Jean Dezert[1], Florentin Smarandache[2]

[1]The French Aerospace Lab, 29 Avenue de la Division Leclerc, 92320 Chatillon, France

[2]Department of Mathematics University of New Mexico Gallup, NM 8730, U.S.A.

Transformations of belief masses into subjective probabilities

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Abstract: *In this chapter, we propose in the DSmT framework, a new probabilistic transformation, called DSmP, in order to build a subjective probability measure from any basic belief assignment defined on any model of the frame of discernment. Several examples are given to show how the DSmP transformation works and we compare it to main existing transformations proposed in the literature so far. We show the advantages of DSmP over classical transformations in term of Probabilistic Information Content (PIC). The direct extension of this transformation for dealing with qualitative belief assignments is also presented. This theoretical work must increase the performances of DSmT-based hard-decision based systems as well as in soft-decision based systems in many fields where it could be used, i.e. in biometrics, medicine, robotics, surveillance and threat assessment, multisensor-multitarget tracking for military and civilian applications, etc.*

3.1 Introduction

In the theories of belief functions, Dempster-Shafer Theory (DST) [10], Transferable Belief Model (TBM) [15] or DSmT [12, 13], the mapping from the belief to the probability domain is a controversial issue. The original purpose of such mappings was to make a (hard) decision, but contrariwise to erroneous widespread idea/claim, this is not the only interest for using such mappings nowadays. Actually the probabilistic transformations of belief mass assignments are very useful in modern multitarget multisensor tracking systems (or in any other systems) where one deals with soft decisions (i.e. where all possible solutions are kept for state estimation with their likelihoods). For example, in a Multiple Hypotheses Tracker using both kinematical and attribute data, one needs to compute all probabilities values for deriving the likelihoods of data association hypotheses and then mixing them altogether to estimate states of targets. Therefore, it is very relevant to use a mapping which provides a high probabilistic information content (PIC) for expecting better performances. This perfectly justifies the theoretical work proposed in this chapter. A classical transformation is the so-called *pignistic probability* [16], denoted $BetP$, which offers a good compromise between the maximum of credibility Bel and the maximum of plausibility Pl for decision support. Unfortunately, $BetP$ doesn't provide the highest PIC in general as pointed out by Sudano [17–19]. We propose hereafter a new generalized pignistic transformation, denoted $DSmP$, which is justified by the maximization of the PIC criterion. An extension of this transformation in the qualitative domain is also presented. This chapter is an extended version of a paper presented at Fusion 2008 conference in Cologne, Germany [7]. An application of $DSmP$ for the Target Type Tracking problem will be presented in Chapter 16.

3.2 Classical and generalized pignistic probabilities

3.2.1 Classical pignistic probability

The basic idea of the classical pignistic probability proposed and coined by Philippe Smets in [14, 16] consists in transfering the positive mass of belief of each non specific element onto the singletons involved in that element split by the cardinality of the proposition when working with normalized basic belief assignments (bba's). The (classical) pignistic probability in TBM framework is given by[1] $BetP(\emptyset) = 0$ and $\forall X \in 2^\Theta \setminus \{\emptyset\}$ by:

[1] We assume that $m(.)$ is of course a non degenerate bba, i.e. $m(\emptyset) \neq 1$.

$$BetP(X) = \sum_{Y \in 2^\Theta, Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)}, \tag{3.1}$$

where 2^Θ is the power set of the finite and discrete frame Θ assuming Shafer's model, i.e. all elements of Θ are assumed truly exclusive.

In Shafer's approach, $m(\emptyset) = 0$ and the formula (3.1) can be rewritten for any singleton $\theta_i \in \Theta$ as

$$BetP(\theta_i) = \sum_{\substack{Y \in 2^\Theta \\ \theta_i \subseteq Y}} \frac{1}{|Y|} m(Y) = m(\theta_i) + \sum_{\substack{Y \in 2^\Theta \\ \theta_i \subset Y}} \frac{1}{|Y|} m(Y) \tag{3.2}$$

3.2.2 Generalized pignistic probability

The classical pignistic probability has been generalized in DSmT framework for any regular bba $m(.) : G^\Theta \mapsto [0, 1]$ (i.e. such that $m(\emptyset) = 0$ and $\sum_{X \in G^\Theta} m(X) = 1$) and for any model of the frame (free DSm model, hybrid DSm model and Shafer's model as well). A detailed presentation of this transformation with several examples can be found in Chapter 7 of [12]. It is given by $BetP(\emptyset) = 0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$ by

$$BetP(X) = \sum_{Y \in G^\Theta} \frac{\mathcal{C}_\mathcal{M}(X \cap Y)}{\mathcal{C}_\mathcal{M}(Y)} m(Y) \tag{3.3}$$

where G^Θ corresponds to the hyper-power set including all the integrity constraints of the model (if any)[2]; $\mathcal{C}_\mathcal{M}(Y)$ denotes the DSm cardinal[3] of the set Y. The formula (3.3) reduces to (3.2) when G^Θ reduces to classical power set 2^Θ when one adopts Shafer's model.

3.3 Sudano's probabilities

indexSudano's probabilities

John Sudano has proposed several transformations for approximating any quantitative belief mass $m(.)$ by a subjective probability measure [21]. These

[2] $G^\Theta = 2^\Theta$ if one adopts Shafer's model for Θ and $G^\Theta = D^\Theta$ (Dedekind's lattice) if one adopts the free DSm model for Θ [12].

[3] $\mathcal{C}_\mathcal{M}(Y)$ is the number of parts of Y in the Venn diagram of the model $\mathcal{M}$ of the frame Θ under consideration [12] (Chap. 7).

transformations were denoted $PrPl$, $PrNPl$, $PraPl$, $PrBel$ and $PrHyb$, and were all defined in DST framework. They use different kinds of mappings either proportional to the plausibility, to the normalized plausibility, to all plausibilities, to the belief, or a hybrid mapping.

$PrPl(.)$ and $PrBel(.)$ transformations are mathematically defined[4] as follows for all $X \neq \emptyset \in \Theta$:

$$PrPl(X) = Pl(X) \cdot \sum_{\substack{Y \in 2^\Theta \\ X \subseteq Y}} \frac{1}{CS[Pl(Y)]} m(Y) \tag{3.4}$$

$$PrBel(X) = Bel(X) \cdot \sum_{\substack{Y \in 2^\Theta \\ X \subseteq Y}} \frac{1}{CS[Bel(Y)]} m(Y) \tag{3.5}$$

where the denominators involved in the formulas are given by the compound-to-sum of singletons $CS[.]$ operator defined by [17]:

$$CS[Pl(Y)] \triangleq \sum_{\substack{Y_i \in 2^\Theta \\ |Y_i|=1 \\ \cup_i Y_i = Y}} Pl(Y_i) \qquad \text{and} \qquad CS[Bel(Y)] \triangleq \sum_{\substack{Y_i \in 2^\Theta \\ |Y_i|=1 \\ \cup_i Y_i = Y}} Bel(Y_i)$$

$PrNPl(.)$, $PraPl(.)$ and $PrHyb(.)$ also proposed by John Sudano [17, 21] are defined as follows:

- The mapping proportional to the *normalized* plausibility

$$PrNPl(X) = \frac{1}{\Delta} \sum_{\substack{Y \in 2^\Theta \\ Y \cap X \neq \emptyset}} m(Y) = \frac{1}{\Delta} \cdot Pl(X) \tag{3.6}$$

  where Δ is a normalization factor such that $\sum_{X \in \Theta} PrNPl(X) = 1$.

- The mapping proportional to *all* plausibilities

$$PraPl(X) = Bel(X) + \epsilon \cdot Pl(X) \tag{3.7}$$

  with

$$\epsilon \triangleq \frac{1 - \sum_{Y \in 2^\Theta} Bel(Y)}{\sum_{Y \in 2^\Theta} Pl(Y)}$$

[4]For notational convenience and simplicity, we use a different but equivalent notation than the one originally proposed by John Sudano in his publications.

- The hybrid pignistic probability

$$PrHyb(X) = PraPl(X) \cdot \sum_{\substack{Y \in 2^\Theta \\ X \subseteq Y}} \frac{1}{CS[PraPl(Y)]} m(Y) \tag{3.8}$$

  with

$$CS[PraPl(Y)] \triangleq \sum_{\substack{Y_i \in 2^\Theta \\ |Y_i|=1 \\ \cup_i Y_i = Y}} PraPl(Y_i)$$

- The pedigree pignistic probability [18]: It is denoted $PrPed(.)$ and was introduced by John Sudano in [18]. $PrPed(.)$ uses the *combined* bba's with the probability proportionally functions to compute a better pignistic probability estimate when used in conjunction with the Generalized belief fusion algorithm [sic [19]]. This kind of transformation is out of the scope of this chapter, since it cannot be applied directly for approximating a bba $m(.)$ without reference to some prior bba's and a fusion rule. Here we search for an efficient approximation of $m(.)$ by a subjective probability measure without any other considerations on how $m(.)$ has been obtained. We just want to use the minimal information available about $m(.)$, i.e. the values of $m(A)$ for all $A \in G^\Theta$.

3.4 Cuzzolin's intersection probability

In 2007, a new transformation has been proposed in [4] by Fabio Cuzzolin in the framework of DST. From a geometric interpretation of Dempster's rule, an *Intersection Probability* measure was proposed from the proportional repartition of the Total Non Specific Mass[5] (TNSM) by each contribution of the non-specific masses involved in it. For notational convenience, we will denote it $CuzzP$ in the sequel.

3.4.1 Definition

$CuzzP(.)$ is defined on any finite and discrete frame $\Theta = \{\theta_1, \ldots, \theta_n\}$, $n \geq 2$, satisfying Shafer's model, by

$$CuzzP(\theta_i) = m(\theta_i) + \frac{\Delta(\theta_i)}{\sum_{j=1}^{n} \Delta(\theta_j)} \times TNSM \tag{3.9}$$

[5] i.e. the mass committed to partial and total ignorances, i.e. to disjunctions of elements of the frame.

with $\Delta(\theta_i) \triangleq Pl(\theta_i) - m(\theta_i)$ and

$$TNSM = 1 - \sum_{j=1}^{n} m(\theta_j) = \sum_{A \in 2^{\Theta}, |A|>1} m(A) \tag{3.10}$$

3.4.2 Remarks

While appealing at the first glance because of its interesting geometric justification, Cuzzolin's transformation seems to be not totally satisfactory in our point of view for approximating any belief mass $m(.)$ into subjective probability for the following reasons:

1. Although (3.9) does not include explicitly Dempster's rule, its geometrical justification [2–4, 6] is strongly conditioned by the acceptance of Dempster's rule as the fusion operator for belief functions. This is a dogmatic point of view we disagree with since it has been recognized for many years by different experts of AI community, that other fusion rules can offer better performances, especially for cases where highly conflicting sources are involved.

2. Some parts of the masses of partial ignorance, say A, involved in the TNSM, are also transferred to singletons, say $\theta_i \in \Theta$ which are not included in A (i.e. such that $\{\theta_i\} \cap A = \emptyset$). Such transfer is not justified and does not make sense in our point of view. To be more clear, let's take $\Theta = \{A, B, C\}$ and $m(.)$ defined on its power set with all masses strictly positive. In that case, $m(A \cup B) > 0$ does count in TNSM and thus it is a bit redistributed back to C with the ratio $\frac{\Delta(C)}{\Delta(A)+\Delta(B)+\Delta(C)}$ through $TNSM > 0$. There is no solid reason for committing partially $m(A \cup B)$ to C since, only A and B are involved in that partial ignorance. Similar remarks hold for the partial redistribution of $m(A \cup C) > 0$.

3. It is easy to verify moreover that $CuzzP(.)$ is mathematically not defined when $m(.)$ is already a probabilistic belief mass because in such case all terms $\Delta(.)$ equal zero in (3.9) so that one gets 0/0 indetermination in Cuzzolin's formula. This remark is important only from the mathematical point of view.

3.5 A new generalized pignistic transformation

We propose a new generalized pignistic transformation, denoted $DSmP$ to avoid confusion with the previous existing transformations, which is straightforward, and also different from Sudano's and Cuzzolin's redistributions which are more refined but less exact in our opinions than what we present here. The basic idea of our $DSmP(.)$ transformation consists in a new way of proportionalizations of the mass of each partial ignorance such as $A_1 \cup A_2$ or $A_1 \cup (A_2 \cap A_3)$ or $(A_1 \cap A_2) \cup (A_3 \cap A_4)$, etc. and the mass of the total ignorance $A_1 \cup A_2 \cup \ldots \cup A_n$, to the elements involved in the ignorances. This new transformation takes into account both the values of the masses and the cardinality of elements in the proportional redistribution process. We first present the general formula for this new transformation, and the numerical examples, and comparisons with respect to other transformations are given in next sections.

3.5.1 The DSmP formula

Let's consider a discrete frame Θ with a given model (free DSm model, hybrid DSm model or Shafer's model), the $DSmP$ mapping is defined by $DSmP_\epsilon(\emptyset) = 0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$ by

$$DSmP_\epsilon(X) = \sum_{Y \in G^\Theta} \frac{\displaystyle\sum_{\substack{Z \subseteq X \cap Y \\ \mathcal{C}(Z)=1}} m(Z) + \epsilon \cdot \mathcal{C}(X \cap Y)}{\displaystyle\sum_{\substack{Z \subseteq Y \\ \mathcal{C}(Z)=1}} m(Z) + \epsilon \cdot \mathcal{C}(Y)} m(Y) \tag{3.11}$$

where $\epsilon \geq 0$ is a tuning parameter and G^Θ corresponds to the hyper-power set including eventually all the integrity constraints (if any) of the model $\mathcal{M}$; $\mathcal{C}(X \cap Y)$ and $\mathcal{C}(Y)$ denote the DSm cardinals[6] of the sets $X \cap Y$ and Y respectively.

The parameter ϵ allows to reach the maximum PIC value of the approximation of $m(.)$ into a subjective probability measure. The smaller ϵ, the better/bigger PIC value. In some particular degenerate cases however, the $DSmP_{\epsilon=0}$ values cannot be derived, but the $DSmP_{\epsilon>0}$ values can however always be derived by choosing ϵ as a very small positive number, say $\epsilon = 1/1000$ for example in order to be as close as we want to the maximum of the PIC (see the next sections for details and examples).

