Multifractal Geometry and Standard Model Symmetries

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Abstract

Despite being supported by overwhelming evidence, the Standard Model (SM) of particle physics is challenged by many foundational questions. The root cause of its gauge structure and of discrete symmetry breaking continues to be unknown. Here we show how these questions may be approached using the multifractal geometry of the SM near the electroweak scale.

Key words: Models beyond the Standard Model, Lie groups, gauge symmetries, discrete symmetries, multifractals, minimal fractal manifolds.

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1. Introduction

For the last few decades, the Standard Model of electroweak and strong interaction (SM) has been tested with remarkable accuracy. Although the SM has been successful in describing experimental observations so far, it is confronted by many foundational questions which resist explanation. The prevailing view is that the SM embodies an effective Lagrangian to a more fundamental theory, whose formulation is yet to be developed. The hope is that future extension(s) of the SM will satisfactorily address these puzzles, while recovering the SM physics in the low-energy limit. A prime example of long-standing issues relates to the origin of SM symmetries. Drawing on the *multifractal geometry* of the SM near the electroweak scale [1-3], here we focus on seeking clues to the following questions:

1) Why is the SM described by the *SU(3) x SU(2) x U(1)* gauge group?
2) Why are there discrete symmetries in the SM (*C, P* and *T*) and why are these symmetries broken while the overall *CPT* symmetry remains exact?

The paper is organized the following way: the first section contains a brief introduction to multifractal theory; the second and third sections elaborate on the connection between multifractals, statistical physics and perturbative quantum field theory (QFT). Section four outlines the set of assumptions on which the paper is built. This enables development of next sections, where scaling flows are shown to generate Lie groups and, implicitly, be linked to the *continuous* and *discrete symmetries* of the SM. A discussion of results and open questions follows in section eight. Two appendix sections are also included. Appendix A deals with *discrete scale invariance* (DSI) in statistical mechanics and Renormalization Group (RG), whereas Appendix B considers the link between multifractal geometry of the SM and *stochastic quantization* [4, 5]. The paper ends with a list of abbreviations used throughout the text.

The reader is cautioned that this work is solely meant to be a preliminary investigation. As such, it lacks the rigor and level of completion that one should expect from a comprehensive research project. Our only goal is to sketch up a novel perspective on the gauge and discrete symmetries of the SM, with no attempt to cover all technical aspects of these topics. Further analysis, on both theoretical and experimental sides, is needed to substantiate, reject or expand the body of ideas presented here.

2. Basics of multifractal theory

This section contains a brief overview of fractal and multifractal sets, with the intent of making the paper accessible and self-contained.

The box-counting dimension defines the main scaling property of fractal sets and is a measure of their *self-similarity*. Consider a fractal set S covered with a collection of boxes characterized by a *local* size r. The number of boxes needed to cover the set is given by

$$N(r) \sim r^{-D} \tag{1}$$

where D stands for the scaling dimension of the set, $D = \log N(r) \cdot \log(r^{-1})$. If S is instead composed by a *global* mixture of fractal subsets, the single scaling dimension is replaced by a continuous spectrum of dimensions $\alpha_{\min} < \alpha < \alpha_{\max}$ such that, for each value $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, the number of boxes of size r covering the set takes the form

$$N_r(\alpha) \sim r^{-f_r(\alpha)} \tag{2}$$

Relation (2) shows that self-similarity of multifractals is defined in terms of a *multifractal spectrum* $f_r(\alpha)$ describing the overall distribution of dimensions α.

