Qualitative Properties of Nonlinear Oscillations in Hamiltonian Systems Having Exponential-Type Restoring Force

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Abstract

In this paper the qualitative properties of a family of an anharmonic oscillator equations of motion were carried out with phase portraits.

1- Local stability analysis

Consider the following generalized equation of motion of a particle subjected to an exponential-type restoring force.

$$\ddot{x} + \omega_0^2 h(x) \exp\left(2\gamma\varphi(x)\right) = 0 \tag{1}$$

where γ, and ω_0 are arbitrary parameter. $h(x)$ and $\varphi(x)$ are arbitrary functions of x. It is worth nothing that equation (1) is enough powerful to give Duffing –type oscillator equations of higher order terms than three by using a Taylor expansion [1].

Let $h(x) = x$. Then (1) takes the form [2]

$$\ddot{x} + \omega_0^2 x \exp\left(2\gamma\varphi(x)\right) = 0 \tag{2}$$

Equation (2) reduces to the harmonic oscillator equation if the parametric choice $\gamma = 0$. Let us now consider, some specific examples of (2). Let $\varphi(x) = \frac{1}{2}x^2$. Then (2) becomes

$$\ddot{x} + \omega_0^2 x \exp\left(\gamma x^2\right) = 0 \tag{3}$$

By imposing $\dot{x} = y$, the equation (3) can be written in a system of two first order equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega_0^2 x \exp\left(\gamma x^2\right) \end{cases} \tag{4}$$

The system (4) admits as equilibrium point

$$M_0 \begin{cases} x_0 = 0 \\ y_0 = 0 \end{cases}$$

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At present, we propose to investigate the stability of equilibrium point.
The Jacobi matrix

$$J = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\ \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{pmatrix} \qquad (5)$$

gives for the system (4)

$$J = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 e^{\gamma x^2} - 2\gamma\omega_0^2 x^2 e^{\gamma x^2} & 0 \end{pmatrix} \qquad (6)$$

Following the equilibrium point M_0, the Jacobi matrix (6) takes the form

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \qquad (7)$$

The corresponding characteristic equation may be written in the form

$$\lambda^2 - \lambda tr(A) + \det(A) = 0 \qquad (8)$$

so that the eigenvalues are

$$\lambda = \pm i\omega_0 \qquad (9)$$

These eigenvalues are pure imaginary numbers. Thus the fixed point is a centre.

Let $\varphi(x) = x$. Then (2) becomes

$$\ddot{x} + \omega_0^2 x \exp(2\gamma x) = 0 \qquad (10)$$

As previously, by noting $\dot{x} = y$, the equation (10) can be written in a system of two first order equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega_0^2 x \exp(2\gamma x) \end{cases} \qquad (11)$$

The system (11) admits as singular point

$$M_1 \begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases}$$

By using the equation (5), we obtain for the system (11) the following Jacobi matrix

$$J = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 e^{2\gamma x} - 2\gamma\omega_0^2 x e^{2\gamma x} & 0 \end{pmatrix} \qquad (12)$$

Introducing the equilibrium point M_1 into the equation (12), the Jacobi matrix takes the form

$$B = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \qquad (13)$$

The Jacobi matrix (7) and (13) are the same. Consequently, the eigenvalues for (13) are pure imaginary numbers. Thus the equilibrium point M_1 is a centre.

2- Phase portraits

This section aims to show graphically the phase portraits of equation (3) and (10) using Matlab's routine ode 45.

Case 1: $\varphi(x) = \frac{1}{2}x^2$

Figure 1 represents the phase portrait of equation (3) for the parameter values $\gamma = 0.5; \dot{x}(t=0) = v_0 = 0.1; \omega_0 = 0.125$ and $x_0 = 0.5; 1.5; 1.8; 2; 2.5$

