Neutrosophic Sets and Systems

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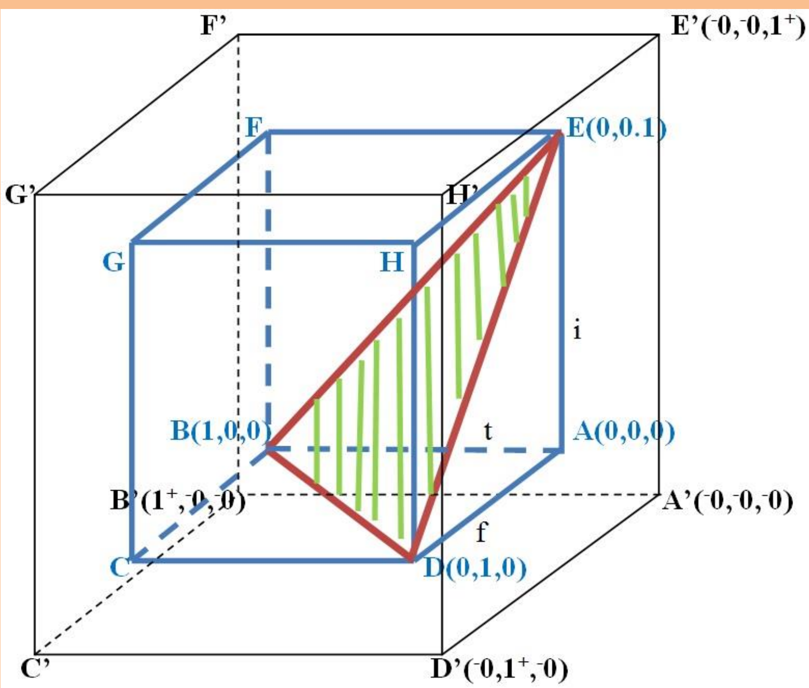

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Contents

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"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeter

minacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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University of New Mexico

Degrees of Membership > 1 and < 0 of the Elements With Respect to a Neutrosophic OffSet

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Abstract. We have defined the Neutrosophic Over-/Under-/Off-Set and -Logic for the first time in 1995 and published in 2007. During 1995-2016 we presented them to various national and international conferences and seminars ([16]-[37]) and did more publishing during 2007-2016 ([1]-[15]). These new notions are totally different from other sets/logics/probabilities.
We extended the neutrosophic set respectively to Neutrosophic Overset {when some neutrosophic component is > 1}, to Neutrosophic Underset {when some neutrosophic component is < 0}, and to Neutrosophic Offset {when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and other neutrosophic component < 0}. This is no surprise since our real-world has numerous examples and applications of over-/under-/off-neutrosophic components.

Keywords: Neutrosophic overset, neutrosophic underset, neutrosophic offset, neutrosophic overlogic, neutrosophic underlogic, neutrosophic offlogic, neutrosophic overprobability, neutrosophic underprobability, neutrosophic offprobability, overmembership (membership degree > 1), undermembership (membership degree < 0), offmembership (membership degree off the interval [0, 1]).

1. Introduction

In the classical set and logic theories, in the fuzzy set and logic, and in intuitionistic fuzzy set and logic, the degree of membership and degree of nonmembership have to belong to, or be included in, the interval [0, 1]. Similarly, in the classical probability and in imprecise probability the probability of an event has to belong to, or respectively be included in, the interval [0, 1].

Yet, we have observed and presented to many conferences and seminars around the globe {see [16]-[37]} and published {see [1]-[15]} that in our real world there are many cases when the degree of membership is greater than 1. The set, which has elements whose membership is over 1, we called it *Overset*.

Even worst, we observed elements whose membership with respect to a set is under 0, and we called it *Underset*.

In general, a set that has elements whose membership is above 1 and elements whose membership is below 0, we called it *Offset* (i.e. there are elements whose memberships are off (over and under) the interval [0, 1]).