It is interesting to note also that when $\epsilon = 1$ and when the masses of all elements Z having $\mathcal{C}(Z) = 1$ are zero, (3.11) reduces to (3.3), i.e. $DSmP_{\epsilon=1} = BetP$. The passage from a free DSm model to a Shafer's model induces a

[6] We have omitted the index of the model $\mathcal{M}$ for notational convenience.

change in the Venn diagram representation, and so the cardinals change as well in the formula (3.11).

If one works on a (ultimate refined) frame Θ, which implies that Shafer's model holds, then the $DSmP_\epsilon(\theta_i)$ probability of any element θ_i, $i = 1, 2, \dots, n$ of the frame $\Theta = \{\theta_1, \dots, \theta_n\}$ can be directly obtained by:

$$DSmP_\epsilon(\theta_i) = m(\theta_i) + (m(\theta_i) + \epsilon) \sum_{\substack{X \in 2^\Theta \\ X \supset \theta_i \\ \mathcal{C}(X) \geq 2}} \frac{m(X)}{\sum_{\substack{Y \in 2^\Theta \\ Y \subset X \\ \mathcal{C}(Y) = 1}} m(Y) + \epsilon \cdot \mathcal{C}(X)} \tag{3.12}$$

The probabilities of (partial or total) ignorances are then obtained from the additivity property of the probabilities of elementary exclusive elements, i.e. for $i, j = 1, \dots, n$, $i \neq j$, $DSmP_\epsilon(\theta_i \cup \theta_j) = DSmP_\epsilon(\theta_i) + DSmP_\epsilon(\theta_j)$, etc.

3.5.2 Advantages of DSmP

DSmP works for all models (free, hybrid and Shafer's). In order to apply classical *BetP*, *CuzzP* or Sudano's mappings, we need at first to refine the frame (on the cases when it is possible!) in order to work with Shafer's model, and then apply their formulas. In the case where refinement makes sense, then one can apply the other subjective probabilities on the refined frame. *DSmP* works on the refined frame as well and gives the same result as it does on the non-refined frame. Thus *DSmP* with $\epsilon > 0$ works on any model and so is very general and appealing. It is a combination of *PrBel* and *BetP*. *PrBel* performs a redistribution of an ignorance mass to the singletons involved in that ignorance proportionally with respect to the singleton masses. While *BetP* also does a redistribution of an ignorance mass to the singletons involved in that ignorance but proportionally with respect to the singleton cardinals. *PrBel* does not work when the masses of all singletons involved in an ignorance are null since it gives the indetermination 0/0; and in the case when at least one singleton mass involved in an ignorance is zero, that singleton does not receive any mass from the distribution even if it was involved in an ignorance, which is not fair/good. *BetP* works all the time, but the redistribution is rough and does not take into account the masses of the singletons.

So, *DSmP* solves the *PrBel* problem by doing a redistribution of the ignorance mass with respect to both the singleton masses and the singletons' cardinals in the same time. Now, if all masses of singletons involved in all ignorances are different from zero, then we can take $\epsilon = 0$, and $DSmP$ coincides with $PrBel$ and both of them give the best result, i.e. the best PIC value.

$PrNPl$ is not satisfactory since it yields an abnormal behavior. Indeed, in any model, when a bba $m(.)$ is transformed into a probability, normally (we mean it is logically that) the masses of ignorances are transferred to the masses of elements of cardinal 1 (in Shafer's model these elements are singletons). Thus, the resulting probability of an element whose cardinal is 1 should be greater than or equal to the mass of that element. In other words, if A in G^Θ and $\mathcal{C}(A) = 1$, then $P(A) \geq m(A)$ for any probability transformation $P(.)$. This legitimate property is not satisfied by $PrNPl$ as seen in the following example.

Example: Let's consider Shafer's model with $\Theta = \{A, B, C\}$ and $m(A) = 0.2$, $m(B) = m(C) = 0$ and $m(B \cup C) = 0.8$, then the $DSmP$ transformation provides for any $\epsilon > 0$:

$$
\begin{aligned}
DSmP_\epsilon(A) &= 0.2 = BetP(A)\\
DSmP_\epsilon(B) &= 0.4 = BetP(B)\\
DSmP_\epsilon(C) &= 0.4 = BetP(C)
\end{aligned}
$$

Applying Sudano's probabilities formulas (3.4)-(3.8), one gets[7]:

- *Probability $PrPl(.)$:*

$$
\begin{aligned}
PrPl(A) &= 0.2 \cdot [0.2/0.2] = 0.2\\
PrPl(B) &= 0.8 \cdot [0.8/(0.8 + 0.8)] = 0.4\\
PrPl(C) &= 0.8 \cdot [0.8/(0.8 + 0.8)] = 0.4
\end{aligned}
$$

- *Probability $PrBel(.)$:*

$$
\begin{aligned}
PrBel(A) &= 0.2 \cdot [0.2/0.2] = 0.2\\
PrBel(B) &= 0 \cdot [0.8/(0 + 0)] = NaN\\
PrBel(C) &= 0 \cdot [0.8/(0 + 0)] = NaN
\end{aligned}
$$

- *Probability $PrNPl(.)$:*

$$
\begin{aligned}
PrNPl(A) &= 0.2/(0.2 + 0.8 + 0.8) \approx 0.1112\\
PrNPl(B) &= 0.8/(0.2 + 0.8 + 0.8) \approx 0.4444\\
PrNPl(C) &= 0.8/(0.2 + 0.8 + 0.8) \approx 0.4444
\end{aligned}
$$

[7] We use NaN acronym here standing for *Not a Number*. We could also use the standard "N/A" standing for "does not apply".

- *Probability* $PraPl(.)$: $\epsilon = \frac{1-0.2-0-0}{0.2+0.8+0.8} \approx 0.4444$

$$\begin{aligned}
PraPl(A) &= 0.2 + 0.4444 \cdot 0.2 \approx 0.2890\\
PraPl(B) &= 0 + 0.4444 \cdot 0.8 \approx 0.3555\\
PraPl(C) &= 0 + 0.4444 \cdot 0.8 \approx 0.3555
\end{aligned}$$

- *Probability* $PrHyb(.)$:

$$\begin{aligned}
PrHyb(A) &= 0.2890 \cdot [\frac{0.2}{0.2890}] = 0.2\\
PrHyb(B) &= 0.3555 \cdot [\frac{0.8}{0.3555 + 0.3555}] = 0.4\\
PrHyb(C) &= 0.3555 \cdot [\frac{0.8}{0.3555 + 0.3555}] = 0.4
\end{aligned}$$

Applying Cuzzolin's probabilities formula (3.9), one gets:

$$\begin{aligned}
CuzzP(A) &= m(A) + \frac{\Delta(A)}{\Delta(A) + \Delta(B) + \Delta(C)} \cdot TNSM\\
&= 0.2 + \frac{0}{0 + 0.8 + 0.8} \cdot 0.8 = 0.2\\
CuzzP(B) &= m(B) + \frac{\Delta(B)}{\Delta(A) + \Delta(B) + \Delta(C)} \cdot TNSM\\
&= 0 + \frac{0.8}{0 + 0.8 + 0.8} \cdot 0.8 = 0.4\\
CuzzP(C) &= m(C) + \frac{\Delta(C)}{\Delta(A) + \Delta(B) + \Delta(C)} \cdot TNSM\\
&= 0 + \frac{0.8}{0 + 0.8 + 0.8} \cdot 0.8 = 0.4
\end{aligned}$$

since $TNSM = m(B \cup C) = 0.8$, $\Delta(A) = Pl(A) - m(A) = 0$, $\Delta(B) = Pl(B) - m(B) = 0.8$ and $\Delta(C) = Pl(C) - m(C) = 0.8$.

In such a particular example, $BetP$, $PrPl$, $CuzzP$, $PrHyb$ and $DSmP_{\epsilon>0}$ transformations coincide. $PrBel(.)$ is mathematically not defined. Such conclusion is not valid in general as we will show in the next examples of this chapter. From this very simple example, one sees clearly the abnormal behavior of $PrNPl(.)$ transformation because $PrNPl(A) = 0.1112 < m(A) = 0.2$; it is not normal that singleton A looses mass when $m(.)$ is transformed into a subjective probability since the resulted subjective probability of an element whose cardinal is 1 should be greater than or equal to the mass of that element.

In summary, $DSmP$ does an improvement of all Sudano, Cuzzolin, and BetP formulas, in the sense that $DSmP$ mathematically makes a more accurate redistribution of the ignorance masses to the singletons involved in ignorances. $DSmP$ and $BetP$ work in both theories: DST (= Shafer's model) and DSmT (= free or hybrid models) as well. In order to use Sudano's and Cuzzolin's in DSmT models, we have to refine the frame (see Example 3.7.5).

3.6 PIC metric for the evaluation of the transformations

Following Sudano's approach [17, 18, 21], we adopt the Probabilistic Information Content (PIC) criterion as a metric depicting the strength of a critical decision by a specific probability distribution. It is an essential measure in any threshold-driven automated decision system. The PIC is the dual of the normalized Shannon entropy. A PIC value of one indicates the total knowledge (i.e. minimal entropy) or information to make a correct decision (one hypothesis has a probability value of one and the rest are zero). A PIC value of zero indicates that the knowledge or information to make a correct decision does not exist (all the hypothesis have an equal probability value), i.e. one has the maximal entropy. The PIC is used in our analysis to sort the performances of the different pignistic transformations through several numerical examples. We first recall what Shannon entropy and PIC measure are and their tight relationship.

3.6.1 Shannon entropy

Shannon entropy, usually expressed in bits (binary digits), of a discrete probability measure $P\{.\}$ over a discrete finite set $\Theta = \{\theta_1, \ldots, \theta_n\}$ is defined by[8] [11]:

$$H(P) \triangleq -\sum_{i=1}^{n} P\{\theta_i\} \log_2(P\{\theta_i\}) \tag{3.13}$$

$H(P)$ measures the randomness carried by any discrete probability measure $P\{.\}$. $H(P)$ is maximal for the uniform probability measure over Θ, i.e. when $P\{\theta_i\} = 1/n$ for $i = 1, 2, \ldots, n$. In that case, one gets:

[8]with common convention $0 \log_2 0 = 0$ as in [1].

$$H(P) = H_{\max} = -\sum_{i=1}^{n} \frac{1}{n} \log_2(\frac{1}{n}) = \log_2(n)$$

$H(P)$ is minimal for a totally *deterministic probability* measure, i.e. for any $P\{.\}$ such that $P\{\theta_i\} = 1$ for some $i \in \{1, 2, \ldots, n\}$ and $P\{\theta_j\} = 0$ for $j \neq i$.

3.6.2 The probabilistic information content

The Probabilistic Information Content (PIC) of a discrete probability measure $P\{.\}$ over a discrete finite set $\Theta = \{\theta_1, \ldots, \theta_n\}$ is defined by [18]:

$$PIC(P) = 1 + \frac{1}{H_{\max}} \cdot \sum_{i=1}^{n} P\{\theta_i\} \log_2(P\{\theta_i\}) \tag{3.14}$$

The PIC metric is nothing but the dual of the normalized Shannon entropy and is actually unitless. It actually measures the information content of a probabilistic source characterized by the probability measure $P\{.\}$. The $PIC(P)$ metric takes its values in $[0, 1]$ and is maximum, i.e. $PIC(P) = PIC_{\max} = 1$ with any *deterministic probability* measures. $PIC(P) = PIC_{\min} = 0$ when the probability measure is uniform over the frame Θ, i.e. $P\{\theta_i\} = 1/n$ for $i = 1, 2, \ldots, n$. The simple relationships between $H(P)$ and $PIC(P)$ are :

$$PIC(P) = 1 - \frac{H(P)}{H_{\max}} \tag{3.15}$$

$$H(P) = H_{\max} \cdot (1 - PIC(P)) \tag{3.16}$$

3.7 Examples and comparisons on a 2D frame

3.7.1 Example 1: Shafer's model with a general source

Let's consider the 2D frame $\Theta = \{A, B\}$ with Shafer's model (i.e. $A \cap B = \emptyset$) and the non-Bayesian quantitative belief assignment (mass) given in Table 3.1. In this example since one adopts Shafer's model for the frame Θ, G^Θ coincides with 2^Θ, i.e. $G^\Theta = 2^\Theta = \{\emptyset, A, B, A \cup B\}$.

	A	B	$A \cup B$
$m(.)$	0.3	0.1	0.6

Table 3.1: Quantitative input for example 3.7.1

Let's explain in details the derivations of the different transformations[9]:

- **With the pignistic probability**:

$$BetP(A) = m(A) + \frac{1}{2}m(A \cup B) = 0.3 + (0.6/2) = 0.60$$
$$BetP(B) = m(B) + \frac{1}{2}m(A \cup B) = 0.1 + (0.6/2) = 0.40$$

Since we are working with Shafer's model, the generalized pignistic probability given by (3.3) coincides with the classical pignistic probability.

- **With Sudano's probabilities**:

Applying Sudano's probabilities formulas (3.4)-(3.8), one gets:

- *With the probability* $PrPl(.)$:

$$PrPl(A) = 0.9 \cdot [0.3/0.9 + 0.6/(0.9 + 0.7)] = 0.6375$$
$$PrPl(B) = 0.7 \cdot [0.1/0.7 + 0.6/(0.9 + 0.7)] = 0.3625$$

- *With the probability* $PrBel(.)$:

$$PrBel(A) = 0.3 \cdot [0.3/0.3 + 0.6/(0.3 + 0.1)] = 0.7500$$
$$PrBel(B) = 0.1 \cdot [0.1/0.1 + 0.6/(0.3 + 0.1)] = 0.2500$$

- *With the probability* $PrNPl(.)$:

$$PrNPl(A) = 0.9/(0.9 + 0.7) = 0.5625$$
$$PrNPl(B) = 0.7/(0.9 + 0.7) = 0.4375$$

- *With the probability* $PraPl(.)$: $\epsilon = \frac{1-0.3-0.1}{0.9+0.7} = 0.375$

$$PraPl(A) = 0.3 + 0.375 \cdot 0.9 = 0.6375$$
$$PraPl(B) = 0.1 + 0.375 \cdot 0.7 = 0.3625$$

- *With the probability* $PrHyb(.)$:

$$PrHyb(A) = 0.6375 \cdot [\frac{0.3}{0.6375} + \frac{0.6}{0.6375 + 0.3625}] = 0.6825$$
$$PrHyb(B) = 0.3625 \cdot [\frac{0.1}{0.3625} + \frac{0.6}{0.6375 + 0.3625}] = 0.3175$$

[9] All results presented here are rounded to their fourth decimal place for convenience.

