Line-Surface Formulation of the Electromagnetic-Power-based Characteristic Mode Theory for Metal-Material Combined Objects

Renzun Lian

***Abstract*—An ElectroMagnetic-Power-based Characteristic Mode Theory (CMT) for Metal-Material combined objects (MM-EMP-CMT) was built by expressing the various electromagnetic powers as the functions of the line current on metal line, the surface current on metal surface, the surface current on the boundary of metal volume, and the total field in material volume, so it can be simply called as the Line-Surface-Volume formulation for the MM-EMP-CMT (LSV-MM-EMP-CMT). As a companion to the LSV-MM-EMP-CMT, a Line-Surface formulation for the MM-EMP-CMT (LS-MM-EMP-CMT) is established in this paper by expressing the various powers as the functions of the line and surface currents on metal part and the surface equivalent current on the boundary of material part.**

The physical essence of LS-MM-EMP-CMT is the same as LSV-MM-EMP-CMT, i.e., to construct the various power-based Characteristic Mode (CM) sets for metal-material combined objects, but the LS-MM-EMP-CMT is more advantageous than the LSV-MM-EMP-CMT in some aspects. For example, the former saves computational resources; the former avoids to calculate the modal scattering field in source region; the field-based definitions for the impedance and admittance of metal-material combined electromagnetic systems can be easily introduced into the former.

***Index Terms*—Admittance, Characteristic Mode (CM), Electromagnetic Power, Impedance, Input Power, Interaction, Metal-Material Combined Object, Output Power, Surface Equivalent Principle.**

I. Introduction

THE Characteristic Mode Theory (CMT) was firstly introduced by R. J. Garbacz in 1965 [1]. Subsequently, R. F. Harrington *et al.* built a series of MoM-based CMTs, such as the Surface EFIE-based CMT for PEC systems (PEC-SEFIE-CMT) [2], the Volume Integral Equation-based CMT for Material bodies (Mat-VIE-CMT) [3], and the Surface Integral Equation-based CMT for Material bodies (Mat-SIE-CMT) [4].

R. Z. Lian is with the School of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China. (e-mail: rzlian@vip.163.com).

The Poynting's theorem-based interpretations for the power characteristics of the Characteristic Mode (CM) sets derived from the PEC-SEFIE-CMT and Mat-SIE-CMT are provided in [5] and [6], such that the physical pictures of these two MoM-based CMTs become clearer.

Recently, some ElectroMagnetic-Power-based CMTs (EMP-CMT) are established, such as the EMP-CMT for PEC systems (PEC-EMP-CMT) [7], the EMP-CMT for Material bodies (Mat-EMP-CMT) [8]-[9], and the EMP-CMT for Metal-Material combined objects (MM-EMP-CMT) [10]. The metal-material combined objects discussed in [10] can include metal part (metal line, surface, and volume) and material part (material volume).

In [10], the various electromagnetic powers generated by metal-material combined systems are expressed as the functions of the line current on metal line, the surface current on metal surface, the surface current on the boundary of metal volume, and the total field (the summation of incident field and scattering field) in material volume, so the theory developed in [10] can be simply called as the Line-Surface-Volume formulation for the MM-EMP-CMT (LSV-MM-EMP-CMT). As a companion to the LSV-MM-EMP-CMT, a new Line-Surface formulation for the MM-EMP-CMT (LS-MM-EMP-CMT) is provided in this paper by expressing the various powers as the functions of the line and surface currents on metal part and the surface equivalent current on the boundary of material part.

The physical essence of LS-MM-EMP-CMT is the same as LSV-MM-EMP-CMT, i.e., to construct the various power-based CM sets for the metal-material combined objects, which have abilities to depict the inherent power characteristics of the metal-material combined objects. However, the LS-MM-EMP-CMT is more advantageous than the LSV-MM-EMP-CMT in some aspects. For example, the former saves the computational resources; the former avoids to compute the modal scattering field in source region; the field-based definitions for the impedance and admittance introduced in [7] and [9] can be easily generalized to the former.

Although many power-based CM sets can be constructed by optimizing various objective powers, only the theory and formulations corresponding to the input/output power are explicitly provided in this paper because of its notable importance as explained in [8].

This paper is organized as follows. The Secs. II-VII provide the fundamental principles and essential formulations of the LS-MM-EMP-CMT, and the Sec. VIII concludes this paper. In what follows, the $e^{j\omega t}$ convention is used throughout.

II. Scattering Sources, Surface Equivalent Sources, and Basic Variable

In this paper, the metal-material combined object is simply called as scatterer, and the scatterer includes two parts, that are the metal part (including three subparts: metal line part, metal surface part, and metal volume part) and the material part (i.e., material volume part). When an external excitation $\bar{F}^{inc}$ incidents on the scatterer, some scattering sources will be excited on the scatterer, and then the scattering field $\bar{F}^{sca}$ is generated by the scattering sources, here $F = E,H$. The field generated by the scattering sources on the metal part is denoted as $\bar{F}^{sca}_{met}$, and the field generated by the scattering sources on the material part is denoted as $\bar{F}^{sca}_{mat}$; it is obvious that $\bar{F}^{sca} = \bar{F}^{sca}_{met} + \bar{F}^{sca}_{mat}$, because of the superposition principle [11]. The summation of the $\bar{F}^{inc}$ and $\bar{F}^{sca}$ is the total field $\bar{F}^{tot}$, i.e., $\bar{F}^{tot} = \bar{F}^{inc} + \bar{F}^{sca}$.

In addition, it is restricted in this paper that the source of $\bar{F}^{inc}$ doesn't distribute on the scatterer.

A. Various domains and scattering sources.

For the metal-material combined scatterers, the scattering currents include the following kinds: the line electric current $\bar{J}^{l}$ on the metal line part, the surface electric current $\bar{J}^{s}_{met,surf}$ on the metal surface part, the surface electric current $\bar{J}^{s}_{met,vol}$ on the boundary of the metal volume part, the volume ohmic electric current $\bar{J}^{vo}$ on the material part, the volume polarized electric current $\bar{J}^{vp}$ on the material part, and the volume magnetized magnetic current $\bar{M}^{vm}$ on the material part [12]-[14]. In addition, the summation of the $\bar{J}^{vo}$ and $\bar{J}^{vp}$ is denoted as $\bar{J}^{vop}$ in this paper. Various scattering charges are related to the corresponding scattering currents by current continuity equation, so the scattering field can be uniquely determined by the scattering currents mentioned above.

The domains occupied by the metal line part, the metal surface part, the metal volume part, and the material part are respectively denoted as $D^{met,line}$, $D^{met,surf}$, $D^{met,vol}$, and $D^{mat,vol}$, and their boundaries are correspondingly denoted as $\partial D^{met,line}$, $\partial D^{met,surf}$, $\partial D^{met,vol}$, and $\partial D^{mat,vol}$ respectively. In the three-dimensional Euclidean space $\mathbb{R}^3$, it is obvious that [15]

$$D^{met,line} = \partial D^{met,line} \tag{1.1}$$
$$D^{met,surf} = \partial D^{met,surf} \tag{1.2}$$

To simplify the symbolic system of this paper and to efficiently distinguish the different domains from each other, the $D^{met,line}$ and $D^{met,surf}$ are respectively denoted as L^{met} and S^{met}, and their boundaries have the same symbolic representations as themselves because of the relations in (1); the $D^{met,vol}$, $D^{mat,vol}$, $\partial D^{met,vol}$, and $\partial D^{mat,vol}$ are respectively denoted as V^{met}, V^{mat}, ∂V^{met}, and ∂V^{mat}.

When the magnetized magnetic current model is utilized to describe the magnetization phenomenon of material part, there doesn't exist the material-based surface electric current on ∂V^{mat} [12]-[14]. In addition, it is obvious that only the case $\text{cl}\left(S^{met} \setminus \text{cl}V^{met}\right) = S^{met}$ is necessary to be considered, so the metal-based surface electric currents $\bar{J}^{s}_{met,surf}$ and $\bar{J}^{s}_{met,vol}$ can be uniformly denoted as the $\bar{J}^{s}$, i.e.,

$$\bar{J}^{s}(\bar{r}) = \begin{cases} \bar{J}^{s}_{met,surf}(\bar{r}) & , \quad (\bar{r} \in S^{met} \setminus \partial V^{met}) \\ \bar{J}^{s}_{met,vol}(\bar{r}) & , \quad (\bar{r} \in \partial V^{met}) \\ 0 & , \quad (\bar{r} \notin S^{met} \cup \partial V^{met}) \end{cases} \tag{2}$$

The symbol "$\text{cl}S$" represents the closure of set S, and $\text{cl}S = S \cup \partial S = \text{int}S \cup \partial S$ for any set S [15]; the symbol "$\text{int}S$" represents the interior of set S, and $\text{int}S = S \setminus \partial S$ [15]. In fact, the above-mentioned relation $\text{cl}\left(S^{met} \setminus \text{cl}V^{met}\right) = S^{met}$ is equivalent to that $S^{met} \cap \text{int}V^{met} = \varnothing \wedge \text{cl}\left(S^{met} \setminus \partial V^{met}\right) = S^{met}$; the relation $S^{met} \cap \text{int}V^{met} = \varnothing$ means that the metal surface part is neither completely nor partially submerged into the metal volume part; the relation $\text{cl}\left(S^{met} \setminus \partial V^{met}\right) = S^{met}$ means that the set $S^{met} \cap \partial V^{met}$ can only be the $\varnothing$ or some lines, but cannot contain any surface. Based on that $\text{cl}\left(S^{met} \setminus \text{cl}V^{met}\right) = S^{met}$, the domain $S^{met} \cup \partial V^{met}$ in (2) can be equivalently rewritten as follows [15]

$$S^{met} \cup \partial V^{met} = \partial\left(S^{met} \cup V^{met}\right) = \partial\left(D^{met,surf} \cup D^{met,vol}\right) \tag{3}$$

so the domain $S^{met} \cup \partial V^{met}$ in (2) can also be simply denoted as $\partial D^{met,sv}$, and then the domain $S^{met} \cup V^{met}$ is simply denoted as $D^{met,sv}$.

Similarly to the relation $\text{cl}\left(S^{met} \setminus \text{cl}V^{met}\right) = S^{met}$, only the case $\text{cl}\left(L^{met} \setminus \text{cl}D^{met,sv}\right) = L^{met}$ is necessary to be considered, and this means that the metal line part is neither completely nor partially submerged into the metal volume part, and that the set $L^{met} \cap \partial D^{met,sv}$ can only be the $\varnothing$ or some points but cannot contain any line.

Some typical examples of the various scattering currents, domains, and boundaries mentioned above are illustrated in the Fig. 1. In addition, the cases plotted in the Figs. 2 and 3 are not considered in this paper.

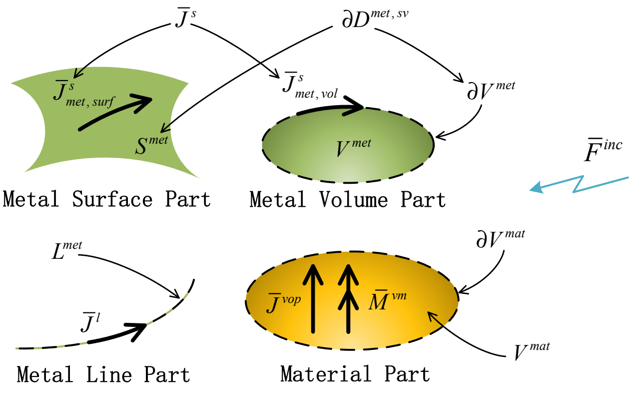

Fig. 1. The metal-material combined object excited by incident field.

