Exact Solutions of a Class of Duffing-van der Pol and Modified Emden Type Equations via Nonlocal Transformation

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Abstract

This paper purposes to show the existence of exact analytical solutions to a class of generalized Duffing-van der Pol and modified Emden type oscillator equations using nonlocal transformation.

1. Let us consider the general class of exactly integrable mixed Liénard-type nonlinear dissipative oscillator equations developed by Monsia et al. [1-2]

$$\ddot{x}+(l\frac{g'(x)}{g(x)}-\gamma\varphi'(x))\dot{x}^2+\mu\dot{x}\exp(\gamma\varphi(x))+\frac{\omega^2\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l}=0 \qquad (1)$$

that transforms under the choice $\varphi(x)=\ln(f(x))$, into [1, 2]

$$\ddot{x}+(l\frac{g'(x)}{g(x)}-\gamma\frac{f'(x)}{f(x)})\dot{x}^2+\mu\dot{x}f(x)^\gamma+\frac{\omega^2 f(x)^{2\gamma}\int g(x)^l dx}{g(x)^l}=0 \qquad (2)$$

were $f(x)$ and $g(x)$ are arbitrary functions of x, and $\ln$ designates the natural logarithm. γ , μ , ω and l are arbitrary parameters. The dot over a symbol denotes differentiation with respect to time and prime means differentiation with respect to x. The parametric choice $l=\gamma$, with $g(x)=f(x)$, leads immediately to the exactly solvable class of Liénard nonlinear differential equations [1, 2]

$$\ddot{x}+\mu\dot{x}f(x)^l+\omega^2 f(x)^l\int f(x)^l dx=0 \qquad (3)$$

By putting $f(x)=a+bx$, with $l=1$, into (3), yields the exactly integrable class of generalized modified Emden type oscillator equations

$$\ddot{x}+\mu\dot{x}(a+bx)+\omega^2(a^2x+\frac{3ab}{2}x^2+\frac{b^2}{2}x^3)=0 \qquad (4)$$

The general solution $x(t)$ of (4) is then given, according to the general nonlocal transformation [1, 2]

$$y=\int g(x)^l dx,\ d\tau=\exp(\gamma\varphi(x))dt \qquad (5)$$

where $y(\tau)$ satisfies the damped linear harmonic oscillator equation

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$$y''+\mu y'+\omega^2 y=0 \quad (6)$$

and prime means here differentiation with respect to τ , by the algebraic relationship

$$bx^2(t)+2ax(t)-2y(\phi(t))=0 \quad (7)$$

and $\tau=\phi(t)$ satisfies

$$\frac{d\tau}{dt}=\pm\sqrt{a^2+2by(\tau)} \quad (8)$$

Thus, the form of the general solution $x(t)$ depends on the sign of the discriminant of the characteristic equation of (6)

$$r^2+\mu r+\omega^2=0 \quad (9)$$

with $y(\tau)=\exp(r\tau)$, which determines the three distinct over-damped, critically damped and under-damped oscillations . It was shown in previous works [2, 3] that the equation (3) for $f(x)=x$, reduces to the generalized modified Emdent type equation

$$\ddot{x}+\mu x^l\dot{x}+\frac{\omega^2}{1+l}x^{2l+1}=0 \quad (10)$$

It is interesting to note that, for $g(x)=1$, the equation (1) takes the form [2-6]

$$\ddot{x}-\gamma\varphi'(x)\dot{x}^2+\mu\dot{x}\exp(\gamma\varphi(x))+\omega^2 x\exp(2\gamma\varphi(x))=0 \quad (11)$$

By substituting $\varphi(x)=x^2$, into (11), yields

$$\ddot{x}-\dot{x}[2\gamma x\dot{x}-\mu\exp(\gamma x^2)]+\omega^2 x\exp(2\gamma x^2)=0 \quad (12)$$

which, by taking $\mu=0$, reduces to

$$\ddot{x}-2\gamma x\dot{x}^2+\omega^2 x\exp(2\gamma x^2)=0 \quad (13)$$

The equation (13) is such that the Taylor expansion of the exponential-type restoring force function

$$\exp(2\gamma x^2)=1+2\gamma x^2+2\gamma^2 x^4+\ldots \quad (14)$$

gives, keeping the first two terms in the series expansion of the exponential function

$$\ddot{x}-2\gamma x\dot{x}^2+\omega^2 x(1+2\gamma x^2)=0 \quad (15)$$

The exact analytical trigonometric, that is harmonic, solution of the equation (13), and consequently, that of the equation (15), is secured by the nonlocal transformation which is previously mentioned. The equation (15) differs only by a sign inversion from the generalized modified Emden type oscillator equation studied in [7]. In this perspective it would be interesting to investigate the dynamics of the following equation

$$\ddot{x} + 2\gamma x\dot{x}^2 + \omega^2 x \exp(2\gamma x^2) = 0 \tag{16}$$

or

$$\ddot{x} + \gamma x\dot{x}^2 + \omega^2 x \exp(\gamma x^2) = 0 \tag{17}$$

in a subsequent work.

2. Consider now $f(x) = a + bx^2$, with $l = 1$. Then the equation (3) yields the exactly solvable class of generalized Duffing-van der Pol nonlinear oscillator equations

$$\ddot{x} + \mu\dot{x}(a + bx^2) + \omega^2(a^2 x + \frac{4ab}{3}x^3 + \frac{b^2}{3}x^5) = 0 \tag{18}$$

The equation (18) may be solved by the same approach as previously. The form of the general solution depends also on the over-damped, critically damped and under-damped solutions of (6) under consideration.

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