Two complementary methods for characterization of multifractals have been devised. The first one is based on the recursive construction of multifractal sets from $i = 1, 2..., N$ local scales r_i with probabilities p_i. Using this method, the definition of the box-counting dimension leads to [6, 7]

$$\sum_{i=1}^{N} p_i^q r_i^{\tau(q)} = 1 \tag{3}$$

in which $p_i = p(r_i)$ and

$$\sum_{i=1}^{N} p_i = 1 \tag{4}$$

Here, q and $\tau(q)$ are two arbitrary exponents and the latter is given by

$$\tau(q) = (1-q)D_q \tag{5}$$

where D_q plays the role of *generalized dimension*. The second method appeals to the $f(\alpha)$ spectrum evaluated as continuous function of α. The two methods are related through the *Legendre transform* [8]

$$f(\alpha(q)) = q\alpha(q) - \tau(q) \tag{6}$$

where the following relations hold

$$\alpha(q) = \frac{\partial \tau(q)}{\partial q} \tag{7}$$

$$q = \left.\frac{\partial f(\alpha)}{\partial \alpha}\right|_{\alpha(q)} \tag{8}$$

In a broader context, multifractal analysis may be regarded as the study of *invariant sets* and is a powerful tool for the characterization of generic dynamical systems. In general, a strange attractor is an attractive limit set with *unstable* trajectories. The emergence of strange attractors is a typical signature of transition to *chaos* in the behavior of nonlinear dynamical systems [8, 9].

3. QFT as analog of multifractal sets

Starting from (3) and (6), it can be shown that classical statistical mechanics offers a straightforward analog of multifractal analysis. In particular, temperature (T), entropy (S), internal energy (U) and free energy (F) are respectively echoed in multifractal theory by $q, \alpha, f(\alpha)$ and $\tau(q)$ [6, 8]. Considering this analogy, the connection

$$F(T,V) = U(S,V) - TS \tag{9}$$

becomes a replica of (6), where V stands for volume and

$$dU = TdS - pdV \Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \tag{10}$$

In what follows, we further expand this analogy by using the known relationship between statistical mechanics and quantum field theory (QFT) [10, 11]. For the sake of simplicity, we focus on a Legendre transform like (6) as applied to four-dimensional *scalar field* theory. This transform connects the generating functional of the theory $W[J]$ to its effective action $\Gamma[\varphi]$ and to the field and current content expressed by $\varphi(x)$ and $J(x) = J[\varphi](x)$, respectively [10, 11]

$$\Gamma[\varphi] = W[J] - \int d^4x\, J(x)\varphi(x) \tag{11}$$

Here, the field is given by the first functional derivative of $W[J]$,

$$\varphi(x) = \frac{\delta W[J]}{\delta J(x)} \tag{12}$$

and the current by the first functional derivative of the effective action,

$$-J(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} \tag{13}$$

Taken together, relations (7) to (13) reveal a meaningful comparison between multifractals, statistical mechanics and QFT, as captured in Tab. 1 below.

Multifractals	**Statistical Mechanics**	**QFT**
q	$T = q^{-1}$	J
α	S	φ
$-f(\alpha)$	$U(S)$	$\Gamma[\varphi]$
$\tau(q)$	$F(T)/T$	$W[J]$

Tab.1: Linking multifractals, statistical mechanics and QFT.

4. Assumptions

The defining attribute of multifractals is that their geometry remains nearly self-similar upon consecutive scaling operations. Likewise, RG leads to a flow of observables from the ultraviolet to the infrared sector of field theory, where the flow is expected to reach a state insensitive to scale transformations. Accordingly, it makes sense to proceed with the following set of assumptions inspired by the route to scale invariance in both multifractal theory and RG:

1. All variables listed in Tab. 1 are considered analytic functions of the energy scale μ. Given an arbitrary reference scale μ_{ref}, the *running scale* is defined as $\lambda = \mu / \mu_{ref}$ with $\lambda \geq 1$.
2. The field entry in Tab.1 represents the vector of field operators $\varphi = \{O_1, O_2, ...\}$. Likewise, the current entry stands for the vector of quantum currents present in the theory $J = \{j_1, j_2, ...\}$.
3. The textbook description of multifractals, statistical mechanics and effective QFT corresponds to the equilibrium regime of low energies ($\lambda \to 1$). An alternative motivation for this ansatz is given in Appendix B.
4. The global nature of multifractal sets implies that the flow of variables occurs on *multiple scales* (λ_i) which are coupled to each other.

