TIME SYMMETRY BREAKING IN PRE-BIG BANG VACUUM STATE

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The vacuum catastrophe refers to the discrepancy between the predicted and detected values of repulsive vacuum energy. To tackle the issue, we start from a model of inflation in a time-symmetric quantistic vacuum, which, due to the Mandelstamm-Tamm inequality, leads to a trapping of a part of the original vacuum during the formation of our closed Universe. We show how the energy of our cosmos is stored in two different compartments, the microscopic quantistic vacuum and the macroscopic detectable cosmos. If we take into account, at the time of the Big Bang, the occurrence of time symmetry breaking into our Universe, we achieve a macroscopic energetic system governed by the second law of thermodynamics, where time acts like a continuous gauge field able to progressively increase the entropy. Our framework makes it possible to assess, among other issues, the vacuum catastrophe, to calculate the minimum three-dimensional volume of the pre-Big Bang vacuum and to make previsions for the future evolution of the two energetic compartments.

Our Universe started with the Big Bang, expanding from a very high density and high temperature state (Penrose, 2011). At the age of 10^{-43} seconds, the Universe was very hot (10^{34} K). By then, the temperature halved every double expansion. The Universe underwent an inflationary expansion at 10^{-35} sec, which is the current explanation for cosmic features such as isotropicity, homogeneity, symmetry and zero curvature. Another gentler inflationary period started approximately 4.5 billion years ago, due to the prevalence of the anti-gravitational vacuum energy (*e.g.*, the repulsive dark energy corresponding to the cosmological constant), that overtook the decrease of matter and radiation due to cosmic dilation (Ellwanger, 2012). The *vacuum catastrophe* refers to the discrepancy between the predicted and real values of the repulsive vacuum energy. Currently, 13.79 billions years after its birth, our Universe is still accelerating, slowly proceeding towards thermal death (Bars and Terning, 2009).
Our Universe displays four known dimensions, three temporal and one spatial. Here we focus on the time. It has been recently suggested that time exists just for observers inside the universe, because it is an emergent phenomenon arising from quantum entanglement (Moreva et al., 2013). Any god-like observer outside the cosmos sees a static, unchanging Universe, just as the Wheeler-DeWitt equations predict. In such a vein, it is noteworthy that time plays a role neither in such equations, nor in the formulation of the entangled states (Peters and Tozzi, 2016). Here we propose a time symmetry-based mechanism, which elucidates how and why, during the Big Bang, part of the original pre-Big Bang vacuum was sequestrated into our time-asymmetrical Universe. The mechanism gave rise to two different types of energy in our cosmos: a microscopic one acting at quantum levels, and a macroscopic one acting at relativistic and non-relativistic scales. Our framework, which focuses on time symmetry breaking, suggests that the arrow of time works like a gauge field, leading to progressive increase of entropy. Furthermore, our approach makes it possible for us to assess different features of our Universe, from the very first phases of the Big Bang to the final doom, from the required cosmic energetic requirements to the vacuum catastrophe. Also, our model makes predictions about the size of the original vacuum that gave rise to our Universe.

Pre-Big Bang vacuum with T-invariance. An overall description of vacuum energy is given as an energy tensor of the form

$$T_v = \lambda g \text{ for } \lambda = \infty, \text{ or } \lambda = 0, \text{ or } \lambda \approx t_v^{-2},$$