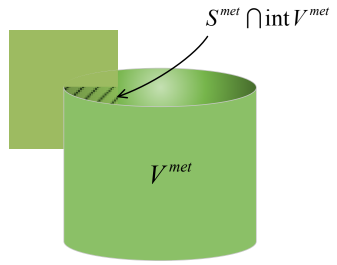

Fig. 2 (a). The case $S^{met} \cap \text{int} V^{met} \neq \varnothing$ which is not considered in this paper.

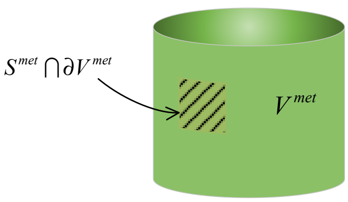

Fig. 2 (b). The case $\text{cl}\left(S^{met} \setminus \partial V^{met}\right) \neq S^{met}$ which is not considered in this paper.

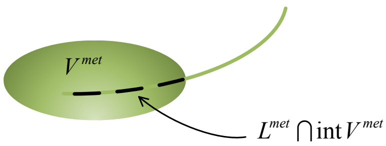

Fig. 3 (a). The case $L^{met} \setminus \text{int} V^{met} \neq \varnothing$ which is not considered in this paper.

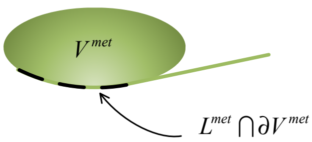

Fig. 3 (b). The case $\text{cl}\left(L^{met} \setminus \partial V^{met}\right) \neq L^{met}$ which is not considered in this paper.

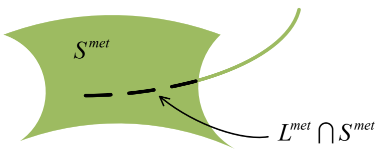

Fig. 3 (c). The case $\text{cl}\left(L^{met} \setminus S^{met}\right) \neq L^{met}$ which is not considered in this paper.

B. The decompositions for the various domains and scattering sources.

Based on the discussions in [10], the L^{met} can be decomposed into two parts, L_{free}^{met} and L_{unfree}^{met}; the $\partial D^{met,sv}$ can be decomposed into two parts, $\partial D_{free}^{met,sv}$ and $\partial D_{unfree}^{met,sv}$; these domains and subdomains satisfy the following relations (4) and (5).

$$L_{free}^{met} \cup L_{unfree}^{met} = L^{met} \tag{4.1}$$
$$L_{free}^{met} \cap L_{unfree}^{met} = \varnothing \tag{4.2}$$

and

$$\partial D_{free}^{met,sv} \cup \partial D_{unfree}^{met,sv} = \partial D^{met,sv} \tag{5.1}$$
$$\partial D_{free}^{met,sv} \cap \partial D_{unfree}^{met,sv} = \varnothing \tag{5.2}$$

The L_{free}^{met} and L_{unfree}^{met} in (4) are defined as follows [10]

$$\begin{aligned} L_{free}^{met} &\triangleq \left\{ \vec{r} : \vec{r} \in L^{met} \setminus \text{int}\left(L^{met} \cup \text{cl} V^{mat}\right) \right\} \\ &= \left\{ \vec{r} : \vec{r} \in L^{met} \setminus \text{int}\left(\text{cl} V^{mat}\right) \right\} \end{aligned} \tag{6.1}$$

$$\begin{aligned} L_{unfree}^{met} &\triangleq \left\{ \vec{r} : \vec{r} \in L^{met} \cap \text{int}\left(L^{met} \cup \text{cl} V^{mat}\right) \right\} \\ &= \left\{ \vec{r} : \vec{r} \in L^{met} \cap \text{int}\left(\text{cl} V^{mat}\right) \right\} \end{aligned} \tag{6.2}$$

and the $\partial D_{free}^{met,sv}$ and $\partial D_{unfree}^{met,sv}$ in (5) are defined as follows [10]

$$\begin{aligned} \partial D_{free}^{met,sv} &\triangleq \left\{ \vec{r} : \vec{r} \in \partial D^{met,sv} \setminus \text{int}\left(S^{met} \cup \text{cl} V^{met} \cup \text{cl} V^{mat}\right) \right\} \\ &= \left\{ \vec{r} : \vec{r} \in \partial D^{met,sv} \setminus \text{int}\left(\text{cl} V^{met} \cup \text{cl} V^{mat}\right) \right\} \end{aligned} \tag{7.1}$$

$$\begin{aligned} \partial D_{unfree}^{met,sv} &\triangleq \left\{ \vec{r} : \vec{r} \in \partial D^{met,sv} \cap \text{int}\left(S^{met} \cup \text{cl} V^{met} \cup \text{cl} V^{mat}\right) \right\} \\ &= \left\{ \vec{r} : \vec{r} \in \partial D^{met,sv} \cap \text{int}\left(\text{cl} V^{met} \cup \text{cl} V^{mat}\right) \right\} \end{aligned} \tag{7.2}$$

Based on the (4) and (5), the scattering currents $\vec{J}^{l}$ and $\vec{J}^{s}$ can be correspondingly decomposed as follows [10]

$$\vec{J}^{l}(\vec{r}) = \vec{J}_{free}^{l}(\vec{r}) + \vec{J}_{unfree}^{l}(\vec{r}) \tag{8}$$
$$\vec{J}^{s}(\vec{r}) = \vec{J}_{free}^{s}(\vec{r}) + \vec{J}_{unfree}^{s}(\vec{r}) \tag{9}$$

here

$$\vec{J}_{free}^{l}(\vec{r}) \triangleq \begin{cases} \vec{J}^{l}(\vec{r}) &, \left(\vec{r} \in L_{free}^{met}\right) \\ 0 &, \left(\vec{r} \in L_{unfree}^{met}\right) \\ 0 &, \left(\vec{r} \notin L^{met}\right) \end{cases} \tag{10.1}$$

$$\vec{J}_{unfree}^{l}(\vec{r}) \triangleq \begin{cases} 0 &, \left(\vec{r} \in L_{free}^{met}\right) \\ \vec{J}^{l}(\vec{r}) &, \left(\vec{r} \in L_{unfree}^{met}\right) \\ 0 &, \left(\vec{r} \notin L^{met}\right) \end{cases} \tag{10.2}$$

and

$$\vec{J}_{free}^{s}(\vec{r}) \triangleq \begin{cases} \vec{J}^{s}(\vec{r}) &, \left(\vec{r} \in \partial D_{free}^{met,sv}\right) \\ 0 &, \left(\vec{r} \in \partial D_{unfree}^{met,sv}\right) \\ 0 &, \left(\vec{r} \notin \partial D^{met,sv}\right) \end{cases} \tag{11.1}$$

$$\vec{J}_{unfree}^{s}(\vec{r}) \triangleq \begin{cases} 0 &, \left(\vec{r} \in \partial D_{free}^{met,sv}\right) \\ \vec{J}^{s}(\vec{r}) &, \left(\vec{r} \in \partial D_{unfree}^{met,sv}\right) \\ 0 &, \left(\vec{r} \notin \partial D^{met,sv}\right) \end{cases} \tag{11.2}$$

It is restricted in this paper that $L_{unfree}^{met} = \varnothing$, and then $L^{met} = L_{free}^{met}$ and $\vec{J}^{l} = \vec{J}_{free}^{l}$. The above restriction implies that the metal line part is neither completely nor partially submerged into the material part. The reason to do the above restriction is that: the case $L_{unfree}^{met} \neq \varnothing$ will lead to that the ∂V^{mat} is not a pure surface, and then leads to that the surface equivalent sources on

∂V^{mat} cannot be easily defined.

Some typical examples are illustrated in the Figs. 4-6. In addition, the case $L^{met}_{unfree} \neq \varnothing$ which is not considered in this paper is plotted in the Fig. 7.

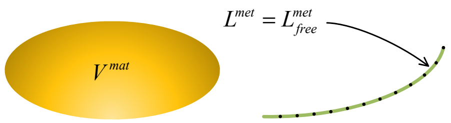

Fig. 4 (a). The metal line and material parts don't contact with each other.

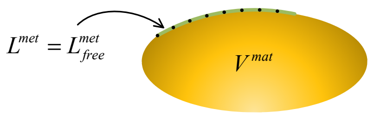

Fig. 4 (b). The metal line and material parts contact with each other.

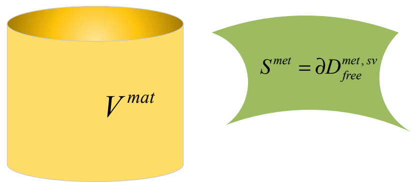

Fig. 5 (a). The metal surface and material parts don't contact with each other.

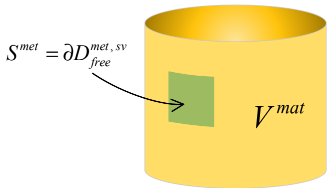

Fig. 5 (b). The metal surface and material parts contact with each other.

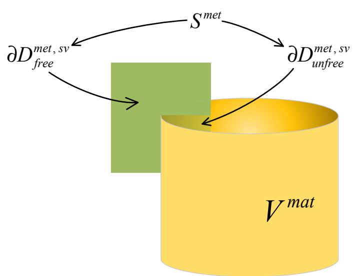

Fig. 5 (c). The metal surface part is partially immerged into the material part.

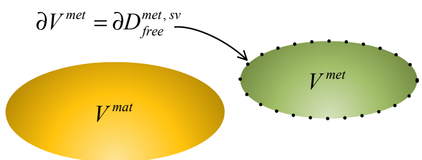

Fig. 6 (a). The metal volume and material parts don't contact with each other.

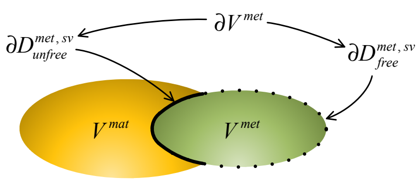

Fig. 6 (b). The metal volume and material parts contact with each other.

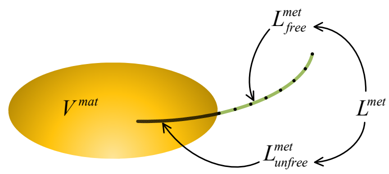

Fig. 7. The case $L^{met}_{unfree} \neq \varnothing$ which is not considered in this paper.

