A TOPOLOGICAL BRAIN ELUCIDATES SYNTACTIC AND SEMANTIC PROCESSING

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When facing a proposition, the brain straightforwardly understands its grammar and discriminates whether it is true or false. Unlike computers, the brain is able to identify signs of sequences in terms of both syntactic symbols and semantic meaning. We show, based on the current literature, that a testable algebraic topological approach gives helpful insights into brain's computational activity during semantic recognition. Indeed, recent suggestions allow us to hypothesize that the semantic properties of a proposition are processed in brain dimensions higher than the syntactic ones. Furthermore, we show how, in a fully reversible process, the syntactic elements embedded in Broca's area might project to scattered semantic cortical zones, where the presence of higher functional dimensions gives rise to an increase in proposition's information content. Taking into account the dictates of novel versions of the Borsuk-Ulam and the fixed-point theorems, we build a framework that provides a feasible explanation for semantics processing in the brain, and also paves the way to novel computers which nodes are built in higher dimensions.

KEYWORDS: logic; meaning; Borsuk-Ulam theorem; computation; truth

Syntax assesses the relationships among the elements of linguistic expressions (Carnie, 2006). A property relative to a proposition is syntactic if it depends just on the symbols by themselves. In philosophical language, syntax stands for grammar, which includes also phonology and orthography, e.g., sounds, spellings and so on (Martinich, 1996). In mathematical logic, a concept related to a symbols sequence is syntactic if it depends just on the symbols forming the sequence, not taking into account the *meaning*. The ambiguous term *meaning* will stand here for the truth or the falseness of an expression or a sequence of symbols. On the other side of the coin, *semantics* assesses the relationships between expressions and extra-linguistic truths (Nerbonne, 1996; Cruse, 2004). A property relative to a proposition is semantic if it depends on the meaning, *e.g.*, if the proposition is true or false. In philosophical language, semantics stand for the study of the meanings of linguistic expressions. A concept related to a symbols sequence is semantic if it depends on the meaning transmitted by the sequence.

While syntactic properties are algorithmically verifiable by computers, semantic ones are not, because the latter are based on the notion of truth, a task that cannot be pursued by current computers. Furthermore, Godel's first incompleteness theorem (Godel, 1931) demonstrates that, if we limit ourselves to syntactic approaches, we will never be able to reach truths. To make an example, Godel states that the phrase: "this enunciate is not demonstrable" could be either a) false, and in this case it absurd, because is false and demonstrable, b) or could be true, and is this case is not demonstrable. Given a group of arithmetical axioms, there is always a true arithmetic proposition that cannot be demonstrated starting from the axioms, if we admit just the syntactic methods dictated by the Hilbert's program (Mancosu, 1998). Therefore, syntactic methods are inadequate in order to understand the whole properties of the same model that we easily understand through semantics. In sum, while the human brain is able to grasp the semantic notion of truth, the latter cannot be verified by computers. Furthermore, Godel's second incompleteness theorem states that is not possible to verify the truth of a group of arithmetic axioms (Godel, 1931). Such problem might be overtaken, because every true arithmetic proposition can be demonstrated starting from Peano axioms (Segre 1994), which use semantic methods. However, we cannot never be sure that the semantic methods we used were truthful. The existence of axioms is based on our extremely questionable intuition that such axioms are true. Therefore, if we use just the unquestionable and consistent syntactic methods, as proposed by Hilbert, we cannot proof all the truths. Vice versa, if we rather use the uncertain semantic methods, we could possibly know all the arithmetic truths, but we cannot be sure that such methods are correct. Here we propose a topological computational method able to throw a bridge between syntactic and semantic expressions, in order to overtake the Godel's incompleteness theorems and to build semantics-solving machines.