- **With Cuzzolin's probability**:

Since $TNSM = m(A \cup B) = 0.6$, $\Delta(A) = Pl(A) - m(A) = 0.6$ and $\Delta(B) = Pl(B) - m(B) = 0.6$, one gets

$$CuzzP(A) = m(A) + \frac{\Delta(A)}{\Delta(A)+\Delta(B)} \cdot TNSM = 0.3 + \frac{0.6}{0.6+0.6} \cdot 0.6 = 0.6000$$
$$CuzzP(B) = m(B) + \frac{\Delta(B)}{\Delta(A)+\Delta(B)} \cdot TNSM = 0.1 + \frac{0.6}{0.6+0.6} \cdot 0.6 = 0.4000$$

- **With $DSmP$ transformation**:

If one uses the DSmP formula (3.11) for this 2D case with Shafer's model, one gets:

$$\begin{aligned} DSmP_\epsilon(A) = & \frac{m(A)+\epsilon\cdot\mathcal{C}(A)}{m(A)+\epsilon\cdot\mathcal{C}(A)} \cdot m(A) + \frac{0}{m(B)+\epsilon\cdot\mathcal{C}(B)} \cdot m(B) \\ & + \frac{m(A)+\epsilon\cdot\mathcal{C}(A)}{m(A)+m(B)+\epsilon\cdot\mathcal{C}(A\cup B)} \cdot m(A\cup B) \end{aligned} \tag{3.17}$$

$$\begin{aligned} DSmP_\epsilon(B) = & \frac{0}{m(A)+\epsilon\cdot\mathcal{C}(A)} \cdot m(A) + \frac{m(B)+\epsilon\cdot\mathcal{C}(B)}{m(B)+\epsilon\cdot\mathcal{C}(B)} \cdot m(B) \\ & + \frac{m(B)+\epsilon\cdot\mathcal{C}(B)}{m(A)+m(B)+\epsilon\cdot\mathcal{C}(A\cup B)} \cdot m(A\cup B) \end{aligned} \tag{3.18}$$

$$\begin{aligned} DSmP_\epsilon(A\cup B) = & \frac{m(A)+\epsilon\cdot\mathcal{C}(A)}{m(A)+\epsilon\cdot\mathcal{C}(A)} \cdot m(A) + \frac{m(B)+\epsilon\cdot\mathcal{C}(B)}{m(B)+\epsilon\cdot\mathcal{C}(B)} \cdot m(B) \\ & + \frac{m(A)+m(B)+\epsilon\cdot\mathcal{C}(A\cup B)}{m(A)+m(B)+\epsilon\cdot\mathcal{C}(A\cup B)} \cdot m(A\cup B) \end{aligned} \tag{3.19}$$

Since we use Shafer's model in this example $\mathcal{C}(A) = \mathcal{C}(B) = 1$ and $\mathcal{C}(A \cup B) = 2$ and finally one gets with the DSmP transformation the following analytical expressions:

$$DSmP_\epsilon(A) = m(A) + \frac{m(A)+\epsilon}{m(A)+m(B)+2\cdot\epsilon} \cdot m(A\cup B)$$

$$DSmP_\epsilon(B) = m(B) + \frac{m(B)+\epsilon}{m(A)+m(B)+2\cdot\epsilon}\cdot m(A\cup B)$$
$$DSmP_\epsilon(A\cup B) = m(A)+m(B)+m(A\cup B) = 1$$

One can verify that the expressions of $DSmP_\epsilon(A)$ and $DSmP_\epsilon(B)$ are also consistent with the formula (3.12) and it can be easily verified that

$$DSmP_\epsilon(A) + DSmP_\epsilon(B) = DSmP_\epsilon(A\cup B) = 1.$$

- Applying formula (3.11) (or equivalently the three previous expressions) for $\epsilon = 0.001$ yields:

$$DSmP_{\epsilon=0.001}(A) \approx 0.3 + 0.4492 = 0.7492$$
$$DSmP_{\epsilon=0.001}(B) \approx 0.1 + 0.1508 = 0.2508$$
$$DSmP_{\epsilon=0.001}(A\cup B) = 1$$

- Applying formula (3.11) for $\epsilon = 0$ yields[10]:

$$DSmP_{\epsilon=0}(A) = 0.3 + 0.45 = 0.75$$
$$DSmP_{\epsilon=0}(B) = 0.1 + 0.15 = 0.25$$
$$DSmP_{\epsilon=0}(A\cup B) = 1$$

	A	B	$PIC(.)$
$PrNPl(.)$	0.5625	0.4375	0.0113
$BetP(.)$	0.6000	0.4000	0.0291
$CuzzP(.)$	0.6000	0.4000	0.0291
$PrPl(.)$	0.6375	0.3625	0.0553
$PraPl(.)$	0.6375	0.3625	0.0553
$PrHyb(.)$	0.6825	0.3175	0.0984
$DSmP_{\epsilon=0.001}(.)$	0.7492	0.2508	0.1875
$PrBel(.)$	0.7500	0.2500	0.1887
$DSmP_{\epsilon=0}(.)$	0.7500	0.2500	0.1887

Table 3.2: Results for example 3.7.1.

Results: We summarize in Table 3.2, the results of the subjective probabilities and their corresponding PIC values sorted by increasing values. It is interesting to note that $DSmP_{\epsilon\to 0}(.)$ provides same result as with $PrBel(.)$ and $PIC(DSmP_{\epsilon\to 0}(.))$ is greater than the PIC values obtained with $PrNPL$, $BetP$, $CuzzP$, $PrPl$ and $PraPl$ transformations.

[10]It is possible since the masses of A and B are not zero, so we actually get a proportionalization with respect to masses only.

3.7.2 Example 2: Shafer's model with the ignorant source

Let's consider the 2D frame $\Theta = \{A, B\}$ with Shafer's model (i.e. $A \cap B = \emptyset$) and the vacuous belief mass characterizing the totally ignorant source given in Table 3.3.

	A	B	$A \cup B$
$m(.)$	0	0	1

Table 3.3: Vacuous belief mass for example 3.7.2

- **With the pignistic probability**:

$$BetP(A) = BetP(B) = 0 + (1/2) = 0.5$$

- **With Sudano's probabilities**:

Applying Sudano's probabilities formulas (3.4)-(3.8), one gets:

- *Probability* $PrPl(.)$:

$$PrPl(A) = PrPl(B) = 1 \cdot [0/1 + 1/(1+1)] = 0.5$$

- *With the probability* $PrBel(.)$:

$$PrBel(A) = PrBel(A) = 0 \cdot [0/0 + 1/(0+0)] = NaN$$

- *With the probability* $PrNPl(.)$:

$$PrNPl(A) = PrNPl(B) = 1/(1+1) = 0.5$$

- *With the probability* $PraPl(.)$: $\epsilon = \frac{1-0-0}{1+1} = 0.5$

$$PraPl(A) = PraPl(B) = 0 + 0.5 \cdot 1 = 0.5$$

- *With the probability* $PrHyb(.)$:

$$PrHyb(A) = PrHyb(B) = 0.5 \cdot [\frac{0}{0.5} + \frac{1}{0.5+0.5}] = 0.5$$

- **With Cuzzolin's probability**:

Since $TNSM = m(A \cup B) = 1$, $\Delta(A) = Pl(A) - m(A) = 1$ and $\Delta(B) = Pl(B) - m(B) = 1$, one gets

$$CuzzP(A) = CuzzP(B) = 0 + \frac{1}{1+1} \cdot 1 = 0.5$$

- **With $DSmP$ transformation**:

Applying formula (3.11) (or (3.12) since we work here with Shafer's model) for $\epsilon > 0$ yields[11]:

$$\begin{aligned} DSmP_{\epsilon>0}(A) &= m(A \cup B)/2 = 0.5 \\ DSmP_{\epsilon>0}(B) &= m(A \cup B)/2 = 0.5 \\ DSmP_{\epsilon>0}(A \cup B) &= 1 \end{aligned}$$

In the particular case of the totally ignorant source characterized by the vacuous belief assignment, all transformations coincide with the uniform probability measure over singletons of Θ, except $PrBel(.)$ which is mathematically not defined in that case. This result can be easily proved for any size of the frame Θ with $|\Theta| > 2$. We summarize in Table 3.4, the results of the subjective probabilities and their corresponding PIC values.

	A	B	$PIC(.)$
$PrBel(.)$	NaN	NaN	NaN
$BetP(.)$	0.5	0.5	0
$PrPl(.)$	0.5	0.5	0
$PrNPl(.)$	0.5	0.5	0
$PraPl(.)$	0.5	0.5	0
$PrHyb(.)$	0.5	0.5	0
$CuzzP(.)$	0.5	0.5	0
$DSmP_{\epsilon>0}(.)$	0.5	0.5	0

Table 3.4: Results for example 3.7.2.

[11] It is not possible to apply the DSmP formula for $\epsilon = 0$ in this particular case, but ϵ can be chosen as small as we want.

3.7.3 Example 3: Shafer's model with a probabilistic source

Let's consider the 2D frame $\Theta = \{A, B\}$ and let's assume Shafer's model and let's see what happens when applying all the transformations on a probabilistic source[12] which commits a belief mass only to singletons of 2^Θ, i.e. a Bayesian mass [10]. It is intuitively expected that all transformations are idempotent when dealing with probabilistic sources, since actually there is no reason/need to modify $m(.)$ (the input mass) to obtain a new subjective probability measure since $Bel(.)$ associated with $m(.)$ is already a probability measure.

If we consider, for example, the uniform probabilistic mass given in Table 3.5, it is very easy to verify in this case, that almost all transformations coincide with the probabilistic input mass as expected, so that the idempotency property is satisfied.

	A	B	$A \cup B$
$m_u(.)$	0.5	0.5	0

Table 3.5: Uniform probabilistic mass for example 3.7.3

Only Cuzzolin's transformation fails to satisfy this property because in $CuzzP(.)$ formula (3.9) one gets 0/0 indetermination since all $\Delta(.) = 0$ in (3.9). This remark is valid whatever the dimension of the frame is, and for any probabilistic mass, not only for uniform belief mass $m_u(.)$. We summarize in Table 3.6, the results of the subjective probabilities and their corresponding PIC values:

	A	B	$PIC(.)$
$CuzzP(.)$	NaN	NaN	NaN
$BetP(.)$	0.5	0.5	0
$PrPl(.)$	0.5	0.5	0
$PrNPl(.)$	0.5	0.5	0
$PraPl(.)$	0.5	0.5	0
$PrHyb(.)$	0.5	0.5	0
$PrBel(.)$	0.5	0.5	0
$DSmP_\epsilon(.)$	0.5	0.5	0

Table 3.6: Results for example 3.7.3.

[12]This has obviously no practical interest since the source already provides a probability measure, nevertheless this is very interesting to see the theoretical behavior of the transformations in such case.

3.7.4 Example 4: Shafer's model with a non-Bayesian mass

Let's assume Shafer's model and the non-Bayesian mass (more precisely the simple support mass) given in Table 3.7. We summarize in Table 3.8, the results obtained with all transformations. One sees that $PIC(DSmP_{\epsilon\to 0})$ is maximum among all PIC values. $PrBel(.)$ does not work correctly since it can not have a division by zero; even overcoming it[13], $PrBel$ does not do a fair redistribution of the ignorance $m(A \cup B) = 0.6$ because B does not receive anything from the mass 0.6, although B is involved in the ignorance $A \cup B$. All $m(A \cup B) = 0.6$ was unfairly redistributed to A only.

	A	B	$A \cup B$
$m(.)$	0.4	0	0.6

Table 3.7: Quantitative input for example 3.7.4

	A	B	$PIC(.)$
$PrBel(.)$	1	NaN	NaN
$PrNPl(.)$	0.6250	0.3750	0.0455
$BetP(.)$	0.7000	0.3000	0.1187
$CuzzP(.)$	0.7000	0.3000	0.1187
$PrPl(.)$	0.7750	0.2250	0.2308
$PraPl(.)$	0.7750	0.2250	0.2308
$PrHyb(.)$	0.8650	0.1350	0.4291
$DSmP_{\epsilon=0.001}(.)$	0.9985	0.0015	0.9838
$DSmP_{\epsilon=0}(.)$	1	0	1

Table 3.8: Results for example 3.7.4.

3.7.5 Example 5: Free DSm model

Let's consider the 2D frame $\Theta = \{A, B\}$ with the free DSm model (i.e. $A \cap B \neq \emptyset$) and the following generalized quantitative belief given in Table 3.9. In the case of free-DSm (or hybrid DSm) models, the pignistic probability $BetP$ and the $DSmP$ can be derived directly from $m(.)$ without the ultimate refinement of the frame Θ whereas Sudano's and Cuzzolin's probabilities cannot be derived

[13]since the *direct* derivation of $PrBel(B)$ cannot be done from the formula (3.5) because of the undefined form 0/0, we could however force it to $PrBel(B) = 0$ since $PrBel(B) = 1 - PrBel(A) = 1 - 1 = 0$, and consequently we indirectly take $PIC(PrBel) = 1$.

directly from the formulas (3.4)-(3.9) in such models. However, Sudano's and Cuzzolin's probabilities can be obtained indirectly after an intermediary step of ultimate refinement of the frame Θ into Θ^{ref} which satisfies Shafer's model. More precisely, instead of working directly on the 2D frame $\Theta = \{A, B\}$ with $m(.)$ given in Table 3.9, we need to work on the 3D frame $\Theta^{\text{ref}} = \{A' \triangleq A \setminus \{A \cap B\}, B' \triangleq B \setminus \{A \cap B\}, C' \triangleq A \cap B\}$ satisfying Shafer's model with the equivalent bba $m(.)$ defined as in Table 3.10.