C. *Surface equivalent sources.*

Based on the discussions in [9]-[10], the surface equivalent sources on ∂V^{mat} can be defined as follows

$$\bar{J}^{SE}(\bar{r}) \; : \; \left\{ \bar{J}^{SE}_0(\bar{r}), \bar{J}^{SE}_+(\bar{r}), \bar{J}^{SE}_-(\bar{r}) \right\} \tag{12}$$

$$\bar{M}^{SE}(\bar{r}) : \; \left\{ \bar{M}^{SE}_0(\bar{r}), \bar{M}^{SE}_+(\bar{r}), \bar{M}^{SE}_-(\bar{r}) \right\} \tag{13}$$

here

$$\bar{J}^{SE}_0(\bar{r}) \triangleq \begin{cases} \left[\hat{n}_{\rightarrow mat}(\bar{r}) \times \bar{H}^{tot}(\bar{r}') \right]_{\bar{r}' \rightarrow \bar{r}} & , \bar{r} \in \partial V^{mat} \setminus \left(S^{met} \cap \partial D^{met,sv}_{unfree} \right) \\ 0 & , \bar{r} \notin \partial V^{mat} \setminus \left(S^{met} \cap \partial D^{met,sv}_{unfree} \right) \end{cases} \tag{14.1}$$

$$\bar{J}^{SE}_+(\bar{r}) \triangleq \begin{cases} \left[\hat{n}_+(\bar{r}) \times \bar{H}^{tot}(\bar{r}_+) \right]_{\bar{r}_+ \rightarrow \bar{r}} & , \bar{r} \in S^{met} \cap \partial D^{met,sv}_{unfree} \\ 0 & , \bar{r} \notin S^{met} \cap \partial D^{met,sv}_{unfree} \end{cases} \tag{14.2}$$

$$\bar{J}^{SE}_-(\bar{r}) \triangleq \begin{cases} \left[\hat{n}_-(\bar{r}) \times \bar{H}^{tot}(\bar{r}_-) \right]_{\bar{r}_- \rightarrow \bar{r}} & , \bar{r} \in S^{met} \cap \partial D^{met,sv}_{unfree} \\ 0 & , \bar{r} \notin S^{met} \cap \partial D^{met,sv}_{unfree} \end{cases} \tag{14.3}$$

and

$$\bar{M}^{SE}_0(\bar{r}) \triangleq \begin{cases} \left[\bar{E}^{tot}(\bar{r}') \times \hat{n}_{\rightarrow mat}(\bar{r}) \right]_{\bar{r}' \rightarrow \bar{r}} & , \bar{r} \in \partial V^{mat} \setminus \left(S^{met} \cap \partial D^{met,sv}_{unfree} \right) \\ 0 & , \bar{r} \notin \partial V^{mat} \setminus \left(S^{met} \cap \partial D^{met,sv}_{unfree} \right) \end{cases} \tag{15.1}$$

$$\bar{M}^{SE}_+(\bar{r}) \triangleq \begin{cases} \left[\bar{E}^{tot}(\bar{r}_+) \times \hat{n}_+(\bar{r}) \right]_{\bar{r}_+ \rightarrow \bar{r}} & , \bar{r} \in S^{met} \cap \partial D^{met,sv}_{unfree} \\ 0 & , \bar{r} \notin S^{met} \cap \partial D^{met,sv}_{unfree} \end{cases} \tag{15.2}$$

$$\bar{M}^{SE}_-(\bar{r}) \triangleq \begin{cases} \left[\bar{E}^{tot}(\bar{r}_-) \times \hat{n}_-(\bar{r}) \right]_{\bar{r}_- \rightarrow \bar{r}} & , \bar{r} \in S^{met} \cap \partial D^{met,sv}_{unfree} \\ 0 & , \bar{r} \notin S^{met} \cap \partial D^{met,sv}_{unfree} \end{cases} \tag{15.3}$$

In (14.1) and (15.1), the subscript "$\bar{r}' \rightarrow \bar{r}$" represents that the $\bar{r}'$ belongs to set $\text{int}V^{mat}$, and that the $\bar{r}'$ approaches to the $\bar{r}$;

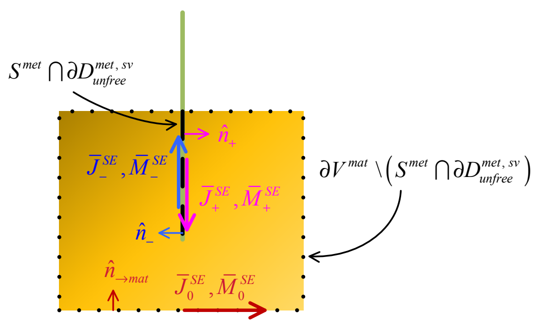

Fig. 8. The sectional view of the Fig. 5 (c), and the surface equivalent sources on ∂V^{mat}.

the $\hat{n}_{\rightarrow mat}$ is the normal direction vector of surface $\partial V^{mat} \setminus \left(S^{met} \cap \partial D_{unfree}^{met,sv}\right)$, and it points to the interior of material part. The two sides of surface $S^{met} \cap \partial D_{unfree}^{met,sv}$ are in the material part, and they are respectively called as the "$+$" side and the "$-$" side of $S^{met} \cap \partial D_{unfree}^{met,sv}$; in (14.2), (14.3), (15.2), and (15.3), the subscript "$\bar{r}_{\pm} \rightarrow \bar{r}$" represents that $\bar{r}_{\pm} \in \operatorname{int} V^{mat}$, and that the $\bar{r}_{\pm}$ approaches to the $\bar{r}$ from the "$\pm$" side.

Taking the Fig. 5 (c) as a typical example, the surface equivalent sources corresponding to this example are illustrated in the Fig. 8.

Because of the (14) and the magnetic field boundary condition on surface $\partial D_{unfree}^{met,sv}$ [12], [14], it can be found out that

$$\bar{J}_{unfree}^{s}\left(\bar{r}\right) = \begin{cases} \bar{J}_{0}^{SE}\left(\bar{r}\right) & , \quad \left(\bar{r} \in \partial D_{unfree}^{met,sv} \cap \partial V^{met}\right) \\ \bar{J}_{+}^{SE}\left(\bar{r}\right) + \bar{J}_{-}^{SE}\left(\bar{r}\right) & , \quad \left(\bar{r} \in \partial D_{unfree}^{met,sv} \cap S^{met}\right) \\ 0 & , \quad \left(\bar{r} \notin \partial D_{unfree}^{met,sv}\right) \end{cases} \tag{16}$$

In fact, the (16) can be concisely written as the following operator form.

$$\bar{J}_{unfree}^{s}\left(\bar{r}\right) = \mathcal{J}_{unfree}^{s}\left(\bar{J}^{SE}\right) \quad , \quad \left(\bar{r} \in \partial D_{unfree}^{met,sv}\right) \tag{17}$$

Based on the discussions in [9], the $\bar{J}^{SE}$ and $\bar{M}^{SE}$ are not independent of each other; in this paper, the $\bar{J}^{SE}$ is selected as a part of *basic variable* [8] because of (17), i.e., the $\bar{M}^{SE}$ is expressed as the function of $\bar{J}^{SE}$ based on the method given in [9], so

$$\bar{M}^{SE}\left(\bar{r}\right) = \mathcal{M}^{SE}\left(\bar{J}^{SE}\right) \quad , \quad \left(\bar{r} \in \partial V^{mat}\right) \tag{18}$$

The specific mathematical expression for the operator (18) can be found in [9].

D. The mathematical expressions of various fields.

Due to the above (18), the $\bar{F}^{tot}$ on $\operatorname{int} V^{mat}$ and the $\bar{F}_{mat}^{sca}$ on $\mathbb{R}^3 \setminus \partial V^{mat}$ can be expressed as the functions of $\bar{J}^{SE}$, and their operator forms are as follows [9]

$$\bar{F}_{int}^{tot}\left(\bar{r}\right) = \mathcal{F}_{int}^{tot}\left(\bar{J}^{SE}\right) \quad , \quad \left(\bar{r} \in \operatorname{int} V^{mat}\right) \tag{19}$$

$$\bar{F}_{mat}^{sca}\left(\bar{r}\right) = \mathcal{F}_{mat}^{sca}\left(\bar{J}^{SE}\right) \quad , \quad \left(\bar{r} \in \mathbb{R}^3 \setminus \partial V^{mat}\right) \tag{20}$$

here $F = E, H$, and correspondingly $\mathcal{F} = \mathcal{E}, \mathcal{H}$. To use the subscript "$int$" in (19) is to emphasize that the operator (19) is only suitable for the total field on $\operatorname{int} V^{mat}$.

Based on the (20), the scattering field $\bar{F}^{sca}$ on $\mathbb{R}^3 \setminus \partial D$ can be expressed as the following linear operator form.

$$\begin{aligned} \bar{F}^{sca}\left(\bar{r}\right) &= \bar{F}_{met}^{sca}\left(\bar{r}\right) + \bar{F}_{mat}^{sca}\left(\bar{r}\right) \\ &= \mathcal{F}\left(\bar{J}^{l}, \bar{J}^{s}\right) + \mathcal{F}_{mat}^{sca}\left(\bar{J}^{SE}\right) \\ &= \mathcal{F}\left(\bar{J}_{free}^{l}, \bar{J}_{free}^{s} + \bar{J}_{unfree}^{s}\right) + \mathcal{F}_{mat}^{sca}\left(\bar{J}^{SE}\right) \\ &= \mathcal{F}\left(\bar{J}_{free}^{l}, \bar{J}_{free}^{s}\right) + \mathcal{F}\left(0, \bar{J}_{unfree}^{s}\right) + \mathcal{F}_{mat}^{sca}\left(\bar{J}^{SE}\right) \\ &= \mathcal{F}\left(\bar{J}_{free}^{l}, \bar{J}_{free}^{s}\right) + \mathcal{F}\left(0, \mathcal{J}_{unfree}^{s}\left(\bar{J}^{SE}\right)\right) + \mathcal{F}_{mat}^{sca}\left(\bar{J}^{SE}\right) \end{aligned} \tag{21}$$

In (21), $\bar{r} \in \mathbb{R}^3 \setminus \partial D$; $F = E, H$, and correspondingly $\mathcal{F} = \mathcal{E}, \mathcal{H}$; the operator $\mathcal{F}\left(\bar{J}^{l}, \bar{J}^{s}\right)$ represents the field generated by the line current $\bar{J}^{l}$ and the surface current $\bar{J}^{s}$ in vacuum, and its mathematical expression can be found in [12] and [14]; the third equality is based on the (9) and that $\bar{J}^{l} = \bar{J}_{free}^{l}$ as explained in the Sec. II-B; the fourth equality is due to the superposition principle [11]; the fifth equality originates from the (17).

Considering of the above (21), the $\bar{F}^{inc}$ on $\operatorname{int} V^{mat}$ can be expressed as the function of $\bar{J}_{free}^{l} = \bar{J}^{l}$, $\bar{J}_{free}^{s}$, and $\bar{J}^{SE}$ as follows

$$\begin{aligned} \bar{F}_{int}^{inc}\left(\bar{r}\right) &= \bar{F}_{int}^{tot}\left(\bar{r}\right) - \bar{F}^{sca}\left(\bar{r}\right) \\ &= \mathcal{F}_{int}^{tot}\left(\bar{J}^{SE}\right) \\ &\quad -\mathcal{F}\left(\bar{J}_{free}^{l}, \bar{J}_{free}^{s}\right) - \mathcal{F}\left(0, \mathcal{J}_{unfree}^{s}\left(\bar{J}^{SE}\right)\right) - \mathcal{F}_{mat}^{sca}\left(\bar{J}^{SE}\right) \end{aligned} \tag{22}$$

here $\bar{r} \in \operatorname{int} V^{mat}$; $F = E, H$, and correspondingly $\mathcal{F} = \mathcal{E}, \mathcal{H}$.