METHODS

StringBUT. Here we introduce a string-based extension of the Borsuk-Ulam theorem, denoted *strBUT* (Peters and Tozzi, 2016a). The Borsuk-Ulam Theorem (BUT) states that a single point on a circumference maps on a sphere to two points equipped with matching description. For technical readers, see: Borsuk (1933), Matousek (2003), Crabb and Jaworowski (2013), Peters (2016), Peters and Tozzi (2016b), Tozzi and Peters (2016a). Continuous projections from an n-dimensional sphere to a n-dimensional Euclidean space lead to a string-based incarnation of the BUT. We consider a geometric structure that has the characteristics of a *string*. By definition, a string on the surface of an n-sphere is a line that represents the path traced by a particle moving along its surface. In an abstract geometric space, a string, also termed *worldline* (Olive, 1987; Olive and Landsberg, 1989) stands for a region of space with either bounded or unbounded lengths. In evaluating a string-based BUT, we take into account antipodal sets instead of antipodal points (Petty, 1971). Indeed, in a point-free geometry (Di Concilio, 2013; Di Concilio and G. Gerla, 2006), regions replace points as the primitives. If we assess a worldline in terms of a spatial region on the surface of an n-sphere, or in an n-dimensional normed linear space, strings can be defined as *antipodal* (**Figure 1**). Strings are antipodal, provided the regions encompassing the strings belong to disjoint parallel hyperplanes: put simply, they have no points in common. A region is called a *worldsheet* if that everyone of its subregions contains at least one string. In other words, the term worldsheet designates a nonempty region of a space completely covered by strings, in which every member is a string. A 2d plane worldsheet can be rolled up to form the lateral surface of a 3d cylinder, termed a worldsheet cylinder. Further, a worldsheet cylinder maps to a worldsheet torus, formed by bending the former until the ends meet (**Figure 1**). In sum, a flattened worldsheet maps to a worldsheet cylinder and a flattened worldsheet cylinder maps to worldsheet torus. It means that a bounded worldsheet cylinder is homotopically equivalent to a worldsheet torus. The strings on different wordsheets are antipodal and descriptively near, e.g., they share matching description. There is however a difference between the strings embedded in wordsheets of diverse dimensions. The higher the dimension of the wordsheet, the more the information encompassed in strings on the same region, because of the higher number of coordinates. Strings contain more information than their projections in a lower dimensions. It means that strBUT allows us to evaluate systems features in higher dimensions, in order to increase the amount of detectable information. To make an example, you might extract the three-dimensional shape of a cat, just looking at its shadow. Vice versa, dropping down a dimension means that each point in the lower dimensional space is simpler. In sum, strBUT provides a way to evaluate changes of information in a topological, other than thermodynamical, fashion.

Now a problem arises. Take the example of a string, say str1, on a n manifold (say a 2D worldsheet). Such str1 is antipodal to str2, e..g. the string embedded on a n+1 manifold (say a 3d worldsheet rolled up into a cylinder). Hovever, for the classical BUT, it must exist on the worlsheet cylinder ALSO another string, say str3, which has matching description with str2. This is because, as BUT dictates, going one dimension higher, a single feature needs to project to TWO matching features. It means that a projection mapping contains multiple mappings in higher levels.

Brouwer's fixed point theorem. Next, consider Brouwer's fixed point theorem (FPT) (Brower, 1906). In simple words, every continuous function from a n-sphere of every dimension (e.g. a disk, or a ball) to itself has at least one fixed point. FPT applies, for example, to any disk-shaped area, where it guarantees the existence of a fixed point, which behaves like a sort of whirlpool attracting moving particles. Su (1997) gives a coffee cup illustration of the FPT. No matter how you continuously slosh the coffee around in a coffee cup, some point is always in the same position that it was before the sloshing began. And if you move this point out of its original position, you will eventually move some other point in the sloshing coffee back into its original position. In strBUT terms, it means that we can always find a n-sphere containing a string. Also, each string has a particular shape and come together with another string, which is termed a *wired friend*. These observations lead to a *wired friend theorem*: every occurrence of a wired friend string with a particular shape on the structure S^n maps to a fixed description, e.g. another string that belongs to another structure. Every wired friend is recognizable by its shape, because the shape of a string is the silhouette of a wired friend string. We achieve a maps of wired friend strings, projecting from the 2D wordsheet, to the cylinder wordsheet, to the woldsheet torus. In sum, we can always find a structure containing a string which is a description of another string in a lower dimensional structure, and vice versa. In the next paragraphs we will discuss how this theorem has important consequences in the study of syntactic and semantic brain processing.

