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Study of Some Improved Ratio Type Estimators Under Second Order Approximation

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Abstract

Chakrabarty (1979), Khoshnevisan et al. (2007), Sahai and Ray (1980), Ismail et al. (2011) and Solanki et al. (2012) proposed estimators for estimating population mean $\overline{Y}$. Up to the first order of approximation and under optimum conditions, the minimum mean squared error (MSE) of all the above estimators is equal to the MSE of the regression estimator. In this paper, we have tried to found out the second order biases and mean square errors of these estimators using information on auxiliary variable based on simple random sampling. Finally, we have compared the performance of these estimators with some numerical illustration.

Keywords: Simple Random Sampling, population mean, study variable, auxiliary variable, exponential ratio type estimator, exponential product estimator, Bias and MSE.

1. Introduction

Let U= (U_1 ,U_2 , U_3, …..,U_i, ….U_N) denotes a finite population of distinct and identifiable units. For estimating the population mean $\overline{Y}$ of a study variable Y, let us consider X be the auxiliary variable that are correlated with study variable Y, taking the corresponding values of the units. Let a sample of size n be drawn from this population using simple random sampling without replacement (SRSWOR) and y_i , x_i (i=1,2,…..n) are the values of the study variable and auxiliary variable respectively for the i-th unit of the sample.

In sampling theory the use of suitable auxiliary information results in considerable reduction in MSE of the ratio estimators. Many authors suggested estimators using some known population parameters of an auxiliary variable. Upadhyaya and Singh (1999), Singh and Tailor (2003), Kadilar and Cingi (2006), Khoshnevisan et al. (2007), Singh et al. (2007), Singh et al. (2008) and Singh and Kumar (2011) suggested estimators in simple random sampling. Most of the authors discussed the properties of estimators along with their first order bias and MSE. Hossain et al. (2006) studied some estimators in second order approximation. In this study we have studied properties of some estimators under second order of approximation.

2. Some Estimators in Simple Random Sampling

For estimating the population mean $\overline{Y}$ of Y, Chakrabarty (1979) proposed ratio type estimator -

$$t_1 = (1-\alpha)\overline{y} + \alpha\overline{y}\frac{\overline{X}}{\overline{x}} \tag{2.1}$$

where $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Khoshnevisan et al. (2007) ratio type estimator is given by

$$t_2 = \overline{y}\left[\frac{\overline{X}}{\beta\overline{x} + (1-\beta)\overline{X}}\right]^g \tag{2.2}$$

where β and g are constants.

Sahai and Ray (1980) proposed an estimator t_3 as

$$t_3 = \overline{y}\left[2 - \left\{\frac{\overline{x}}{\overline{X}}\right\}^W\right] \tag{2.3}$$

Ismail et al. (2011) proposed and estimator t4 for estimating the population mean $\overline{Y}$ of Y as

$$t_4 = \bar{y}\left[\frac{\bar{x} + a(\bar{X} - \bar{x})}{\bar{x} + b(\bar{X} - \bar{x})}\right]^p \tag{2.4}$$

where p, a and b are constant.

Also, for estimating the population mean $\bar{Y}$ of Y, Solanki et al. (2012) proposed an estimator t_5 as

$$t_5 = \bar{y}\left[2 - \left\{\left(\frac{\bar{x}}{\bar{X}}\right)^{\lambda} \exp\frac{\delta(\bar{x} - \bar{X})}{(\bar{x} + \bar{X})}\right\}\right] \tag{2.5}$$

where λ and δ are constants, suitably chosen by minimizing mean square error of the estimator t_5 .

3. Notations used

Let us define, $e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}$ and $e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}$, then $E(e_0) = E(e_1) = 0$.

For obtaining the bias and MSE the following lemmas will be used:

Lemma 3.1

(i) $V(e_0) = E\{(e_0)^2\} = \frac{N-n}{N-1}\frac{1}{n}C_{02} = L_1C_{02}$

(ii) $V(e_1) = E\{(e_1)^2\} = \frac{N-n}{N-1}\frac{1}{n}C_{20} = L_1C_{20}$

(iii) $COV(e_0, e_1) = E\{(e_0e_1)\} = \frac{N-n}{N-1}\frac{1}{n}C_{11} = L_1C_{11}$

Lemma 3.2

(iv) $E\{(e_1{}^2e_0)\} = \frac{(N-n)}{(N-1)}\frac{(N-2n)}{(N-2)}\frac{1}{n^2}C_{21} = L_2C_{21}$

(v) $E\{(e_1{}^3)\} = \frac{(N-n)}{(N-1)}\frac{(N-2n)}{(N-2)}\frac{1}{n^2}C_{30} = L_2C_{30}$