where t_v is Planck time and g is a mathematical tensor quantity (Penrose, 2011). The vacuum catastrophe refers to the huge discrepancy in our Universe between the detected values of vacuum energy, and the theoretical previsions, that suggest a huge value. Indeed, the vacuum energy of free space has been estimated to be 10−9 joules (10−2 ergs), per cubic meter, *e.g.*, $0.683\rho_{crit}$, while theoretical arguments require it to have a much larger value of 10^{113} joules per cubic meter (Hobson et al., 2006). Some Authors believe that our Universe arose from an inflationary event taking place in the quantistic vacuum (Veneziano, 1998). If we assume that an inflative event in the quantistic vacuum gave rise to the Big Bang and our Universe, we are allowed to hypothesize a pre-Big Bang quantistic "original vacuum", e.g., a vacuum state equipped with the massive levels of dark energy predicted by theoreticians. This original vacuum is equipped both with false vacuum, e.g., high-energy fields with vacuum expectation value different from zero, and real vacuum, e.g., low-energy fields that do not allow the production of real particles. The original vacuum is a *n*-manifold, bounded or unbounded. The term manifold is commonly used in this context, referring to a spatial structure with a smooth, rubber sheet geometry. A CPT symmetry occurs in the original vacuum. While real charged particles cannot exist in such an empty vacuum, however charged virtual particles are produced. Every one blinks into existence with its antimatter counterpart, then both quickly annihilate and fall back into the original vacuum. Therefore, the energetic balance in the original vacuum is always zero. In the original quantistic vacuum, a time reversal symmetry (also called time-reversal invariance or *T*-symmetry) occurs, *e.g.*, many variables do not change under a time reversal transformation (right part of **Figure 1**). To make some examples in our observable Universe, time-reversal invariant parameters include: coupling constants except those associated with the weak force, electric potential and field, density of electric charge, electric polarization, some particle features (position in three-space, acceleration, force, energy), microscopic level of physical systems, including microscopic reversibility in physical and chemical kinetics, laws of mechanics and, last but not the least, the same observable Universe in equilibrium states. On the other hand, some variables in our observable Universe change under a time reversal transformation (a feature also called time reversal symmetry violation, T-asymmetry): violations of either C, or, P or T, the time when an event occurs, power, electromagnetic vector potential, magnetic field, density of electric current, magnetization, weak force, some particle features (velocity, linear momentum, angular momentum), macroscopic levels of physical systems and the observable Universe in non-equilibrium states.

In the original vacuum, the Mandelstamm-Tamm inequality for quantistic systems holds (Mandelstam and Tamm, 1945). The inequality assesses the rate evolution of an isolated quantum state, allowing us to calculate how long does it take for a quantum system like the original vacuum to evolve to an orthogonal state (Vaidman 1992). There is a general lower bound for the lifetime of all quantum states in terms of reasonable measures of uncertainty in energy, even if the standard deviation is infinite. Indeed, according to a formulation of the Heisengberg principle, energy and time in the original vacuum can be related by the formula:

$$\Delta E \, \Delta T \geq \frac{h}{4\pi}$$

where ΔE is a non-infinite uncertainty in energy that is independent of time (Uffink, 1993). This means that a significant change in energetic conformation may occur in a very short time interval. The original vacuum is time-invariant, and therefore time can be both positive or negative, or even "frozen" and close to T_0. These short time intervals lead to huge fluctuations in energy. One of such common, almost instantaneous, fluctuations could have produced the inflation that gave rise to our Universe.

Vacuum trapped in an inflationary T-asymmetric Universe. We stated that, in very short times, huge energetic differences may occur in the original vacuum. One (or more) large fluctuations are able to put in motion the well-described process of inflation. In the original vacuum, a high energetic state of false vacuum abruptly falls towards the low energetic basins of real vacuum (left part of **Figure 1**). This process leads to an inflationary mechanism, e.g., the production of a huge amount of anti-gravitational energy that leads to cosmic dilation and the generation of real particles and fields in our Universe. Such inflationary event that occurred in the original vacuum is different from the one that occurred 10^{-35} sec after the birth of Universe. Both might be explained by huge energetic oscillations occurring in almost frozen times. Therefore, our Universe (and probably countless others, equipped with cosmological constants and physical laws different from ours) is born, like a closed bubble arising from the original vacuum. It is noteworthy that, in our framework, the closed inflationary Universe encompasses parts of the original vacuum that was trapped into the bubble during its formation. Indeed, our bubble includes, apart from particles and energy, also a certain amount of vacuum, "stolen" from the original one (right part of **Figure 1**). Such vacuum, that we will term "trapped vacuum", is equipped with a much lower energetic level than the original one. Such a level matches the experimentally detected value of $0.683\rho_{crit}$. Therefore, our closed Universe is equipped with a certain amount of energy, provided by two sources: a) the observable particles, fields and energy produced by the inflationary events occurred in the original vacuum during the

