltonians. Last but not the least present analysis reveals the quantum behaviour in classical differential equation.

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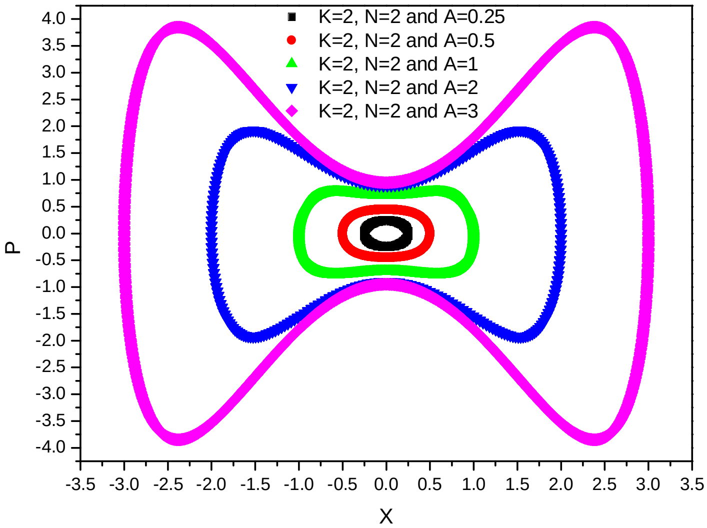


Figure 1: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = K = 2, for various values of A.

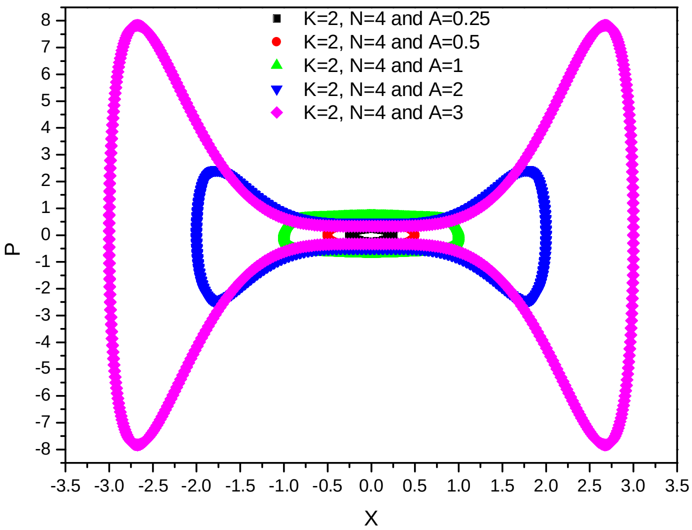
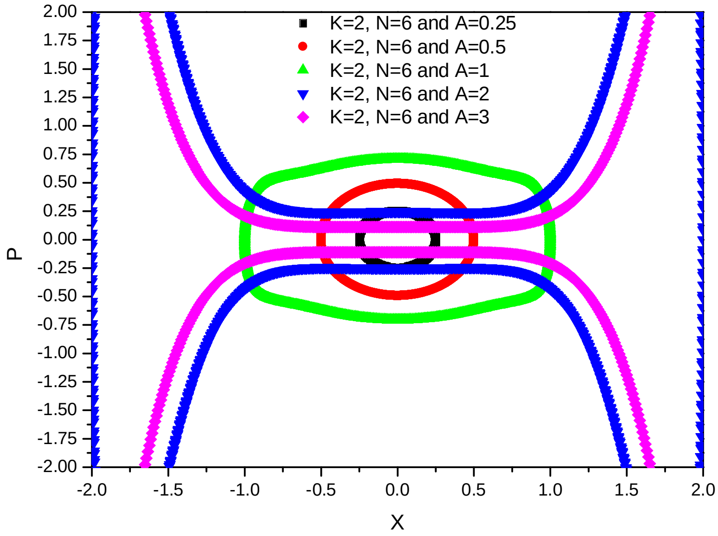


Figure 2: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = 4, K = 2, for various values of A.



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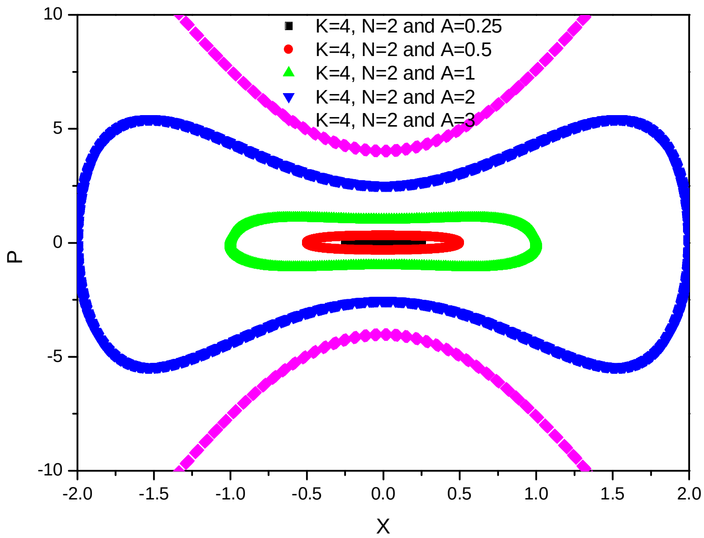