	$A \cap B$	A	B	$A \cup B$
$m(.)$	0.4	0.2	0.1	0.3

Table 3.9: Quantitative input on the original frame Θ

	C'	$A' \cup C'$	$B' \cup C'$	$A' \cup B' \cup C'$
$m(.)$	0.4	0.2	0.1	0.3

Table 3.10: Quantitative equivalent input on the refined frame Θ^{ref}

- **With the pignistic probability**: With the generalized pignistic transformation [12] (Chap.7, p. 148), one gets:

$$\begin{aligned} BetP(A) &= m(A) + \frac{m(B)}{2} + m(A \cap B) + \frac{2}{3} m(A \cup B) \\ &= 0.2 + 0.05 + 0.4 + 0.2 = 0.85 \end{aligned}$$

$$\begin{aligned} BetP(B) &= m(B) + \frac{m(A)}{2} + m(A \cap B) + \frac{2}{3} m(A \cup B) \\ &= 0.1 + 0.1 + 0.4 + 0.2 = 0.80 \end{aligned}$$

$$\begin{aligned} BetP(A \cap B) &= \frac{m(A)}{2} + \frac{m(B)}{2} + m(A \cap B) + \frac{1}{3} m(A \cup B) \\ &= 0.1 + 0.05 + 0.4 + 0.1 = 0.65 \end{aligned}$$

We can easily check that

$$BetP(A \cup B) = BetP(A) + BetP(B) - BetP(A \cap B) = 0.85 + 0.80 - 0.65 = 1$$

- **With Sudano's probabilities**: Working on the refined frame Θ^{ref}, with the bba $m(.)$ defined in Table 3.10, one finally obtains from (3.4)-(3.8):

- *With the probability* $PrPl(.)$:

$$PrPl(A') \approx 0.1456$$
$$PrPl(B') \approx 0.0917$$
$$PrPl(C') \approx 0.7627$$

so that:

$$PrPl(A) = 0.1456 + 0.7627 = 0.9083$$
$$PrPl(B) = 0.0917 + 0.7627 = 0.8544$$
$$PrPl(A \cap B) = PrPl(C') = 0.7627$$

- *With the probability* $PrBel(.)$: It cannot be directly computed by (3.5) because of the division by zero involved in derivation of $PrBel(A')$ and $PrBel(B')$, i.e. formally one gets

$$PrBel(A') = NaN$$
$$PrBel(B') = NaN$$
$$PrBel(C') = 1$$

But because $PrBel(C') = 1$, one can set artificially/indirectly $PrBel(A') = 0$ and $PrBel(B') = 0$, so that:

$$PrBel(A) = NaN + 1 \approx 0 + 1 = 1$$
$$PrBel(B) = NaN + 1 \approx 0 + 1 = 1$$
$$PrBel(A \cap B) = 1$$

but fundamentally, $PrBel(A) = NaN$ and $PrBel(B) = NaN$ from $PrBel(.)$ formula.

- *With the probability* $PrNPl(.)$:

$$PrNPl(A') \approx 0.2632$$
$$PrNPl(B') \approx 0.2105$$
$$PrNPl(C') \approx 0.5263$$

so that:

$$PrNPl(A) = 0.2632 + 0.5263 = 0.7895$$
$$PrNPl(B) = 0.2105 + 0.5263 = 0.7368$$
$$PrNPl(A \cap B) = PrNPl(C') = 0.5263$$

- *With the probability* $PraPl(.)$: $\epsilon \approx 0.3157$

$$PraPl(A') \approx 0.1579$$
$$PraPl(B') \approx 0.1264$$
$$PraPl(C') \approx 0.7157$$

so that:

$$PraPl(A) = 0.1579 + 0.7157 = 0.8736$$
$$PraPl(B) = 0.1264 + 0.7157 = 0.8421$$
$$PraPl(A \cap B) = PraPl(C') = 0.7157$$

- *With the probability* $PrHyb(.)$:

$$PrHyb(A') \approx 0.0835$$
$$PrHyb(B') \approx 0.0529$$
$$PrHyb(C') \approx 0.8636$$

so that:

$$PrHyb(A) = 0.0835 + 0.8636 = 0.9471$$
$$PrHyb(B) = 0.0529 + 0.8636 = 0.9165$$
$$PrHyb(A \cap B) = PrHyb(C') = 0.8636$$

- **With Cuzzolin's probability**: Working on the refined frame Θ^{ref}, with the bba $m(.)$ defined in Table 3.10, one has $TNSM = m(A' \cup C') + m(B' \cup C') + m(A' \cup B' \cup C') = 0.6$, $\Delta(A') = 0.5$, $\Delta(B') = 0.4$ and $\Delta(C') = 0.4$. Therefore:

$$\begin{aligned}
CuzzP(A') &= m(A') + \frac{\Delta(A')}{\Delta(A') + \Delta(B') + \Delta(C')} \cdot TNSM \\
&= 0 + \frac{0.5}{0.5 + 0.4 + 0.6} \cdot 0.6 = 0.20 \\
CuzzP(B') &= m(B') + \frac{\Delta(B')}{\Delta(A') + \Delta(B') + \Delta(C')} \cdot TNSM \\
&= 0 + \frac{0.4}{0.5 + 0.4 + 0.6} \cdot 0.6 = 0.16 \\
CuzzP(C') &= m(C') + \frac{\Delta(C')}{\Delta(A') + \Delta(B') + \Delta(C')} \cdot TNSM \\
&= 0.4 + \frac{0.6}{0.5 + 0.4 + 0.6} \cdot 0.6 = 0.64
\end{aligned}$$

which finally gives:

$$\begin{aligned} CuzzP(A) &= CuzzP(A') + CuzzP(C') = 0.84 \\ CuzzP(B) &= CuzzP(B') + CuzzP(C') = 0.80 \\ CuzzP(A \cap B) &= CuzzP(C') = 0.64 \end{aligned}$$

- **With $DSmP$ transformation**:

If one uses the DSmP formula (3.11) for this 2D case with the free DSm model where $\mathcal{C}(A \cap B) = 1$, $\mathcal{C}(A) = \mathcal{C}(B) = 2$ and $\mathcal{C}(A \cup B) = 3$, one gets the following analytical expressions of $DSmP_\epsilon(.)$ (assuming all denominators are strictly positive):

$$\begin{aligned} DSmP_\epsilon(A \cap B) = m(A \cap B) + \frac{m(A \cap B) + \epsilon}{m(A \cap B) + 2 \cdot \epsilon} \cdot m(A) + \frac{m(A \cap B) + \epsilon}{m(A \cap B) + 2 \cdot \epsilon} \cdot m(B) \\ + \frac{m(A \cap B) + \epsilon}{m(A \cap B) + 3 \cdot \epsilon} \cdot m(A \cup B) \end{aligned} \tag{3.20}$$

$$\begin{aligned} DSmP_\epsilon(A) = m(A \cap B) + m(A) + \frac{m(A \cap B) + \epsilon}{m(A \cap B) + 2 \cdot \epsilon} \cdot m(B) \\ + \frac{m(A \cap B) + 2 \cdot \epsilon}{m(A \cap B) + 3 \cdot \epsilon} \cdot m(A \cup B) \end{aligned} \tag{3.21}$$

$$\begin{aligned} DSmP_\epsilon(B) = m(A \cap B) + m(B) + \frac{m(A \cap B) + \epsilon}{m(A \cap B) + 2 \cdot \epsilon} \cdot m(A) \\ + \frac{m(A \cap B) + 2 \cdot \epsilon}{m(A \cap B) + 3 \cdot \epsilon} \cdot m(A \cup B) \end{aligned} \tag{3.22}$$

$$DSmP_\epsilon(A \cup B) = m(A \cap B) + m(A) + m(B) + m(A \cup B) = 1 \tag{3.23}$$

- Applying formula (3.11) for $\epsilon = 0.001$ yields:

$$\begin{aligned} &DSmP_{\epsilon=0.001}(A \cap B) \approx 0.9978 \\ &DSmP_{\epsilon=0.001}(A) \approx 0.9990 \\ &DSmP_{\epsilon=0.001}(B) \approx 0.9988 \\ &DSmP_{\epsilon=0.001}(A \cup B) = 1 \end{aligned}$$

which induces the underlying probability measure on the refined frame

$$p_1 = P(A \setminus (A \cap B)) \approx 0.0012$$
$$p_2 = P(B \setminus (A \cap B)) \approx 0.0010$$
$$p_3 = P(A \cap B) \approx 0.9978$$

This yields to $PIC \approx 0.9842$.

- Applying formula (3.11) for $\epsilon = 0$ yields[14]: One gets

$$DSmP_{\epsilon=0}(A \cap B) = 1 \qquad DSmP_{\epsilon=0}(A) = 1$$
$$DSmP_{\epsilon=0}(A \cup B) = 1 \qquad DSmP_{\epsilon=0}(B) = 1$$

which induces the underlying probability measure on the refined frame

$$p_1 = P(A \setminus (A \cap B)) = 0$$
$$p_2 = P(B \setminus (A \cap B)) = 0$$
$$p_3 = P(A \cap B) = 1$$

which yields the maximum PIC value, i.e. $PIC = 1$.

We summarize in Table 3.11, the results of the subjective probabilities and their corresponding PIC values sorted in increasing order:

	A	B	$A \cap B$	$PIC(.)$
$PrNPl(.)$	0.7895	0.7368	0.5263	0.0741
$CuzzP(.)$	0.8400	0.8000	0.6400	0.1801
$BetP(.)$	0.8500	0.8000	0.6500	0.1931
$PraPl(.)$	0.8736	0.8421	0.7157	0.2789
$PrPl(.)$	0.9083	0.8544	0.7627	0.3570
$PrHyb(.)$	0.9471	0.9165	0.8636	0.5544
$DSmP_{\epsilon=0.001}(.)$	0.9990	0.9988	0.9978	0.9842
$PrBel(.)$	NaN	NaN	1	1
$DSmP_{\epsilon=0}(.)$	1	1	1	1

Table 3.11: Results for example 3.7.5.

[14] It is possible since the mass of $A \cap B$ is not zero.

From Table 3.11, one sees that $PIC(DSmP_{\epsilon\to 0})$ is the maximum value. $PrBel$ does not work correctly because it cannot be directly evaluated for A and B since the underlying $PrBel(A')$ and $PrBel(B')$ are mathematically undefined in such case.

Remark: If one works on the *refined frame* Θ^{ref} and one applies the $DSmP$ mapping of the bba $m(.)$ defined in Table 3.10, one obtains naturally the same results for $DSmP$ as those given in table 3.11. Of course the results of $BetP$ in Table 3.11 are the same using directly the formula (3.3) as those using (3.1) on Θ^{ref}.

Proof: Applying (3.11) with Shafer's model for $m(.)$ defined in Table 3.10, one gets directly the $DSmP_\epsilon$ values of atomic elements A', B' and C' of the refined frame Θ^{ref}, i.e.:

$$\begin{aligned} DSmP_\epsilon(A') = & \frac{\epsilon\cdot\mathcal{C}(A')}{m(C')+\epsilon\cdot\mathcal{C}(A'\cup C')}\cdot m(A'\cup C') \\ & + \frac{\epsilon\cdot\mathcal{C}(A')}{m(C')+\epsilon\cdot\mathcal{C}(A'\cup B'\cup C')}\cdot m(A'\cup B'\cup C') \end{aligned} \tag{3.24}$$

$$\begin{aligned} DSmP_\epsilon(B') = & \frac{\epsilon\cdot\mathcal{C}(B')}{m(C')+\epsilon\cdot\mathcal{C}(B'\cup C')}\cdot m(B'\cup C') \\ & + \frac{\epsilon\cdot\mathcal{C}(B')}{m(C')+\epsilon\cdot\mathcal{C}(A'\cup B'\cup C')}\cdot m(A'\cup B'\cup C') \end{aligned} \tag{3.25}$$

$$\begin{aligned} DSmP_\epsilon(C') = & \frac{m(C')+\epsilon\cdot\mathcal{C}(C')}{m(C')+\epsilon\cdot\mathcal{C}(C')}\cdot m(C') \\ & + \frac{m(C')+\epsilon\cdot\mathcal{C}(C')}{m(C')+\epsilon\cdot\mathcal{C}(A'\cup C')}\cdot m(A'\cup C') + \frac{m(C')+\epsilon\cdot\mathcal{C}(C')}{m(C')+\epsilon\cdot\mathcal{C}(B'\cup C')}\cdot m(B'\cup C') \\ & + \frac{m(C')+\epsilon\cdot\mathcal{C}(C')}{m(C')+\epsilon\cdot\mathcal{C}(A'\cup B'\cup C')}\cdot m(A'\cup B'\cup C') \end{aligned} \tag{3.26}$$

Since on the refined frame with Shafer's model, $\mathcal{C}(A')=\mathcal{C}(B')=\mathcal{C}(C')=1$, $\mathcal{C}(A'\cup B')=\mathcal{C}(A'\cup C')=\mathcal{C}(B'\cup C')=2$ and $\mathcal{C}(A'\cup B'\cup C')=3$, the previous expressions can be simplified as:

$$DSmP_\epsilon(A') = \frac{\epsilon}{m(C')+2\cdot\epsilon}\cdot m(A'\cup C') + \frac{\epsilon}{m(C')+3\cdot\epsilon}\cdot m(A'\cup B'\cup C') \tag{3.27}$$

$$DSmP_{\epsilon}(B') = \frac{\epsilon}{m(C') + 2\cdot\epsilon}\cdot m(B'\cup C') + \frac{\epsilon}{m(C') + 3\cdot\epsilon}\cdot m(A'\cup B'\cup C') \tag{3.28}$$

$$\begin{aligned} DSmP_{\epsilon}(C') = m(C') &+ \frac{m(C')+\epsilon}{m(C') + 2\cdot\epsilon}\cdot m(A'\cup C') + \frac{m(C')+\epsilon}{m(C') + 2\cdot\epsilon}\cdot m(B'\cup C') \\ &+ \frac{m(C')+\epsilon}{m(C') + 3\cdot\epsilon}\cdot m(A'\cup B'\cup C') \end{aligned} \tag{3.29}$$