Big Bang; and b) the dark energy enclosed in the trapped vacuum. The energetic behaviors of the open original vacuum and the trapped one are different. They display a completely different evolution (**Figure 1**). The cause of this discrepancy lies in the very structure of the closed bubble and its subtle differences from the original vacuum. While the original vacuum displays T-symmetry, in our Universe the time invariance is lost. Indeed, our time, is not anymore symmetric. The arrow of time becomes positive and cannot be reverted, splitting the past from the future. In such a way, the sudden appearance of a novel single parameter, e.g., a T-symmetry violation in the trapped vacuum, is able to explain why high energy particles, once produced in the T-symmetric original vacuum during the Big Bang, give rise to our Universe, instead of promptly disappear. Time, together with other factors, makes it possible for the bubble to not implode, rather progressively to dilate (stretch) indefinitely.

In our Universe, due to time asymmetry, the two energies of the trapped vacuum and the observable Universe have different developments (**Figure 1**). In the macroscopic Universe, the energy becomes enthalpy and drives the cosmos towards progressively higher levels of entropy. Indeed, our observable Universe is subjected to the second law of thermodynamics. It means that our cosmos' asymmetric time dimension acts like a constraint, allowing just the trajectories that lead towards an increase of entropy in the macroscopic Universe. We will describe this macroscopic process in the following paragraphs. On the other side, in the microscopic compartment of the trapped vacuum, energy oscillations progressively decrease with time passing, due to the quantistic Mandelstamm-Tamm inequality. For the first law of thermodynamics, this vacuum energy is conserved. However, the large energetic oscillations that could lead to false vacuums and subsequent inflationary processes disappear, because the energetic variations are progressively flattened. It is noteworthy that, in very long timescales, the Universe reaches a final state of equilibrium, where time symmetry is restored. Indeed, a system at equilibrium displays time reversal invariance.