Because the $\bar{J}^{vop}$ and $\bar{M}^{vm}$ in $\operatorname{int} V^{mat}$ are uniquely determined by the $\bar{F}^{tot}$ in $\operatorname{int} V^{mat}$ [8]-[9], [12]-[14], they can be expressed as the following operator forms.

$$\bar{J}^{vop}\left(\bar{r}\right) = \mathcal{J}^{vop}\left(\bar{J}^{SE}\right) \quad , \quad \left(\bar{r} \in \operatorname{int} V^{mat}\right) \tag{23.1}$$

$$\bar{M}^{vm}\left(\bar{r}\right) = \mathcal{M}^{vm}\left(\bar{J}^{SE}\right) \quad , \quad \left(\bar{r} \in \operatorname{int} V^{mat}\right) \tag{23.2}$$

The specific mathematical expressions for the operators in (19), (20), and (23) can be found in [9], and then the specific mathematical expressions for the operators in (21) and (22) can be easily obtained, and they are not specifically provided here.

E. Basic variable.

As pointed out in [8]-[9], to express the various scattering sources as the functions of some independent variables is indispensable for the EMP-CMT, and the independent variables are called as *basic variables*. Based on the discussions in [9]-[10] and the discussions in the above Sec. II-D, it is obvious that the basic variables for the LS-MM-EMP-CMT can be selected as the $\left\{\bar{J}_{free}^{l} = \bar{J}^{l}, \bar{J}_{free}^{s}, \bar{J}^{SE}\right\}$, and they can be uniformly written as follows

$$\text{Basic Variable } \bar{V}\left(\bar{r}\right) \triangleq \begin{cases} \bar{J}_{free}^{l}\left(\bar{r}\right) = \bar{J}^{l}\left(\bar{r}\right) & , \quad \left(\bar{r} \in L_{free}^{met} = L^{met}\right) \\ \bar{J}_{free}^{s}\left(\bar{r}\right) & , \quad \left(\bar{r} \in \partial D_{free}^{met,sv}\right) \\ \bar{J}^{SE}\left(\bar{r}\right) & , \quad \left(\bar{r} \in \partial V^{mat}\right) \\ 0 & , \quad \left(\bar{r} \notin L^{met} \cup \partial D_{free}^{met,sv} \cup \partial V^{mat}\right) \end{cases} \tag{24}$$

Because $\partial D_{unfree}^{met,sv} \subseteq \partial V^{mat}$ and $\partial D^{met,sv} = \partial D_{free}^{met,sv} \cup \partial D_{unfree}^{met,sv}$ and $L^{met} = \partial D^{met,line}$ and $\partial V^{mat} = \partial D^{mat}$, the set $L^{met} \cup \partial D_{free}^{met,sv} \cup \partial V^{mat}$ in (24) can also be equivalently denoted as the $\partial D^{met,line} \cup \partial D^{met,sv} \cup \partial D^{mat}$, or more simply denoted as the ∂D, i.e.,

$$\begin{aligned} \partial D &\triangleq \partial D^{met,line} \cup \partial D^{met,sv} \cup \partial D^{mat} \\ &= L^{met} \cup \partial D^{met,sv} \cup \partial V^{mat} \\ &= L^{met} \cup \partial D_{free}^{met,sv} \cup \partial V^{mat} \end{aligned} \tag{25}$$

Inserting the (24) into the (9), (17), and (19)-(23), the various fields and scattering currents can be further written as the following linear operator forms.

$$\bar{F}_{int}^{X}(\bar{r}) = \mathcal{F}_{int}^{X}(\bar{V}) \quad , \quad (\bar{r} \in \text{int} V^{mat}) \tag{26}$$
$$\bar{F}^{sca}(\bar{r}) = \mathcal{F}^{sca}(\bar{V}) \quad , \quad (\bar{r} \in \mathbb{R}^3 \setminus \partial D) \tag{27}$$

and

$$\bar{J}^{l}(\bar{r}) = \mathcal{J}^{l}(\bar{V}) \quad , \quad (\bar{r} \in L^{met}) \tag{28}$$
$$\bar{J}^{s}(\bar{r}) = \mathcal{J}^{s}(\bar{V}) \quad , \quad (\bar{r} \in \partial D^{met,sv}) \tag{29}$$
$$\bar{J}^{vop}(\bar{r}) = \mathcal{J}^{vop}(\bar{V}) \quad , \quad (\bar{r} \in \text{int} V^{mat}) \tag{30.1}$$
$$\bar{M}^{vm}(\bar{r}) = \mathcal{M}^{vm}(\bar{V}) \quad , \quad (\bar{r} \in \text{int} V^{mat}) \tag{30.2}$$

here $X = inc,tot$; $F = E,H$, and correspondingly $\mathcal{F} = \mathcal{E},\mathcal{H}$.

In fact, the tangential component of $\bar{E}^{sca}$ on the domain ∂D can be determined as the following (31), because there don't exist the line and surface magnetic currents on ∂D .

$$\bar{E}^{sca,tan}(\bar{r}) = \mathcal{E}^{sca,tan}(\bar{V}) = \begin{cases} \lim_{\bar{r}' \to \bar{r}} \hat{\tau}^{l}(\bar{r})\left[\hat{\tau}^{l}(\bar{r}) \cdot \bar{E}^{sca}(\bar{r}')\right] & , \quad (\bar{r} \in L^{met}) \\ \lim_{\bar{r}' \to \bar{r}} \sum_{i=1,2} \hat{\tau}_{i}^{s}(\bar{r})\left[\hat{\tau}_{i}^{s}(\bar{r}) \cdot \bar{E}^{sca}(\bar{r}')\right], & (\bar{r} \in \partial D \setminus L^{met}) \end{cases} \tag{31}$$

here $\bar{r}' \notin \partial D$; the $\hat{\tau}^{l}$ is the tangential direction of L^{met} ; the $\hat{\tau}_{1}^{s}$ and $\hat{\tau}_{2}^{s}$ are parallel to the surface $\partial D \setminus L^{met} = \partial D^{met,sv} \bigcup \partial V^{mat}$, and they are orthogonal to each other.

III. Interaction, Output Power, and Input Power

In this section, the mathematical expression and physical meaning of the interaction between the incident field and the metal-material combined scatterer is discussed, and then the mathematical expressions for the output and input powers are provided.

A. Interaction.

The interaction between incident field and scatterer is just the interaction between $\bar{F}^{inc}$ and $\{\bar{J}^{l},\bar{J}^{s},\bar{J}^{vop},\bar{M}^{vm}\}$, and its mathematical expression is as follows

$$\mathcal{I} = \mathcal{I}^{met,line} + \mathcal{I}^{met,sv} + \mathcal{I}^{mat} \tag{32}$$

The $\mathcal{I}^{met,line}$ in (32) is the interaction between $\bar{E}^{inc}$ and $\bar{J}^{l}$, and

$$\begin{aligned} \mathcal{I}^{met,line} &= \frac{1}{2}\left\langle \bar{J}^{l},\bar{E}^{inc}\right\rangle_{L^{met}} = \frac{1}{2}\left\langle \bar{J}^{l},\bar{E}^{inc,tan}\right\rangle_{L^{met}} \\ &= -\frac{1}{2}\left\langle \bar{J}^{l},\bar{E}^{sca,tan}\right\rangle_{L^{met}} = -\frac{1}{2}\left\langle \bar{J}^{l},\bar{E}^{sca}\right\rangle_{L^{met}} \end{aligned} \tag{33.1}$$

here $\bar{E}^{inc,tan}$ is the tangential component of the $\bar{E}^{inc}$ on L^{met} ; the inner product is defined as $< \bar{g},\bar{h} >_{\Omega} \triangleq \int_{\Omega} \bar{g}^{*} \cdot \bar{h} \, d\Omega$, and the symbol " $*$ " denotes the complex conjugate of relevant quantity, and the symbol " $\cdot$ " is the scalar product for field vectors. The $\mathcal{I}^{met,sv}$ in (32) is the interaction between $\bar{E}^{inc}$ and $\bar{J}^{s}$, and [7]

$$\begin{aligned} \mathcal{I}^{met,sv} &= \frac{1}{2}\left\langle \bar{J}^{s},\bar{E}^{inc}\right\rangle_{\partial D^{met,sv}} = \frac{1}{2}\left\langle \bar{J}^{s},\bar{E}^{inc,tan}\right\rangle_{\partial D^{met,sv}} \\ &= -\frac{1}{2}\left\langle \bar{J}^{s},\bar{E}^{sca,tan}\right\rangle_{\partial D^{met,sv}} = -\frac{1}{2}\left\langle \bar{J}^{s},\bar{E}^{sca}\right\rangle_{\partial D^{met,sv}} \end{aligned} \tag{33.2}$$

here $\bar{E}^{inc,tan}$ is the tangential component of the $\bar{E}^{inc}$ on $\partial D^{met,sv}$, and the third equality is based on the surface EFIE [12], [14]. The $\mathcal{I}^{mat}$ in (32) is the interaction between $\{\bar{E}^{inc},\bar{H}^{inc}\}$ and $\{\bar{J}^{vop},\bar{M}^{vm}\}$, and [8]-[9]

$$\begin{aligned} \mathcal{I}^{mat} &= (1/2)\left\langle \bar{J}^{vop},\bar{E}^{inc}\right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{inc},\bar{M}^{vm}\right\rangle_{V^{mat}} \\ &= (1/2)\left\langle \bar{J}^{vop},\bar{E}^{inc}\right\rangle_{\text{int} V^{mat}} + (1/2)\left\langle \bar{H}^{inc},\bar{M}^{vm}\right\rangle_{\text{int} V^{mat}} \\ &= (1/2)\left\langle \bar{J}^{vop},\bar{E}^{tot}\right\rangle_{\text{int} V^{mat}} + (1/2)\left\langle \bar{H}^{tot},\bar{M}^{vm}\right\rangle_{\text{int} V^{mat}} \\ &\quad -(1/2)\left\langle \bar{J}^{vop},\bar{E}^{sca}\right\rangle_{\text{int} V^{mat}} - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{vm}\right\rangle_{\text{int} V^{mat}} \\ &= -(1/2)\left\langle \bar{J}^{vop},\bar{E}^{sca}\right\rangle_{V^{mat}} - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{vm}\right\rangle_{V^{mat}} \\ &\quad +(1/2)\left\langle \bar{J}^{vop},\bar{E}^{tot}\right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{tot},\bar{M}^{vm}\right\rangle_{V^{mat}} \end{aligned} \tag{33.3}$$

here the second equality is due to that $(1/2) < \bar{J}^{vop},\bar{E}^{inc} >_{\partial V^{mat}} = 0 = (1/2) < \bar{H}^{inc},\bar{M}^{vm} >_{\partial V^{mat}}$, because there doesn't exist the material-based surface scattering current on ∂V^{mat} [12]-[14]; the third equality is due to that $\bar{F}^{inc} = \bar{F}^{tot} - \bar{F}^{sca}$; the fourth equality is due to the same reason as the second equality. The reason to use the second equality in (33.3) is that the specific mathematical expressions for the $\bar{F}^{tot}$, $\bar{H}^{sca}$, and $\bar{E}^{sca,norm} = \bar{E}^{sca} - \bar{E}^{sca,tan}$ on ∂V^{mat} are not provided in this paper.

