Exact Analytical Periodic Solutions with Sinusoidal Form to a Class of Position-Dependent Mass Liénard-Type Oscillator Equations

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Abstract

This letter consists of additions to the paper '' A Class of Position-Dependent Mass Liénard Differential Equations via a General Nonlocal Transformation''. The objective is to highlight the fact that the general second-order nonlinear differential equation theory of position-dependent mass oscillators developed previously has the ability to provide exact analytical periodic solutions with sinusoidal form to the class of quadratic Liénard-type equations, like the motion of a particle on a rotating parabola and Morse- type oscillator equation, under question.

1. Consider, firstly, the general class of exactly solvable mixed Liénard-type nonlinear differential equations [1]

$$\ddot{x}+(l\frac{g'(x)}{g(x)}-\gamma\varphi'(x))\dot{x}^2+\mu\dot{x}\exp(\gamma\varphi(x))+\frac{\omega^2\exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l}=0 \quad (1)$$

obtained by applying the nonlocal transformation

$$y(\tau)=\int g(x)^l dx \;,\; d\tau=\exp(\gamma\varphi(x))dt \quad (2)$$

to the damped linear harmonic oscillator equation

$$y''(\tau)+\mu y'(\tau)+\omega^2 y(\tau)=0 \quad (3)$$

where prime means differentiation with respect to the independent variable τ . μ , ω , l and γ are arbitrary parameters and, $g(x)$ and $\varphi(x)$ are also arbitrary functions of x . This transformation ensures exact analytical solutions to any equation that belongs to (1) as a function of the solution to damped linear harmonic oscillator equation.

2. Consider now the case where $\mu=0$, and $g(x)=1$. Then equation (1) reduces to

$$\ddot{x}-\gamma\varphi'(x)\dot{x}^2+\omega^2 x\exp(2\gamma\varphi(x))=0 \quad (4)$$

where prime denotes here differentiation with respect to the dependent variable $x(t)$, and dot over a symbol means differentiation with respect to time. It is noted in [2] that for $\varphi(x)=\ln(f(x))$, equation (4) becomes

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$$\ddot{x}-\gamma\frac{f'(x)}{f(x)}\dot{x}^2+\omega^2 x f(x)^{2\gamma}=0 \qquad (5)$$

Setting $f(x)=\frac{1}{\sqrt{1+\lambda x^2}}$, and $\gamma=1$, into (5), yields the equation of motion of a particle moving on a rotating parabola [2]

$$\ddot{x}+\frac{\lambda x}{1+\lambda x^2}\dot{x}^2+\frac{\omega^2 x}{1+\lambda x^2}=0 \qquad (6)$$

In this context the nonlocal transformation (2) becomes

$$y(\tau)=x(t)\ ,\ d\tau=\frac{dt}{\sqrt{1+\lambda x^2}} \qquad (7)$$

and, if the solution to linear harmonic oscillator equation according to (3) reads

$$y(\tau)=A_0\sin(\omega\tau+\theta_0) \qquad (8)$$

where A_0 and θ_0 are arbitrary constants, then the desired exact analytical periodic solution with sinusoidal form to equation (6) of a particle moving on a rotating parabola takes the form

$$x(t)=A_0\sin(\omega\phi(t)+\theta_0) \qquad (9)$$

where the function $\tau=\phi(t)$, satisfies

$$\frac{d\tau}{dt}=\frac{1}{\sqrt{1+\lambda A_0^2\sin^2(\omega\tau+\theta_0)}} \qquad (10)$$

3. Let $\varphi(x)=x$. Then, equation (4) reduces to the Morse-type position-dependent mass oscillator equation

$$\ddot{x}-\gamma\dot{x}^2+\omega^2 x\exp(2\gamma x)=0 \qquad (11)$$

that admits the exact analytical periodic solutions of trigonometric form

$$x(t)=A_0\sin(\omega\psi(t)+\theta_0) \qquad (12)$$

where the function $\tau=\psi(t)$, obeys

$$\frac{d\tau}{dt}=\exp[\gamma A_0\sin(\omega\tau+\theta_0)] \qquad (13)$$

That being so, exact analytical periodic solutions for many other quadratic Liénard-type oscillator equations may be expressed as a sinusoidal function of time, as previously shown for the Mathews-Lakshmanan oscillator equations by applying the theory of position-dependent mass oscillator via a nonlocal transformation [1].

References

[1] M. D. Monsia, J. Akande, D. K. K. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou, A class of position-dependent mass Liénard differential equations via a general nonlocal transformation, viXra:1608.0226v1.(2016).

[2] M. D. Monsia, J. Akande, D. K. K. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou, Additions to ‘’ A Class of Position-Dependent Mass Liénard Differential Equations via a General Nonlocal Transformation’’ , viXra :1608.0266v1.(2016).