5. Lie Groups from scaling flows

Previous considerations and the content of Tab. 1 point out that multifractals (as well as their QFT counterparts) are *composite structures* built from interconnected entities. Continuous transformations within multifractals or QFT variables may be broadly viewed as flows in an appropriate multi-dimensional phase-space. There is a four-parameter group of transformations on a four-dimensional space defined as

$$x_i' = f_i(x_1, x_2, x_3, x_4; \lambda_1, \lambda_2, \lambda_3, \lambda_4), \quad i = 1, 2, 3, 4 \tag{14}$$

where the phase-space coordinates are given by

$$x = (\alpha, q, \tau(q), f(\alpha)) \text{ or } x = (J, \varphi, W[J], \Gamma[\varphi]) \tag{15}$$

Here, per the first assumption above, each evolution parameter λ_i represents the running energy scale normalized to a reference energy scale, that is,

$$\lambda_i = \mu_i / \mu_{ref} \tag{16}$$

In what follows, we refer to (14) as *scaling flows,* asymptotically reaching stationarity in the limit $\lambda_i = 1$. The aim of this section is to explicitly show how Lie groups arise from (14).

To this end, assume that one of the running scales λ_i varies from some fixed value λ_0 while all the other scales are held constant [12]. This is equivalent to stating that the flow relative to λ_i is *faster* than the flow relative to all other scales, that is, $\Delta\lambda_i >> \Delta\lambda_k$ for $i \neq k$. Next, examine the effect of infinitesimal coordinate transformations on an arbitrary differentiable function F

$$dF = \sum_{j=1}^{4} \frac{\partial F}{\partial x_j} dx_j = \sum_{j=1}^{4} \frac{\partial F}{\partial x_j} (\sum_{i=1}^{4} \frac{\partial f_j}{\partial \lambda_i}\bigg|_{\lambda_i=\lambda_0} d\lambda_i) = \sum_{i=1}^{4} d\lambda_i (\sum_{j=1}^{4} \frac{\partial f_j}{\partial \lambda_i}\bigg|_{\lambda_i=\lambda_0} \frac{\partial}{\partial x_j}) F \tag{17}$$

The coefficient multiplying $d\lambda_i$ in this expression can be identified as the differential operator

$$X_i = \sum_{j=1}^{4} \frac{\partial f_j}{\partial \lambda_i}\bigg|_{\lambda_i=\lambda_0} \frac{\partial}{\partial x_j} \quad , \quad i = 1, 2, 3, 4 \tag{18}$$

X_i represent the generators of the Lie group, which are non-commuting via the closure relation

$$\left[X_l, X_k\right] = c_{lk}^m X_m, \quad l,k,m = 1,2,3,4 \tag{19}$$

where c_{lk}^m are the structure constants. Additional insights into the topic of flows in phase space and their association to Lie groups may be found in [13, 19].

6. Connection to the SM gauge group

Any new framework of ideas built exclusively on scalar field theory, such as the one developed in section 5, cannot be a realistic expansion of QFT and SM. Spin 1 and spin ½ fields and their interactions must be obviously included in the picture and properly accounted for. However, the Landau-Ginzburg theory of critical behavior, used in conjunction with the mapping theorem, enable a reduction in complexity near the infrared limit of field theory and provides a reasonable baseline for model building [1]. Pursuing this line of inquiry, it can be shown that the repetitive structure of the particle masses and flavors arises from the *universal period-doubling route to chaos in nonlinear dynamics* [1-3, 14]. Close to the electroweak scale, SM behaves like a tightly constrained *multifractal set*, with field and coupling components acting as generators of the set.

These observations suggest that it makes sense to generalize (14) to (19) to $p > 4$ and arrive at a framework that can accommodate Lie groups of dimensionality 1, 2 and 3. The aim of this section is to explore this scenario.

