Generalized Modified Emden Equation Mapped into the Linear Harmonic Oscillator Equation

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Abstract

The generalized modified Emden equation also known as the generalized second order Riccati equation, is exactly solved in terms of the periodic solution of the linear harmonic oscillator. The solutions for specific values of parameters are discussed. The conditions for isochronous oscillations are also investigated.

1-Introduction

The modified Emden-type equation

$$\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0, \tag{1}$$

where overdot means differentiation with respect to time, and α and β are arbitrary constants, is one of the most nonlinear dissipative oscillator equations investigated as a physically important dynamical system. This may be attributed to the fact that the modified Emden equation arises in modeling of many physical and engineering nonlinear problems. The modified Emden equation has been studied by several authors following different analytical approaches. In [1], a general solution for arbitrary values of α and β is developed from appropriate canonical transformations. In [2] the equation has been mapped into the Abel equations to secure its exact integrability. The modified Emden equation has also been explored in [3] from the Lagrangian method of constants variation and a factorization technique of differential operators. However, if the preceding equation has been with more or less complexity analytically integrated, its generalized form

$$\ddot{x} + \alpha x^l \dot{x} + \beta x^{2l+1} = 0 \tag{2}$$

where l is an arbitrary parameter, has appeared more hard to be exactly solved for arbitrary values of l, α and β. In [4], an exact expression for the solution of (2) has been formulated from an Hamiltonian point of view, that is through suitable canonical equations of motion. Recently, the present authors [5] by

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mapping the equation (2) into the damped linear harmonic oscillator equation through a nonlocal transformation developed a simple analytical expression for the exact solution to equation (2). But, the equation (2) may also be generalized with a linear forcing function as

$$\ddot{x}+\alpha x^{l}\dot{x}+\beta x^{2l+1}+\omega^{2}x=0 \tag{3}$$

where ω is an arbitrary parameter. In [6], this equation is solved under arbitrary values of l, α, β and ω, and it is shown that (3) may exhibit isochronous oscillations as the linear harmonic oscillator. As well the generalized form

$$\ddot{x}+\left(a_{1}x^{l}+a_{2}\right)\dot{x}+a_{3}x^{2l+1}+a_{4}x^{l+1}+\omega^{2}x=0 \tag{4}$$

has been investigated in [7] through the Prelle-Singer (PS) analytical technique. In this work a more generalization of (4), viz

$$\ddot{x}+(a_{1}x^{l}+a_{2}x^{l+1}+a_{3})\dot{x}+a_{4}x^{2l+1}+\omega^{2}x+a_{5}x^{2l+2}+a_{6}x^{2l+3}+a_{7}x^{l+1}+a_{8}x^{l+2}=0 \tag{5}$$

is considered from the standpoint of exact integrability. The equations (3)-(5) belong to the class of Liénard equations

$$\ddot{x}+P(x)\dot{x}+Q(x)=0 \tag{6}$$

where $P(x)$ and $Q(x)$ are functions of x. If the question of existence of periodic solution of (6) is extensively studied in the literature [8], the problem of finding appropriate explicit periodic and isochronous solutions for physical and engineering calculations, that is in terms of simple elementary functions, remains still an open mathematical investigation field. In this context, the following question is imposed: Can the explicit periodic and isochronous solutions for (5) be formulated in terms of elementary mathematical functions? The pertinence of this question results from the fact that periodic solutions consist of an important feature of dynamical systems. It is easy to note in the literature that a lot of works on the dynamics of physical and engineering systems is devoted to the study of periodic response. Isochronous solutions are particularly interesting for physical and engineering practices since the frequency of oscillations is independent of the amplitude as it is the case of the linear harmonic oscillator solution so that the stability of the system motion is secured. In this work it is assumed that periodic and isochronous oscillations can be analytically computed in terms of elementary functions for the equation (5) when parameters satisfy some restrictive conditions. This analytical prediction has the advantage to enable one to not only better understanding the dynamics of the system under question, but also to controlling this dynamics through system parameters. To that end, the equation (5) is mapped into the linear harmonic oscillator equation through a simple variable transformation (section 2) so that the general solution for the equation (5) may be formulated in terms of periodic solution to the linear harmonic oscillator as a simple analytical expression (section 3). Finally the predicted solutions are discussed (section 4) and conclusions are presented in the last section.