Inserting the last equalities of (33) into the (32), the interaction $\mathcal{I}$ can be written as follows

$$\begin{aligned} \mathcal{I} &= -(1/2)\left\langle \bar{J}^{l},\bar{E}^{sca}\right\rangle_{L^{met}} \\ &\quad -(1/2)\left\langle \bar{J}^{s},\bar{E}^{sca}\right\rangle_{\partial D^{met,sv}} \\ &\quad -(1/2)\left\langle \bar{J}^{vop},\bar{E}^{sca}\right\rangle_{V^{mat}} - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{vm}\right\rangle_{V^{mat}} \\ &\quad +(1/2)\left\langle \bar{J}^{vop},\bar{E}^{tot}\right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{tot},\bar{M}^{vm}\right\rangle_{V^{mat}} \end{aligned} \tag{34}$$

Based on the source Poynting's theorem, the first three lines in (34) can be rewritten as follows

$$\begin{aligned} &-(1/2)\left\langle \bar{J}^{l},\bar{E}^{sca}\right\rangle_{L^{met}} \\ &-(1/2)\left\langle \bar{J}^{s},\bar{E}^{sca}\right\rangle_{\partial D^{met,sv}} \\ &-(1/2)\left\langle \bar{J}^{vop},\bar{E}^{sca}\right\rangle_{V^{mat}} - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{vm}\right\rangle_{V^{mat}} \\ &= P^{sca,rad} + j\, P^{sac,react,vac} \end{aligned} \tag{35}$$

here

$$P^{sca,rad} = \frac{1}{2}\oiint_{S_{\infty}}\left[\bar{E}^{sca} \times \left(\bar{H}^{sca}\right)^{*}\right] \cdot d\bar{S} \tag{36.1}$$

$$\begin{aligned} P^{sac,react,vac} &= 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{sca},\mu_0\bar{H}^{sca}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{sca},\bar{E}^{sca}\right\rangle_{\mathbb{R}^3}\right] \\ &= 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{sca},\mu_0\bar{H}^{sca}\right\rangle_{\mathbb{R}^3\backslash\partial D} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{sca},\bar{E}^{sca}\right\rangle_{\mathbb{R}^3\backslash\partial D}\right] \end{aligned} \tag{36.2}$$

The symbol " S_∞ " in (36.1) represents a closed spherical surface at infinity; the second equality in (36.2) is due to that $(1/4) < \bar{H}^{sca},\mu_0\bar{H}^{sca} >_{\partial D} = 0 = (1/4) < \varepsilon_0\bar{E}^{sca},\bar{E}^{sca} >_{\partial D}$.

Based on that $\bar{J}^{vop} = j\omega\Delta\varepsilon_c\bar{E}^{tot}$ and $\bar{M}^{vm} = j\omega\Delta\mu\bar{H}^{tot}$ on V^{mat} [8]-[9], and employing that $\varepsilon_c = \varepsilon + \sigma/j\omega$ and $\Delta\varepsilon_c = \varepsilon_c - \varepsilon_0$ and $\Delta\varepsilon = \varepsilon - \varepsilon_0$ and $\Delta\mu = \mu - \mu_0$, the fourth line in (34) can be rewritten as follows

$$\begin{aligned} &(1/2)\left\langle \bar{J}^{vop},\bar{E}^{tot}\right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{tot},\bar{M}^{vm}\right\rangle_{V^{mat}} \\ &= P^{tot,loss} + j\,P^{tot,react,mat} \end{aligned} \tag{37}$$

here

$$\begin{aligned} P^{tot,loss} &= \frac{1}{2}\left\langle \sigma\bar{E}^{tot},\bar{E}^{tot}\right\rangle_{V^{mat}} \\ &= \frac{1}{2}\left\langle \sigma\bar{E}^{tot},\bar{E}^{tot}\right\rangle_{\text{int}\,V^{mat}} \end{aligned} \tag{38.1}$$

$$\begin{aligned} P^{tot,react,mat} &= 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{tot},\Delta\mu\bar{H}^{tot}\right\rangle_{V^{mat}} - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^{tot},\bar{E}^{tot}\right\rangle_{V^{mat}}\right] \\ &= 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{tot},\Delta\mu\bar{H}^{tot}\right\rangle_{\text{int}\,V^{mat}} - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^{tot},\bar{E}^{tot}\right\rangle_{\text{int}\,V^{mat}}\right] \end{aligned} \tag{38.2}$$

In (38.1), the second equality originates from that $(1/2) < \sigma\bar{E}^{tot},\bar{E}^{tot} >_{\partial V^{mat}} = 0$. The second equality in (38.2) is due to that $(1/4) < \bar{H}^{tot},\Delta\mu\bar{H}^{tot} >_{\partial V^{mat}} = 0 = (1/4) < \Delta\varepsilon\bar{E}^{tot},\bar{E}^{tot} >_{\partial V^{mat}}$.

Inserting the (35) and (37) into the (34), the interaction $\mathcal{I}$ can be written as follows

$$\mathcal{I} = P^{sca,rad} + P^{tot,loss} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right) \tag{39}$$

B. Output power and input power.

Based on the discussions in above Sec. III-A and the conclusions given in [7]-[10], the output power P^{out} and input power P^{inp} are respectively as follows

$$\begin{aligned} P^{out} &= P^{sca,rad} + P^{tot,loss} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right) \\ &= -(1/2)\left\langle \bar{J}^{l},\bar{E}^{sca,tan}\right\rangle_{L^{met}} \\ &\quad -(1/2)\left\langle \bar{J}^{s},\bar{E}^{sca,tan}\right\rangle_{\partial D^{met,sv}} \\ &\quad -(1/2)\left\langle \bar{J}^{vop},\bar{E}^{sca}\right\rangle_{\text{int}\,V^{mat}} - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{vm}\right\rangle_{\text{int}\,V^{mat}} \\ &\quad +(1/2)\left\langle \bar{J}^{vop},\bar{E}^{tot}\right\rangle_{\text{int}\,V^{mat}} + (1/2)\left\langle \bar{H}^{tot},\bar{M}^{vm}\right\rangle_{\text{int}\,V^{mat}} \end{aligned} \tag{40}$$

and

$$\begin{aligned} P^{inp} = \mathcal{I} &= (1/2)\left\langle \bar{J}^{l},\bar{E}^{inc}\right\rangle_{L^{met}} \\ &\quad +(1/2)\left\langle \bar{J}^{s},\bar{E}^{inc}\right\rangle_{\partial D^{met,sv}} \\ &\quad +(1/2)\left\langle \bar{J}^{vop},\bar{E}^{inc}\right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{inc},\bar{M}^{vm}\right\rangle_{V^{mat}} \end{aligned} \tag{41}$$

and the conservation law of energy [11] corresponding to the electromagnetic power version is as follows

$$P^{out} = \mathcal{I} = P^{inp} \tag{42}$$

Inserting the (26)-(31) into the (40)-(41), the output and input powers can also be written as the following operator forms.

$$P^{out} = \mathcal{P}^{out}\left(\bar{V}\right) \tag{43}$$

$$P^{inp} = \mathcal{P}^{inp}\left(\bar{V}\right) \tag{44}$$

IV. THE MATRIX FORM FOR OUTPUT POWER

The basic variable $\bar{V}$ can be expanded in terms of the basis function set $\{\bar{b}_\xi(\bar{r})\}$ as follows

$$\bar{V}(\bar{r}) = \sum_{\xi=1}^{\Xi} a_\xi \bar{b}_\xi(\bar{r}) = \bar{B}\cdot\bar{a} \quad , \quad (\bar{r}\in\partial D) \tag{45}$$

here $\bar{B} = [\bar{b}_1(\bar{r}),\bar{b}_2(\bar{r}),\cdots,\bar{b}_\Xi(\bar{r})]$, and $\bar{a} = [a_1,a_2,\cdots,a_\Xi]^T$, and the superscript " T " represents matrix transposition; the symbol " $\cdot$ " represents matrix multiplication.

Inserting the (45) into (43) and employing that $\bar{F}^{inc} = \bar{F}^{tot} - \bar{F}^{sca}$, the matrix form for output power P^{out} can be written as follows

$$P^{out} = \mathcal{P}^{out}\left(\bar{B}\cdot\bar{a}\right) = \bar{a}^H\cdot\bar{\bar{P}}^{out}\cdot\bar{a} \tag{46.1}$$

here $\bar{\bar{P}}^{out} = [p^{out}_{\xi\zeta}]_{\Xi\times\Xi}$, and

$$\begin{aligned} p^{out}_{\xi\zeta} &= -\frac{1}{2}\left\langle \mathcal{J}^{l}(\bar{b}_\xi),\mathcal{E}^{sca,tan}(\bar{b}_\zeta)\right\rangle_{L^{met}} \\ &\quad -\frac{1}{2}\left\langle \mathcal{J}^{s}(\bar{b}_\xi),\mathcal{E}^{sca,tan}(\bar{b}_\zeta)\right\rangle_{\partial D^{met,sv}} \\ &\quad +\frac{1}{2}\left\langle \mathcal{J}^{vop}(\bar{b}_\xi),\mathcal{E}^{inc}_{int}(\bar{b}_\zeta)\right\rangle_{\text{int}\,V^{mat}} + \frac{1}{2}\left\langle \mathcal{H}^{inc}_{int}(\bar{b}_\xi),\mathcal{M}^{vm}(\bar{b}_\zeta)\right\rangle_{\text{int}\,V^{mat}} \end{aligned} \tag{46.2}$$

for any $\xi,\zeta = 1,2,\cdots,\Xi$. The superscript " H " in (46.1) is the transpose conjugate of relevant matrix.

The matrix $\bar{\bar{P}}^{out}$ in (46.1) can be decomposed as [7]-[10]

$$\bar{\bar{P}}^{out} = \bar{\bar{P}}^{out}_{+} + j\,\bar{\bar{P}}^{out}_{-} \tag{47.1}$$

here

$$\begin{aligned} \bar{\bar{P}}^{out}_{+} &= \frac{1}{2}\left[\bar{\bar{P}}^{out} + \left(\bar{\bar{P}}^{out}\right)^H\right] \\ \bar{\bar{P}}^{out}_{-} &= \frac{1}{2j}\left[\bar{\bar{P}}^{out} - \left(\bar{\bar{P}}^{out}\right)^H\right] \end{aligned} \tag{47.2}$$

Obviously, the matrices $\bar{\bar{P}}^{out}_{+}$ and $\bar{\bar{P}}^{out}_{-}$ are Hermitian, so the $\bar{a}^H\cdot\bar{\bar{P}}^{out}_{+}\cdot\bar{a}$ and $\bar{a}^H\cdot\bar{\bar{P}}^{out}_{-}\cdot\bar{a}$ are always real numbers for any vector $\bar{a}\in\mathbb{C}^\Xi$ [16], and then [8]-[9]

$$\begin{aligned}\bar{a}^H \cdot \bar{\bar{P}}_+^{out} \cdot \bar{a} &= \mathrm{Re}\left\{\mathcal{P}^{out}\left(\bar{B}\cdot\bar{a}\right)\right\} \\ &= \mathcal{P}^{sca,rad}\left(\bar{B}\cdot\bar{a}\right)+\mathcal{P}^{tot,loss}\left(\bar{B}\cdot\bar{a}\right)\end{aligned} \tag{48.1}$$

$$\begin{aligned}\bar{a}^H \cdot \bar{\bar{P}}_-^{out} \cdot \bar{a} &= \mathrm{Im}\left\{\mathcal{P}^{out}\left(\bar{B}\cdot\bar{a}\right)\right\} \\ &= \mathcal{P}^{sca,react,vac}\left(\bar{B}\cdot\bar{a}\right)+\mathcal{P}^{tot,react,mat}\left(\bar{B}\cdot\bar{a}\right)\end{aligned} \tag{48.2}$$

In (48), the $\mathcal{P}^{sca,rad}\left(\bar{B}\cdot\bar{a}\right)$ is the operator form of the power $P^{sca,rad}$ generated by the currents corresponding to the $\bar{V}=\bar{B}\cdot\bar{a}$, and the other symbols can be similarly explained.