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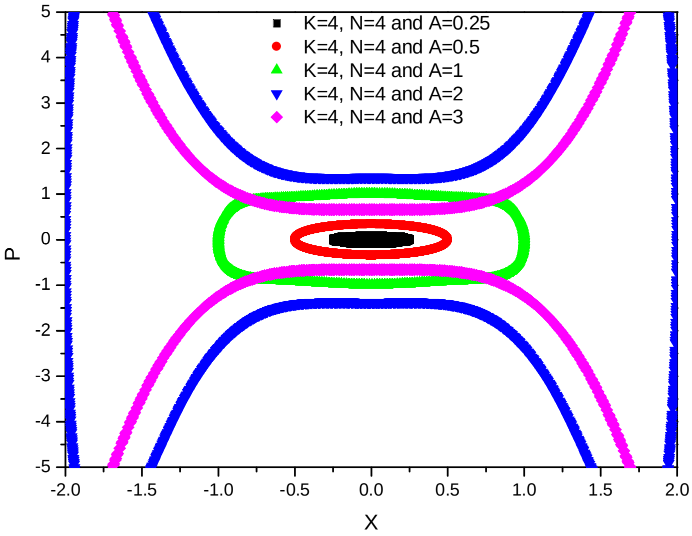


Figure 5: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = K = 4, for various values of A.

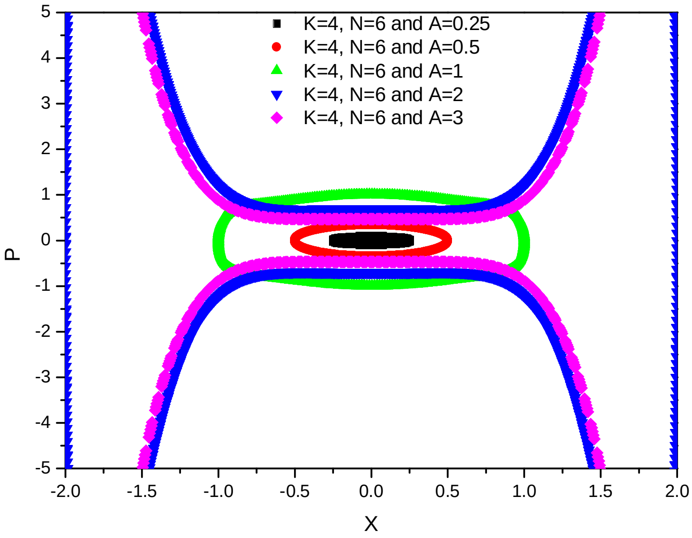


Figure 6: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = 6, K = 4, for various values of A.

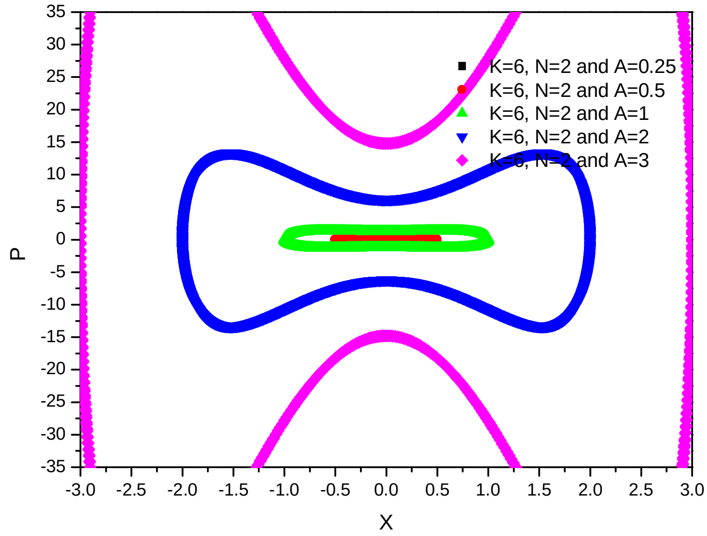


Figure 7: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = 2, K = 6, for various values of A.