Lemma 3.3

(vi) $E(e_1{}^3e_0) = L_3C_{31} + 3L_4C_{20}C_{11}$

(vii) $E\{(e_1{}^4)\} = \frac{(N-n))(N^2 + N - 6nN + 6n^2)}{(N-1)(N-2)(N-3)}\frac{1}{n^3}C_{30} = L_3C_{40} + 3L_4C_{20}{}^2$

(viii) $E(e_1{}^2e_0{}^2) = L_3C_{40} + 3L_4C_{20}$

Where $L_3 = \frac{(N-n))(N^2 + N - 6nN + 6n^2)}{(N-1)(N-2)(N-3)}\frac{1}{n^3}$, $L_4 = \frac{N(N-n))(N-n-1)(n-1)}{(N-1)(N-2)(N-3)}\frac{1}{n^3}$

and $Cpq = \frac{(Xi - \overline{X})^p}{\overline{X}^p} \frac{(Y_i - \overline{Y})^q}{\overline{Y}^q}$.

Proof of these lemma's are straight forward by using SRSWOR (see Sukhatme and Sukhatme (1970)).

4. First Order Biases and Mean Squared Errors

The expression for the biases of the estimators t_1, t_2, t_3, t_4 and t_5 are respectively given by

$$Bias(t_1) = \overline{Y}\left[\frac{1}{2}\alpha L_1 C_{20} - \alpha L_1 C_{11}\right] \quad (4.1)$$

$$Bias(t_2) = \overline{Y}\left[\frac{g(g+1)}{2} L_1 C_{20} - g\beta L_1 C_{11}\right] \quad (4.2)$$

$$Bias(t_3) = \overline{Y}\left[-\frac{w(w-1)}{2} L_1 C_{20} - wL_1 C_{11}\right] \quad (4.3)$$

$$Bias(t_4) = \overline{Y}\left[bDL_1 C_{20} - DL_1 C_{11} + \frac{D(b-a)(p-1)}{2} L_1 C_{20}\right] \quad (4.4)$$

$$Bias(t_5) = \overline{Y}\left[-\frac{K(K-1)}{2} L_1 C_{20} - KL_1 C_{11}\right] \quad (4.5)$$

where, D =p (b-a) and $k = \frac{(\delta + 2\lambda)}{2}$.

Expression for the MSE's of the estimators t_1, t_2 t_3 t_4 and t_5 are, respectively given by

$$MSE(t_1) = \overline{Y}^2\left[L_1 C_{02} + \alpha^2 L_1 C_{20} - 2\alpha L_1 C_{11}\right] \quad (4.6)$$

$$MSE(t_2) = \overline{Y}^2\left[L_1 C_{02} + g^2 \beta^2 L_1 C_{20} - 2g\beta L_1 C_{11}\right] \quad (4.7)$$

$$MSE(t_3) = \overline{Y}^2\left[L_1 C_{02} + w^2 L_1 C_{20} - 2wL_1 C_{11}\right] \quad (4.8)$$

$$MSE(t_4) = \overline{Y}^2\left[L_1 C_{02} + DL_1 C_{20} - 2DL_1 C_{11}\right] \quad (4.9)$$

$$MSE(t_5) = \overline{Y}^2\left[L_1 C_{02} + k^2 L_1 C_{20} - 2kL_1 C_{11}\right] \quad (4.10)$$

5. Second Order Biases and Mean Squared Errors

Expressing estimator t_i's (i=1,2,3,4) in terms of e's (i=0,1), we get

$$t_1 = \overline{Y}(1+e_0)\{(1-\alpha) + \alpha(1+e_1)^{-1}\}$$

Or

$$t_1 - \overline{Y} = \overline{Y}\left\{e_0 + \frac{e_1}{2} + \frac{\alpha}{2}e_1{}^2 - \alpha e_0 e_1 + \alpha e_0 e_1{}^2 - \frac{\alpha}{6}e^3 - \frac{\alpha}{6}e_0 e_1{}^3 + \frac{\alpha}{24}e^4\right\} \qquad (5.1)$$

Taking expectations, we get the bias of the estimator t_1 up to the second order of approximation as

$$\text{Bias}_2(t_1) = \overline{Y}\left[\frac{\alpha}{2}L_1C_{20} - \alpha L_1C_{11} - \frac{\alpha}{6}L_2C_{30} + \alpha L_2C_{21} - \frac{\alpha}{6}\left(L_3C_{31} + 3L_4C_{20}C_{11}\right) + \frac{\alpha}{24}(L_3C_{40} + 3L_4C_{20}{}^2)\right] \qquad (5.2)$$