When topology meets logic. In this section, a sample application of strBUT and FTP is given in terms of syntactics and semantics. We hypothesize that the syntactic elements of a language lie on a n-manifold, while the semantic ones lie on an n+1 manifold. A syntactic proposition becomes semantic when brain mechanisms projects the former one dimension higher, and vice versa. In logical terms, we may state that p=q is syntactic, while $(p=q)^{n+1}$ is semantic. The higher dimension stands for the truth function, e.g., whether a proposition in true or false. To make an example,

consider the syntactic preposition: “the-skin-is-tender” and the semantic one, equipped with a truth value of TRUE (or YES): “*the skin is tender*”. The truth value, of course, could be FALSE (or NOT), in case of a disease affecting the skin. We write (the-skin-is-tender)^{n} and (*the skin is tender*)$^{n+1}$, where n stands for the abstract dimensions in which the proposition is embedded. The two prepositions also stand for two groups, where the n+1 group encompasses the n-group. The n+1 group, equipped with a truth function (YES and NOT), cannot be fully detected if we just look at the n-group, because the latter displays less dimensions. A part of information is lost in syntactics, due to the lower number of coordinates. It means that the whole semantic group can be assessed just in case we are in dimensions higher that the syntactic ones. In topological terms, syntactic operations take place when strings, e.g, linguistic prepositions, are placed in a 2D wordsheet, while semantic operations take place when strings are placed in wordsheet cylinders and toruses lying in higher dimensions. Therefore, a group equipped with the further dimensions of TRUE or FALSE is formed through a projective, continuous mechanism (**Figure 1B**).
For the classical BUT, semantic elements encompass not just one, but two antipodal strings, equipped with matching description. Such framework will be useful in the next paragraphs, when we will assess the cortical counterparts of propositions. Are there differences between the two points lying in higher dimensions and the single, lower-dimensional point? The foremost distinction is that the two antipodal points, lying one dimension higher, form a group by themselves. It means that semantic processing can be studied just in a context equipped with dimensions higher than syntactic symbols. Furthermore, when n increases, syntactic structures can be compared with models existing one dimension higher. What does the dimension n stand for? The n exponent might simply stand for a further spatial brain dimension. In this case, computers equipped with nodes endowed in higher spatial dimensions might be able to understand semantics.