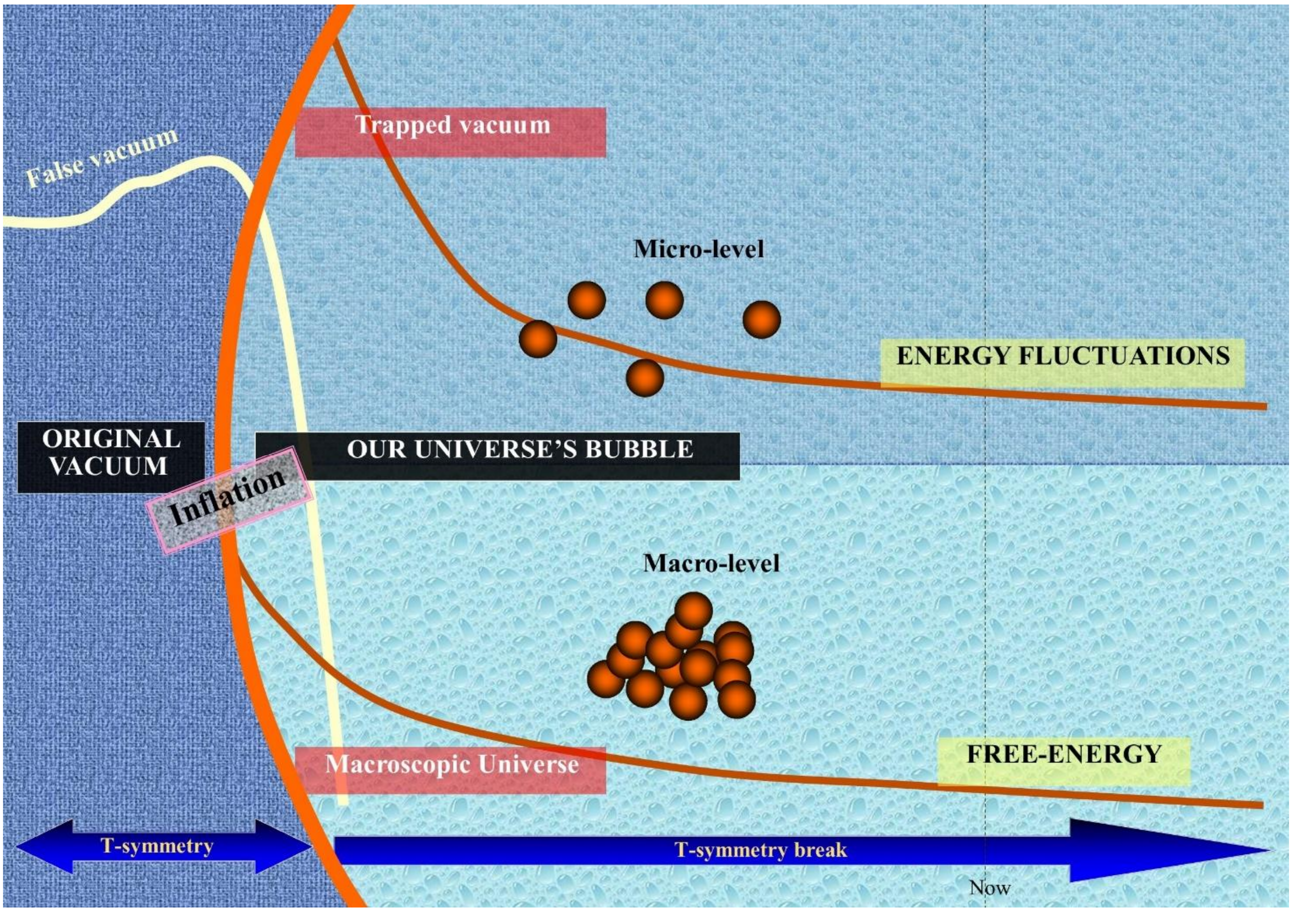

Figure 1. Evolution from the original time symmetric vacuum to our time asymmetric Universe. Our Universe is splitted into two compartments with different energetic features: the trapped vacuum and the macroscopic Universe. See the text for further details.

Energetic evolution of the macroscopic Universe. Macroscopic physical systems like the observable Universe are not just regulated by stochastic variables and random fluctuations[4], but also by constraints given by the arrow of time. The same stands for biological system: to make an example, protein-folding final conformation is dictated by the minimum frustration principle on long evolutionary timescales, which states that proteins' energy decreases more than expected, as they assume conformations progressively more like the native state (Bryngelson and Wolynes, 1987; Ferreiro et al., 2011; Tozzi et al., 2016). Despite the large number of different scenarios, the processes governing time constraints of physical and biological systems may be generalized, taking into account a universal principle: the second law of thermodynamics, which states that "*every process occurring in nature proceeds in the sense in which the sum of the entropies of all bodies taking part in the process is increased" (*Planck's formulation). Therefore, the positive arrow of time observed in the macroscopic Universe (due to the time-reversal symmetry violation) is strictly correlated with the second law of thermodynamics. In other words, the arrow of time acts, in long timescales, like a constraint dictating the evolution of the Universe towards an increase of entropy.