One sees that the expressions of $DSmP_{\epsilon}(A')$, $DSmP_{\epsilon}(B')$ and $DSmP_{\epsilon}(C')$ we obtain here, coincide with the expressions that one would obtain by applying directly the formula (3.12) specifically when Shafer's model holds (i.e. when a ultimate refined frame is used). It can be easily verified that:

$$DSmP_{\epsilon}(A') + DSmP_{\epsilon}(B') + DSmP_{\epsilon}(C') = 1$$

Replacing ϵ and $m(C')$, $m(A'\cup C')$, $m(B'\cup C')$ and $m(A'\cup B'\cup C')$ by their numerical values, one gets the same numerical values as those given by p_1, p_2 and p_3. For example if $\epsilon = 0.001$, one obtains from the previous expressions:

$$DSmP_{\epsilon=0.001}(A') = \frac{0.001}{0.4 + 2\cdot 0.001}\cdot 0.2 + \frac{0.001}{0.4 + 3\cdot 0.001}\cdot 0.3 \approx 0.0012$$

$$DSmP_{\epsilon=0.001}(B') = \frac{0.001}{0.4 + 2\cdot 0.001}\cdot 0.1 + \frac{0.001}{0.4 + 3\cdot 0.001}\cdot 0.3 \approx 0.0010$$

$$\begin{aligned} DSmP_{\epsilon=0.001}(C') = 0.4 &+ \frac{0.4+0.001}{0.4 + 2\cdot 0.001}\cdot 0.2 \\ &+ \frac{0.4+0.001}{0.4 + 2\cdot 0.001}\cdot 0.1 + \frac{0.4+0.001}{0.4 + 3\cdot 0.001}\cdot 0.3 \approx 0.9978 \end{aligned}$$

From the probabilities of these atomic elements A', B' and C', one can easily compute the probability of $A\cap B = C'$, $A = A'\cup C'$, $B = B'\cup C'$ and $A\cup B = A'\cup B'\cup C'$ by:

$$DSmP_{\epsilon}(A\cap B) = DSmP_{\epsilon}(C')$$

$$DSmP_{\epsilon}(A) = DSmP_{\epsilon}(A') + DSmP_{\epsilon}(C')$$

$$DSmP_\epsilon(B) = DSmP_\epsilon(B') + DSmP_\epsilon(C')$$

$$DSmP_\epsilon(A \cup B) = DSmP_\epsilon(A') + DSmP_\epsilon(B') + DSmP_\epsilon(C')$$

Therefore, for $\epsilon = 0.001$, one obtains:

$$DSmP_{\epsilon=0.1}(A \cap B) = 0.9978$$

$$DSmP_{\epsilon=0.1}(A) = 0.0012 + 0.9978 = 0.9990$$

$$DSmP_{\epsilon=0.1}(B) = 0.0010 + 0.9978 = 0.9988$$

$$DSmP_{\epsilon=0.1}(A \cup B) = 0.0012 + 0.0010 + 0.9978 = 1$$

We can verify that this result is the same result as the one obtained directly with formula 3.11 when one uses the free DSm model (see 7th row of the Table 3.11). This completes the proof.

3.8 Examples on a 3D frame

3.8.1 Example 6: Shafer's model with a non-Bayesian mass

This example is drawn from [21]. Let's consider the 3D frame $\Theta = \{A, B, C\}$ with Shafer's model and the following non-Bayesian quantitative belief mass.

	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
$m(.)$	0.35	0.25	0.02	0.20	0.07	0.05	0.06

Table 3.12: Quantitative input for example 3.8.1

- **With the pignistic probability**: Applying formula (3.1), one gets

$$BetP(A) = 0.35 + \frac{0.20}{2} + \frac{0.07}{2} + \frac{0.06}{3} = 0.5050$$
$$BetP(B) = 0.25 + \frac{0.20}{2} + \frac{0.05}{2} + \frac{0.06}{3} = 0.3950$$
$$BetP(C) = 0.02 + \frac{0.07}{2} + \frac{0.05}{2} + \frac{0.06}{3} = 0.1000$$

- **With Sudano's probabilities**: The belief and plausibility ao A, B and C are

$$Bel(A) = 0.35 \qquad Bel(B) = 0.25 \qquad Bel(C) = 0.02$$

$$Pl(A) = 0.68 \qquad Pl(B) = 0.56 \qquad Pl(C) = 0.20$$

Applying formulas (3.4)-(3.8), one obtains the following Sudano's probabilities:

– *With the probability $PrPl(.)$*:

$$PrPl(A) = Pl(A) \cdot [\frac{m(A)}{Pl(A)} + \frac{m(A \cup B)}{Pl(A) + Pl(B)} + \frac{m(A \cup C)}{Pl(A) + Pl(C)} + \frac{m(A \cup B \cup C)}{Pl(A) + Pl(B) + Pl(C)}]$$
$$= 0.68 \cdot [\frac{0.35}{0.68} + \frac{0.20}{1.24} + \frac{0.07}{0.88} + \frac{0.06}{1.44}] \approx 0.5421$$

and similarly,

$$PrPl(B) \approx 0.4005 \qquad PrPl(C) \approx 0.0574$$

– *With the probability $PrBel(.)$*:

$$PrBel(A) = Bel(A) \cdot [\frac{m(A)}{Bel(A)} + \frac{m(A \cup B)}{Bel(A) + Bel(B)} + \frac{m(A \cup C)}{Bel(A) + Bel(C)} + \frac{m(A \cup B \cup C)}{Bel(A) + Bel(B) + Bel(C)}]$$
$$= 0.35 \cdot [\frac{0.35}{0.35} + \frac{0.20}{0.60} + \frac{0.07}{0.37} + \frac{0.06}{0.62}] \approx 0.5668$$

and similarly,

$$PrBel(B) \approx 0.4038 \qquad PrBel(C) \approx 0.0294$$

– *With the probability $PrNPl(.)$*:

$$PrNPl(A) = \frac{1}{\Delta} Pl(A) = \frac{Pl(A)}{Pl(A) + Pl(B) + Pl(C)} = \frac{0.68}{1.44} \approx 0.4722$$

and similarly,

$$PrNPl(B) \approx 0.3889 \qquad PrNPl(C) \approx 0.1389$$

– *With the probability $PraPl(.)$*: Applying formula (3.7), one gets

$$\epsilon = \frac{1 - Bel(A) - Bel(B) - Bel(C)}{Pl(A) + Pl(B) + Pl(C)} = \frac{0.38}{1.44}$$

$$PraPl(A) = Bel(A) + \epsilon \cdot Pl(A) = 0.35 + \frac{0.38}{1.44} \cdot 0.68 \approx 0.5294$$

and similarly,

$$PraPl(B) \approx 0.3978 \qquad PraPl(C) \approx 0.0728$$

– *With the probability* $PrHyb(.)$:

$$PrHyb(A) = PraPl(A) \cdot [\frac{m(A)}{PraPl(A)} + \frac{m(A \cup B)}{PraPl(A) + PraPl(B)} + \frac{m(A \cup C)}{PraPl(A) + PraPl(C)} + \frac{m(A \cup B \cup C)}{PraPl(A) + PraPl(B) + PraPl(C)}] \approx 0.5575$$

and similarly,

$$PrHyb(B) \approx 0.4019 \qquad PrHyb(C) \approx 0.0406$$

- **With Cuzzolin's probability**: Since $TNSM = m(A \cup B) = 0.38$, $\Delta(A) = Pl(A) - m(A) = 0.33$, $\Delta(B) = Pl(B) - m(B) = 0.31$ and $\Delta(C) = Pl(C) - m(C) = 0.18$, one gets:

$$CuzzP(A) = 0.35 + \frac{0.33}{0.33 + 0.31 + 0.18} \cdot 0.38 \approx 0.5029$$
$$CuzzP(B) = 0.25 + \frac{0.31}{0.33 + 0.31 + 0.18} \cdot 0.38 \approx 0.3937$$
$$CuzzP(C) = 0.02 + \frac{0.18}{0.33 + 0.31 + 0.18} \cdot 0.38 \approx 0.1034$$

- **With $DSmP$ transformation**:

- Applying formula (3.11) for $\epsilon = 0.001$ yields:

$$DSmP_{\epsilon=0.001}(A) \approx 0.5665$$
$$DSmP_{\epsilon=0.001}(B) \approx 0.4037$$
$$DSmP_{\epsilon=0.001}(C) \approx 0.0298$$
$$DSmP_{\epsilon=0.001}(A \cup B) \approx 0.9702$$
$$DSmP_{\epsilon=0.001}(A \cup C) \approx 0.5963$$
$$DSmP_{\epsilon=0.001}(B \cup C) \approx 0.4335$$
$$DSmP_{\epsilon=0.001}(A \cup B \cup C) = 1$$

- Applying formula (3.11) for $\epsilon = 0$ yields:

$$
\begin{aligned}
DSmP_{\epsilon=0}(A) &\approx 0.5668\\
DSmP_{\epsilon=0}(B) &\approx 0.4038\\
DSmP_{\epsilon=0}(C) &\approx 0.0294\\
DSmP_{\epsilon=0}(A\cup B) &\approx 0.9706\\
DSmP_{\epsilon=0}(A\cup C) &\approx 0.5962\\
DSmP_{\epsilon=0}(B\cup C) &\approx 0.4332\\
DSmP_{\epsilon=0}(A\cup B\cup C) &= 1
\end{aligned}
$$

We summarize in Table 3.13, the results of the subjective probabilities and their corresponding PIC values sorted in increasing order.

	A	B	C	$PIC(.)$
$PrNPl(.)$	0.4722	0.3889	0.1389	0.0936
$CuzzP(.)$	0.5029	0.3937	0.1034	0.1377
$BetP(.)$	0.5050	0.3950	0.1000	0.1424
$PraPl(.)$	0.5294	0.3978	0.0728	0.1861
$PrPl(.)$	0.5421	0.4005	0.0574	0.2149
$PrHyb(.)$	0.5575	0.4019	0.0406	0.2517
$DSmP_{\epsilon=0.001}(.)$	0.5665	0.4037	0.0298	0.2783
$PrBel(.)$	0.5668	0.4038	0.0294	0.2793
$DSmP_{\epsilon=0}(.)$	0.5668	0.4038	0.0294	0.2793

Table 3.13: Results for example 3.8.1.

One sees that $DSmP_{\epsilon\to 0}$ provides the same result as $PrBel$ which corresponds the best result in term of PIC for this example.

3.8.2 Example 7: Shafer's model with another non-Bayesian mass

Let's consider the 3D frame $\Theta = \{A, B, C\}$ with Shafer's model and the following non-Bayesian quantitative belief mass:

	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
$m(.)$	0.1	0	0.2	0.3	0.1	0	0.3

Table 3.14: Quantitative input for example 3.8.2

- **With the pignistic probability**:

$$BetP(A) = 0.1 + \frac{0.3}{2} + \frac{0.1}{2} + \frac{0.3}{3} = 0.40$$
$$BetP(B) = 0 + \frac{0.3}{2} + \frac{0.3}{3} = 0.25$$
$$BetP(C) = 0.2 + \frac{0.1}{2} + \frac{0.3}{3} = 0.35$$

- **With Sudano's probabilities**: The belief and plausibility of A, B and C are

$$Bel(A) = 0.10 \qquad Bel(B) = 0 \qquad Bel(C) = 0.20$$
$$Pl(A) = 0.80 \qquad Pl(B) = 0.60 \qquad Pl(C) = 0.60$$

Applying formulas (3.4)-(3.8), one obtains:

  – *With the probability $PrPl(.)$*:

$$\begin{aligned} PrPl(A) = Pl(A) \cdot [&\frac{m(A)}{Pl(A)} + \frac{m(A \cup B)}{Pl(A) + Pl(B)} \\ &+ \frac{m(A \cup C)}{Pl(A) + Pl(C)} + \frac{m(A \cup B \cup C)}{Pl(A) + Pl(B) + Pl(C)}] \\ = 0.80 \cdot [&\frac{0.10}{0.80} + \frac{0.30}{1.40} + \frac{0.10}{1.40} + \frac{0.30}{2}] \approx 0.4486 \end{aligned}$$

  and similarly,

$$PrPl(B) \approx 0.2186 \qquad PrPl(C) \approx 0.3328$$

  – *With the probability $PrBel(.)$*:

$$\begin{aligned} PrBel(A) = Bel(A) \cdot [&\frac{m(A)}{Bel(A)} + \frac{m(A \cup B)}{Bel(A) + Bel(B)} \\ &+ \frac{m(A \cup C)}{Bel(A) + Bel(C)} + \frac{m(A \cup B \cup C)}{Bel(A) + Bel(B) + Bel(C)}] \\ = 0.10 \cdot [&\frac{0.10}{0.80} + \frac{0.30}{0.10} + \frac{0.10}{0.30} + \frac{0.30}{0.30}] \approx 0.5333 \end{aligned}$$

$PrBel(C) \approx 0.4667$ but from the formula (3.5), one gets $PrBel(B) = NaN$ because of the division by zero. Since $PrBel(A)+PrBel(B)+PrBel(C)$ must be one, one could circumvent the problem by taking $PrBel(B) = 0$.