We begin by extending (14) to

$$x_i' = f_i(x_1, x_2, ..., x_p; \lambda_1, \lambda_2, ..., \lambda_p), \quad i = 1,2,...,p, \quad p > 4 \tag{20}$$

and recalling that elements of the Lie group can be written as

$$E(\theta_1,\theta_2,....,\theta_p)=\exp(\sum_{j=1}^{p} i\,\theta_j X_j) \tag{21}$$

where θ_j are continuous parameters and X_j the group generators [12]. In particular, the group *SU(2)* is the set of all two-dimensional, complex unitary matrices with unit determinant. The group elements are defined by three generators and three parameters,

$$E(\theta_1,\theta_2,\theta_3)=\exp(-i\,\theta_j\,X_j)\,,\quad j=1,2,3 \tag{22}$$

where the summation convention has been applied. The generators of *SU(2)* form a set of linearly independent, traceless 2 x 2 Hermitian matrices

$$X_1=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\qquad X_2=\frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\qquad X_3=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{23}$$

Likewise, the group $SU(3)$ is the set of all three dimensional, complex unitary matrices having unit determinant. The group is defined by eight parameters and generators

$$E(\theta_1,\theta_2,...,\theta_8)=\exp(-i\,\theta_j\,X_j)\,,\quad j=1,2,...,8 \tag{24}$$

The generators of $SU(3)$ form a set of eight linearly independent and traceless Hermitian matrices and are a generalization of (23).

The onset of gauge symmetries is reflected in the covariant derivative operator D^{μ} entering the SM Lagrangian. This correspondence can be presented as

$$U(1):\quad D^{\mu}=\partial^{\mu}-ig_1\frac{Y}{2}B^{\mu}\,,\quad g_1Y\Leftrightarrow X_1\,,\quad j=1$$

$$SU(2): \quad D^{\mu} = \partial^{\mu} - ig_2 \frac{\tau_j \cdot W_j^{\mu}}{2}, \quad g_2 \tau_j \Leftrightarrow X_j, \quad j = 1,2,3 \tag{25}$$

$$SU(3): \quad D^{\mu} = \partial^{\mu} - ig_3 \frac{\omega_j}{2} \cdot G_j^{\mu}, \quad g_3 \omega_j \Leftrightarrow X_j, \quad j = 1,2,...,8$$

Here, Y, τ_j, ω_j stand for the generators of the $U(1)$, $SU(2)$ and $SU(3)$ groups, B^{μ}, W_j^{μ} and G_j^{μ} are the gauge bosons of the electroweak model and quantum chromodynamics (QCD), coupled to their respective generators via g_1, g_2 and g_3. Comparing (18) to (25) hints that the *three gauge symmetries of the SM derive from the scaling flows (20)*. The multifractal content of the SM, derived in [1, 3, 20] from the Feigenbaum scenario, is organized around (25). This is the main result of our work.

7. <u>**Connection to the discrete symmetries of the SM**</u>

To simplify the ensuing derivation, we now return to scalar field theory (relations (14) to (19)) and take the fastest running scale to be λ_4. In addition, we assume that, near the equilibrium regime $\lambda_4 \to 1$, the effective action retains its dependence on currents, fields and generating functional as in

$$\Gamma = \Gamma(\lambda_4, J, \varphi, W[J]), \quad \lambda_4 > 1 \tag{26}$$

with the provision that the Legendre transform (11) holds exactly at $\lambda_4 = 1$. Demanding that the effective action approaches scale invariance in this limit amounts to

$$\frac{d}{d\lambda_4}\Gamma = (\beta_1 + \beta_2 \frac{\partial}{\partial J} + \beta_3 \frac{\partial}{\partial \varphi} + \beta_4 \frac{\partial}{\partial W})\Gamma = 0, \quad \lambda_4 \to 1 \tag{27}$$

where

$$\beta_1(\lambda_4) = \frac{\partial \Gamma}{\partial \lambda_4}, \quad \beta_4(\lambda_4) = \frac{\partial W}{\partial \lambda_4} \tag{28a}$$

$$\beta_2(\lambda_4) = \frac{\partial J}{\partial \lambda_4}, \quad \beta_3(\lambda_4) = \frac{\partial \varphi}{\partial \lambda_4} \tag{28b}$$