In this framework, it is interesting to see what happens to physical paths when time is kept fixed (**Figure 2A**). In a graph plotting time t on the X-axis and the space x on the Y-axis, take two trajectories which both display a starting position at x_0 and an ending position at x_2. In **Figure 2A**, the black solid curve x(t) stands for the trajectory describing the real path followed by a particle in the macroscopic Universe. This path is dictated by the second law of thermodynamics and the arrow of time. Note that such a real trajectory is constrained along the line of the time: it cannot go back from t1 to t0, e.g., time cannot be reversed (Goldstein, 1980; Torby, 1984). On the contrary, the dotted black curve x^I displays one of the possible virtual trajectories, e.g., a trajectory different from the real one, that might take place when the time is kept fixed and $\delta t = 0$ (Sommerfeld, 1952; Landau and Lifshitz, 1976). Starting from the position x_1 and time t_1, the virtual displacement δx, e.g., from x_1 to the point c, is shown in **Figure 2A** (green arrow). Therefore, in very short timescales, virtual changes in physical/biological trajectories might occur far from the real path. It might be also hypothesized that virtual trajectories stand for singularities and timeless perturbations in our Universe: they could be regarded, for example, as places in which life occurs. Virtual constraints are central, because they allow us to understand the current state of our Universe. Indeed, an observer cannot understand whether a trajectory she is observing lies on a virtual or a real path, therefore she cannot predict the real, final trajectory that will necessarily take place according to the second law of thermodynamics. In very short timescales, the macroscopic paths bear a resemblance to Feynman's diagrams describing subatomic particles' behaviour: the possible trajectories are countless, although just a very few are the most probable in the real life. The next paragraph, just for technical readers, provides a mathematical explanation of virtual constraints.

Technical interlude: virtual constraints. In analytical mechanics, a virtual displacement is an assumed change of system coordinates occurring while time is held constant. It is called "virtual" rather than "real", since no actual displacement takes place without the passage of time. The key concept of virtual constraints is a dynamically imposed outer feedback control, so that the trajectory of a particle or an agent in the system's phase space can be "forced" towards the desired orbits and outputs (Canudas-de-Wit, 2004). Virtual constraints reduce the degrees of freedom, coordinating the evolution of the various links throughout a single variable. A closed-loop mechanism is achieved, wherein dynamic behaviour is fully determined by the evolution of simplest lower-dimension system (Stepp et al., 2010). The resulting system is called a "virtual limit system".

In mathematical terms, we define a set of n – 1 outputs (or constraints):

$$y = \varphi(p,q) = \vec{q} - h(\theta, p) = \vec{q} - h(\theta, p(t)),$$

where *y* and $\varphi(p,q)$ are the outputs or constraints, $\vec{q} \in R^{n-1}$ is a vector describing the actuated coordinates and velocities, *p* is the set of the design parameters, $\theta \in R$ is the unactuated variable, θ(q) is a function of the generalized coordinates of*q*. The latter equation describes the most general condition.

An inner-feedback loop is used to perform output feedback linearization in a local domain, where the matrix is invertible:

$$\psi(q)u = k(q;\dot{q}) + v,$$

where *v* is the outer feedback loop. Note that the equation includes a term $\dot{q}$ which depends on time, where the upper dot stands for the partial time derivative, i.e., $\dot{q} = \frac{\partial q}{\partial t}$ (Canudas-de-Wit et al., 2003).

If an outer feedback loop *v* is designed to zeroing the output *y*, we get a partially linearized system in the form:

$$\ddot{y} = v.$$

Then the full system dynamic is captured by the solutions of:

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0,$$

together with the imposed constraint for mean *q*-value:

$$\overline{q} = h(\theta, p),$$

where $\theta(q)$, $\alpha(\theta)$, $\beta(\theta)$ and $\gamma(\theta)$ are scalar functions depending on the inner feedback loop.

In conclusion, the virtual constraints are forces external to the system's phase space, able to modify an internal trajectory towards the required one. This process allows one to deal with high-dimensional systems with underactuated degree one, by only analyzing this second-order nonlinear equation.
In analytical mechanics the researchers cope with under-actuated Lagrangian systems of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=B(q)u,$$

where q and $\dot{q}$ are vectors of generalized coordinates and velocities, $L(q;\dot{q})$ is a Lagrangian of the system, $B(q)$ is a matrix function of an appropriate dimension, with rank equal to the number of inputs and u is a vector of independent control inputs. The under-actuation means that dim u< dim q, i.e., the number of actuators is less than the number of its degrees of freedom.