– *With the probability $PrNPl(.)$:*

$$PrNPl(A) = \frac{Pl(A)}{Pl(A)+Pl(B)+Pl(C)} = \frac{0.80}{2} = 0.40$$
$$PrNPl(B) = \frac{Pl(B)}{Pl(A)+Pl(B)+Pl(C)} = \frac{0.60}{2} = 0.30$$
$$PrNPl(C) = \frac{Pl(C)}{Pl(A)+Pl(B)+Pl(C)} = \frac{0.60}{2} = 0.30$$

– *With the probability $PraPl(.)$:* Applying formula (3.7), one gets

$$\epsilon = \frac{1 - Bel(A) - Bel(B) - Bel(C)}{Pl(A)+Pl(B)+Pl(C)} = 0.35$$

$$PraPl(A) = 0.10 + 0.35 \cdot 0.80 = 0.38$$
$$PraPl(B) = 0 + 0.35 \cdot 0.60 = 0.21$$
$$PraPl(C) = 0.20 + 0.35 \cdot 0.60 = 0.41$$

– *With the probability $PrHyb(.)$:*

$$PrHyb(A) \approx 0.4553 \quad PrHyb(B) \approx 0.1698 \quad PrHyb(C) \approx 0.3749$$

- **With Cuzzolin's probability**: Since $TNSM = 0.3 + 0.1 + 0.3 = 0.7$, $\Delta(A) = 0.7$, $\Delta(B) = 0.6$ and $\Delta(C) = 0.4$, one gets:

$$CuzzP(A) = 0.1 + \frac{0.7}{1.7} \times 0.7 = 0.388$$
$$CuzzP(B) = 0 + \frac{0.6}{1.7} \times 0.7 = 0.247$$
$$CuzzP(C) = 0.2 + \frac{0.4}{1.7} \times 0.7 = 0.365$$

- **With $DSmP$ transformation**:

- Applying formula (3.11) for $\epsilon = 0.001$ yields:

$$\begin{aligned} DSmP_{\epsilon=0.001}(A) &\approx 0.5305 \\ DSmP_{\epsilon=0.001}(B) &\approx 0.0039 \\ DSmP_{\epsilon=0.001}(C) &\approx 0.4656 \\ DSmP_{\epsilon=0.001}(A \cup B) &\approx 0.5344 \\ DSmP_{\epsilon=0.001}(A \cup C) &\approx 0.9961 \\ DSmP_{\epsilon=0.001}(B \cup C) &\approx 0.4695 \\ DSmP_{\epsilon=0.001}(A \cup B \cup C) &= 1 \end{aligned}$$

- The formula (3.11) for $\epsilon = 0$ cannot be applied in this example because of 0/0 indetermination, but one can always choose ϵ arbitrary small in order to evaluate $DSmP_{\epsilon \to 0}(.)$.

	A	B	C	$PIC(.)$
$PrBel(.)$	0.5333	NaN	0.4667	NaN
$PrNPl(.)$	0.4000	0.3000	0.3000	0.0088
$CuzzP(.)$	0.3880	0.2470	0.3650	0.0163
$BetP(.)$	0.4000	0.2500	0.3500	0.0164
$PraPl(.)$	0.3800	0.2100	0.4100	0.0342
$PrPl(.)$	0.4486	0.2186	0.3328	0.0368
$PrHyb(.)$	0.4553	0.1698	0.3749	0.0650
$DSmP_{\epsilon=0.001}(.)$	0.5305	0.0039	0.4656	0.3500

Table 3.15: Results for example 3.8.2.

We summarize in Table 3.15, the results of the subjective probabilities and their corresponding PIC values sorted in increasing order. One sees that $DSmP_{\epsilon \to 0}$ provides the highest PIC and $PrBel$ is mathematically undefined. If one set artificially $PrBel(B) = 0$, one will get the same result with $PrBel$ as with $DSmP_{\epsilon \to 0}$.

3.8.3 Example 8: Shafer's model with yet another non-Bayesian mass

Let's modify a bit the previous and consider the 3D frame $\Theta = \{A, B, C\}$ with Shafer's model and the following non-Bayesian quantitative belief assignments (mass) having masses on B and C equal zero and according to Table 3.16.

	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
$m(.)$	0.1	0	0	0.2	0	0.3	0.4

Table 3.16: Quantitative input for example 3.8.3

	A	B	C	$PIC(.)$
$PrBel(.)$	0.7000	NaN	NaN	NaN
$CuzzP(.)$	0.3455	0.3681	0.2864	0.0049
$PrNPl(.)$	0.3043	0.3913	0.3044	0.0067
$BetP(.)$	0.3333	0.3833	0.2834	0.0068
$PraPl(.)$	0.3739	0.3522	0.2739	0.0077
$PrHyb(.)$	0.3526	0.4066	0.2408	0.0203
$PrPl(.)$	0.3093	0.4377	0.2530	0.0239
$DSmP_{\epsilon=0.001}(.)$	0.6903	0.1558	0.1539	0.2413

Table 3.17: Results for example 3.8.3.

We summarize in Table 3.17, the results of the subjective probabilities and their corresponding PIC values sorted in increasing order. $DSmP_{\epsilon\to 0}$ provides here the best results in term of PIC metric with respect to all other transformations. $PrBel$ doesn't work here because the two values $PrBel(B)$ and $PrBel(C)$ are mathematically undefined. Of course if we set artificially $PrBel(B) = PrBel(C) = (1 - PrBel(A))/2 = 0.15$, then we will obtain same result as with $DSmP_{\epsilon\to 0}$, but there is no solid reason for using such artificial trick for circumventing the inherent limitation of the $PrBel$ transformation.

3.8.4 Example 9: Shafer's model with yet another non-Bayesian mass

Here is an example where the $PrBel(.)$ provides a counter intuitive result. Let's consider again Shafer's model for $\Theta = \{A, B, C\}$ with the following bba

In this example $PrBel(B)$ and $PrBel(C)$ require division by zero which is impossible. Even if in $PrBel$ formula we force the mass $m(B \cup C) = 0.9$ to

	A	B	C	$B \cup C$
$m(.)$	0.1	0	0	0.9

Table 3.18: Quantitative input for example 3.8.4

be transferred to A, we get $PrBel(A) = 1$, but it is not fair nor intuitive to have the mass of $B \cup C$ transferred to A, since A was not at all involved in the ignorance $B \cup C$. Using $DSmP_\epsilon$ we get for the elements of Θ: $DSmP_\epsilon(A) = 0.1$, $DSmP_\epsilon(B) = 0.45$ and $DSmP_\epsilon(C) = 0.45$ no matter what $\epsilon > 0$ is equal to. We summarize in Table 3.19, the results of the subjective probabilities and their corresponding PIC values sorted in increasing order (the verification is left to the reader):

	A	B	C	$PIC(.)$
$PrBel(.)$	0.1000	NaN	NaN	NaN
$PraPl(.)$	0.1474	0.4263	0.4263	0.0814
$BetP(.)$	0.1000	0.4500	0.4500	0.1362
$CuzzP(.)$	0.1000	0.4500	0.4500	0.1362
$PrPl(.)$	0.1000	0.4500	0.4500	0.1362
$PrHyb(.)$	0.1000	0.4500	0.4500	0.1362
$DSmP_\epsilon(.)$	0.1000	0.4500	0.4500	0.1362
$PrNPl(.)$	0.0526	0.4737	0.4737	0.2146

Table 3.19: Results for example 3.8.4.

One sees that $DSmP_\epsilon$ coincides with $BetP$, $CuzzP$, $PrPl(.)$ and $PrHyb(.)$ in this special case. $PrBel(.)$ is mathematically undefined. If one forces artificially $PrBel(B) = PrBel(C) = 0$, one gets $PrBel(A) = 1$ which does not make sense. $PrNPl$ provides a *better* PIC than other transformations here only because it is subject to an abnormal behavior as already explained in section 3.5.2, and therefore it cannot be considered as a serious candidate for transforming any bba into a subjective probability.

3.8.5 Example 10: Hybrid DSm model

We consider here the hybrid DSm model for the frame $\Theta = \{A, B, C\}$ in which we force all possible intersection of elements of Θ to be empty, except $A \cap B$. In this case the hyper-power set D^Θ reduces to 9 elements $\{\emptyset, A \cap B, A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C\}$. The quantitative belief masses are chosen according to Table 3.20 (the mass of elements not included in the Table are equal to zero).

	$A \cap B$	A	C
$m(.)$	0.2	0.1	0.2
	$A \cup B$	$A \cup C$	$A \cup B \cup C$
$m(.)$	0.3	0.1	0.1

Table 3.20: Quantitative input for example 3.8.5

One has according to Figure 3.1 (see [12], page 55): $\mathcal{C}(A \cap B) = 1$, $\mathcal{C}(A) = 2$, $\mathcal{C}(B) = 2$, $\mathcal{C}(C) = 1$, $\mathcal{C}(A \cup B) = 3$, $\mathcal{C}(A \cup C) = 3$, $\mathcal{C}(B \cap C) = 3$ and $\mathcal{C}(A \cup B \cup C) = 4$.

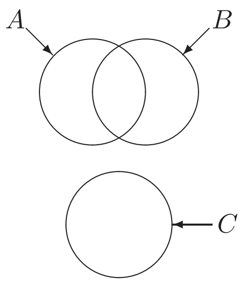

Figure 3.1: Hybrid DSm model for example 3.8.4

In order to apply Sudano's and Cuzzolin's transformations, we need to work on the refined frame Θ^{ref} with Shafer's model as depicted on Figure 3.2:

$$\Theta^{\text{ref}} = \{A' \triangleq A \setminus (A \cap B), B' \triangleq B \setminus (A \cap B), C' \triangleq C, D' \triangleq A \cap B\}$$

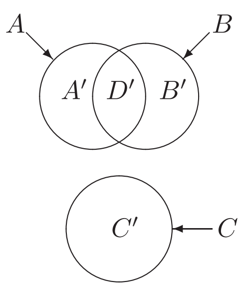

Figure 3.2: Refined 3D frame

For Sudano's and Cuzzolin's transformations, we use the following equivalent bba as numerical input of the transformations:

	D'	$A' \cup D'$	C'
$m(.)$	0.2	0.1	0.2
	$A' \cup B' \cup D'$	$A' \cup C' \cup D'$	$A' \cup B' \cup C' \cup D'$
$m(.)$	0.3	0.1	0.1

Table 3.21: Quantitative equivalent input on refined frame for example 3.8.5

- **With the pignistic probability**:

Applying the generalized pignistic transform (3.3) directly on Θ with $m(.)$ given in Table 3.20, one gets:

$BetP\{A \cap B\} =$	$BetP\{A\} =$	$BetP\{B\} =$	$BetP\{C\} =$
$(1/1) \cdot 0.2$	$(1/1) \cdot 0.2$	$(1/1) \cdot 0.2$	$(0/1) \cdot 0.2$
$+(1/2) \cdot 0.1$	$+(2/2) \cdot 0.1$	$+(1/2) \cdot 0.1$	$+(0/2) \cdot 0.1$
$+(1/2) \cdot 0$	$+(1/2) \cdot 0$	$+(2/2) \cdot 0$	$+(0/2) \cdot 0$
$+(0/1) \cdot 0.2$	$+(0/2) \cdot 0.2$	$+(0/1) \cdot 0.2$	$+(1/1) \cdot 0.2$
$+(1/3) \cdot 0.3$	$+(2/3) \cdot 0.3$	$+(2/3) \cdot 0.3$	$+(0/3) \cdot 0.3$
$+(1/3) \cdot 0.1$	$+(2/3) \cdot 0.1$	$+(1/3) \cdot 0.1$	$+(1/3) \cdot 0.1$
$+(1/3) \cdot 0$	$+(1/3) \cdot 0$	$+(2/3) \cdot 0$	$+(1/3) \cdot 0$
$+(1/4) \cdot 0.1$	$+(2/4) \cdot 0.1$	$+(2/4) \cdot 0.1$	$+(1/4) \cdot 0.1$
≈ 0.408333	≈ 0.616666	≈ 0.533333	≈ 0.258333

Table 3.22: Derivation of $BetP\{A \cap B\}$, $BetP\{A\}$, $BetP\{B\}$ and $BetP\{C\}$

It is easy to verify that the pignistic probability of the whole frame Θ is one since one has $BetP\{A \cup B \cup C\} = (1/1) \cdot 0.2 + (2/2) \cdot 0.1 + (2/2) \cdot 0 + (2/2) \cdot 0.2 + (3/3) \cdot 0.3 + (3/3) \cdot 0.1 + (3/3) \cdot 0 + (4/4) \cdot 0.1 = 0.2 + 0.1 + 0.2 + 0.3 + 0.1 + 0.1 = 1$. Moreover, one can verify also that the classical equality $BetP\{A \cup B\} = BetP\{A\} + BetP\{B\} - BetP\{A \cap B\}$ is satisfied since $BetP(.)$ is a (subjective) probability measure, similarly for $BetP\{A \cup C\}$ and for $BetP\{B \cup C\}$. $BetP\{A \cap C\}$ and $BetP\{B \cap C\}$ equal zero in this example.

$BetP\{A \cup B\} =$	$BetP\{A \cup C\} =$	$BetP\{B \cup C\} =$
$(1/1) \cdot 0.2$	$(1/1) \cdot 0.2$	$(1/1) \cdot 0.2$
$+(2/2) \cdot 0.1$	$+(2/2) \cdot 0.1$	$+(1/2) \cdot 0.1$
$+(2/2) \cdot 0$	$+(1/2) \cdot 0$	$+(2/2) \cdot 0$
$+(0/1) \cdot 0.2$	$+(1/1) \cdot 0.2$	$+(2/2) \cdot 0.2$
$+(3/3) \cdot 0.3$	$+(2/3) \cdot 0.3$	$+(2/3) \cdot 0.3$
$+(2/3) \cdot 0.1$	$+(3/3) \cdot 0.1$	$+(2/3) \cdot 0.1$
$+(2/3) \cdot 0$	$+(2/3) \cdot 0$	$+(3/3) \cdot 0$
$+(3/4) \cdot 0.1$	$+(3/4) \cdot 0.1$	$+(3/4) \cdot 0.1$
≈ 0.741666	$= 0.875000$	≈ 0.791666

Table 3.23: Derivation of $BetP\{A \cup B\}$, $BetP\{A \cup C\}$ and $BetP\{B \cup C\}$

The underlying probability measure of the atomic elements of Θ^{ref} is then given by:

$$\begin{aligned}
BetP\{A'\} &= BetP\{A\} - BetP\{A \cap B\} \approx 0.2084 \\
BetP\{B'\} &= BetP\{B\} - BetP\{A \cap B\} \approx 0.1250 \\
BetP\{C'\} &= BetP\{C\} \approx 0.2583 \\
BetP\{D'\} &= BetP\{A \cap B\} \approx 0.4083
\end{aligned}$$

- **With Sudano's probabilities**: The belief and plausibility of elements of Θ^{ref} are

$$\begin{aligned}
Bel(A') &= 0 & Pl(A') &= 0.6 \\
Bel(B') &= 0 & Pl(B') &= 0.4 \\
Bel(C') &= 0.2 & Pl(C') &= 0.4 \\
Bel(D') &= 0.2 & Pl(D') &= 0.8
\end{aligned}$$

Applying the formulas (3.4)-(3.8) on Θ^{ref} with the masses given in Table 3.21, one gets:

- *With the probability $PrPl(.)$*:

$$\begin{aligned}
PrPl(A') &\approx 0.2035 & PrPl(B') &\approx 0.0848 \\
PrPl(C') &\approx 0.2404 & PrPl(D') &\approx 0.4713
\end{aligned}$$

- *With the probability $PrBel(.)$*: One cannot directly apply (3.5) because of the division by zero involved in derivation of $PrBel(A')$ and $PrBel(B')$, i.e. formally one gets

$$PrBel(A') = NaN \qquad PrBel(B') = NaN$$
$$PrBel(C') = 0.3000 \qquad PrBel(D') = 0.7000$$

But because $PrBel(C') + PrBel(D') = 1$, one can set artificially/indirectly $PrBel(A') = PrBel(B') = 0$ and this would yield to $PIC \approx 0.5593$, but fundamentally, $PrBel(A') = NaN$ and $PrBel(B') = NaN$ from $PrBel(.)$ formula, so that PIC is mathematically inderterminate.