Similarly, we get the biases of the estimator's t_2, t_3, t_4 and t_5 up to second order of approximation, respectively as

$$\text{Bias}_2(t_2) = \overline{Y}\left[\frac{g(g+1)}{2}\beta^2 L_1C_{20} - g\beta L_1C_{11} - \frac{g(g+1)}{2}\beta^2 L_2C_{21} - \frac{g(g+1)(g+2)}{6}\beta^3 L_2C_{30} - \frac{g(g+1)(g+2)}{6}\beta^3\left(L_3C_{31} + 3L_4C_{20}C_{11}\right) + \frac{g(g+1)(g+2)(g+3)}{24}\beta^4(L_3C_{40} + 3L_4C_{20}{}^2)\right] \qquad (5.3)$$

$$\text{Bias}_2(t_3) = \overline{Y}\left[-\frac{w(w-1)}{2}L_1C_{20} - wL_1C_{11} - \frac{w(w-1)}{2}L_2C_{21} - \frac{w(w-1)(w-2)}{6}L_2C_{30} - \frac{w(w-1)(w-2)}{6}\left(L_3C_{31} + 3L_4C_{20}C_{11}\right) - \frac{w(w-1)(w-2)(w-3)}{24}(L_3C_{40} + 3L_4C_{20}{}^2)\right] \qquad (5.4)$$

$$\text{Bias}_2(t_4) = \overline{Y}\left[\frac{(D_1 bD)}{2}L_1C_{20} - DL_1C_{11} + \frac{(bD + D_1)}{}L_2C_{21} - \frac{(b^2D + 2bD_1 + D_2)}{}L_2C_{30} - (b^2D + 2bD_1)\left(L_3C_{31} + 3L_4C_{20}C_{11}\right) + \frac{(b^3D + 3b^2D_1 + 3bD_2 + D_3)}{}(L_3C_{40} + 3L_4C_{20}{}^2)\right] \qquad (5.5)$$

where, $D_1 = D\frac{(b-a)(p-1)}{2}$ $\qquad D_2 = D_1\frac{(b-a)(p-2)}{3}$.

$$Bias_2(t_5) = \overline{Y}\Big[-\frac{k(k-1)}{2}L_1C_{20} - kL_1C_{11} - \frac{k(k-1)}{2}L_2C_{21} - ML_2C_{30} - M(L_3C_{31} + 3L_4C_{20}C_{11})$$

$$- N(L_3C_{40} + 3L_4C_{20}{}^2)\Big] \tag{5.6}$$

Where, $M=\frac{1}{2}\left\{\frac{(\delta^3-6\delta^2)}{24} + \frac{(\alpha(\delta^2-2\delta)}{4} + \frac{\lambda(\lambda-1)}{2}\delta + \frac{\lambda(\lambda-1)(\lambda-2)}{3}\right\}$,

$k = \frac{(\delta+2\lambda)}{2}$,

$N=\frac{1}{8}\left\{\frac{(\delta^4-12\delta^3+12\delta^2)}{48} + \frac{(\alpha(\delta^3-6\delta)}{6} + \frac{\lambda(\lambda-1)}{2}(\delta^2-2\delta) + \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{3}\right\}$.

The MSE's of the estimators t_1, t_2, t_3, t_4 and t_5 up to the second order of approximation are, respectively given by

$$MSE_2(t_1) = \overline{Y}^2\Big[L_1C_{02} + \alpha^2L_1C_{20} - 2\alpha L_1C_{11} - \alpha^2L_2C_{30} + (2\alpha^2+\alpha)L_2C_{21}$$
$$- 2\alpha^2(L_3C_{31} + 3L_4C_{20}C_{11})$$
$$+ \alpha(\alpha+1)(L_3C_{22} + 3L_4(C_{20}C_{02} + C_{11}^2)) + \frac{5}{24}\alpha^2(L_3C_{40} + 3L_4C_{20}{}^2)\Big] \tag{5.7}$$

$$MSE_2(t_2) = \overline{Y}^2\Big[L_1C_{02} + g^2\beta^2L_1C_{20} - 2\beta gL_1C_{11} - \beta^3g^2(g+1)L_2C_{30} + g(3g+1)\beta^2L_2C_{21}$$
$$- 2\beta gL_2C_{12} - \left\{\frac{7g^3+9g^2+2g}{3}\right\}\beta^3(L_3C_{31} + 3L_4C_{20}C_{11})$$
$$+ g(2g+1)\beta^2(L_3C_{22} + 3L_4(C_{20}C_{02} + C_{11}^2))$$
$$+ \left\{\frac{2g^3+9g^2+10g+3}{6}\right\}\beta^4(L_3C_{40} + 3L_4C_{20}{}^2\Big] \tag{5.8}$$