"Neutrosophic" means based on three components *T (truth-membership)*, *I (indeterminacy)*, and *F (falsehood-nonmembership)*. And "over" means above 1, "under" means below 0, while "offset" means behind/beside the set on both sides of the interval [0, 1], over and under, more and less, supra and below, out of, off the set. Similarly, for "offlogic", "offmeasure", "offprobability", "offstatistics" etc.

It is like a pot with boiling liquid, on a gas stove, when the liquid swells up and leaks out of pot. The pot (the interval [0, 1]) can no longer contain all liquid (i.e., all neutrosophic truth / indeterminate / falsehood values), and therefore some of them fall out of the pot (i.e., one gets neutrosophic truth / indeterminate / falsehood values which are > 1), or the pot cracks on the bottom and the liquid pours down (i.e., one gets neutrosophic truth / indeterminate / falsehood values which are < 0).

Mathematically, they mean getting values off the interval [0, 1].

The American aphorism "think outside the box" has a perfect resonance to the neutrosophic offset, where the box is the interval [0, 1], yet values outside of this interval are permitted.

2. Example of Overmembership and Undermembership.

In a given company a full-time employer works 40 hours per week. Let's consider the last week period. Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was 30/40 = 0.75 < 1.

John worked full-time, 40 hours, so he had the membership degree 40/40 = 1, with respect to this company.

But George worked overtime 5 hours, so his membership degree was (40+5)/40 = 45/40 = 1.125 > 1. Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was 0/40 = 0.

Yet, Richard, who was also hired as a full-time, not only didn't come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was - 20/40 = - 0.50 < 0.

Consequently, we need to make distinction between employees who produce damage, and those who produce profit, or produce neither damage no profit to the company.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to take them into consideration.

Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively *Neutrosophic Over-/Under-/Off-Logic, -Measure, -Probability, -Statistics* etc. [Smarandache, 2007].

Another Example of Membership Above 1 and Membership Below 0.

Let's consider a spy agency S = {S_1, S_2, …, S_{1000}} of a country Atara against its enemy country Batara. Each agent S_j, j ∈ {1, 2, …, 1000}, was required last week to accomplish 5 missions, which represent the full-time contribution/membership.

Last week agent S_{27} has successfully accomplished his 5 missions, so his membership was $T(A_{27})$ = 5/5 = 1 = 100% (full-time membership).

Agent S_{32} has accomplished only 3 missions, so his membership is $T(S_{32})$ = 3/5 = 0.6 = 60% (part-time membership).

Agent S_{41} was absent, without pay, due to his health problems; thus $T(S_{41})$ = 0/5 = 0 = 0% (null-membership).

Agent S_{53} has successfully accomplished his 5 required missions, plus an extra mission of another agent that was absent due to sickness, therefore $T(S_{53})$ = (5+1)/5 = 6/5 = 1.2 > 1 (therefore, he has membership above 1, called over-membership).

Yet, agent S_{75} is a double-agent, and he leaks highly confidential information about country Atara to the enemy country Batara, while simultaneously providing misleading information to the country Atara about the enemy country Batara. Therefore S_{75} is a negative agent with respect to his country Atara, since he produces damage to Atara, he was estimated to having intentionally done wrongly all his 5 missions, in addition of compromising a mission of another agent of country Atara, thus his membership $T(S_{75})$ = - (5+1)/5 = - 6/5 = -1.2 < 0 (therefore, he has a membership below 0, called under-membership).

3. Definitions and the main work

1. **Definition of Single-Valued Neutrosophic Overset.**

Let U be a universe of discourse and the neutrosophic set $A_1 \subset U$.

Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element x ∈ U, with respect to the neutrosophic set A_1:

$T(x), I(x), F(x) : U \rightarrow [0, \Omega]$

where $0 < 1 < \Omega$, and Ω is called overlimit,

$T(x), I(x), F(x) \in [0, \Omega]$.