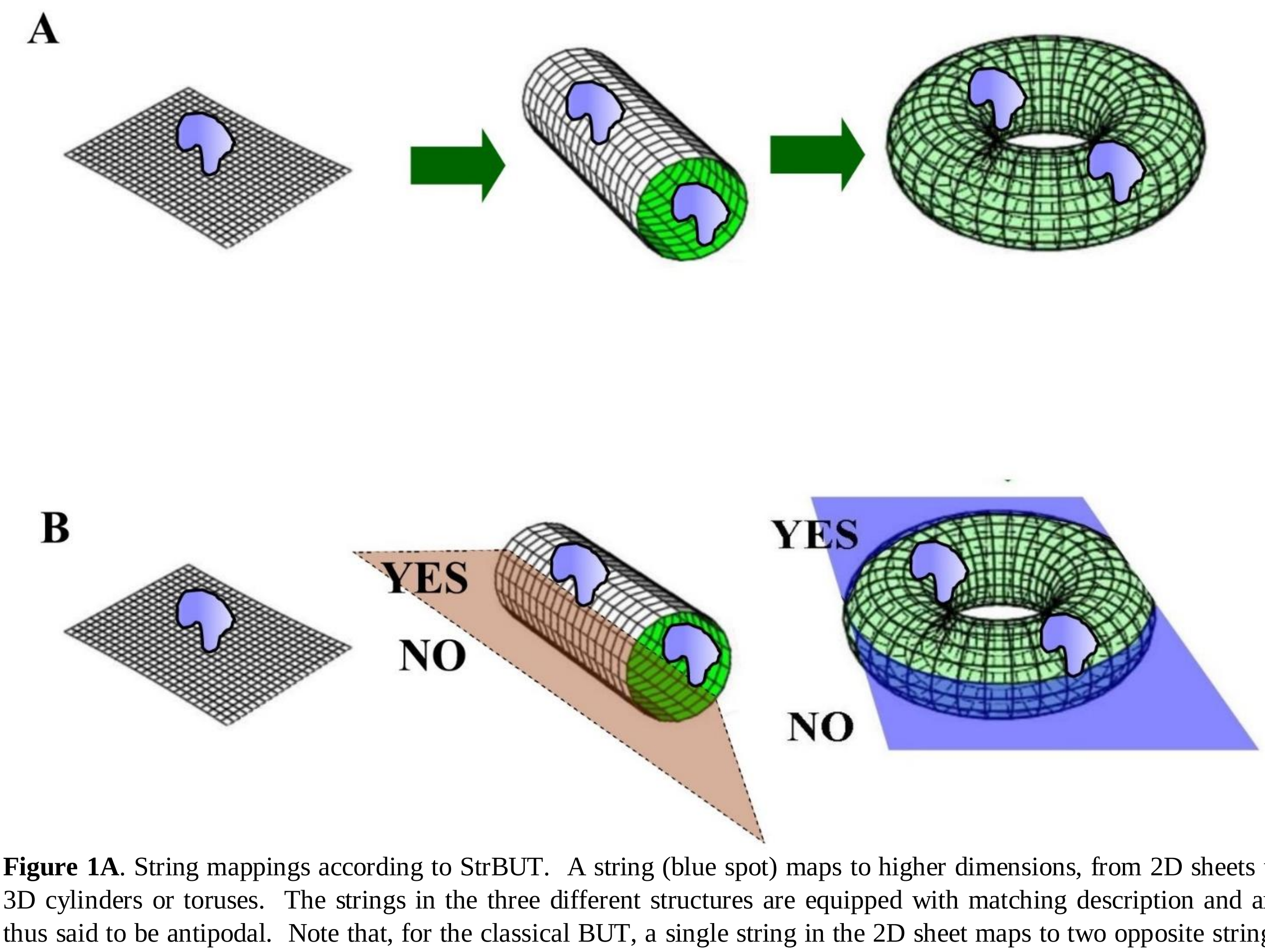

Figure 1A. String mappings according to StrBUT. A string (blue spot) maps to higher dimensions, from 2D sheets to 3D cylinders or toruses. The strings in the three different structures are equipped with matching description and are thus said to be antipodal. Note that, for the classical BUT, a single string in the 2D sheet maps to two opposite strings on the 3D structures.
Figure 1B. Syntactic and semantic counterparts of strBUT. On the left, the blue spot stands for a bidimensional syntactic construct. Going one dimension higher, we achieve a third dimension. This further dimension stands for the semantic concept of truth value (YES or NO). In sum, strings gain novel information when projected into higher-dimensional structures.

RESULTS

In touch with predictions from the above mentioned linguistic-topological model, we hypothesize that our brain processes syntactic and semantic elements through a change in functional or spatial dimensions. The syntactic processing that occurs at one level of brain activity is projected to a higher level, giving rise to semantic recognition, which encompasses more information. In the next paragraphs, taking into account the recent literature, we will go through possible physical brain counterparts.