$$MSE_2(t_3)=\overline{Y}^2\Big[L_1C_{02}+w^2L_1C_{20}-2wL_1C_{11}-w^2(w-1)L_2C_{30}+w(w+1)L_2C_{21}-2wL_2C_{12}$$
$$+\left\{\frac{5w^3-3w^2-2w}{3}\right\}\left(L_3C_{31}+3L_4C_{20}C_{11}\right)$$
$$+w\left(L_3C_{22}+3L_4(C_{20}C_{02}+C_{11}^2)\right)+\left\{\frac{7w^4-18w^3+11w^2}{24}\right\}(L_3C_{40}+3L_4{C_{20}}^2)\Big] \tag{5.9}$$

$$MSE_2(t_4)=\overline{Y}^2\Big[L_1C_{02}+D^2L_1C_{20}-2DL_1C_{11}-4DD_1L_2C_{30}+(2bD+2D_1+2D^2)L_2C_{21}-2DL_2C_{12}$$
$$+\left\{2D^2+2b^2D+2DD_1+4bD_1+4bD^2\right\}\left(L_3C_{31}+3L_4C_{20}C_{11}\right)$$
$$+\left\{D^2+2D_1+2bD\right\}\left(L_3C_{22}+3L_4(C_{20}C_{02}+C_{11}^2)\right)$$
$$+\left\{3b^2D^2+D_1^2+2DD_2+12bDD_1\right\}(L_3C_{40}+3L_4C_{20}^2)\Big] \tag{5.10}$$

$$MSE_2(t_5)=\overline{Y}^2\Big[L_1C_{02}+k^2L_1C_{20}-2kL_1C_{11}+kL_2C_{21}-2kL_2C_{12}+k^2(k-1)L_2C_{30}$$
$$+2k^2(k-1)\left(L_3C_{31}+3L_4C_{20}C_{11}\right)+k\left(L_3C_{22}+3L_4(C_{20}C_{02}+C_{11}^2)\right)$$
$$+\frac{(k^2-k)^2}{4}(L_3C_{40}+3L_4{C_{20}}^2)\Big] \tag{5.11}$$

6. Numerical Illustration

For a natural population data, we have calculated the biases and the mean square error's of the estimator's and compare these biases and MSE's of the estimator's under first and second order of approximations.

Data Set

The data for the empirical analysis are taken from 1981, Utter Pradesh District Census Handbook, Aligar. The population consist of 340 villages under koil police station, with Y=Number of agricultural labour in 1981 and X=Area of the villages (in acre) in 1981. The following values are obtained

$\overline{Y} = 73.76765$, $\overline{X} = 2419.04$, $N = 340, n = 70, n' = 120$, n=70, C_{02}=0.7614, C_{11}=0.2667, C_{03}=2.6942, C_{12}=0.0747, C_{12}=0.1589, C_{30}=0.7877, C_{13}=0.1321, C_{31}=0.8851, C_{04}=17.4275 C_{22}=0.8424, C_{40}=1.3051

Table 6.1: Biases and MSE's of the estimators

Estimator	Bias		MSE	
	First order	**Second order**	**First order**	**Second order**
t_1	0.0044915	0.004424	39.217225	39.45222
t_2	0	-0.00036	39.217225 (for g=1)	39.33552 (for g=1)
t_3	-0.04922	-0.04935	39.217225	39.29102
t_4	0.2809243	-0.60428	39.217225	39.44855
t_5	-0.027679	-0.04911	39.217225	39.27187

In the Table 6.1 the biases and MSE's of the estimators t_1, t_2, t_3, t_4 and t_5 are written under first order and second order of approximations. For all the estimators t1, t_2, t_3, t_4 and t_5, it was observed that the value of the biases decreased and the value of the MSE's increased for second order approximation. MSE's of the estimators up to the first order of approximation under optimum conditions are same. From Table 6.1 we observe that under second order of approximation the estimator t_5 is best followed by t_3,and t_2 among the estimators considered here for the given data set.

7. Estimators under stratified random sampling

In survey sampling, it is well established that the use of auxiliary information results in substantial gain in efficiency over the estimators which do not use such information. However, in planning surveys, the stratified sampling has often proved needful in improving the precision of estimates over simple random sampling. Assume that the population U consist of L strata as $U=U_1, U_2,\ldots,U_L$. Here the size of the stratum U_h is N_h, and the size of simple random sample in stratum U_h is n_h, where h=1, 2,---,L.