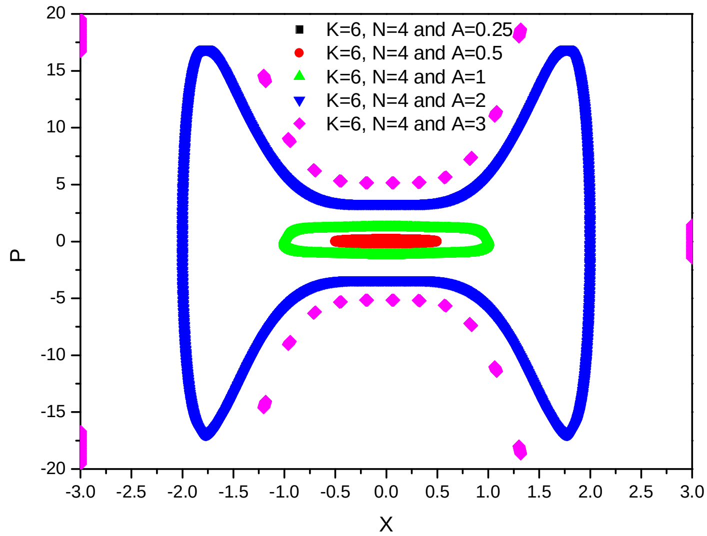
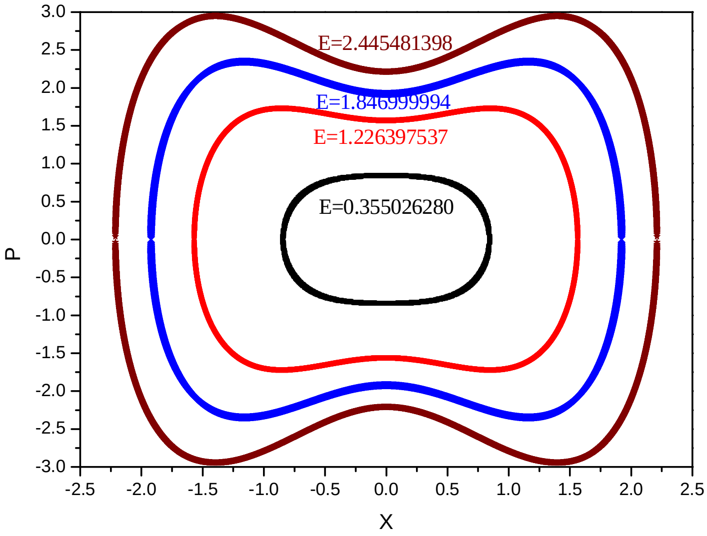


Figure 8: Classical phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = 4, K = 6, for various values of A.



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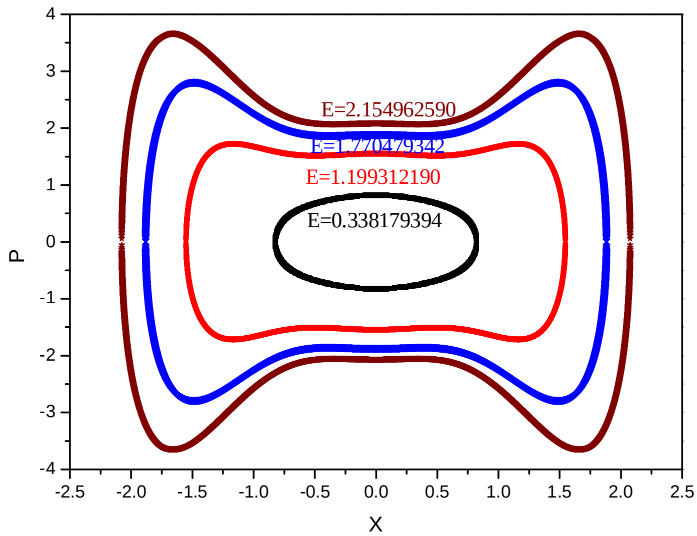

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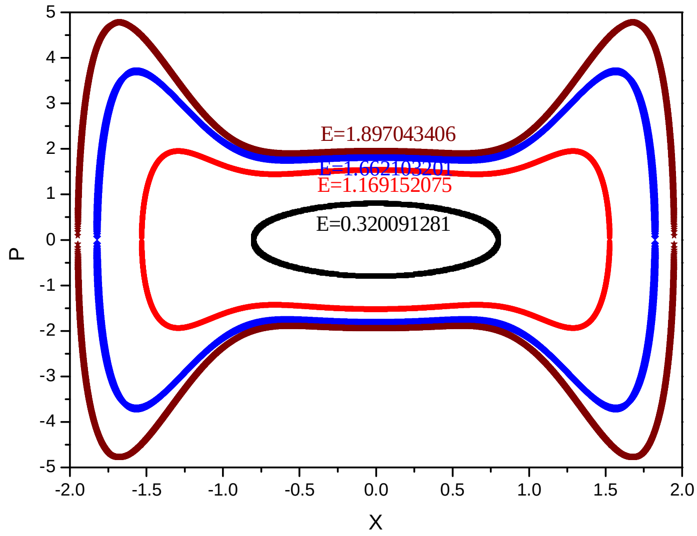
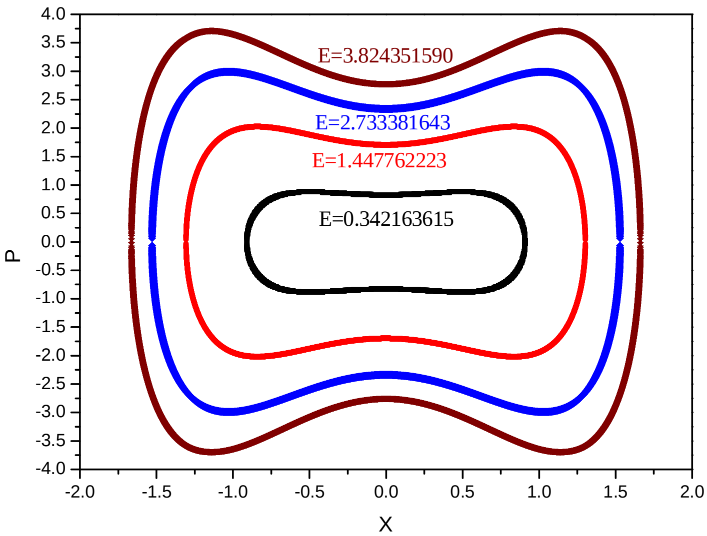


Figure 11: Phase trajectories of the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = 6, K = 2, for various values of E = H.