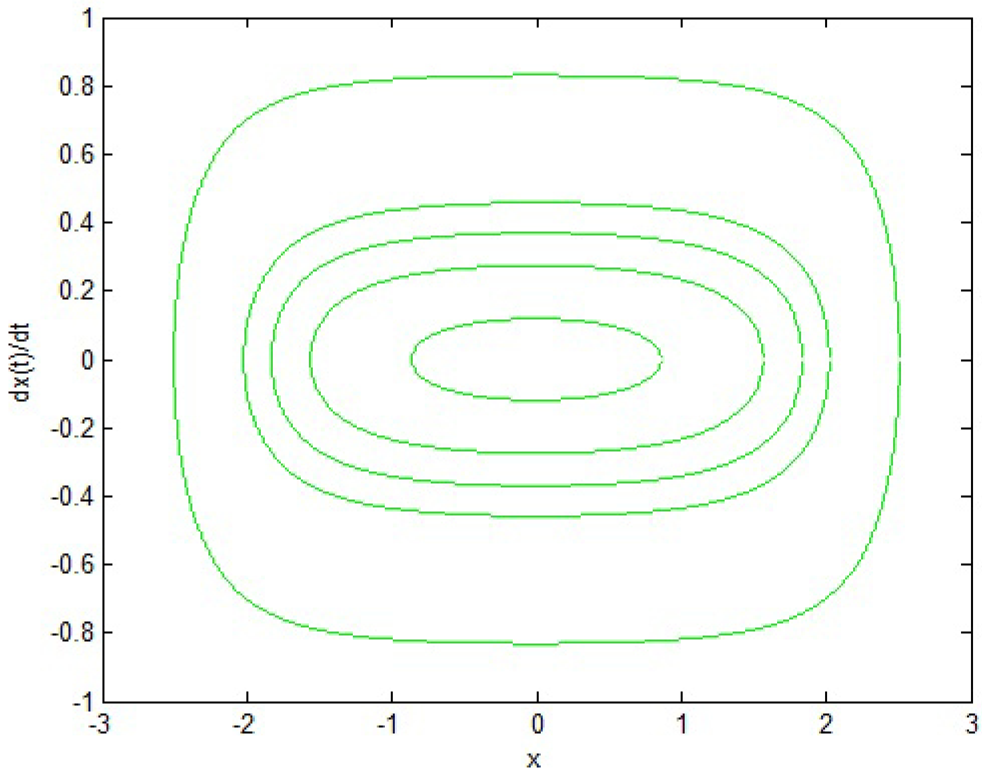

Fig 1: Graphical representation showing the phase portrait of equation (3).

Case 2: $\varphi(x) = x$. The phase portrait has been obtained for $\gamma = 10^{-7}; x_0 = 1.2; \dot{x}(t=0) = v_0 = 10^{-3}$ and $\omega_0 = 0.125$.

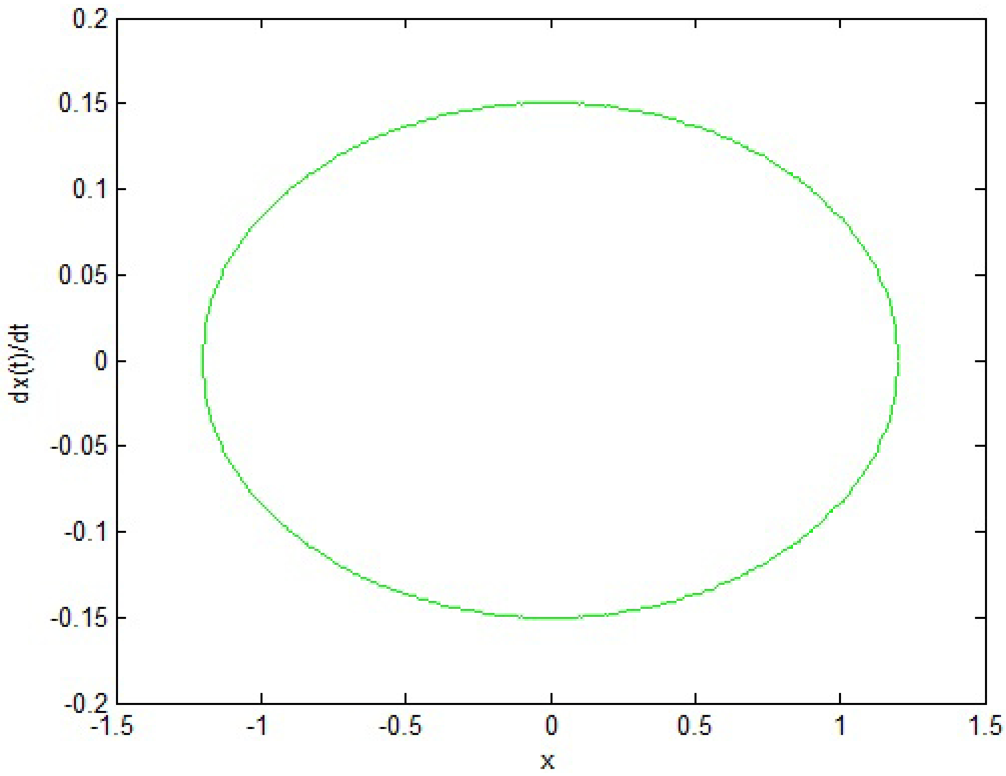

Fig 2: Graphical representation showing the phase portrait of equation (10).

References

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