The Chakrabarty(1979) ratio- type estimator under stratified random sampling is given by

$$t_1' = (1-\alpha)\bar{y}_{st} + \alpha\bar{y}_{st}\frac{\bar{X}}{\bar{x}_{st}} \tag{7.1}$$

where,

$$\bar{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi}, \qquad \bar{x}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} x_{hi},$$

$$\bar{y}_{st} = \sum_{h=1}^{L} w_h \bar{y}_h, \quad \bar{x}_{st} = \sum_{h=1}^{L} w_h \bar{x}_h, \text{ and } \quad \bar{X} = \sum_{h=1}^{L} w_h \bar{X}_h.$$

Khoshnevisan et al. (2007) ratio- type estimator under stratified random sampling is given by

$$t_2' = \bar{y}_{st}\left[\frac{\bar{X}}{\beta\bar{x}_{st} + (1-\beta)\bar{X}}\right]^g \tag{7.2}$$

where g is a constant, for g=1 , t_2' is same as conventional ratio estimator whereas for g = -1, it becomes conventional product type estimator.

Sahai and Ray (1980) estimator t_3 under stratified random sampling is given by

$$t_3' = \bar{y}_{st}\left[2 - \left\{\frac{\bar{x}_{st}}{\bar{X}}\right\}^W\right] \tag{7.3}$$

Ismail et al. (2011) estimator under stratified random sampling t_4' is given by

$$t'_4 = \bar{y}_{st}\left[\frac{\bar{x} + a(\bar{X} - \bar{x}_{st})}{\bar{x} + b(\bar{X} - \bar{x}_{st})}\right]^p \tag{7.4}$$

Solanki et al. (2012) estimator under stratified random sampling is given as

$$t'_5 = \bar{y}_{st}\left[2 - \left\{\left(\frac{\bar{x}_{st}}{\bar{X}}\right)^{\lambda} \exp\frac{\delta(\bar{x}_{st} - \bar{X})}{(\bar{x}_{st} + \bar{X})}\right\}\right] \tag{7.5}$$

where λ and δ are the constants, suitably chosen by minimizing MSE of the estimator t'_5.

8. Notations used under stratified random sampling

Let us define, $e_0 = \frac{\bar{y}_{st} - \bar{y}}{\bar{y}}$ and $e_1 = \frac{\bar{x}_{st} - \bar{x}}{\bar{x}}$, then $E(e_0) = E(e_1) = 0$.

To obtain the bias and MSE of the proposed estimators, we use the following notations in the rest of the article:

$$\bar{y}_{st} = \sum_{h=1}^{L} w_h \bar{y}_h = \bar{Y}(1 + e_0),$$

$$\bar{x}_{st} = \sum_{h=1}^{L} w_h \bar{x}_h = \bar{X}(1 + e_1),$$

such that,

$E(e_0) = E(e_1) = E(e_2) = 0,$ and

$$V_{rs} = \sum_{h=1}^{L} W_h^{r+s} E\left[(\bar{x}_h - \bar{X}_h)^r (\bar{y}_h - \bar{Y}_h)^s\right]$$

Also

$$E(e_0^2) = \frac{\sum_{h=1}^{L} w_h^2 \gamma_h S_{yh}^2}{\bar{Y}^2} = V_{20}$$

$$E(e_1^2) = \frac{\sum_{h=1}^{L} w_h^2 \gamma_h S_{xyh}^2}{\bar{X}^2} = V_{02}$$

$$E(e_0 e_1) = \frac{\sum_{h=1}^{L} w_h^2 \gamma_h S_{xyh}^2}{\overline{X}\overline{Y}} = V_{11}$$

Where

$$S_{yh}^2 = \frac{\sum_{i=1}^{N_h} (\overline{y}_h - \overline{Y}_h)^2}{N_h - 1} \quad , \quad S_{xh}^2 = \frac{\sum_{i=1}^{N_h} (\overline{x}_h - \overline{X}_h)^2}{N_h - 1} \quad , \quad S_{xyh} = \frac{\sum_{i=1}^{N_h} (\overline{x}_h - \overline{X}_h)(\overline{y}_h - \overline{Y}_h)}{N_h - 1}$$

$$\gamma_h = \frac{1 - f_h}{n_h}, \quad f_h = \frac{n_h}{N_h}, \quad w_h = \frac{N_h}{n_h}.$$

Some additional notations for second order approximation,

$$V_{rs} = \sum_{h=1}^{L} W_h^{r+s} \frac{1}{\overline{Y}^r \overline{X}^s} E\left[\left(\overline{y}_h - \overline{Y}_h\right)^s \left(\overline{x}_h - \overline{X}_h\right)^r\right]$$