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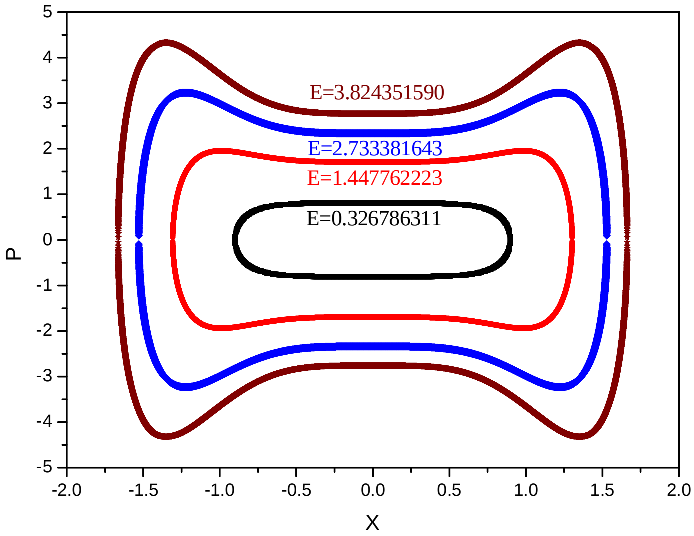

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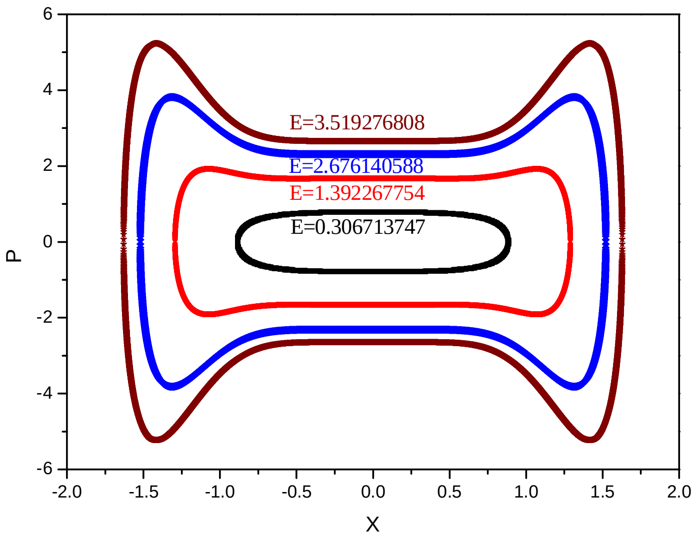

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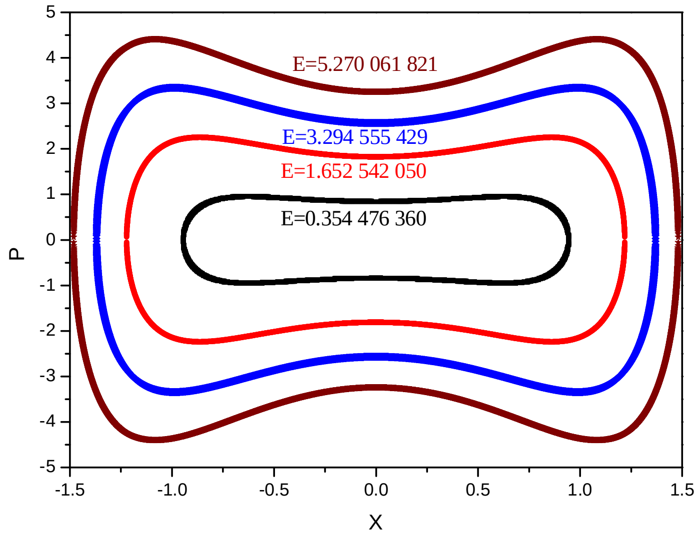

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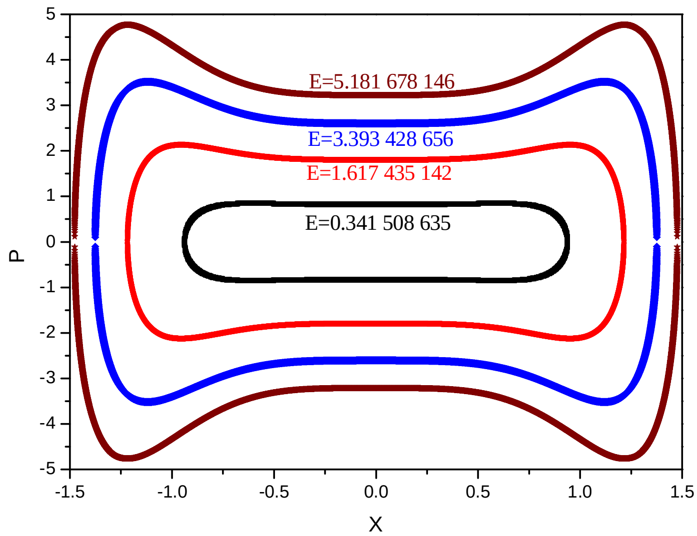