Prediction of macroscopic trajectories in long cosmic timescales. Here we, elucidating the close relationships among virtual displacements, probabilities and time, show how we can assess whether observed trajectories are real or virtual and, in case they are virtual, how long does it take to become real. Real displacements are governed by the second law of thermodynamics because in every closed system, either physical or biological, the thermodynamical entropy relentlessly increases from time T_0 to $T_2 = \infty$, until its maximum value. It is however preferable to use the informational entropy, instead of the thermodynamical one. Indeed, the two entropies are linked through the formula:
$S = k\,H$
in which S is the thermodynamical entropy, k is the Bolztmann constant and H is the Shannon informational entropy. The informational entropy, apart from the invaluable advantage of quantifying the macroscopic states without a perfect knowledge of the microscopic ones, is not linked with time. Therefore, Shannon entropy allows us to neglect the parameter time from our system, in order to assess its behavior not taking into account its evolution. However, to evaluate whether the trajectory we observe in different time frames is real or virtual, we need to superimpose the arrow of time on the "classic" two-dimensional plane containing the Shannon curve. Indeed, we are allowed to add a third dimension to the 2-D plot of Shannon entropy (**Figure 2B**). The vector of time ζ lies in a plane forming an angle A with the 2-D plane of Shannon entropy. This procedure allows the observer to evaluate how much time is needed, for the particle she is observing, to reach the real final state. To make an example, in the system Universe, T_0 on the vector of time ζ stands for the state of minimum entropy, e.g., the initial Big Bang, while $T_2 = \infty$ stands for the state of maximum entropy, e.g., the hypothetical final macroscopic state of the Universe. The real trajectory follows the Shannon curve: the highest probabilities can be found at p=0 and 1, while the maximum entropy at p =0.5- Starting from the probability of a virtual constraint observed in a point c, the corresponding point T_1 on the arrow of time can be detected. Therefore, we can calculate how much time is required to the vector of time ζ to reach $T_2 = \infty$, which stands for the Universes's "real" final state at the energetic equilibrium. In order to proceed, there is a still unknown parameter left: the value of the angle A. The latter can be calculated by sketching a differential geometry-based theory (Tozzi and Peters, 2016). We need to ponder the Universe as a system where real displacements, e.g., the real trajectory of particles or events, stand for the continuous, global symmetry. This symmetry stands for the energetic constraints dictated by the second law of thermodynamics, e.g., an energetic gradient flow occurring just in long timescales. In turn, in the very instant in which T is "frozen", fixed and equals to zero, virtual displacements occur. The latter stand for continuous groups of local transformations, able to "break" the above mentioned symmetry. The local loss of symmetry, e.g., a disturbance of the gradient flow, can be restored by introducing a continuous field, the time, able to re-establish the gradient flow. The time, in such a framework, stands for a field which is continuous, by the point of view of an inertial observer. There are many possible ways to deal with real and virtual displacements in a differential geometric sense, for example by analysing them in terms of sections of fibre bundles, jet manifolds and Ehresmann connections (Ehresmann, 1950; Abraham and Marsden, 1978; Lang, 1995; Kolar and Michor, 1993). One of the procedures is described in Sengupta et al. (2016). In sum, we can make accurate predictions about the time required for a trajectory in order to converge towards the most probable pattern, e.g., to increase its thermodynamical entropy. Our scheme resembles a physical gauge theory, retaining its tenets (Higgs, 1964; DeWitt, 1967;'t Hooft, 1971; Zeidler, 2011):

a) The system is equipped with a continuous, preserved "global" symmetry (and a corresponding Lagrangian).
b) The system displays a continuous group of "local" transformations, equipped with a Lie group.
c) The Lagrangian is kept invariant under such local transformations by a "gauge field", i.e. a continuous force acting on the system.