To maintain internal consistency of the model defined by (26), in the following we demand that, regardless of λ_4, the effective action stays *strictly scale invariant* as the scaling flow approaches the equilibrium point, i.e.

$$\boxed{\frac{d\Gamma}{d\lambda_4} = \frac{\partial \Gamma}{\partial \lambda_4} = 0, \quad \lambda_4 \to 1} \tag{29a}$$

or,

$$(\beta_2 \frac{\partial}{\partial J} + \beta_3 \frac{\partial}{\partial \varphi} + \beta_4 \frac{\partial}{\partial W})\Gamma = 0 \tag{29b}$$

The number of independent beta-functions describing the field theory can be reduced from four to three on account of the Legendre transform (11). Further demanding that the remaining $\beta_{2,3,4}$ flows stay *nearly insensitive* to λ_4 as $\lambda_4 \to 1$ can be expressed generically as [15, 16, Appendix A]

$$y(\mu) \approx \frac{1}{a} y(\lambda_4 \mu) + y_0(\mu) \tag{30}$$

Here, the observable vector is $y=(J,\varphi,W[J])$, $\mu=\lambda_4\mu_{ref}$ is the running energy scale associated with $\Gamma[\varphi]$, μ_0 is an initial energy scale and a a scaling factor. It is helpful to work with normalized quantities, hence we rewrite (30) as

$$y(z)\approx\frac{1}{a}y(\lambda_4 z)+y_0(z)\,,\qquad z=\frac{\mu}{\mu_0} \tag{31}$$

For the specific choice $y_0(z)=0$, (31) has a power-law solution of the form

$$y(z)\approx Cz^s\,,\qquad s=-\frac{\log(a^{-1})}{\log\lambda_4} \tag{32}$$

It is seen that $\frac{y(\lambda_4 z)}{y(z)}\approx\lambda_4^s$, which means that the relative value of y is nearly independent of the normalized energy scale z. This feature is an essential attribute that links power laws to *self-similarity of multifractal structures* and the onset of critical behavior [8, 15]. Inserting (32) into (31) yields the condition

$$1\approx\frac{\lambda_4^s}{a}\Rightarrow\exp(i2\pi n)\approx\frac{\lambda_4^s}{a}\,,\qquad n=0,\pm1,\pm2,\ldots \tag{33}$$

from which we obtain

$$s\approx-\frac{\log(a^{-1})}{\log\lambda_4}+i\frac{2\pi n}{\log\lambda_4} \tag{34}$$

The case $n=0$ corresponds to the traditional continuous scale invariance encountered in critical phenomena. For $n\neq0$, the solution (34) is complex and it relates to *discrete scale invariance* (DSI) with the *preferred scale* λ_4 [16]. The rationale for the emergence of DSI lies in the linearization of the RG equation describing the evolution of the observable y near the fixed point. Typically, DSI exhibits *lacunarity*, in the sense that it leads to a geometric series of preferred scales $\{\lambda_i\}$ [16]. Appendix A outlines the justification for DSI in terms of the Mellin transform.

It is apparent that a preferred energy scale $\mu_4 = \lambda_4 \mu_{ref}$ automatically induces a non-vanishing cutoff in the spacetime domain, $\Delta x = O(\mu_4^{-1})$. This cutoff leads to a violation of Lorentz invariance if the resolution of spacetime measurements is on the order of Δx. The situation replicates the way discrete spacetime near the Planck scale breaks relativistic invariance in Loop Quantum Gravity [17]. If λ_4 does not fall too far from 1 and if the reference scale is substantially closer to the electroweak scale, $\mu_{ref} = O(M_{EW})$, the onset of Δx necessarily breaks the *translation and reflection symmetries* of spacetime coordinates as the scaling flow converges to the electroweak scale M_{EW}.