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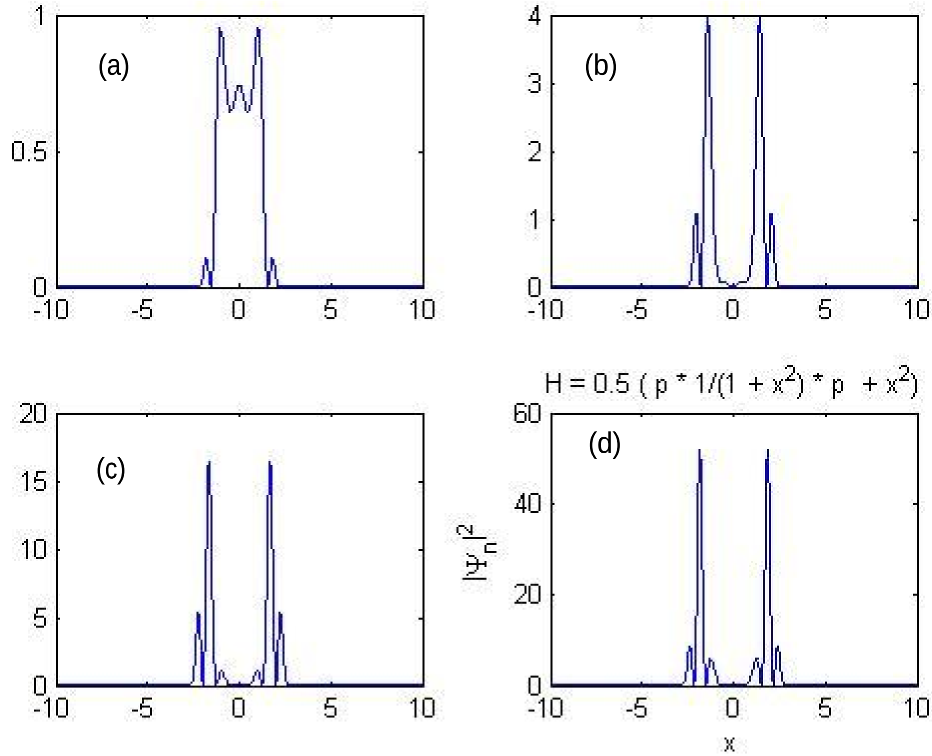
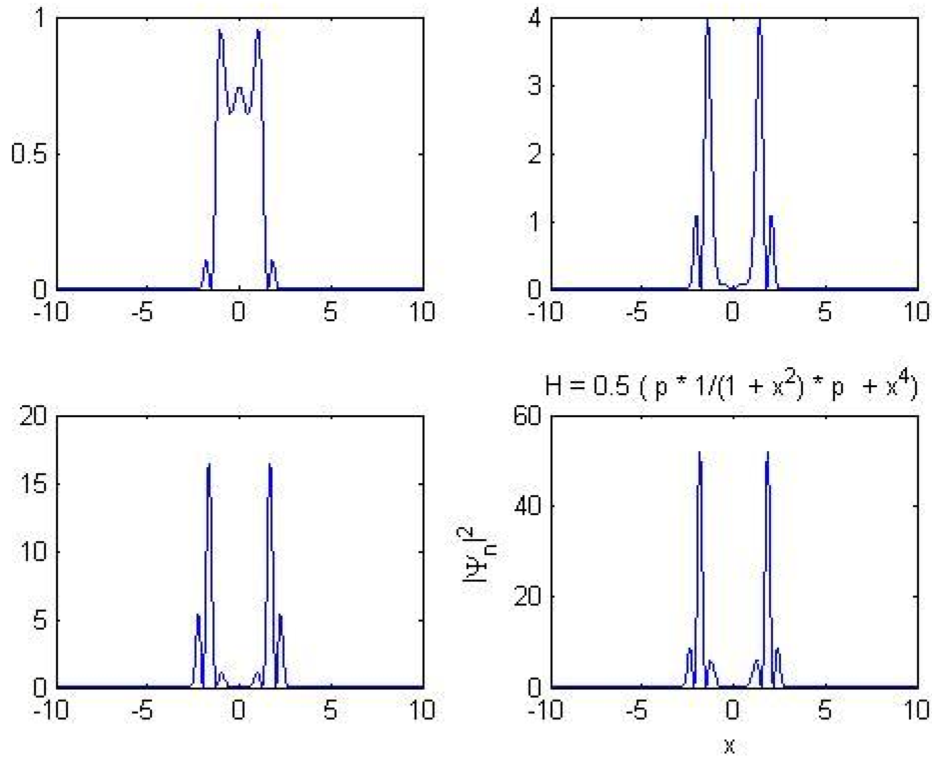


Figure 17: Wavefunction for the Hamiltonian system (12) with $\omega_0 = \lambda = 1$, N = K = 2, for (a) n=0, (b) n=1, (c) n=2 and (d) n=3.