V. Output power CM (OutCM) Set and OutCM-based Modal Expansion

Similarly to the PEC-EMP-CMT [7], the Mat-EMP-CMT [8]-[9], and the LSV-MM-EMP-CMT [10], a new line-surface formulation of the Output power CM (OutCM) set and the corresponding modal expansion method for the metal-material combined objects are discussed in this section.

A. Output power CM (OutCM) set.

When the matrix $\bar{\bar{P}}_+^{out}$ is positive definite at frequency f, the OutCM set can be obtained by solving the following generalized characteristic equation [7]-[10], [16].

$$\bar{\bar{P}}_-^{out}(f)\cdot\bar{a}_\xi(f) = \lambda_\xi(f)\,\bar{\bar{P}}_+^{out}(f)\cdot\bar{a}_\xi(f) \tag{49.1}$$

When the matrix $\bar{\bar{P}}_+^{out}$ is positive semi-definite at frequency f_0, the modal vectors can be obtained by using the following limitations for any $\xi=1,2,\cdots,\Xi$ [7]-[10].

$$\bar{a}_\xi(f_0) = \lim_{f\to f_0}\bar{a}_\xi(f) \tag{49.2}$$

The modal basic variables are as follows for any $\xi=1,2,\cdots,\Xi$

$$\bar{V}_\xi(\bar{r}) = \bar{B}\cdot\bar{a}_\xi \quad , \quad (\bar{r}\in\partial D) \tag{50}$$

The modal scattering currents are as follows

$$\bar{J}_\xi^l(\bar{r}) = \mathcal{J}^l\left(\bar{V}_\xi\right) \quad , \quad \left(\bar{r}\in L^{met}\right) \tag{51}$$

$$\bar{J}_\xi^s(\bar{r}) = \mathcal{J}^s\left(\bar{V}_\xi\right) \quad , \quad \left(\bar{r}\in \partial D^{met,sv}\right) \tag{52}$$

$$\bar{J}_\xi^{vop}(\bar{r}) = \mathcal{J}^{vop}\left(\bar{V}_\xi\right) \quad , \quad \left(\bar{r}\in \mathrm{int}V^{mat}\right) \tag{53.1}$$

$$\bar{M}_\xi^{vm}(\bar{r}) = \mathcal{M}^{vm}\left(\bar{V}_\xi\right) \quad , \quad \left(\bar{r}\in \mathrm{int}V^{mat}\right) \tag{53.2}$$

for any $\xi=1,2,\cdots,\Xi$, and the relevant operators are defined as (28)-(30).

The various modal fields corresponding to the above modal currents are as follows

$$\bar{F}_{int,\xi}^X(\bar{r}) = \mathcal{F}_{int}^X\left(\bar{V}_\xi\right) \quad , \quad \left(\bar{r}\in \mathrm{int}V^{mat}\right) \tag{54}$$

$$\bar{F}_\xi^{sca}(\bar{r}) = \mathcal{F}^{sca}\left(\bar{V}_\xi\right) \quad , \quad \left(\bar{r}\in \mathbb{R}^3\setminus\partial D\right) \tag{55}$$

for any $\xi=1,2,\cdots,\Xi$. In (54) and (55), $X=inc,tot$; $F=E,H$, and correspondingly $\mathcal{F}=\mathcal{E},\mathcal{H}$; the relevant operators are defined as (26) and (27). Based on the (31),

$$\bar{E}_\xi^{sca,tan}(\bar{r}) = \mathcal{E}^{sca,tan}\left(\bar{V}_\xi\right) \quad , \quad (\bar{r}\in\partial D) \tag{56}$$

for any $\xi=1,2,\cdots,\Xi$. In addition, the following relation (57) is valid for the modal fields for any $\xi=1,2,\cdots,\Xi$.

$$\bar{F}_{int,\xi}^{inc}(\bar{r}) = \bar{F}_{int,\xi}^{tot}(\bar{r})-\bar{F}_\xi^{sca}(\bar{r}) \quad , \quad \left(\bar{r}\in \mathrm{int}V^{mat}\right) \tag{57}$$

The above modal currents and modal fields satisfy the following power orthogonality [7]-[10].

$$\begin{aligned}P_\xi^{out}\delta_{\xi\zeta} &= P_{\xi\zeta}^{out} \\ &= P_{\xi\zeta;sca,rad}^{out}+P_{\xi\zeta;tot,loss}^{out}+j\left(P_{\xi\zeta;sca,react,vac}^{out}+P_{\xi\zeta;tot,react,mat}^{out}\right)\end{aligned} \tag{58}$$

In (58), the $\delta_{\xi\zeta}$ is Kronecker delta symbol, and

$$\begin{aligned}P_\xi^{out} &= \mathrm{Re}\left\{P_\xi^{out}\right\}+j\,\mathrm{Im}\left\{P_\xi^{out}\right\} \\ &= P_{\xi;sca,rad}^{out}+P_{\xi;tot,loss}^{out}+j\left(P_{\xi;sca,react,vac}^{out}+P_{\xi;tot,react,mat}^{out}\right)\end{aligned} \tag{59}$$

and

$$\begin{aligned}P_{\xi\zeta}^{out} = &-\frac{1}{2}\left\langle\bar{J}_\xi^l,\bar{E}_\zeta^{sca,tan}\right\rangle_{L^{met}} \\ &-\frac{1}{2}\left\langle\bar{J}_\xi^s,\bar{E}_\zeta^{sca,tan}\right\rangle_{\partial D^{met,sv}} \\ &+\frac{1}{2}\left\langle\bar{J}_\xi^{vop},\bar{E}_{int,\zeta}^{inc}\right\rangle_{\mathrm{int}V^{mat}}+\frac{1}{2}\left\langle\bar{H}_{int,\xi}^{inc},\bar{M}_\zeta^{vm}\right\rangle_{\mathrm{int}V^{mat}}\end{aligned} \tag{60}$$

and

$$P_{\xi\zeta;sca,rad}^{out} = \frac{1}{2}\oint\!\!\!\oint_{S_\infty}\left[\bar{E}_\zeta^{sca}\times\left(\bar{H}_\xi^{sca}\right)^*\right]\cdot d\bar{S} \tag{61.1}$$

$$P_{\xi\zeta;tot,loss}^{out} = \frac{1}{2}\left\langle\sigma\bar{E}_{int,\xi}^{tot},\bar{E}_{int,\zeta}^{tot}\right\rangle_{\mathrm{int}V^{mat}} \tag{61.2}$$

$$P_{\xi\zeta;sca,react,vac}^{out} = 2\omega\left[\frac{1}{4}\left\langle\bar{H}_\xi^{sca},\mu_0\bar{H}_\zeta^{sca}\right\rangle_{\mathbb{R}^3\setminus\partial D}-\frac{1}{4}\left\langle\varepsilon_0\bar{E}_\xi^{sca},\bar{E}_\zeta^{sca}\right\rangle_{\mathbb{R}^3\setminus\partial D}\right] \tag{61.3}$$

$$P_{\xi\zeta;tot,react,mat}^{out} = 2\omega\left[\frac{1}{4}\left\langle\bar{H}_{int,\xi}^{tot},\Delta\mu\bar{H}_{int,\zeta}^{tot}\right\rangle_{\mathrm{int}V^{mat}}-\frac{1}{4}\left\langle\Delta\varepsilon\bar{E}_{int,\xi}^{tot},\bar{E}_{int,\zeta}^{tot}\right\rangle_{\mathrm{int}V^{mat}}\right] \tag{61.4}$$

In the (59), $P_{\xi;sca,rad}^{out}=P_{\xi\xi;sca,rad}^{out}$, and $P_{\xi;tot,loss}^{out}=P_{\xi\xi;tot,loss}^{out}$, and $P_{\xi;sca,react,vac}^{out}=P_{\xi\xi;sca,react,vac}^{out}$, and $P_{\xi;tot,react,mat}^{out}=P_{\xi\xi;tot,react,mat}^{out}$.

B. OutCM-based modal expansion.

Because of the completeness of the OutCM set [7]-[10], the basic variable $\bar{V}$ on ∂D, the scattering currents $\{\bar{J}^l,\bar{J}^s,\bar{J}^{vop},\bar{M}^{vm}\}$ on scatterer, the scattering fields $\{\bar{E}^{sca},\bar{H}^{sca}\}$ on $\mathbb{R}^3\setminus\partial D$, the tangential scattering electric field $\bar{E}^{sca,tan}$ on ∂D, and the fields $\{\bar{E}^{inc},\bar{H}^{inc}\}$ and $\{\bar{E}^{tot},\bar{H}^{tot}\}$ on $\mathrm{int}V^{mat}$ can be expanded in terms of the OutCM set as follows

$$\bar{V}(\bar{r}) = \sum_{\xi=1}^{\Xi}c_\xi\bar{V}_\xi(\bar{r}) \quad , \quad (\bar{r}\in\partial D) \tag{62}$$

and

$$\bar{J}^{l}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_{\xi}\bar{J}_{\xi}^{l}(\bar{r}) \quad , \quad (\bar{r} \in L^{met}) \tag{63}$$

$$\bar{J}^{s}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_{\xi}\bar{J}_{\xi}^{s}(\bar{r}) \quad , \quad (\bar{r} \in \partial D^{met,sv}) \tag{64}$$

$$\bar{J}^{vop}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_{\xi}\bar{J}_{\xi}^{vop}(\bar{r}) \quad , \quad (\bar{r} \in \text{int}V^{mat}) \tag{65.1}$$

$$\bar{M}^{vm}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_{\xi}\bar{M}_{\xi}^{vm}(\bar{r}) \quad , \quad (\bar{r} \in \text{int}V^{mat}) \tag{65.2}$$

and

$$\bar{F}^{sca}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_{\xi}\bar{F}_{\xi}^{sca}(\bar{r}) \quad , \quad (\bar{r} \in \mathbb{R}^3 \setminus \partial D) \tag{66}$$

$$\bar{E}^{sca,tan}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_{\xi}\bar{E}_{\xi}^{sca,tan}(\bar{r}) \quad , \quad (\bar{r} \in \partial D) \tag{67}$$

here $F = E,H$, and

$$\bar{F}_{int}^{X}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_{\xi}\bar{F}_{int,\xi}^{X}(\bar{r}) \quad , \quad (\bar{r} \in \text{int}V^{mat}) \tag{68}$$

here $X = inc,tot$, and $F = E,H$.

Based on the power orthogonality (58) for OutCM set, the system output power P^{out} can be expanded in terms of the modal powers as follows

$$\begin{aligned} P^{out} &= \sum_{\xi=1}^{\Xi} |c_{\xi}|^2 P_{\xi}^{out} \\ &= \left(\sum_{\xi=1}^{\Xi} |c_{\xi}|^2 P_{\xi;sca,rad}^{out} + \sum_{\xi=1}^{\Xi} |c_{\xi}|^2 P_{\xi;tot,loss}^{out} \right) \\ &\quad + j\left(\sum_{\xi=1}^{\Xi} |c_{\xi}|^2 P_{\xi;sca,react,vac}^{out} + \sum_{\xi=1}^{\Xi} |c_{\xi}|^2 P_{\xi;tot,react,mat}^{out} \right) \end{aligned} \tag{69}$$

In (69), the terms corresponding to loss will disappear, if the material part is lossless.