However, our framework draws an important distinction from a classical gauge theory. Our concept of Lagrangian is slightly different: instead of referring to the principle of least action and the "preservation" of a physical quantity as usual in gauge theories and in Noether theorem, our Lagrangian refers to the "dissipation" of a physical quantity through a gradient flow. In other words, we provide a framework in which a Universe trajectory can be predicted in long timescales, by introducing the arrow of time in guise of a gauge field that keeps invariant the second law of thermodynamics. Such gauge field counterbalances the effects of local energetic peaks caused by the virtual transformations that might occur in very short timescales.

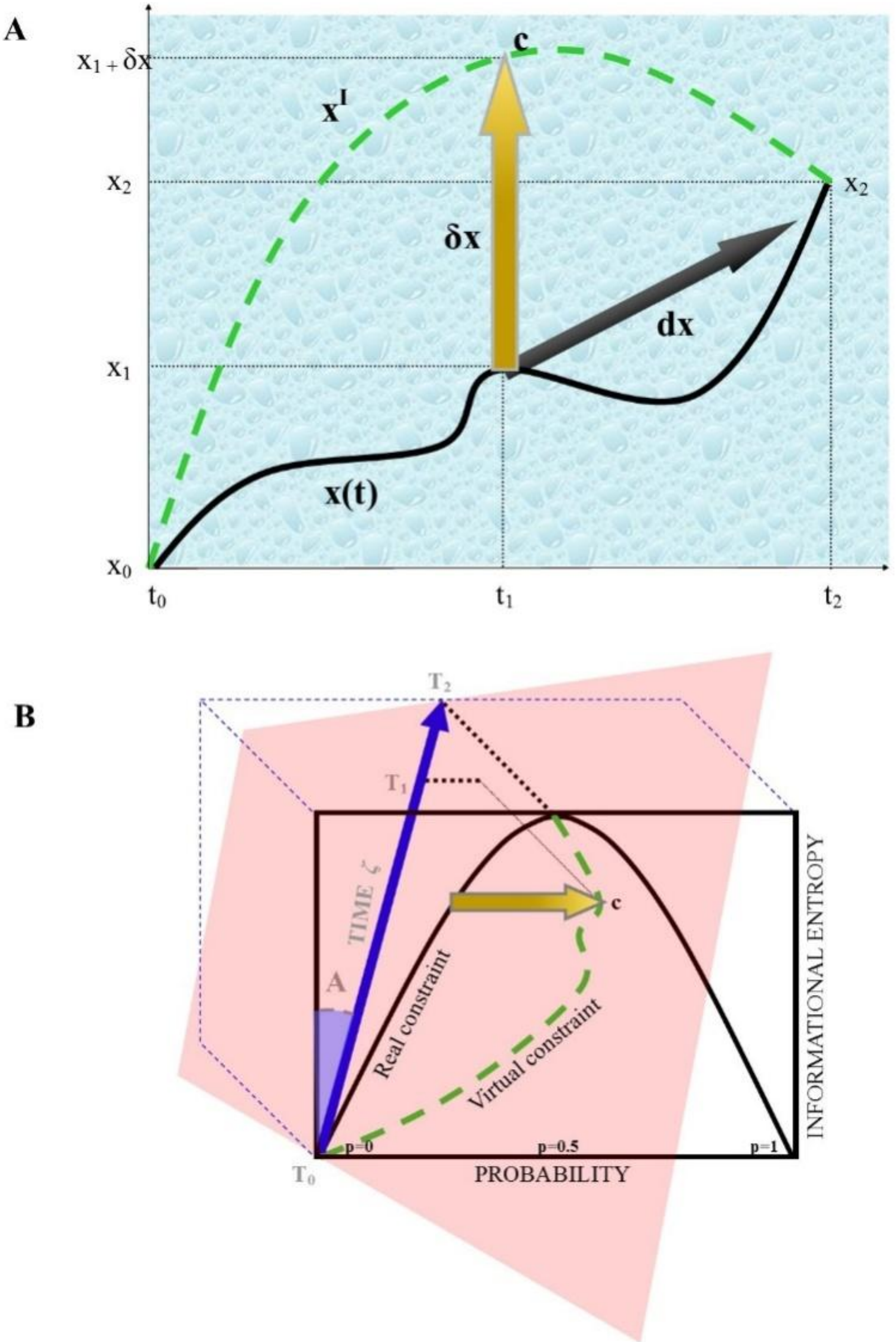

Figure 2A. Graph of the macroscopic observable Universe plotting time t on the X-axis and the space x on the Y-axis. The regular displacement dx is a vector pointing in the direction of the motion (black arrow), which arises from differentiating with respect to time parameter along the path of the motion. In contrast, the virtual displacement δx is a tangent vector to the constraining manifold at a fixed time, because it arises from the differentiation with respect to the enumerating paths of the motion, varied in a manner consistent with the constraints. See text for further details.

Figure 2B. Informational entropy is plotted as a function of the random variable p, in the case of two possibilities with probabilities p and (1-p). The solid line black stands for the Shannon entropy (under ergodic conditions). The values p=0 and p=1 stand for the minimum entropy, p= 0.5 for the maximum entropy. The arrow of time ζ (blue solid line) lies on a third coordinate of the phase space and forms the angle A. Due to the energetic gradient flows dictated by the second law of thermodynamics, time, in its real route from T_0 to $T_2 = \infty$, must be correlated with an increase in informational entropy. Given a virtual displacement c on the virtual trajectory (dotted black line), the corresponding value of T_1 on the arrow of time can be calculated, provided the value of the angle A is known.

CONCLUSIONS

We described the peculiar features of energy and time in the original vacuum and in our Universe. The latter encompasses trapped vacuum and is equipped with T-symmetry break. Further, we showed that the vector of time may stand for the macroscopic Universe's energetic gradient, locally "broken" by virtual timeless perturbations. Our framework allows us to elucidate controversial issues in cosmology and to make a few empirically testable previsions.