A Single-Valued Neutrosophic Overset A_1 is defined as:

$A_1 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,

such that there exists at least one element in A_1 that has at least one neutrosophic component that is > 1, and no element has neutrosophic components that are < 0.

For **example**: $A_1 = \{(x_1, <1.3, 0.5, 0.1>), (x_2, <0.2, 1.1, 0.2>)\}$, since $T(x_1) = 1.3 > 1$, $I(x_2) = 1.1 > 1$, and no neutrosophic component is < 0.

Also $O_2 = \{(a, <0.3, -0.1, 1.1>)\}$, since I(a) = - 0.1 < 0 and F(a) = 1.1 > 1.

2. **Definition of Single-Valued Neutrosophic Underset.**

Let U be a universe of discourse and the neutrosophic set $A_2 \subset U$.

Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element x ∈ U, with respect to the neutrosophic set A_2:

$T(x), I(x), F(x) : U \rightarrow [\Psi, 1]$

where $\Psi < 0 < 1$, and Ψ is called underlimit,

$T(x), I(x), F(x) \in [\Psi, 1]$.

A Single-Valued Neutrosophic Underset A_2 is defined as:
$A_2 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exists at least one element in A_2 that has at least one neutrosophic component that is < 0, and no element has neutrosophic components that are > 1.
For **example**: $A_2 = \{(x_1, <-0.4, 0.5, 0.3>), (x_2, <0.2, 0.5, -0.2>)\}$, since $T(x_1) = -0.4 < 0$, $F(x_2) = -0.2 < 0$, and no neutrosophic component is > 1.

3. Definition of Single-Valued Neutrosophic Offset.

Let U be a universe of discourse and the neutrosophic set $A_3 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the set A_3:
$T(x), I(x), F(x) : U \rightarrow [\Psi, \Omega]$
where $\Psi < 0 < 1 < \Omega$, and Ψ is called underlimit, while Ω is called overlimit,
$T(x), I(x), F(x) \in [\Psi, \Omega]$.
A Single-Valued Neutrosophic Offset A_3 is defined as:
$A_3 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exist some elements in A_3 that have at least one neutrosophic component that is > 1, and at least another neutrosophic component that is < 0.
For **examples**: $A_3 = \{(x_1, <1.2, 0.4, 0.1>), (x_2, <0.2, 0.3, -0.7>)\}$, since $T(x_1) = 1.2 > 1$ and $F(x_2) = -0.7 < 0$.
Also $B_3 = \{(a, <0.3, -0.1, 1.1>)\}$, since $I(a) = -0.1 < 0$ and $F(a) = 1.1 > 1$.

4. Single Valued Neutrosophic Overset / Underset / Offset Operators.

Let U be a universe of discourse and $A = \{(x, <T_A(x), I_A(x), F_A(x)>), x \in U\}$ and
and $B = \{(x, <T_B(x), I_B(x), F_B(x)>), x \in U\}$ be two single-valued neutrosophic oversets / undersets / offsets.
$T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x): U \rightarrow [\Psi, \Omega]$
where $\Psi \leq 0 < 1 \leq \Omega$, and Ψ is called underlimit, while Ω is called overlimit,
$T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x) \in [\Psi, \Omega]$.
We take the inequality sign $\leq$ instead of < on both extremes above, in order to comprise all three cases: overset {when $\Psi = 0$, and $1 < \Omega$}, underset {when $\Psi < 0$, and $1 = \Omega$}, and offset {when $\Psi < 0$, and $1 < \Omega$}.

4.1. Single Valued Neutrosophic Overset / Underset / Offset Union.

Then $A \cup B = \{(x, <\max\{T_A(x), T_B(x)\}, \min\{I_A(x), I_B(x)\}, \min\{F_A(x), F_B(x)\}>), x \in U\}$

4.2. Single Valued Neutrosophic Overset / Underset / Offset Intersection.

Then $A \cap B = \{(x, <\min\{T_A(x), T_B(x)\}, \max\{I_A(x), I_B(x)\}, \max\{F_A(x), F_B(x)\}>), x \in U\}$

4.3. Single Valued Neutrosophic Overset / Underset / Offset Complement.

The neutrosophic complement of the neutrosophic set A is $C(A) = \{(x, <F_A(x), \Psi + \Omega - I_A(x), T_A(x)>), x \in U\}$.