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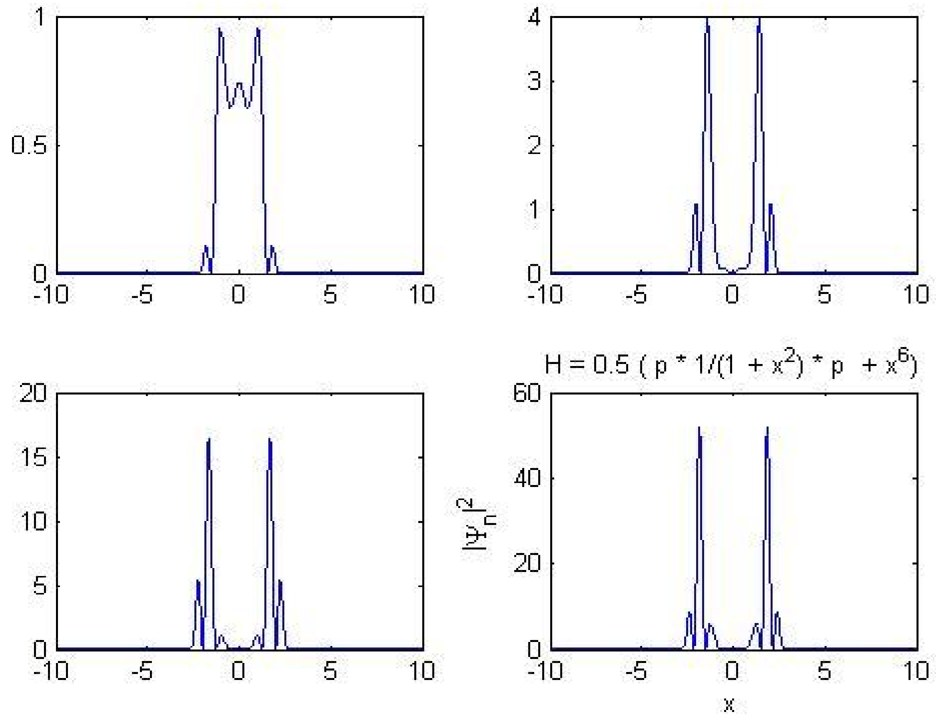

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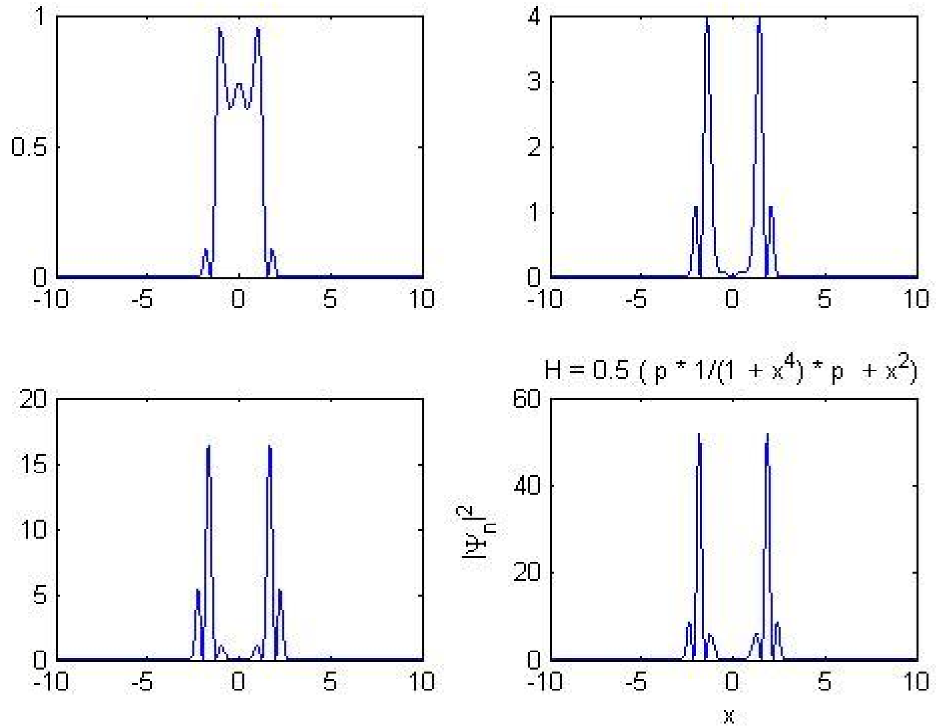