Hence, we arrive at a couple of new conclusions:

1. DSI offers a plausible explanation of *discrete symmetry breaking* in particle reactions involving the electroweak interaction.
2. While partial reflection symmetries P and T are broken because of DSI, the overall *Lorentz and CPT symmetries must stay unbroken*. This conclusion can be traced back to the constraint (29), which forces the effective action to be insensitive to any choice of λ_4.

8. Discussion

Our tentative findings may be summarized as follows:

1. The SM gauge group *SU(3) x SU(2) x U(1)* arises from the infrared regime of scaling flows (20).
2. DSI generates (at least) one preferred energy scale near the equilibrium point of (20) and is the root cause of discrete symmetry breaking in the SM.

There is a host of open questions requiring further scrutiny and clarifications. Here are some of them:

a. The amplitude of log-periodic scale corrections is assumed to decay exponentially fast as a function of harmonics [16]. Does this remain true for the onset of DSI in the SM?

b. Does the lacunarity property of DSI imply that there are other scales beside the electroweak scale that are bound to show up below M_{EW} or be entirely suppressed?

c. Are these additional scales related to the full spectrum of SM particle masses, gauge charges and flavors? Note that this question arises because log-periodic corrections are consistent with the Feigenbaum scenario of bifurcations in nonlinear dynamics [14, 16].

d. QCD exhibits *chiral symmetry breaking*, in which the light u, d and s quarks develop condensates in the vacuum and acquire larger "constituent masses". How is the emergence of preferred scales in DSI linked to this mechanism? Is DSI a substitute of *dimensional transmutation* in the SM?

e. What other insights are there, as related to RG and its unexplored implications [19]?

Table 2 below contains a condensed description of continuous scaling flows and their associated discrete symmetries in multifractals and QFT (SM).

Continuous	$x'_{\lambda_4} = f(x,\lambda_4);\ x=(\alpha, q, \tau(q), f(\alpha))$	$x'_{\lambda_4} = f(x,\lambda_4);\ x=(J,\varphi, W[J], \Gamma[\varphi])$
Discrete	$\lambda_4 \to \{\lambda_i\},\ i=1,2,\ldots$	C, P, T and CPT

Tab. 2: Scaling flows and discrete symmetries in multifractals and QFT(SM).

<u>Appendix </u>A

Following [18], let $y(z)$ denote a generic field dependent on the set of variables indexed by z. Renormalization of $y(z)$ implies the existence of an equation of the form

$$y(z) = \mathrm{T}\{y(z)\} + u(z) \tag{A1}$$

in which T is a linear renormalization operator and $u(z)$ a smooth field. The fixed point solution of (A1) after n operator iterations is denoted by $y^*(z)$ and satisfies

$$y(z) = \sum_{n=1}^{\infty} \mathrm{T}^n\{u(z)\} + y^*(z) \tag{A2}$$

It can be shown that the solution of (A2) displays DSI. The proof is based on the Mellin transform, which is used in the study of functions exhibiting *scaling symmetry*. The Mellin transform of (A1) reads

$$Y(s) \equiv \int_0^{\infty} z^{s-1} y(z)\,dz \tag{A3}$$

where the variable s has the meaning of a relative scale ($s = z/z'$). Close to a fixed point z_0 of the renormalization sequence (A2), the operator T is well approximated by

$$(\mathrm{T}y)(z) \approx \frac{1}{a} y(z_0 + \lambda_0(z - z_0)) \tag{A4}$$

Assuming $z_0 = 0$ for simplicity, the Mellin transform (A3) takes the form

$$Y(s) = \frac{1}{1 - a^{-1}\lambda_0^s} U(s) + Y^*(s) \tag{A5}$$

The first term of (A5) presents a series of poles in Mellin scales for integer n

$$s_n = -\frac{\log(a^{-1})}{\log \lambda_0} + i\frac{2\pi n}{\log \lambda_0} \tag{A6}$$

indicating the onset of the *preferred scale* λ_0. The behavior of $y(z)$ is determined by the complex poles of (A5), which have been identified with the *complex valued dimensions of fractal structures* [15]. It is instructive to note that the spectrum of the Mellin transform is singular, the same way the Fourier spectrum of periodic signals is also singular.