Where,

$$C_{rs(h)} = \frac{1}{N_h} \sum_{i=1}^{N_h} \left[\left(\overline{y}_h - \overline{Y}_h\right)^s \left(\overline{x}_h - \overline{X}_h\right)^r\right]$$

$$V_{12} = \sum_{h=1}^{L} W_h^3 \frac{k_{1(h)} C_{12(h)}}{\overline{Y}\overline{X}^2} \qquad V_{21} = \sum_{h=1}^{L} W_h^3 \frac{k_{1(h)} C_{21(h)}}{\overline{Y}^2 \overline{X}} \qquad V_{30} = \sum_{h=1}^{L} W_h^3 \frac{k_{1(h)} C_{30(h)}}{\overline{Y}^3}$$

$$V_{03} = \sum_{h=1}^{L} W_h^3 \frac{k_{1(h)} C_{03(h)}}{\overline{X}^3} \qquad V_{13} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)} C_{13(h)} + 3k_{3(h)} C_{01(h)} C_{02(h)}}{\overline{Y}\overline{X}^3}$$

$$V_{04} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)} C_{04(h)} + 3k_{3(h)} C_{02(h)}^2}{\overline{X}^4}$$

$$V_{22} = \sum_{h=1}^{L} W_h^4 \frac{k_{2(h)} C_{22(h)} + k_{3(h)} \left(C_{01(h)} C_{02(h)} + 2C_{11(h)}^2\right)}{\overline{Y}^2 \overline{X}^2}$$

Where

$$k_{1(h)} = \frac{(N_h - n_h)(N_h - 2n_h)}{n^2(N_h - 1)(N_h - 2)}$$

$$k_{2(h)} = \frac{(N_h - n_h)(N_h + 1)N_h - 6n_h(N_h - n_h)}{n^3(N_h - 1)(N_h - 2)(N_h - 3)}$$

$$k_{3(h)} = \frac{(N_h - n_h)N_h(N_h - n_h - 1)(n_h - 1)}{n^3(N_h - 1)(N_h - 2)(N_h - 3)}$$

9. First Order Biases and Mean Squared Errors

The biases of the estimators t_1', t_2', t_3', t_4' and t_5' are respectively given by

$$\text{Bias}(t_1') = \overline{Y}\left[\frac{1}{2}\alpha L_1 V_{02} - \alpha V_{11}\right] \tag{9.1}$$

$$\text{Bias}(t_2') = \overline{Y}\left[\frac{g(g+1)}{2}\beta^2 V_{02} - g\beta V_{11}\right] \tag{9.2}$$

$$\text{Bias}(t_3') = \overline{Y}\left[-\frac{w(w-1)}{2}V_{02} - wV_{11}\right] \tag{9.3}$$

$$\text{Bias}(t_4') = \overline{Y}\left[bDV_{02} - DV_{11} + \frac{D(b-a)(p-1)}{2}V_{02}\right] \tag{9.4}$$

$$\text{Bias}(t_5') = \overline{Y}\left[-\frac{K(K-1)}{2}V_{02} - KV_{11}\right] \tag{9.5}$$

Where, D =p(b-a) and $k = \frac{(\delta + 2\lambda)}{2}$.

The MSE's of the estimators t_1', t_2', t_3', t_4' and t_5' are respectively given by

$$\text{MSE}(t_1') = \overline{Y}^2\left[V_{20} + \alpha^2 V_{02} - 2\alpha V_{11}\right] \tag{9.6}$$

$$\text{MSE}(t_2') = \overline{Y}^2\left[V_{20} + g^2\beta^2 V_{02} - 2g\beta V_{11}\right] \tag{9.7}$$

$$\text{MSE}(t_3') = \overline{Y}^2\left[V_{20} + w^2 V_{02} - 2wV_{11}\right] \tag{9.8}$$

$$\text{MSE}(t_4') = \overline{Y}^2\left[V_{20} + DV_{02} - 2DV_{11}\right] \tag{9.9}$$

$$\text{MSE}(t_5') = \overline{Y}^2\left[V_{20} + k^2 V_{02} - 2kV_{11}\right] \tag{9.10}$$

10. Second Order Biases and Mean Squared Errors

Expressing estimator t_i's (i=1,2,3,4) in terms of e's (i=0,1), we get

$$t_1' = \overline{Y}(1+e_0)\{(1-\alpha)+\alpha(1+e_1)^{-1}\}$$

Or

$$t_1' - \overline{Y} = \overline{Y}\left\{e_0 + \frac{e_1}{2} + \frac{\alpha}{2}e_1{}^2 - \alpha e_0 e_1 + \alpha e_0 e_1{}^2 - \frac{\alpha}{6}e^3 - \frac{\alpha}{6}e_0 e_1{}^3 + \frac{\alpha}{24}e^4\right\} \tag{10.1}$$

Taking expectations, we get the bias of the estimator t_1' up to the second order of approximation as

$$\text{Bias}_2(t_1') = \overline{Y}\left[\frac{\alpha}{2}V_{02} - \alpha V_{11} - \frac{\alpha}{6}V_{03} + \alpha V_{12} - \frac{\alpha}{6}V_{13} + \frac{\alpha}{24}V_{04}\right] \tag{10.2}$$