- *With the probability $PrNPl(.)$:*

$$PrNPl(A') \approx 0.2728 \qquad PrNPl(B') \approx 0.1818$$
$$PrNPl(C') \approx 0.1818 \qquad PrNPl(D') \approx 0.3636$$

- *With the probability $PraPl(.)$:* $\epsilon \approx 0.2727$

$$PraPl(A') \approx 0.1636 \qquad PraPl(B') \approx 0.1091$$
$$PraPl(C') \approx 0.3091 \qquad PraPl(D') \approx 0.4182$$

- *With the probability $PrHyb(.)$:*

$$PrHyb(A') \approx 0.1339 \qquad PrHyb(B') \approx 0.0583$$
$$PrHyb(C') \approx 0.2656 \qquad PrHyb(D') \approx 0.5422$$

- **With Cuzzolin's probability**: Working on the refined frame Θ^{ref}, with the bba $m(.)$ defined in Table 3.21, one has $TNSM = 0.6$, $\Delta(A') = 0.6$, $\Delta(B') = 0.4$, $\Delta(C') = 0.2$ and $\Delta(D') = 0.6$. Therefore:

$$\begin{aligned} CuzzP(A') &= m(A') + \frac{\Delta(A') \cdot TNSM}{\Delta(A') + \Delta(B') + \Delta(C') + \Delta(D')} \\ &= 0 + \frac{0.6 \cdot 0.6}{0.6 + 0.4 + 0.2 + 06} = 0.2000 \end{aligned}$$

$$\begin{aligned} CuzzP(B') &= m(B') + \frac{\Delta(B') \cdot TNSM}{\Delta(A') + \Delta(B') + \Delta(C') + \Delta(D')} \\ &= 0 + \frac{0.4 \cdot 0.6}{0.6 + 0.4 + 0.2 + 06} \approx 0.1333 \end{aligned}$$

$$CuzzP(C') = m(C') + \frac{\Delta(C')\cdot TNSM}{\Delta(A')+\Delta(B')+\Delta(C')+\Delta(D')}$$
$$= 0.2 + \frac{0.2\cdot 0.6}{0.6+0.4+0.2+06} \approx 0.2667$$

$$CuzzP(D') = m(D') + \frac{\Delta(D')\cdot TNSM}{\Delta(A')+\Delta(B')+\Delta(C')+\Delta(D')}$$
$$= 0.2 + \frac{0.6\cdot 0.6}{0.6+0.4+0.2+06} = 0.4000$$

- **With $DSmP$ transformation**: Applying directly the formula (3.11) on the frame Θ with this hybrid model and for the chosen bba $m(.)$, yield to the following analytical expressions:

$$DSmP_\epsilon(A\cap B) = \frac{m((A\cap B)\cap(A\cap B))+\epsilon\cdot\mathcal{C}((A\cap B)\cap(A\cap B))}{m(A\cap B)+\epsilon\cdot\mathcal{C}(A\cap B)}\cdot m(A\cap B)$$
$$+\frac{m(A\cap(A\cap B))+\epsilon\cdot\mathcal{C}(A\cap(A\cap B))}{m(A\cap B)+\epsilon\cdot\mathcal{C}(A)}\cdot m(A)$$
$$+\frac{m((A\cup B)\cap(A\cap B))+\epsilon\cdot\mathcal{C}((A\cup B)\cap(A\cap B))}{m(A\cap B)+\epsilon\cdot\mathcal{C}(A\cup B)}\cdot m(A\cup B)$$
$$+\frac{m((A\cup C)\cap(A\cap B))+\epsilon\cdot\mathcal{C}((A\cup C)\cap(A\cap B))}{m(A\cap B)+m(C)+\epsilon\cdot\mathcal{C}(A\cup C)}\cdot m(A\cup C)$$
$$+\frac{m((A\cup B\cup C)\cap(A\cap B))+\epsilon\cdot\mathcal{C}((A\cup B\cup C)\cap(A\cap B))}{m(A\cap B)+m(C)+\epsilon\cdot\mathcal{C}(A\cup B\cup C)}\cdot m(A\cup B\cup C)$$

Since we work with this hybrid DSm model, one has $\mathcal{C}(A\cap B)=1$, $\mathcal{C}(C)=1$, $\mathcal{C}(A)=\mathcal{C}(B)=2$, $\mathcal{C}(A\cup B)=\mathcal{C}(A\cup C)=\mathcal{C}(B\cup C)=3$ and $\mathcal{C}(A\cup B\cup C)=4$. So that the previous expression can be simplified as:

$$DSmP_\epsilon(A\cap B) = \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B) + \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+\frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B) + \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)$$
$$+\frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

Similarly, one gets:

$$DSmP_\epsilon(A) = \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B) + \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+ \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B) + \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)$$
$$+ \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

$$DSmP_\epsilon(B) = \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B) + \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+ \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B) + \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)$$
$$+ \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

$$DSmP_\epsilon(C) = \frac{m(C)+\epsilon\cdot 1}{m(C)+\epsilon\cdot 1}\cdot m(C) + \frac{m(C)+\epsilon\cdot 1}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)$$
$$+ \frac{m(C)+\epsilon\cdot 1}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

$$DSmP_\epsilon((A\cap B)\cup C) = \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B) + \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+ \frac{m(C)+\epsilon\cdot 1}{m(C)+\epsilon\cdot 1}\cdot m(C) + \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B)$$
$$+ \frac{m(A\cap B)+m(C)+\epsilon\cdot 2}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C) + \frac{m(A\cap B)+m(C)+\epsilon\cdot 3}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

$$DSmP_\epsilon(A\cup B) = \frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B) + \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+ \frac{m(A\cap B)+\epsilon\cdot 3}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B) + \frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)$$
$$+ \frac{m(A\cap B)+\epsilon\cdot 3}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

$$DSmP_\epsilon(A\cup C)=\frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B)+\frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+\frac{m(C)+\epsilon\cdot 1}{m(C)+\epsilon\cdot 1}\cdot m(C)+\frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B)$$
$$+\frac{m(A\cap B)+m(C)+\epsilon\cdot 3}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)+\frac{m(A\cap B)+m(C)+\epsilon\cdot 3}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

$$DSmP_\epsilon(B\cup C)=\frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B)+\frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+\frac{m(C)+\epsilon\cdot 1}{m(C)+\epsilon\cdot 1}\cdot m(C)+\frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B)$$
$$+\frac{m(A\cap B)+m(C)+\epsilon\cdot 2}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)+\frac{m(A\cap B)+m(C)+\epsilon\cdot 3}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)$$

$$DSmP_\epsilon(A\cup B\cup C)=\frac{m(A\cap B)+\epsilon\cdot 1}{m(A\cap B)+\epsilon\cdot 1}\cdot m(A\cap B)+\frac{m(A\cap B)+\epsilon\cdot 2}{m(A\cap B)+\epsilon\cdot 2}\cdot m(A)$$
$$+\frac{m(C)+\epsilon\cdot 1}{m(C)+\epsilon\cdot 1}\cdot m(C)+\frac{m(A\cap B)+\epsilon\cdot 3}{m(A\cap B)+\epsilon\cdot 3}\cdot m(A\cup B)$$
$$+\frac{m(A\cap B)+m(C)+\epsilon\cdot 3}{m(A\cap B)+m(C)+\epsilon\cdot 3}\cdot m(A\cup C)$$
$$+\frac{m(A\cap B)+m(C)+\epsilon\cdot 4}{m(A\cap B)+m(C)+\epsilon\cdot 4}\cdot m(A\cup B\cup C)=1$$

- Applying formula (3.11) for $\epsilon=0.001$ yields:

$$DSmP_{\epsilon=0.001}(A\cap B)\approx 0.6962$$
$$DSmP_{\epsilon=0.001}(A)\approx 0.6987$$
$$DSmP_{\epsilon=0.001}(B)\approx 0.6979$$
$$DSmP_{\epsilon=0.001}(C)\approx 0.2996$$
$$DSmP_{\epsilon=0.001}((A\cap B)\cup C)\approx 0.9958$$
$$DSmP_{\epsilon=0.001}(A\cup B)\approx 0.7004$$
$$DSmP_{\epsilon=0.001}(A\cup C)\approx 0.9983$$
$$DSmP_{\epsilon=0.001}(B\cup C)\approx 0.9975$$
$$DSmP_{\epsilon=0.001}(A\cup B\cup C)=1$$

which induces the underlying probability measure on the refined frame

$$P(A')\approx 0.0025\quad P(B')\approx 0.0017\quad P(C')\approx 0.2996\quad P(D')\approx 0.6962$$

	A'	B'	C'	D'	$PIC(.)$
$PrBel(.)$	NaN	NaN	0.3000	0.7000	NaN
$PrNPl(.)$	0.2728	0.1818	0.1818	0.3636	0.0318
$CuzzP(.)$	0.2000	0.1333	0.2667	0.4000	0.0553
$BetP(.)$	0.2084	0.1250	0.2583	0.4083	0.0607
$PraPl(.)$	0.1636	0.1091	0.3091	0.4182	0.0872
$PrPl(.)$	0.2035	0.0848	0.2404	0.4713	0.1124
$PrHyb(.)$	0.1339	0.0583	0.2656	0.5422	0.1928
$DSmP_{\epsilon=0.001}(.)$	0.0025	0.0017	0.2996	0.6962	0.5390

Table 3.24: Results for example 3.8.5.

We summarize in Table 3.24, the results on the refined frame for the subjective probabilities and their corresponding PIC values sorted in increasing order. $DSmP_{\epsilon\to 0}$ provides here the best result in term of PIC metric with respect to all other transformations.

3.8.6 Example 11: Free DSm model

We consider the free DSm model as Figure 3.3 for $\Theta = \{A, B, C\}$ with the bba given in Table 3.25.

	$A \cap B \cap C$	$A \cap B$	A
$m(.)$	0.1	0.2	0.3
	$A \cup B$	$A \cup B \cup C$	
$m(.)$	0.1	0.3	

Table 3.25: Quantitative input for example 3.8.6

In order to apply Sudano's and Cuzzolin's transformations, we need to work one the refined frame

$$\Theta^{\text{ref}} = \{A', B', C', D', E', F', G'\}$$

where elements of Θ^{ref} corresponds to separate parts (assuming such refinement makes physically sense/meaning - sometimes depending on the nature of elements A, B and C the refinement has no physical sense but can just be seen as a mathematical abstract refined frame) of the Venn Diagram of Figure 3.3.

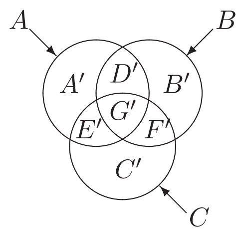

Figure 3.3: Free DSm model for a 3D frame.

- **With the pignistic probabilities**: One gets for singletons of Θ^{ref}.