C. Expansion coefficients.

When the external excitation is given, the interaction $\mathcal{I}$ and output power P^{out} can be respectively written as the following (70) and (71) based on the discussions in Sec. III.

$$\begin{aligned} \mathcal{I} &= \mathcal{I}(\bar{V}) \\ &= (1/2)\langle \mathcal{J}^{l}(\bar{V}), \bar{E}^{inc} \rangle_{L^{met}} + (1/2)\langle \mathcal{J}^{s}(\bar{V}), \bar{E}^{inc} \rangle_{\partial D^{met,sv}} \\ &\quad + (1/2)\langle \mathcal{J}^{vop}(\bar{V}), \bar{E}^{inc} \rangle_{\text{int}V^{mat}} + (1/2)\langle \bar{H}^{inc}, \mathcal{M}^{vm}(\bar{V}) \rangle_{\text{int}V^{mat}} \end{aligned} \tag{70}$$

$$P^{out} = \mathcal{P}^{out}(\bar{V}) \tag{71}$$

In (70), the $\bar{E}^{inc}$ and $\bar{H}^{inc}$ are known.

Based on the conservation law of energy (42) and the variational principle [17], the $\bar{V}$ will make the following functional $\mathfrak{F}$ be zero and stationary.

$$\mathfrak{F}(\bar{V}) = \mathcal{I}(\bar{V}) - \mathcal{P}^{out}(\bar{V}) \tag{72}$$

Inserting the (62) and (70)-(71) into the (72) and employing the Ritz's procedure [18], the following simultaneous equations for the expansion coefficients $\{c_{\xi}\}$ in (62)-(69) are derived for any $\xi = 1,2,\cdots,\Xi$.

$$\begin{aligned} &(1/2)\langle \bar{J}_{\xi}^{l}, \bar{E}^{inc} \rangle_{L^{met}} + (1/2)\langle \bar{J}_{\xi}^{s}, \bar{E}^{inc} \rangle_{\partial D^{met,sv}} \\ &+ (1/2)\langle \bar{J}_{\xi}^{vop}, \bar{E}^{inc} \rangle_{\text{int}V^{mat}} + (1/2)\langle \bar{H}^{inc}, \bar{M}_{\xi}^{vm} \rangle_{\text{int}V^{mat}} \\ = &-(1/2)\langle \bar{J}_{\xi}^{l}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{E}_{\zeta}^{sca,tan} \rangle_{L^{met}} - (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{J}_{\zeta}^{l}, \bar{E}_{\xi}^{sca,tan} \rangle_{L^{met}} \\ &-(1/2)\langle \bar{J}_{\xi}^{s}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{E}_{\zeta}^{sca,tan} \rangle_{\partial D^{met,sv}} - (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{J}_{\zeta}^{s}, \bar{E}_{\xi}^{sca,tan} \rangle_{\partial D^{met,sv}} \\ &+(1/2)\langle \bar{J}_{\xi}^{vop}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{E}_{int,\zeta}^{inc} \rangle_{\text{int}V^{mat}} + (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{J}_{\zeta}^{vop}, \bar{E}_{int,\xi}^{inc} \rangle_{\text{int}V^{mat}} \\ &+(1/2)\langle \bar{H}_{int,\xi}^{inc}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{M}_{\zeta}^{vm} \rangle_{\text{int}V^{mat}} + (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{H}_{int,\zeta}^{inc}, \bar{M}_{\xi}^{vm} \rangle_{\text{int}V^{mat}} \end{aligned} \tag{73.1}$$

and

$$\begin{aligned} &-(1/2)\langle \bar{J}_{\xi}^{l}, \bar{E}^{inc} \rangle_{L^{met}} - (1/2)\langle \bar{J}_{\xi}^{s}, \bar{E}^{inc} \rangle_{\partial D^{met,sv}} \\ &-(1/2)\langle \bar{J}_{\xi}^{vop}, \bar{E}^{inc} \rangle_{\text{int}V^{mat}} + (1/2)\langle \bar{H}^{inc}, \bar{M}_{\xi}^{vm} \rangle_{\text{int}V^{mat}} \\ = &(1/2)\langle \bar{J}_{\xi}^{l}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{E}_{\zeta}^{sca,tan} \rangle_{L^{met}} - (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{J}_{\zeta}^{l}, \bar{E}_{\xi}^{sca,tan} \rangle_{L^{met}} \\ &+(1/2)\langle \bar{J}_{\xi}^{s}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{E}_{\zeta}^{sca,tan} \rangle_{\partial D^{met,sv}} - (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{J}_{\zeta}^{s}, \bar{E}_{\xi}^{sca,tan} \rangle_{\partial D^{met,sv}} \\ &-(1/2)\langle \bar{J}_{\xi}^{vop}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{E}_{int,\zeta}^{inc} \rangle_{\text{int}V^{mat}} + (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{J}_{\zeta}^{vop}, \bar{E}_{int,\xi}^{inc} \rangle_{\text{int}V^{mat}} \\ &-(1/2)\langle \bar{H}_{int,\xi}^{inc}, \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{M}_{\zeta}^{vm} \rangle_{\text{int}V^{mat}} + (1/2)\langle \textstyle\sum_{\zeta=1}^{\Xi} c_{\zeta}\bar{H}_{int,\zeta}^{inc}, \bar{M}_{\xi}^{vm} \rangle_{\text{int}V^{mat}} \end{aligned} \tag{73.2}$$

In (73), the relation (57) has been utilized.

By solving the (73), the coefficient $\{c_{\xi}\}$ can be determined. If the orthogonality of (58) is utilized in (73), the coefficient $\{c_{\xi}\}$ can be concisely written as the (74) for any $\xi = 1,2,\cdots,\Xi$.

$$c_{\xi} = \begin{cases} \dfrac{1}{P_{\xi}^{out}} \cdot \left[\dfrac{1}{2}\langle \bar{J}_{\xi}^{l}, \bar{E}^{inc} \rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}_{\xi}^{s}, \bar{E}^{inc} \rangle_{\partial D^{met,sv}} \right] & , \quad (\Delta\mu, \Delta\varepsilon_c = 0) \\ \dfrac{1}{P_{\xi}^{out}} \cdot \left[\dfrac{1}{2}\langle \bar{J}_{\xi}^{l}, \bar{E}^{inc} \rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}_{\xi}^{s}, \bar{E}^{inc} \rangle_{\partial D^{met,sv}} \right] = \left[\dfrac{1}{P_{\xi}^{out}} \cdot \dfrac{1}{2}\langle \bar{H}^{inc}, \bar{M}_{\xi}^{vm} \rangle_{\text{int}V^{mat}} \right]^{*} & , \quad (\Delta\mu \neq 0, \Delta\varepsilon_c = 0) \\ \dfrac{1}{P_{\xi}^{out}} \cdot \left[\dfrac{1}{2}\langle \bar{J}_{\xi}^{l}, \bar{E}^{inc} \rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}_{\xi}^{s}, \bar{E}^{inc} \rangle_{\partial D^{met,sv}} + \dfrac{1}{2}\langle \bar{J}_{\xi}^{vop}, \bar{E}^{inc} \rangle_{\text{int}V^{mat}} \right] & , \quad (\Delta\mu = 0, \Delta\varepsilon_c \neq 0) \\ \dfrac{1}{P_{\xi}^{out}} \cdot \left[\dfrac{1}{2}\langle \bar{J}_{\xi}^{l}, \bar{E}^{inc} \rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}_{\xi}^{s}, \bar{E}^{inc} \rangle_{\partial D^{met,sv}} + \dfrac{1}{2}\langle \bar{J}_{\xi}^{vop}, \bar{E}^{inc} \rangle_{\text{int}V^{mat}} \right] = \left[\dfrac{1}{P_{\xi}^{out}} \cdot \dfrac{1}{2}\langle \bar{H}^{inc}, \bar{M}_{\xi}^{vm} \rangle_{\text{int}V^{mat}} \right]^{*} & , \quad (\Delta\mu, \Delta\varepsilon_c \neq 0) \end{cases} \tag{74}$$

The symbol " $\cdot$ " in (74) represents the ordinary scalar multiplication.

VI. Impedance and Admittance of Metal-Material combined Electromagnetic Systems

In this section, the field-based definitions for the impedance and admittance introduced in [7] and [9] are generalized to the metal-material combined electromagnetic systems.

A. System impedance and admittance.

Following the ideas of [7] and [9], the field-based definitions for the system impedance Z and the system admittance Y of the metal-material combined electromagnetic systems are defined as follows

$$Z = \mathcal{Z}(\bar{V}) \triangleq \frac{\mathcal{P}^{inp}(\bar{V})}{N^J} = \frac{\mathcal{P}^{out}(\bar{V})}{N^J} \tag{75.1}$$

here

$$\begin{aligned} N^J = & \frac{1}{2L}\langle \bar{J}^l, \bar{J}^l \rangle_{L^{met}} + \frac{1}{2}\langle \bar{J}^s_{free}, \bar{J}^s_{free} \rangle_{\partial D^{met,sv}_{free}} \\ & + \frac{1}{2}\langle \bar{J}^{SE}_0, \bar{J}^{SE}_0 \rangle_{\partial V^{mat} \setminus (S^{met} \cap \partial D^{met,sv}_{unfree})} \\ & + \frac{1}{2}\langle \bar{J}^{SE}_+, \bar{J}^{SE}_+ \rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} + \frac{1}{2}\langle \bar{J}^{SE}_-, \bar{J}^{SE}_- \rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} \end{aligned} \tag{75.2}$$

and

$$Y = \mathcal{Y}(\bar{V}) \triangleq \frac{\mathcal{P}^{inp}(\bar{V})}{N^\rho} = \frac{\mathcal{P}^{out}(\bar{V})}{N^\rho} \tag{76.1}$$

here

$$\begin{aligned} N^\rho = & \frac{1}{2L}\left\langle \frac{\rho^l}{\varepsilon_0}, \frac{\rho^l}{\varepsilon_0} \right\rangle_{L^{met}} + \frac{1}{2}\left\langle \frac{\rho^s_{free}}{\varepsilon_0}, \frac{\rho^s_{free}}{\varepsilon_0} \right\rangle_{\partial D^{met,sv}_{free}} \\ & + \frac{1}{2}\left\langle \frac{\rho^{SE}_0}{\varepsilon_0}, \frac{\rho^{SE}_0}{\varepsilon_0} \right\rangle_{\partial V^{mat} \setminus (S^{met} \cap \partial D^{met,sv}_{unfree})} \\ & + \frac{1}{2}\left\langle \frac{\rho^{SE}_+}{\varepsilon_0}, \frac{\rho^{SE}_+}{\varepsilon_0} \right\rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} + \frac{1}{2}\left\langle \frac{\rho^{SE}_-}{\varepsilon_0}, \frac{\rho^{SE}_-}{\varepsilon_0} \right\rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} \end{aligned} \tag{76.2}$$

In the (75) and (76), the L is the length of L^{met}; the ρ^l is the line electric charge on L^{met}, and the ρ^s_{free} is the surface electric charge on $\partial D^{met,sv}_{free}$, and the ρ^{SE}_0 and $\rho^{SE}_\pm$ are respectively the surface equivalent electric charges on $\partial V^{mat} \setminus (S^{met} \cap \partial D^{met,sv}_{unfree})$ and $S^{met} \cap \partial D^{met,sv}_{unfree}$; the various charges and the related currents satisfy the corresponding current continuity equations; the inner product for the scalars is defined as $< g,h >_\Omega \triangleq \int_\Omega g^* \cdot h \, d\Omega$.