Similarly we get the Biases of the estimator's t_2', t_3', t_4' and t_5' up to second order of approximation, respectively as

$$\text{Bias}_2(t_2') = \overline{Y}\left[\frac{g(g+1)}{2}\beta^2 V_{02} - g\beta V_{11} - \frac{g(g+1)}{2}\beta^2 V_{12} - \frac{g(g+1)(g+2)}{6}\beta^3 V_{30}\right.$$
$$\left. - \frac{g(g+1)(g+2)}{6}\beta^3 V_{31} + \frac{g(g+1)(g+2)(g+3)}{24}\beta^4 V_{04}\right] \tag{10.3}$$

$$\text{Bias}_2(t_3') = \overline{Y}\left[-\frac{w(w-1)}{2}V_{02} - wV_{11} - \frac{w(w-1)}{2}V_{12} - \frac{w(w-1)(w-2)}{6}V_{03}\right.$$
$$\left. - \frac{w(w-1)(w-2)}{6}V_{31} - \frac{w(w-1)(w-2)(w-3)}{24}V_{04}\right] \tag{10.4}$$

$$\text{Bias}_2(t_4') = \overline{Y}\left[\frac{(D_1 bD)}{2}V_{02} - DV_{11} + (bD+D_1)V_{12} - (b^2D + 2bD_1 + D_2)V_{03}\right.$$
$$\left. - (b^2D + 2bD_1)V_{31} + (b^3D + 3b^2D_1 + 3bD_2 + D_3)V_{04}\right] \tag{10.5}$$

Where, D =p(b-a) $\quad D_1 = D\frac{(b-a)(p-1)}{2} \quad D_2 = D_1\frac{(b-a)(p-2)}{3}$

$$\text{Bias}_2(t_5') = \overline{Y}\left[\left[-\frac{k(k-1)}{2}V_{02} - kV_{11} - \frac{k(k-1)}{2}V_{12} - MV_{03} - MV_{31} - NV_{04}\right] \tag{10.6}\right.$$

Where, $M=\frac{1}{2}\left\{\frac{(\delta^3-6\delta^2)}{24}+\frac{(\alpha(\delta^2-2\delta)}{4}+\frac{\lambda(\lambda-1)}{2}\delta+\frac{\lambda(\lambda-1)(\lambda-2)}{3}\right\}$, $k=\frac{(\delta+2\lambda)}{2}$

$$N=\frac{1}{8}\left\{\frac{(\delta^4-12\delta^3+12\delta^2)}{48}+\frac{(\alpha(\delta^3-6\delta)}{6}+\frac{\lambda(\lambda-1)}{2}(\delta^2-2\delta)+\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{3}\right\}$$

Following are the MSE of the estimators $t'_{1,}t'_2,t'_3,t'_4$ and t'_5 up to second order of approximation

$$MSE_2(t'_1)=\overline{Y}^2\Big[V_{20}+\alpha^2V_{02}-2\alpha V_{11}-\alpha^2V_{03}+(2\alpha^2+\alpha)V_{12} \\ -2\alpha^2V_{31}+\alpha(\alpha+1)V_{22}+\frac{5}{24}\alpha^2V_{04}\Big] \tag{10.7}$$

$$MSE_2(t'_2)=\overline{Y}^2\Big[V_{20}+g^2\beta^2V_{02}-2\beta gV_{11}-\beta^3g^2(g+1)V_{03}+g(3g+1)\beta^2V_{12} \\ -2\beta gV_{21}-\left\{\frac{7g^3+9g^2+2g}{3}\right\}\beta^3V_{31}+g(2g+1)\beta^2V_{22} \\ +\left\{\frac{2g^3+9g^2+10g+3}{6}\right\}\beta^4V_{04}\Big] \tag{10.8}$$

$$MSE_2(t'_3)=\overline{Y}^2\Big[V_{20}+w^2V_{02}-2wV_{11}-w^2(w-1)V_{03}+w(w+1)V_{12}-2wV_{21} \\ +\left\{\frac{5w^3-3w^2-2w}{3}\right\}V_{31}+wV_{22}+\left\{\frac{7w^4-18w^3+11w^2}{24}\right\}V_{04}\Big] \tag{10.9}$$

$$MSE_2(t'_4)=\overline{Y}^2\Big[V_{20}+D^2V_{02}-2DV_{11}-4DD_1V_{03}+(2bD+2D_1+2D^2)V_{12}-2DV_{21} \\ +\left\{2D^2+2b^2D+2DD_1+4bD_1+4bD^2\right\}V_{31}+\left\{D^2+2D_1+2bD\right\}V_{22} \\ +\left\{3b^2D^2+D_1^2+2DD_2+12bDD_1\right\}V_{04}\Big] \tag{10.10}$$

$$MSE_2(t'_5)=\overline{Y}^2\Big[V_{20}+k^2V_{02}-2kV_{11}+kV_{12}-2kV_{21}+k^2(k-1)V_{03} \\ +2k^2(k-1)V_{31}+kV_{22}+\frac{(k^2-k)^2}{4}V_{04}\Big] \tag{10.11}$$

11. Numerical Illustration

For the natural population data, we shall calculate the bias and the mean square error of the estimator and compare Bias and MSE for the first and second order of approximation.