Brain dimensions. In the novel framework, syntax displays less dimensions than semantics in the central nervous system, so that semantic inputs have increased complexity. The first step is to evaluate whether the brain encompasses different dimensions. The term brain *dimension* may reflect either functional activity, or anatomical connections between cortical areas. The functional approach, based on complex network analysis of brain signals, assesses neural space's multidimensionality in a brain conceived as a complex dynamical system (Kida et al, 2016; Giusti et al., 2016; Simas et al., 2015). It allows us to describe nervous dynamics as vectors or tensors in pluri-dimensional phase spaces. Apart from the *canonical* three dimensions, the technique is also able to assess other neural features, such as, e.g., frequency and magnitude, each one standing for other possible dimensions (Kida et al, 2016).

The anatomical approach to brain dimensionality evaluates cortical locations. Tozzi and Peters (2016a; 2016b) recently suggested that brain trajectories might display four spatial dimensions, instead of the canonical three, during cortical spontaneous activity. Such trajectories can be described in terms of torus-shaped structures. Stemmler et al. (2015) showed how animals can navigate by reading out a simple population vector of grid cell activity across multiple spatial scales. It means that the behavior of population vectors, each one lying in different anatomical dimensions embedded on functional toruses, predicts neural and behavioral correlates of grid cell readout. Benson et al. (2016) developed an algorithmic framework for studying how complex networks are organized by higher-order connectivity patterns, revealing hubs and geographical elements not readily achievable by other methods. They showed that information propagation units exhibit rich higher-order organizational structures. In such a vein, Kleinberg et al. (2016) recently proposed that real networks are not just random combinations of single network layers, but are instead organized in specific ways dictated by hidden geometric correlations between layers, which allow the detection of multidimensional communities. Crucial for our strBUT arguments, such multidimensionality enables trans-layer link prediction, in order that connections in one layer can be predicted by observing the hidden geometric space of another (Kleineberg et al, 2016).

Localizations. The brain localizations of syntactic and semantic processing described in literature seem to confirm our topological framework. The left anterior language areas, e.g. Broca's area (left Brodmann area 44), is crucial for syntactic processing in speech production and perception (Skeide and Friederici, 2016; Stromswold et al, 1996), despite the wide variability between individuals and the presence of diverse circuits, not dedicated just to a single kind of linguistic information processing (Sahin et al., 2009). While syntax seems to be localized in relatively fewer and smaller brain areas, semantics is instead scattered throughout vaste areas of the cortical surface (Huth et al, 2016). It is in touch with a model encompassing StrBUT and the fixed-point theorem, which regards semantic concepts as multiple antipodal strings on a torus surface and syntactic concepts as single strings located in specific brain zones (**Figure 2**). It also means that syntactic elements lie on brain dimensions higher than syntactic ones.

According to strBUT dictates, single strings grasp less information than their matching descriptions embedded in higher projections. Note also that semantic information is not aggregated, rather it is scattered in the brain. Syntactic symbols stand for single sets of objects, while semantic meanings for numerous sets of objects with matching description.

Recent suggestion let us hypothesize that, during human brain development, semantic processes precede syntactic ones. Indeed, Skeide and Friederici (2016) demonstrated in children a slow developmental segregation of syntax from semantics. The human embryo can already distinguish vowels *in utero*, but grammatical complexity is usually not fully mastered until at least 7 years. In the first three years, children rapidly acquire bottom-up processing skills, primarily implemented in the temporal areas. In a further stage, until the adolescence, top-down processes emerge gradually with the increasing structural connectivity of the left inferior frontal cortex. It means that childen are equipped with more semantics than adults, in touch with the hypothesis of decrease of functional brain dimensions with time passing. In a topological framework, it might be hypothesized that the FPT holds with time passing, leading to convergence of semantic concepts into fully developed syntactic ones.