The problem of the vacuum catastrophe is solved straightforwardly: just a small amount of the original vacuum is embedded in our Universe. It means that the trapped vacuum displays the low experimentally detected values of dark energy, while the original vacuum the very high predicted ones. Therefore, we are allowed to calculate the size of the original vacuum that gave rise to our Universe: if 10−9 joules (10−2 ergs) per cubic meter are trapped in the size of our Universe before inflation, e.g., 3 x 10^{-25} cm, it means that the total dark energy of 10^{113} joules per cubic meter is trapped in an original vacuum with spatial dimensions (in cubic meters) of more than 100 orders of magnitude. Such calculation does not take in to account the possible decrease of dark energy occurred in the Universe lifetime until the current age. However, this loss of energy is not likely to happen in the trapped vacuum, because of the first law of thermodynamics.

The concepts of "constraints" and "virtual displacement" from analytical mechanics shed new light on the role of time and timescales in physical systems such as the Universe. Here we proposed a covariant version of a gauge theory, in which the required global symmetry stands for the real constrained trajectories, i.e., the energetic gradient flows dictated by the second law of thermodynamics. The virtual displacements, occurring while time is held constant, stand for the local transformations acting on the Universe. In this framework, time stands for a continuous gauge field. At the end of the Universe, before the thermal death, the quantum fluctuations in the trapped vacuum are close to zero. Indeed, our framework suggests that, with time passing, the vacuum will progressively loose its dark energy oscillations. It means that the spontaneous production of real matter and energy from quantistic vacuum will decrease. The energy located in the false vacuum will not have further possibilities to fluctuate, then it will not be able to give rise to novel inflationary events. Therefore, other inflations will not be possible when our Universe will approximate its end. However, in the very last phases of thermal death, a state of energetic equilibrium will be achieved, both in the trapped vacuum and in the observable Universe. This means that an almost complete T-symmetry could be restored. Some questions remain unanswered: at thermal death, does another Big Bang take place? Or does the trapped vacuum join the original one? Further studies will elucidate such issues.

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