Data Set-1

To illustrate the performance of above estimators, we have considered the natural Data given in *Singh and Chaudhary (1986, p.162)*.

The data were collected in a pilot survey for estimating the extent of cultivation and production of fresh fruits in three districts of Uttar- Pradesh in the year 1976-1977.

Table 11.1: Biases and MSE's of the estimators

Estimator	Bias		MSE	
	First order	Second order	First order	second order
t_1'	-10.82707903	-13.65734654	1299.110219	1372.906438
t_2'	-10.82707903	6.543275811	1299.110219	1367.548263
t_3'	-27.05776113	-27.0653128	1299.110219	1417.2785
t_4'	11.69553975	-41.84516913	1299.110219	2605.736045
t_5'	-22.38574093	-14.95110301	1299.110219	2440.644397

From Table 11.1 we observe that the MSE's of the estimators t'_1, t'_2, t'_3, t'_4 and t'_5 are same up to the first order of approximation but the biases are different. The MSE of the estimator t'_2 is minimum under second order of approximation followed by the estimator t'_1 and other estimators.

Conclusion

In this study we have considered some estimators whose MSE's are same up to the first order of approximation. We have extended the study to second order of approximation to search for best estimator in the sense of minimum variance. The properties of the estimators are studied under SRSWOR and stratified random sampling. We have observed from Table 6.1 and Table 11.1 that the behavior of the estimators changes dramatically when we consider the terms up to second order of approximation.

REFERENCES

Bahl, S. and Tuteja, R.K. (1991) : Ratio and product type exponential estimator. Information and Optimization Science XIII 159-163.

Chakrabarty, R.P. (1979) : Some ratio estimators, Journal of the Indian Society of Agricultural Statistics **31**(1), 49–57.

Hossain, M.I., Rahman, M.I. and Tareq, M. (2006) : Second order biases and mean squared errors of some estimators using auxiliary variable. SSRN.

Ismail, M., Shahbaz, M.Q. and Hanif, M. (2011) : A general class of estimator of population mean in presence of non–response. Pak. J. Statist. 27(4), 467-476.

Khoshnevisan, M., Singh, R., Chauhan, P., Sawan, N., and Smarandache, F. (2007). A general family of estimators for estimating population mean using known value of some population parameter(s), Far East Journal of Theoretical Statistics 22 181–191.

Ray, S.K. and Sahai, A (1980) : Efficient families of ratio and product type estimators, Biometrika **67**(1) , 211–215.

Singh, D. and Chudhary, F.S. (1986): *Theory and analysis of sample survey designs*. Wiley Eastern Limited, New Delhi.

Singh, H.P. and Tailor, R. (2003). Use of known correlation coefficient in estimating the finite population mean. Statistics in Transition 6, 555-560.

Singh, R., Cauhan, P., Sawan, N., and Smarandache, F. (2007): *Auxiliary Information and A Priori Values in Construction of Improved Estimators.* Renaissance High Press.

Singh, R., Chauhan, P. and Sawan, N. (2008): On linear combination of Ratio-product type exponential estimator for estimating finite population mean. Statistics in Transition,9(1),105-115.

Singh, R., Kumar, M. and Smarandache, F. (2008): Almost Unbiased Estimator for Estimating Population Mean Using Known Value of Some Population Parameter(s). Pak. J. Stat. Oper. Res., 4(2) pp63-76.

Singh, R. and Kumar, M. (2011): A note on transformations on auxiliary variable in survey sampling. MASA, 6:1, 17-19.

Solanki, R.S., Singh, H. P. and Rathour, A. (2012) : An alternative estimator for estimating the finite population mean using auxiliary information in sample surveys. ISRN Probability and Statistics doi:10.5402/2012/657682

Srivastava, S.K. (1967) : An estimator using auxiliary information in sample surveys. Cal. Stat. Ass. Bull. 15:127-134.

Sukhatme, P.V. and Sukhatme, B.V. (1970): Sampling theory of surveys with applications. Iowa State University Press, Ames, U.S.A.

Upadhyaya, L. N. and Singh, H. P. (199): Use of transformed auxiliary variable in estimating the finite population mean. Biom. Jour., 41, 627-636.