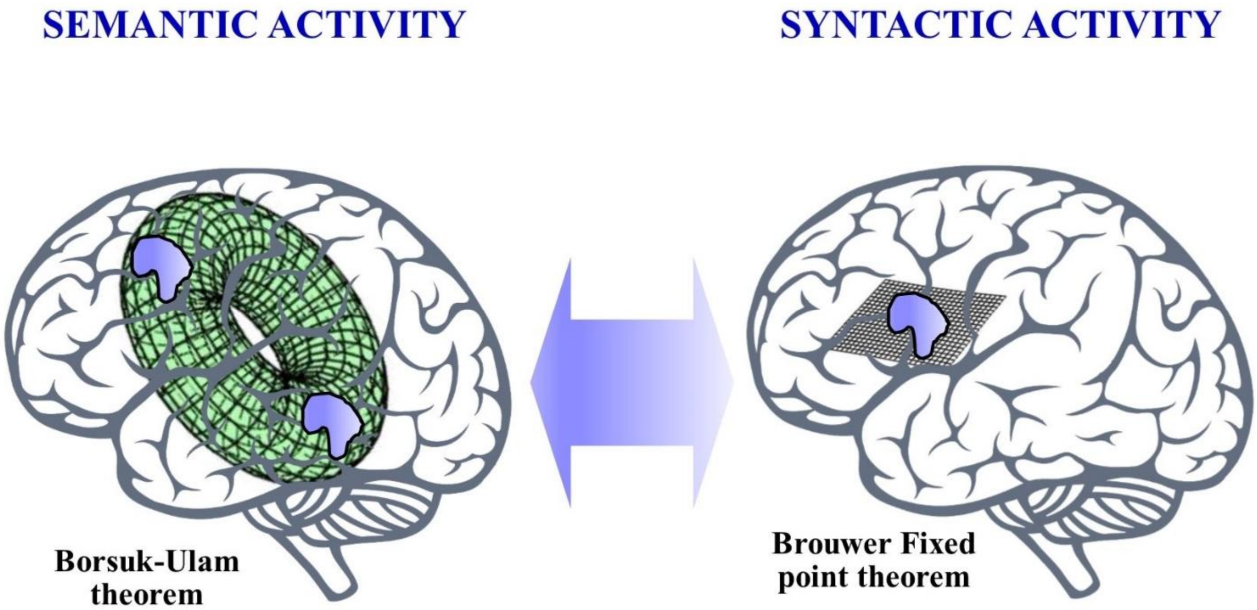

Figure 2. Semantic and syntactic brain activity in a topological framework. Mappings occur among brain regions which are temporarily equipped with different functional dimensions. According to the strBUT dictates, higher dimensional activities lie on a wordsheet torus, where scattered antipodal strings stand for the cortical areas activated during semantic activity. On the other side, syntactic processes lie on a 2D wordsheet, where the movements of the more localized strings endowed in the Broca's area are dictated by the FPT. Note that the process is full reversible, depending on the direction of the continuous mapping from one dimension to another.

DISCUSSION

We showed how semantic and syntactic brain abilities can be investigated in terms of algebraic topology, in particular through strBUT and FPT. Our data gave some understanding of how the connectivity among the brain active centres might change during syntactic or semantic inputs. These findings pave the way to novel computational and logic approaches to brain functions. Semantic concepts are based on the notion of truth; consequently, computers which carry inputs just on a 2D surface cannot evaluate whether a preposition is true or false. Despite massively parallel computers have become available (Angel and Leong, 2014), such kind of multidimensionality does not tackle the long-standing problem of semantics. Indeed, current multidimensional efforts just concern either access methods, in which sets of multidimensional points give computers support at the physical level (Gaede and Günther, 1998), or digital signal processing (Pastizzo et al, 2002) and data sampling (Mersereau and Speake, 1983). Multiple dimensions are merely used as representational devices for data analysis, limiting the potential of the current artificial intelligence studies. The fact that brain nodes can be embedded in higher dimensional spaces suggests the possibility to build computers equipped with pluridimensional nodes endowed in 3D geometric spaces. Indeed, a computer can be hypothesized having connections that are topologically 4D, even if embedded in a simple 3D space. Such computers, implemented in ordinary 3D space, but equipped with nodes with the same number of neighbors as a points in a 4D cube, could be able to perform semantic operations, due to the more degrees of freedom. Recent studies explored this approach. For example, Ursino et al. (2009; 2010; 2011; 2014) investigated how complex semantics can be extracted from the statistics of input features, using attractor neural networks. They focused on how similarity among objects, feature dominance and distinctiveness can be naturally coded using Hebbian training. Their model includes a lexical network, representing word-forms, and a semantic network composed of several areas, each one coding for a different feature. They use a simple taxonomy of schematic objects, i.e., a vector of features, with shared features (categories) and distinctive features (individual members) (Ursino et al., 2014). However, their framework differs from ours, because the former takes into account the hypothesis, opposite to ours, that concepts are represented in memory as a collection of features that *shrinks* the sum of previous sensory, motor and emotional experiences.