The system resistance, reactance, conductance, and susceptance of metal-material combined systems are respectively as follows

$$R = \mathcal{R}(\bar{V}) = \operatorname{Re}\{\mathcal{Z}(\bar{V})\} \tag{77.1}$$

$$X = \mathcal{X}(\bar{V}) = \operatorname{Im}\{\mathcal{Z}(\bar{V})\} \tag{77.2}$$

and

$$G = \mathcal{G}(\bar{V}) = \operatorname{Re}\{\mathcal{Y}(\bar{V})\} \tag{78.1}$$

$$B = \mathcal{B}(\bar{V}) = \operatorname{Im}\{\mathcal{Y}(\bar{V})\} \tag{78.2}$$

B. Modal impedance and admittance.

The field-based definitions for the modal impedance, resistance, reactance, admittance, conductance, and susceptance introduced in the papers [7] and [9] can be generalized to the OutCMs of metal-material combined objects as follows

$$Z_\xi \triangleq P^{out}_\xi / N^J_\xi \tag{79}$$

$$R_\xi = \operatorname{Re}\{Z_\xi\} \tag{80.1}$$

$$X_\xi = \operatorname{Im}\{Z_\xi\} \tag{80.2}$$

here

$$\begin{aligned} N^J_\xi = & \frac{1}{2L}\langle \bar{J}^l_\xi, \bar{J}^l_\xi \rangle_{L^{met}} + \frac{1}{2}\langle \bar{J}^s_{free,\xi}, \bar{J}^s_{free,\xi} \rangle_{\partial D^{met,sv}_{free}} \\ & + \frac{1}{2}\langle \bar{J}^{SE}_{0,\xi}, \bar{J}^{SE}_{0,\xi} \rangle_{\partial V^{mat} \setminus (S^{met} \cap \partial D^{met,sv}_{unfree})} \\ & + \frac{1}{2}\langle \bar{J}^{SE}_{+,\xi}, \bar{J}^{SE}_{+,\xi} \rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} + \frac{1}{2}\langle \bar{J}^{SE}_{-,\xi}, \bar{J}^{SE}_{-,\xi} \rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} \end{aligned} \tag{81}$$

here $\bar{J}^s_{free,\xi} = \bar{J}^s_\xi$ on $\partial D^{met,sv}_{free}$, and

$$Y_\xi \triangleq P^{out}_\xi / N^\rho_\xi \tag{82}$$

$$G_\xi = \operatorname{Re}\{Y_\xi\} \tag{83.1}$$

$$B_\xi = \operatorname{Im}\{Y_\xi\} \tag{83.2}$$

here

$$\begin{aligned} N^\rho_\xi = & \frac{1}{2L}\left\langle \frac{\rho^l_\xi}{\varepsilon_0}, \frac{\rho^l_\xi}{\varepsilon_0} \right\rangle_{L^{met}} + \frac{1}{2}\left\langle \frac{\rho^s_{free,\xi}}{\varepsilon_0}, \frac{\rho^s_{free,\xi}}{\varepsilon_0} \right\rangle_{\partial D^{met,sv}_{free}} \\ & + \frac{1}{2}\left\langle \frac{\rho^{SE}_{0,\xi}}{\varepsilon_0}, \frac{\rho^{SE}_{0,\xi}}{\varepsilon_0} \right\rangle_{\partial V^{mat} \setminus (S^{met} \cap \partial D^{met,sv}_{unfree})} \\ & + \frac{1}{2}\left\langle \frac{\rho^{SE}_{+,\xi}}{\varepsilon_0}, \frac{\rho^{SE}_{+,\xi}}{\varepsilon_0} \right\rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} + \frac{1}{2}\left\langle \frac{\rho^{SE}_{-,\xi}}{\varepsilon_0}, \frac{\rho^{SE}_{-,\xi}}{\varepsilon_0} \right\rangle_{S^{met} \cap \partial D^{met,sv}_{unfree}} \end{aligned} \tag{84}$$

In (84), the ρ^l_ξ is the modal line electric charge on L^{met}, and the $\rho^s_{free,\xi}$ is the modal surface electric charge on $\partial D^{met,sv}_{free}$, and the $\rho^{SE}_{0,\xi}$ and $\rho^{SE}_{\pm,\xi}$ are respectively the modal surface equivalent electric charges on $\partial V^{mat} \setminus (S^{met} \cap \partial D^{met,sv}_{unfree})$ and $S^{met} \cap \partial D^{met,sv}_{unfree}$; the various modal charges and the related modal currents satisfy the corresponding current continuity equations.

VII. Modal Quantities corresponding to the OutCM Set of Metal-Material combined Objects

In fact, the introductions for the modal impedance (79) and admittance (82) provide an efficient way to define the various *modal quantities* (introduced in [7]-[8]) for the OutCM set of metal-material combined objects, and they are discussed in this section.

The $\Delta\varepsilon_c \neq 0$ case is considered here, so the formulation $c_\xi = (1/P_\xi^{out}) \cdot [(1/2)\langle \bar{J}_\xi^l, \bar{E}^{inc}\rangle_{L^{met}} + (1/2)\langle \bar{J}_\xi^s, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} + (1/2)\langle \bar{J}_\xi^{vop}, \bar{E}^{inc}\rangle_{\text{int}V^{mat}}]$ is utilized in the following discussions. Of cause, the case $(\Delta\mu \neq 0, \Delta\varepsilon_c = 0)$ can be similarly discussed, but it will not be repeated in this paper.

A. Modal normalization.

The field-based expression for the normalized basic variable is as follows

$$\tilde{\bar{V}}_\xi(\bar{r}) = \frac{\bar{V}_\xi(\bar{r})}{\sqrt{N_\xi^J}} \quad , \quad (\bar{r} \in \partial D) \tag{85}$$

for any $\xi = 1,2,\cdots,\Xi$. The field-based expressions for the normalized modal currents and fields can be similarly obtained.

B. Modal quantities.

The normalized version of expansion formulation (69) is as follows

$$\begin{aligned} P^{out} &= \sum_{\xi=1}^{\Xi} |\tilde{c}_\xi|^2 Z_\xi \\ &= \sum_{\xi=1}^{\Xi} \left\{ \frac{1}{|Z_\xi|} \cdot \left| \frac{1}{2}\langle \tilde{\bar{J}}_\xi^l, \bar{E}^{inc}\rangle_{L^{met}} + \frac{1}{2}\langle \tilde{\bar{J}}_\xi^s, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} \right.\right. \\ &\qquad \left.\left. + \frac{1}{2}\langle \tilde{\bar{J}}_\xi^{vop}, \bar{E}^{inc}\rangle_{\text{int}V^{mat}} \right|^2 \cdot \frac{Z_\xi}{|Z_\xi|} \right\} \end{aligned} \tag{86}$$

Based on the above (86), the various modal quantities can be defined for the metal-material combined objects as the following (87)-(92) for any $\xi = 1,2,\cdots,\Xi$ [7]-[8].

$$\text{GMS}_\xi^{sys,tot} \triangleq \frac{1}{|Z_\xi|} \tag{87}$$

and

$$\text{MACE}_\xi^{mod} \triangleq \left| \frac{1}{2}\langle \tilde{\bar{J}}_\xi^l, \bar{E}^{inc}\rangle_{L^{met}} + \frac{1}{2}\langle \tilde{\bar{J}}_\xi^s, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} + \frac{1}{2}\langle \tilde{\bar{J}}_\xi^{vop}, \bar{E}^{inc}\rangle_{\text{int}V^{mat}} \right|^2 \tag{88}$$

and

$$\text{MAOP}_\xi^{mod,act} \triangleq \frac{R_\xi}{|Z_\xi|} \tag{89.1}$$

$$\text{MAOP}_\xi^{mod,react} \triangleq \frac{X_\xi}{|Z_\xi|} \tag{89.2}$$

and

$$\text{SMS}_\xi^{sys,tot} \triangleq \text{GMS}_\xi^{sys,tot} \cdot \text{MACE}_\xi^{mod} \tag{90}$$

and

$$\text{SMS}_\xi^{sys,act} \triangleq \text{SMS}_\xi^{sys,tot} \cdot \text{MAOP}_\xi^{mod,act} \tag{91.1}$$

$$\text{GMS}_\xi^{sys,act} \triangleq \text{GMS}_\xi^{sys,tot} \cdot \text{MAOP}_\xi^{mod,act} \tag{91.2}$$

and

$$\text{SMS}_\xi^{sys,react} \triangleq \text{SMS}_\xi^{sys,tot} \cdot \text{MAOP}_\xi^{mod,react} \tag{92.1}$$

$$\text{GMS}_\xi^{sys,react} \triangleq \text{GMS}_\xi^{sys,tot} \cdot \text{MAOP}_\xi^{mod,react} \tag{92.2}$$

The various modal quantities defined in above (87)-(92) and the modal component $|\tilde{c}_\xi|^2 Z_\xi$ in (86) satisfy the relation (93) for any $\xi = 1,2,\cdots,\Xi$.

VIII. Conclusions

A new line-surface formulation of MM-EMP-CMT is established in this paper, and it is simply denoted as LS-MM-EMP-CMT. Just like the previous PEC-EMP-CMT, Mat-EMP-CMT, and LSV-MM-EMP-CMT, the CM sets derived from LS-MM-EMP-CMT can reveal the inherent power characteristics of metal-material combined electromagnetic systems.

The physical effectiveness of LS-MM-EMP-CMT is the same as the LSV-MM-EMP-CMT. However, the former is more advantageous than the latter in some aspects, such as saving computational resources and avoiding to compute the modal scattering field in source region etc.

In addition, a variational formulation for the scattering problem of metal-material combined objects is provided based on the conservation law of energy, and the unknowns only include the line and surface sources; the field-based definitions for the impedance and admittance of metal-material combined electromagnetic systems are introduced in this paper.

$$\begin{aligned} |\tilde{c}_\xi|^2 Z_\xi &= \text{SMS}_\xi^{sys,act} + j\,\text{SMS}_\xi^{sys,react} = \overbrace{\text{MACE}_\xi^{mod} \cdot \text{GMS}_\xi^{sys,tot}}^{\text{SMS}_\xi^{sys,tot}} \cdot \text{MAOP}_\xi^{mod,act} + j\,\overbrace{\text{MACE}_\xi^{mod} \cdot \text{GMS}_\xi^{sys,tot}}^{\text{SMS}_\xi^{sys,tot}} \cdot \text{MAOP}_\xi^{mod,react} \\ &= \text{MACE}_\xi^{mod} \cdot \underbrace{\text{GMS}_\xi^{sys,tot} \cdot \text{MAOP}_\xi^{mod,act}}_{\text{GMS}_\xi^{sys,act}} + j\,\text{MACE}_\xi^{mod} \cdot \underbrace{\text{GMS}_\xi^{sys,tot} \cdot \text{MAOP}_\xi^{mod,react}}_{\text{GMS}_\xi^{sys,react}} \end{aligned} \tag{93}$$

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