$BetP\{A'\} \approx 0.1345$ $BetP\{B'\} \approx 0.0595$ $BetP\{C'\} \approx 0.0429$
$BetP\{D'\} \approx 0.2345$ $BetP\{E'\} \approx 0.1345$ $BetP\{F'\} \approx 0.0595$
$BetP\{G'\} \approx 0.3345$

- **With Sudano's probabilities**: The belief and plausibility of elements of Θ^{ref} are

$$
\begin{aligned}
Bel(A') &= 0 & Pl(A') &= 0.7\\
Bel(B') &= 0 & Pl(B') &= 0.4\\
Bel(C') &= 0 & Pl(C') &= 0.3\\
Bel(D') &= 0 & Pl(D') &= 0.9\\
Bel(E') &= 0 & Pl(E') &= 0.7
\end{aligned}
$$

$$
\begin{aligned}
Bel(F') &= 0 & Pl(F') &= 0.4\\
Bel(G') &= 0.1 & Pl(G') &= 1
\end{aligned}
$$

Applying the formulas (3.4)-(3.8) on Θ^{ref} one obtains:

- *With the probability $PrPl(.)$*:

$PrPl(A') \approx 0.1284$ $PrPl(B') \approx 0.0370$ $PrPl(C') \approx 0.0205$
$PrPl(D') \approx 0.2599$ $PrPl(E') \approx 0.1284$ $PrPl(F') \approx 0.0370$
$PrPl(G') \approx 0.3887$

- *With the probability $PrBel(.)$*: If one applies $PrBel(.)$ formula restricted only with positive masses (to circumvent 0/0 undeterminations, one obtains $PrBel(G') = 1$ and $PIC = 1$. But if one strictly applies $PrBel(.)$ formula which normally must include all masses (even those taking zero values), then $PrBel(.)$ yields to 0/0 indeterminations and thus $PIC = NaN$.
- *With the probability $PrNPl(.)$*:

$$
\begin{array}{ll}
PrNPl(A') \approx 0.1591 & PrNPl(B') \approx 0.0909 \\
PrNPl(C') \approx 0.0682 & PrNPl(D') \approx 0.2045 \\
PrNPl(E') \approx 0.1591 & PrNPl(F') \approx 0.0909 \\
PrNPl(G') \approx 0.2273 &
\end{array}
$$

- *With the probability $PraPl(.)$*: $\epsilon \approx 0.2045$

$$
\begin{array}{ll}
PraPl(A') \approx 0.1432 & PraPl(B') \approx 0.0818 \\
PraPl(C') \approx 0.0614 & PraPl(D') \approx 0.1841 \\
PraPl(E') \approx 0.1432 & PraPl(F') \approx 0.0818 \\
PraPl(G') \approx 0.3045 &
\end{array}
$$

- *With the probability $PrHyb(.)$*:

$$
\begin{array}{ll}
PrHyb(A') \approx 0.1136 & PrHyb(B') \approx 0.0333 \\
PrHyb(C') \approx 0.0184 & PrHyb(D') \approx 0.2214 \\
PrHyb(E') \approx 0.1136 & PrHyb(F') \approx 0.0333 \\
PrHyb(G') \approx 0.4663 &
\end{array}
$$

- **With Cuzzolin's probability**: Working on the refined frame Θ^{ref}, one has $TNSM = 0.9$, $\Delta(A') = 0.7$, $\Delta(B') = 0.4$, $\Delta(C') = 0.3$, $\Delta(D') = 0.9$, $\Delta(E') = 0.7$, $\Delta(F') = 0.4$ and $\Delta(G') = 0.9$. Therefore:

$$
\begin{array}{ll}
CuzzP(A') \approx 0.1465 & CuzzP(B') \approx 0.0837 \\
CuzzP(C') \approx 0.0628 & CuzzP(D') \approx 0.1884 \\
CuzzP(E') \approx 0.1465 & CuzzP(F') \approx 0.0837 \\
CuzzP(G') \approx 0.2884 &
\end{array}
$$

- **With $DSmP$ transformation**:

  Applying formula (3.11) for $\epsilon = 0.001$ yields[15]:

$$
\begin{aligned}
&DSmP_{\epsilon=0.001}(A\cap B\cap C) \approx 0.9678\\
&DSmP_{\epsilon=0.001}(A\cap B) \approx 0.9764\\
&DSmP_{\epsilon=0.001}(A\cap C) \approx 0.9745\\
&DSmP_{\epsilon=0.001}(B\cap C) \approx 0.9716\\
&DSmP_{\epsilon=0.001}((A\cup B)\cap C) \approx 0.9782\\
&DSmP_{\epsilon=0.001}((A\cup C)\cap B) \approx 0.9802\\
&DSmP_{\epsilon=0.001}((B\cup C)\cap A) \approx 0.9831\\
&DSmP_{\epsilon=0.001}((A\cap B)\cup(A\cap C)\cup(B\cap C)) \approx 0.9868\\
&DSmP_{\epsilon=0.001}(A) \approx 0.9897\\
&DSmP_{\epsilon=0.001}(B) \approx 0.9839\\
&DSmP_{\epsilon=0.001}(C) \approx 0.9810\\
&DSmP_{\epsilon=0.001}((A\cap B)\cup C) \approx 0.9896\\
&DSmP_{\epsilon=0.001}((A\cap C)\cup B) \approx 0.9963\\
&DSmP_{\epsilon=0.001}((B\cap C)\cup A) \approx 0.9935\\
&DSmP_{\epsilon=0.001}(A\cup B) \approx 0.9972\\
&DSmP_{\epsilon=0.001}(A\cup C) \approx 0.9963\\
&DSmP_{\epsilon=0.001}(B\cup C) \approx 0.9934\\
&DSmP_{\epsilon=0.001}(A\cup B\cup C) = 1
\end{aligned}
$$

  which induces the underlying probability measure on the refined frame[16]

$$
\begin{aligned}
P(A') = DSmP(A) - DSmP(A\cap B) - DSmP(A\cap C)&\\
+ DSmP(A\cap B\cap C) \approx 0.0066&
\end{aligned}
$$

$$
\begin{aligned}
P(B') = DSmP(B) - DSmP(A\cap B) - DSmP(B\cap C)&\\
+ DSmP(A\cap B\cap C) \approx 0.0038&
\end{aligned}
$$

$$
\begin{aligned}
P(C') = DSmP(C) - DSmP(A\cap C) - DSmP(B\cap C)&\\
+ DSmP(A\cap B\cap C) \approx 0.0028&
\end{aligned}
$$

[15] The verification is left to the reader.

[16] Here we use the Poincaré's formula. The index ϵ has been omitted due to space limitation for notational convenience.

$$
\begin{aligned}
P(D') &= DSmP(A \cap B) - DSmP(A \cap B \cap C) \approx 0.0086 \\
P(E') &= DSmP(A \cap C) - DSmP(A \cap B \cap C) \approx 0.0067 \\
P(F') &= DSmP(B \cap C) - DSmP(A \cap B \cap C) \approx 0.0037 \\
P(G') &= DSmP(A \cap B \cap C) \approx 0.9678
\end{aligned}
$$

Note that these probabilities can also be computed directly by the formula (3.12) using the proper bba defined on the refined frame. For example by applying (3.12), one gets for this example (with $m(A) = m(A' \cup D' \cup E' \cup G') = 0.3$, $m(A \cup B) = m(A' \cup B' \cup D' \cup E' \cup F' \cup G') = 0.1$ and $m(A \cup B \cup C) = m(A' \cup B' \cup C' \cup D' \cup E' \cup F' \cup G') = 0.2$)

$$P(A') = \frac{\epsilon \cdot 0.3}{0.1 + \epsilon \cdot 4} + \frac{\epsilon \cdot 0.1}{0.1 + \epsilon \cdot 6} + \frac{\epsilon \cdot 0.3}{0.1 + \epsilon \cdot 7}$$

which is equal to 0.0066 when $\epsilon = 0.001$. Similar derivations can be done using (3.12) to obtain directly the probabilities of the other elements of the refined frame.

We summarize in Table 3.26, the PIC values obtained with the different transformations sorted in increasing order. $DSmP_{\epsilon \to 0}$ provides here again the best result in term of PIC metric with respect to all other transformations.

Transformations	$PIC(.)$
$PrBel(.)$	NaN
$PrNPl(.)$	0.0414
$CuzzP(.)$	0.0621
$PraPl(.)$	0.0693
$BetP(.)$	0.1176
$DSmP_{\epsilon=0.1}(.)$	0.1854
$PrPl(.)$	0.1940
$PrHyb(.)$	0.2375
$DSmP_{\epsilon=0.001}(.)$	0.8986

Table 3.26: Results for example 3.8.6.

3.9 Extension of DSmP for qualitative belief

In order to compute directly with words (linguistic labels), Smarandache and Dezert have defined in [13] a *qualitative basic belief assignment* $qm(.)$ as a mapping function from G^{Θ} into a set of linguistic labels $L = \{L_0, \tilde{L}, L_{m+1}\}$

where $\tilde{L} = \{L_1, \cdots, L_m\}$ is a finite set of linguistic labels and where $m \geq 2$ is an integer. For example, L_1 can take the linguistic value "poor", L_2 the linguistic value "good", etc. $\tilde{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \cdots \prec L_n$. To work on a true closed linguistic set L under linguistic operators, $\tilde{L}$ is extended with two extreme values $L_0 = L_{\min}$ and $L_{m+1} = L_{\max}$, where L_0 corresponds to the minimal qualitative value and L_{m+1} corresponds to the maximal qualitative value, in such a way that $L_0 \prec L_1 \prec L_2 \prec \cdots \prec L_m \prec L_{m+1}$, where $\prec$ means inferior to, or less (in quality) than, or smaller than, etc.

From the isomorphism between the set of linguistic equidistant labels and a set of numbers in the interval $[0, 1]$ and the DSm Field and Linear Algebra of Refined Labels (FLARL) proposed in Chapter 2, one disposes of a set of precise operators on linguistic labels (addition, subtraction, multiplication, division, etc) which allows a direct extension of (quantitative) DSmP formula to its qualitative version as follows: $qDSmP_\epsilon(\emptyset) = L_0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$ by

$$qDSmP_\epsilon(X) = \sum_{Y \in G^\Theta} \frac{\sum\limits_{\substack{Z \subseteq X \cap Y \\ \mathcal{C}(Z)=1}} qm(Z) + \epsilon \cdot \mathcal{C}(X \cap Y)}{\sum\limits_{\substack{Z \subseteq Y \\ \mathcal{C}(Z)=1}} qm(Z) + \epsilon \cdot \mathcal{C}(Y)} qm(Y) \tag{3.30}$$

where all operations[17] in (3.30) are referred to labels as explained in Chapter 2.

The derivation of a qualitative PIC from qualitative DSmP can be also obtained as follows: Let's consider a finite space of discrete exclusive events $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ and a subjective qualitative alike probability measure $qP(.) : \Theta \mapsto L = \{L_0, L_1, \ldots, L_m, L_{m+1}\}$. Then one defines the entropy and PIC metrics from $qP(.)$ as

$$H(qP) \triangleq -\sum_{i=1}^{n} qP\{\theta_i\} \log_2(qP\{\theta_i\}) \tag{3.31}$$

$$PIC(qP) = 1 + \frac{1}{H_{\max}} \cdot \sum_{i=1}^{n} qP\{\theta_i\} \log_2(qP\{\theta_i\}) \tag{3.32}$$

where $H_{\max} = \log_2(n)$ and in order to compute the logarithms, one utilizes the isomorphism $L_i = i/(m + 1)$.

[17] In our previous papers, we used only approximate operators for labels. In working with FLARL, we use precise operators for labels.

3.10 Example for qualitative DSmP

Let's consider the frame $\Theta = \{A, B, C\}$ with Shafer's model and the following set of linguistic labels $L = \{L_0, L_1, L_2, L_3, L_4, L_5\}$ $(m = 4)$ with $L_0 = L_{\min}$ and $L_5 = L_{\max}$. Let's consider the following qualitative belief assignment $qm(A) = L_1$, $qm(B \cup C) = L_4$ and $qm(X) = L_0$ for all $X \in 2^\Theta \setminus \{A, B \cup C\}$. $qm(.)$ is quasi-normalized since $\sum_{X \in 2^\Theta} qm(X) = L_5 = L_{\max}$. In this example, $qm(B \cup C) = L_4$ is redistributed by $qDSmP_\epsilon(.)$ to B and C only, since B and C were involved in the ignorance, proportionally with respect to their cardinals (since their masses are $L_0 \equiv 0$). Applying $qDSmP_\epsilon(.)$ formula (3.30), one gets for this example:

$$qDSmP_\epsilon(A) = L_1$$

$$\begin{aligned}
qDSmP_\epsilon(B) &= \frac{qm(B) + \epsilon \cdot \mathcal{C}(B)}{qm(B) + qm(C) + \epsilon \cdot \mathcal{C}(B \cup C)} qm(B \cup C) \\
&= \frac{L_0 + \epsilon \cdot 1}{L_0 + L_0 + \epsilon \cdot 2} \cdot L_4 = \frac{L_{0+(\epsilon \cdot 1) \cdot 5}}{L_{0+0+(\epsilon \cdot 2) \cdot 5}} \cdot L_4 \\
&= \frac{L_{\epsilon \cdot 5}}{L_{\epsilon \cdot 10}} \cdot L_4 = L_{\frac{5\epsilon}{10\epsilon} \cdot 5} \cdot L_4 = L_{2.5} \cdot L_4 \\
&= L_{2.5 \cdot 4/5} = L_{10/5} = L_2
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
qDSmP_\epsilon(C) &= \frac{qm(C) + \epsilon \cdot \mathcal{C}(C)}{qm(B) + qm(C) + \epsilon \cdot \mathcal{C}(B \cup C)} qm(B \cup C) \\
&= \frac{L_0 + \epsilon \cdot 1}{L_0 + L_0 + \epsilon \cdot 2} \cdot L_4 = L_2
\end{aligned}$$

Thanks to the isomorphism between labels and numbers, all the properties of operations with numbers are transmitted to the operations with labels. $qDSmP_\epsilon(.)$ is normalized since $qDSmP_\epsilon(A) + qDSmP_\epsilon(B) + qDSmP_\epsilon(C)$ equals $L_1 + L_2 + L_2 = L_5 = L_{\max}$. Applying the PIC formula (3.32), one obtains (here $n = \mid \Theta \mid = 3$):

$$PIC(qDSmP_\epsilon) = 1 + \frac{1}{\log_2 3}(L_1 \log_2(L_1) + L_2 \log_2(L_2) + L_2 \log_2(L_2) \approx \frac{1}{5} L_1$$

where in order to compute the qualitative logarithms, one utilized the isomorphism $L_i = \frac{i}{m+1}$.

3.11 Conclusions

Motivated by the necessity to use a better (more informational) probabilistic approximation of belief assignment $m(.)$ for applications involving hard and/or soft decisions, we have developed in this chapter a new probabilistic transformation, called $DSmP$, for approximating $m(.)$ into a subjective probability measure. $DSmP$ provides the maximum of the Probabilistic Information Content (PIC) of the source because it is based on proportional redistribution of partial and total uncertainty masses to elements of cardinal 1 with respect to their corresponding masses and cardinalities. $DSmP$ works with any model (Shafer's, hybrid, or free DSm model) of the frame. $DSmP_{\epsilon=0}$ coincides with Sudano's $PrBel$ transformation for the cases when all masses of singletons involved in ignorances are nonzero. $PrBel$ formula is restricted to work on Shafer's model only while $DSmP_{\epsilon>0}$ is always defined and for any model. We have clearly shown through simple examples that the classical $BetP$ and Cuzzolin's transformations do not perform well in term of PIC criterion. It has been shown also how $DSmP$ can be extended to the qualitative domain to approximate qualitative belief assignments provided by human sources in natural language.

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