2- Mapping (5) to the linear harmonic oscillator equation

2.1 Ansatz for the variable transformation

Recently, exploring the exact integrability of a class of quadratic Liénard-type differential equations following the three distinct damped dynamical regimes the present author have shown that the variable transformation [9]

$$\frac{\dot{y}}{y} = a\dot{x}g(x) + b\,f(x) \tag{7}$$

where $f(x)$ and $g(x) \neq 0$ are arbitrary functions of x, and a and b are arbitrary parameters, has the ability to map the damped linear harmonic oscillator equation

$$\ddot{y} + \lambda\dot{y} + \omega_0{}^2 y = 0 \tag{8}$$

to the class of mixed Liénard nonlinear dissipative oscillator equation

$$\ddot{x} + \left[\frac{g'(x)}{g(x)} + ag(x)\right]\dot{x}^2 + \left[\frac{b}{a}\frac{f'(x)}{g(x)} + 2bf(x) + \lambda\right]\dot{x} + \frac{b^2}{a}\frac{(f(x))^2}{g(x)} + \frac{\lambda b}{a}\frac{f(x)}{g(x)} + \frac{\omega_0^2}{ag(x)} = 0 \tag{9}$$

and vice versa, the transformation (7) reduces (9) to (8). It is now possible to map the generalized modified Emden equation with forcing term (5) onto the linear harmonic oscillator equation.

2.2 Reduction of (5) to the linear harmonic oscillator equation

To do this reduction, let $g(x) = \dfrac{1}{ax}$, with $a \neq 0$.

Then (9) reduces to

$$\ddot{x}+\left(2b\,f(x)+b\,x\,f'(x)+\lambda\right)\dot{x}+\left(b^2\,f(x)+\lambda b\right)x\,f(x)+{\omega_0}^2\,x=0 \tag{10}$$

The equation (10) for $f(x)=x^l+dx^{l+1}$, becomes

$$\ddot{x}+\left[b(l+2)x^l+bd\,(l+3)x^{l+1}+\lambda\right]\dot{x}+{\omega_0}^2x+b^2x^{2l+1}+2b^2d\,x^{2l+2}+b^2d^2x^{2l+3}+b\lambda x^{l+1}+bd\lambda\,x^{l+2}=0 \tag{11}$$

which takes the form

$$\ddot{x}+\left[b(l+2)x^l+bd(l+3)x^{l+1}\right]\dot{x}+{\omega_0}^2x+b^2x^{2l+1}+2b^2d\,x^{2l+2}+b^2d^2x^{2l+3}=0 \tag{12}$$

for $\lambda=0$.

So the equation (5) according to (7) is mapped to the linear harmonic oscillator equation

$$\ddot{y}+{\omega_0}^2\,y=0 \tag{13}$$

under the conditions that

$$a_1=b(l+2) \tag{14.a}$$

$$a_2=b\,d(l+3) \tag{14.b}$$

$$a_4=b^2 \tag{14.c}$$

$$a_5=2b^2d \tag{14.d}$$

$$a_6=b^2d^2 \tag{14.e}$$

$$a_3=a_7=a_8=0 \tag{14.f}$$

In other words, the equation (11) becomes in terms of a_1, a_2, …. and a_8

$$\ddot{x}+\left[a_1\,x^l+a_2\,x^{l+1}\right]\dot{x}+\frac{{a_1}^2}{(l+2)^2}x^{2l+1}+\frac{2a_1\,a_2}{(l+2)(l+3)}x^{2l+2}+\frac{{a_2}^2}{(l+3)^2}x^{2l+3}+\omega_0x=0 \tag{15}$$

In this perspective the general solution for (15) may exactly be expressed as a function of the periodic solution to (13) in a suitable explicit form.

3. Exact solution of (15) in terms of elementary functions

Let $y=A\sin(\omega_0\,t+\varphi)$, be the solution to (13), where A and φ are arbitrary constants. Then the desired solution $x(t)$ for (15) satisfies the first order non-linear differential equation

$$\dot{x} - \omega_0\, x \cot(\omega_0\, t + \varphi) + \frac{a_1}{(l+2)} x^{l+1} + \frac{a_2}{(l+3)} x^{l+2} = 0 \qquad (16)$$

The exact solution for (16) secures the explicit solution to (15).

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