On Generations in Elementary Particle Physics

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11.30.-j Symmetry and conservation laws

Abstract

Elementary Fermions come in generations, e.g. (electron, muon, tau), neutrino-triplet, etc., which is an experimental fact and sometimes apostrophized as a mystery [1] because a theoretical explanation is missing. That there are more than three generations is considered possible but unlikely. We show that generations follow from group-theoretical arguments and that their number is determined by the number of space dimensions.

Group-Theoretical Remarks

The Lorentz group SO(3,1) conserves the 4-dimensional indefinite (3,1) metric and can be related to the group SO(4), which conserves the positive-definite 4-dimensional metric, by replacing the time coordinate, *t,* with *it*. Correspondingly, the representations of the Lorentz group are conventionally derived from the ones of the universal cover group of SO(4), which is isomorphic to the direct product of SU(2)xSU(2) [2], through Weyl's 'unitary trick' [3]. This trick yields an equivalence of the representations of a semisimple group G, whose Lie algebra has a Cartan decomposition *k*+*p*, with the unitary group U belonging to the Lie algebra *k*+*ip*. In our case *k* corresponds to the three infinitesimal rotations of 3-space and *p* to the infinitesimal boosts along the coordinate axes. Under the equivalence these boosts transform to the three infinitesimal rotations about the space axes and the imaginary time axis.

For the trick to hold, the group U = SU(2)xSU(2) must be simply connected, which it is, and G must be connected. This latter condition is not fulfilled, since SO(3,1) decays into a connected subgroup of orthochronous transformations, which preserve the direction of time, and a second, disconnected part which is obtained from this subgroup by multiplication with the total inversion, called *PT*, with space inversion (parity) *P* and time inversion *T*. In SO(4) the corresponding transformation is continuously reached from the identity transformation.

If the conditions for the unitary trick would be fulfilled the finite dimensional representations of SO(3,1) would correspond to the (*k*,*l*) representations of SO(4) as is commonly assumed. Here *k* and *l* can take on non-negative half-integer values and label the well known representations of the two SU(2) factors [2]. In the following we will illustrate how this picture must be modified.

We call the infinitesimal rotations of SO(4)

$$S_{mn} = -S_{nm} \quad , \qquad (1)$$

with m, n = 0,...,3; $m<n$.

The generators of the two SU(2) factors are obtained as [2]

$$T_j^{1,2} = \frac{1}{2}(S_{mn} \pm S_{0,j}) \quad , \ (j, m, n) = (1, 2, 3) \text{ and cyclic.} \tag{2}$$

The two *S*-operators on the right hand side commute.

As an example we consider a (½, 0) representation which acts on a two-dimensional space, which is conventionally taken as representation of elementary fermions [4]. Transformations in the T^2- generated SU(2) do not affect the representation, but a φ-'rotation' generated by T^1 leads to the well known spin-½ representation, in exponentiated form:

$$D^j(\phi) = e^{\frac{\phi}{2}(S_{mn}+S_{0j})} = e^{\frac{\phi}{2}S_{mn}} e^{\frac{\phi}{2}S_{0j}} = e^{\frac{\phi}{4i}P_{mn}} e^{\frac{\phi}{4i}P_{0j}} = (\cos(\frac{\phi}{4}) - i\sin(\frac{\phi}{4})P_{mn})(\cos(\frac{\phi}{4}) - i\sin(\frac{\phi}{4})P_{0j}) \quad , \tag{3}$$

where we have used the fact that S_{mn} and S_{0j} commute and where we have replaced the *S*-generators by the Pauli matrices with their property $P^2 = 1$. The double indices (m, n) and $(0, j)$ in these Pauli matrices have to be identified with j ($j, m, n = 1, 2, 3$ with cyclic permutation) as is understood from the T^1 generators (2). A rotation by 8π leads back to the identity, in accordance with the fact that SO(4) is quadruply covered by SU(2)xSU(2). The accidental 4π-periodicity is due to independence on T^2.

At this point we can apply the 'unitary trick' to proceed to the corresponding representation of SO(3,1). This amounts to replacing S_{0j} by iS_{0j} [3] which leads to

$$D^j(\phi) = (\cos(\frac{\phi}{4}) - i\sin(\frac{\phi}{4})P_{mn})(\cosh(\frac{\phi}{4}) + \sinh(\frac{\phi}{4})P_{0j}) \quad . \tag{4}$$

where the index 0 now labels the time. The important difference to (3) is, that the periodicity has disappeared. Consequently, there must be three representations, corresponding to j = 1, 2, 3, (or x, y, z) such that a boost in an arbitrary direction will be represented in the subspace provided by the combination of the basis spaces, corresponding to the boost direction. Therefore, for SO(3,1) there are no single spin-½ representations but they must occur in families of three, owing to the three space dimensions, precisely as the phenomenology of the standard model finds.

Dynamical aspects of these generation-triplets must follow from a treatment of the appropriate field equations as outlined in references [5-7].

References

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