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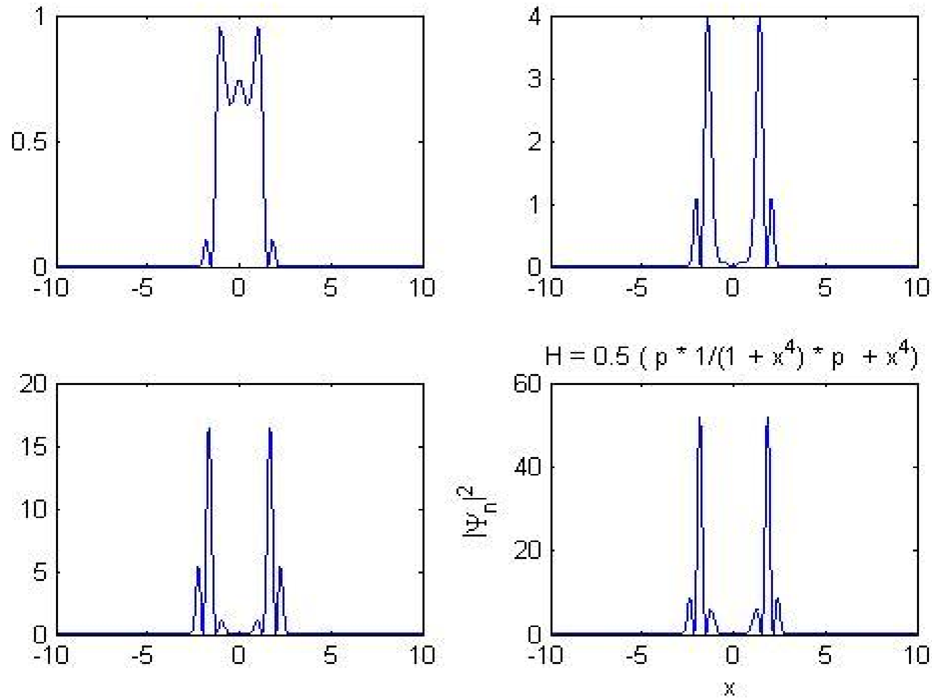

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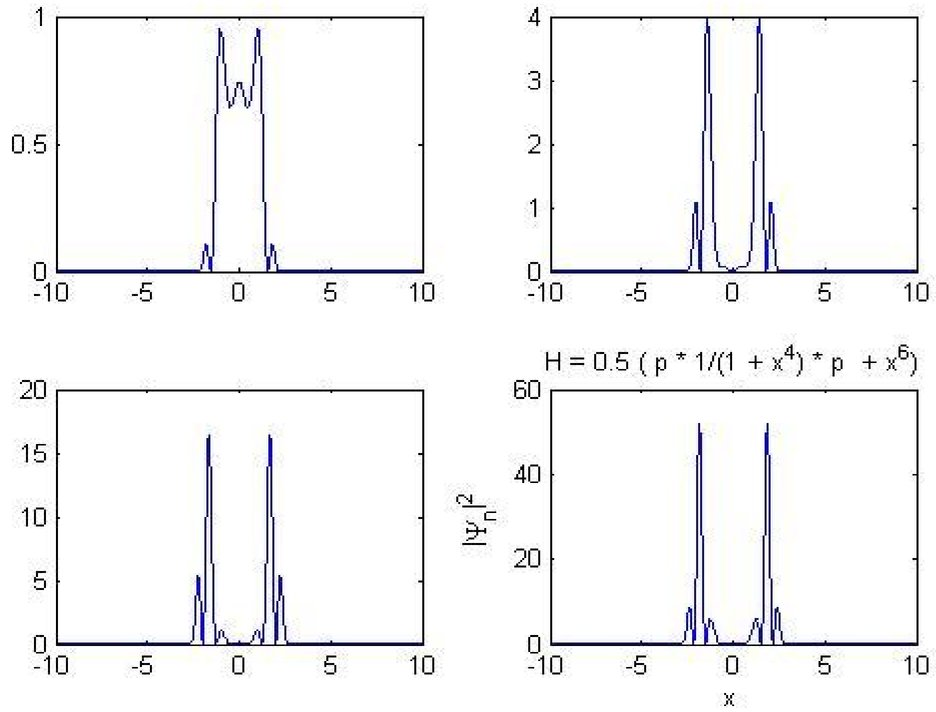

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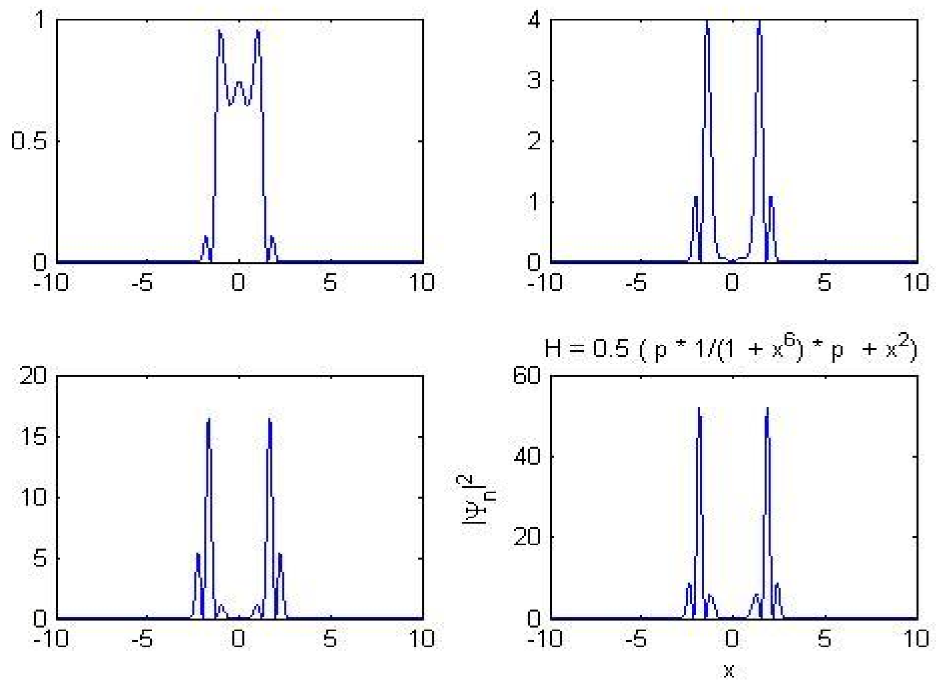