<u>Appendix B</u>

The object of this Appendix section is to delve into the connection between multifractal geometry of the SM and *stochastic quantization*, the latter being founded on the analogy between Euclidean QFT and equilibrium statistical mechanics [4-5]. Stochastic quantization identifies the Euclidean path integral measure of QFT,

$\exp\{-S_E\}/\mathrm{D}\varphi\exp\{-S_E\}$, (with $\hbar = 1$) with the stationary distribution of a stochastic process. This interpretation implies that Euclidean Green functions of QFT are indistinguishable from to the correlation functions of equilibrium statistical mechanics (see B4 below).

The key premise of stochastic quantization is that the fields $\varphi(x)$ are supplemented with an additional coordinate called "*fictitious*" time τ. Fields become coupled to a fictitious heat reservoir which relaxes to thermal equilibrium as $\tau \to \infty$. In this picture, the fictitious time evolution of $\varphi(x)$ in four-dimensional Euclidean space resembles a *continuous random walk*. It is described by a stochastic differential equation of the Langevin or Fokker-Planck type. For example,

$$\frac{\partial\varphi(x,\tau)}{\partial\tau} = -\frac{\partial S_E}{\partial\varphi(x,\tau)} + \eta(x,\tau) \tag{B1}$$

Here, S_E is the Euclidean action of the system, obtained by integration over τ

$$S_E = \int d\tau\, d^4x\, L(\varphi(x,\tau), \frac{\partial\varphi(x,\tau)}{\partial x}) \tag{B2}$$

and $\eta(x,\tau)$ stands for delta-correlated Gaussian noise. In the equilibrium limit $\tau \to \infty$, equal time correlation functions of the fields are shown to be identical to the corresponding quantum Green functions, i.e.

$$\lim_{\tau\to\infty}\langle\varphi(x_1,\tau)...\varphi(x_k,\tau)\rangle = \langle\varphi(x_1)...\varphi(x_k)\rangle \tag{B3}$$

where

$$\langle \varphi(x_1)...\varphi(x_k) \rangle = \frac{\int D\varphi \exp(-S_E)\varphi(x_1)...\varphi(x_k)}{\int D\varphi \exp(-S_E)} \tag{B4}$$

It was shown in [1, 2] that, near the electroweak scale M_{EW}, the spectrum of particle masses m_i entering the SM satisfies the closure constraint

$$\sum_{i=1}^{16} r_i^2 = \sum_{i=1}^{16} (\frac{m_i}{M_{EW}})^2 = 1 \tag{B5}$$

Since (B5) reflects a typical relationship in the theory of multifractal sets, it allows for a direct connection between multifractal geometry of the SM and stochastic quantization. To this end, let us start from the reciprocal of the evolution parameters defined in section four and (16). It reads

$$\lambda_i^{-1} = 1 - \varepsilon_i = 1 - (\frac{m_i}{M_{EW}})^2 (\frac{M_{EW}}{\Lambda_{UV}})^2 = 1 - r_i^2 \varepsilon_0 = O(1) \,, \quad \varepsilon_i = r_i^2 \varepsilon_0 = O(4 - D) << 1 \tag{B6}$$

in which ε_0 represents an arbitrarily small deviation from the four-dimensionality of classical spacetime and Λ_{UV} is the high-energy scale [1, 2]. The fictitious time of stochastic quantization can be then interpreted as $\tau = \langle \tau_i \rangle$, with $\tau_i = O(\varepsilon_i^{-1})$. Conventional formulation of QFT is recovered in the deep infrared limit $\tau = \langle \tau_i \rangle \to \infty$, $\varepsilon_i \to 0$, where spacetime dimensionality settles at $D = 4$.

Abbreviations list

SM = Standard Model of High-Energy Physics

QFT = Quantum Field Theory

RG = Renormalization Group

DSI = Discrete Scale Invariance

EW = Electroweak

QCD = Quantum Chromodynamics

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