In logic, our framework elucidates several previous suggestions by various Authors. Hilbert's program states that every true proposition must be demonstrated starting from one of the available axioms. Except for the axioms, no other proposition can be considered true by itself, but must be necessarily assessed through algorythms and computational processing. It means that a demonstrable proposition is syntactic, because it could be verified by inspecting every symbol, with no need to keep into account its meaning. However, the Godel's first incompleteness theorem states that, given a group of arithmetical axioms, there will always be a true arithmetic enunciate that cannot be demonstrated starting from the axioms, if we just admit the methods suggested by the Hilbert's program. The theorem states that, if we limit ourselves to syntactic methods of reasoning, there will always be truths that are not accessible to us. Syntactic methods are inadequate in order to understand all the properties of an otherwise semantically understandable model. To make an example, our mental model of natural numbers cannot be completely characterized through syntactic methods. Furthermore, how can we be sure that all the axioms are true? Their truth depends just on the discourse universe, because the concept of truth has to do with semantics. According to formalists, axioms are indeed just starting points of linguistic jokes (Wittgenstein, 1953): the choice of an axiom looks like writing the rules of a table game. Therefore, every proposition, even if entirely false, is theoretically demonstrable, if we start from an erratic group of axioms. Our topological model sheds new light on some of these issues. While Hilbert describes real objects in terms of mathematical axioms, we describe them in terms of projections and mappings. Maps from the lower syntactic dimensions to the higher semantic ones ride out the caveats cast by Godel's theorems. In a language, paraphrasing Charles Ball, groups of words and expressions are correlated both with their shapes and meanings, which pertain to the same conceptual sphere, but to different dimensions. We could imagine that syntax works with individual symbols, while semantics works instead with groups of symbols.

For sake of brevity, here we did not tackle the issue of the third linguistic subfield (together with syntax and semantics), e.g., *pragmatics*, that investigates the ways in which context and use contributes to meaning (Mey 2001). However, our data pave also the way to a forthcoming interpretation of pragmatics, in terms of dimensions even higher than the semantic ones. We would like to bring to an end with a few excerpts from Wittgenstein's Philosophical Investigations (1953), where the author seems to foresight the topological framework in which propositions are embedded: "*I said to myself "I wonder what time it is?"—And if this sentence has a particular atmosphere, how am I to separate it from the sentence itself? It would never have occurred to me to think the sentence had such an aura if I had not thought of how one might say it differently—as a quotation, as a joke, as practice in elocution, and so on...* (I, 607). *Suppose someone said: every familiar word, in a book for example, actually carries an atmosphere with it in our minds, a 'corona' of lightly indicated uses.—Just as if each figure in a painting were surrounded by delicate shadowy drawings of scenes, as it were in another dimension, and in them we saw the figures in different contexts* (II_{VI}). *But how is it possible to see an object according to an interpretation? — The question represents it as a queer fact; as if something were being forced into a form it did not really fit. But no squeezing, no forcing took place here. When it looks as if there were no room for such a form between other ones you have to look for it in another dimension. If there is no room here, there is room in another dimension* (II_{XI}).

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