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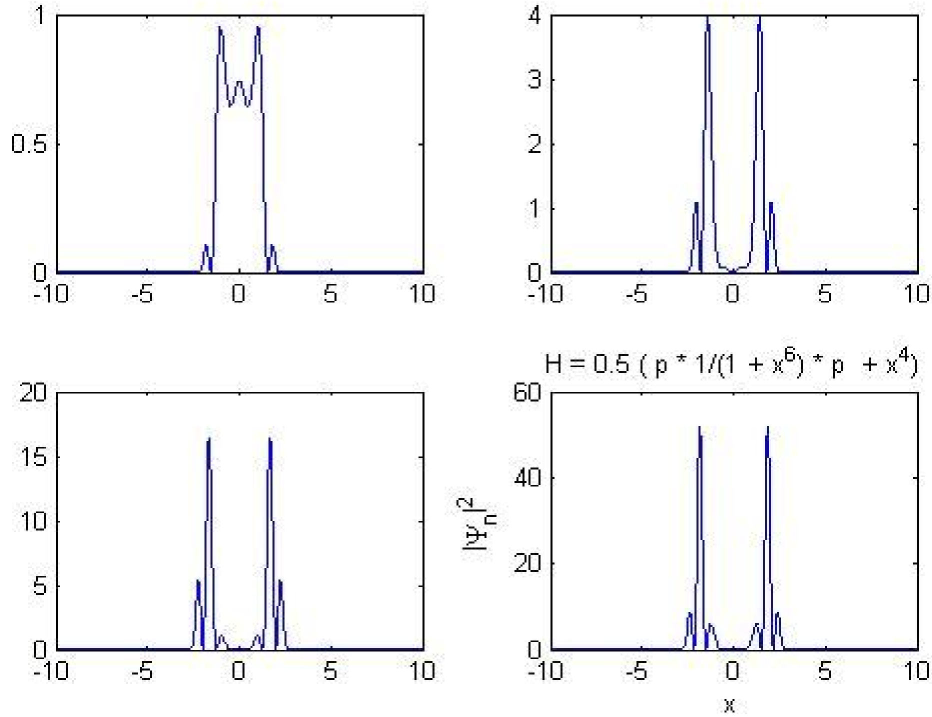

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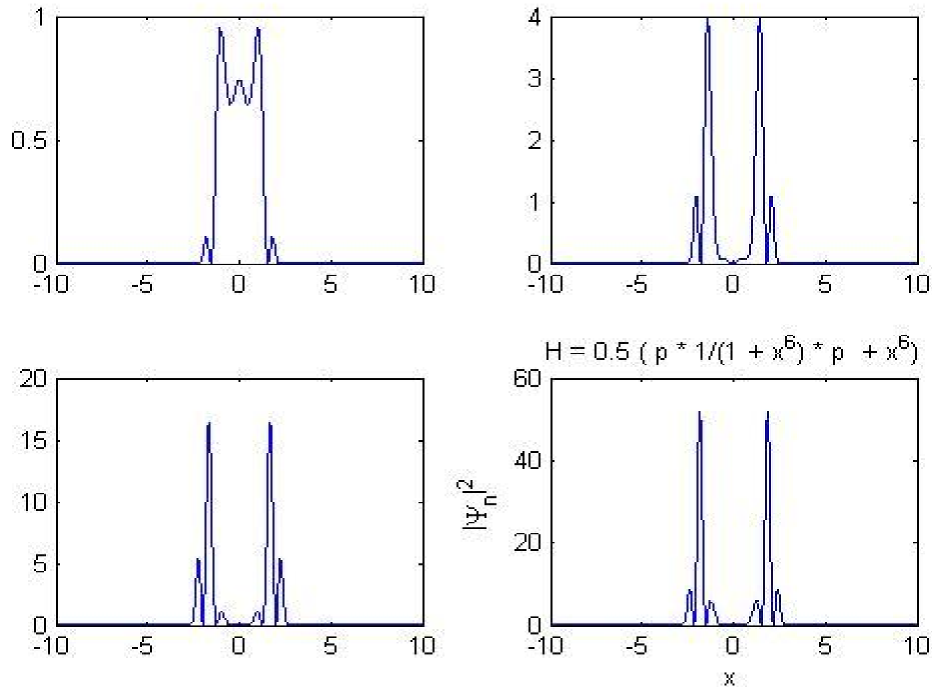

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