5. Definition of Interval-Valued Neutrosophic Overset.

Let U be a universe of discourse and the neutrosophic set $A_1 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_1:
$T(x), I(x), F(x) : U \rightarrow P([0, \Omega])$,
where $0 < 1 < \Omega$, and Ω is called overlimit,
$T(x), I(x), F(x) \subseteq [0, \Omega]$, and $P([0, \Omega])$ is the set of all subsets of $[0, \Omega]$.
An Interval-Valued Neutrosophic Overset A_1 is defined as:
$A_1 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exists at least one element in A_1 that has at least one neutrosophic component that is partially or totally above 1, and no element has neutrosophic components that is partially or totally below 0.
For **example**: $A_1 = \{(x_1, <(1, 1.4], 0.1, 0.2>), (x_2, <0.2, [0.9, 1.1], 0.2>)\}$, since $T(x_1) = (1, 1.4]$ is totally above 1, $I(x_2) = [0.9, 1.1]$ is partially above 1, and no neutrosophic component is partially or totally below 0.

6. Definition of Interval-Valued Neutrosophic Underset.

Let U be a universe of discourse and the neutrosophic set $A_2 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_2:
$T(x), I(x), F(x) : U \rightarrow [\Psi, 1]$,
where $\Psi < 0 < 1$, and Ψ is called underlimit,
$T(x), I(x), F(x) \subseteq [\Psi, 1]$, and $P([\Psi, 1])$ is the set of all subsets of $[\Psi, 1]$.
An Interval-Valued Neutrosophic Underset A_2 is defined as:
$A_2 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exists at least one element in A_2 that has at least one neutrosophic component that is partially or totally below 0, and no element has neutrosophic components that are partially or totally above 1.
For **example**: $A_2 = \{(x_1, <(-0.5,-0.4), 0.6, 0.3>), (x_2, <0.2, 0.5, [-0.2, 0.2]>)\}$, since $T(x_1) = (-0.5, -0.4)$ is totally below 0, $F(x_2) = [-0.2, 0.2]$ is partially below 0, and no neutrosophic component is partially or totally above 1.

7. Definition of Interval-Valued Neutrosophic Offset.

Let U be a universe of discourse and the neutrosophic set $A_3 \subset U$.

Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element x ∈ U, with respect to the set A_3:
T(x), I(x), F(x) : U → P($[\Psi,\Omega]$),
where $\Psi < 0 < 1 < \Omega$, and Ψ is called underlimit, while Ω is called overlimit,
T(x), I(x), F(x) ⊆ $[\Psi,\Omega]$, and P($[\Psi,\Omega]$) is the set of all subsets of $[\Psi,\Omega]$.
An Interval-Valued Neutrosophic Offset A_3 is defined as:
A_3 = {(x, <T(x), I(x), F(x)>), x ∈ U},
such that there exist some elements in A_3 that have at least one neutrosophic component that is partially or totally above 1, and at least another neutrosophic component that is partially or totally below 0.
For **examples**: A_3 = {(x_1, <[1.1, 1.2], 0.4, 0.1>), (x_2, <0.2, 0.3, (-0.7, -0.3)>)}, since T(x_1) = [1.1, 1.2] that is totally above 1, and F(x_2) = (-0.7, -0.3) that is totally below 0.
Also B_3 = {(a, <0.3, [-0.1, 0.1], [1.05, 1.10]>)}, since I(a) = [- 0.1, 0.1] that is partially below 0, and F(a) = [1.05, 1.10] that is totally above 1.

8. Interval-Valued Neutrosophic Overset / Underset / Offset Operators.

Let U be a universe of discourse and A = {(x, <T_A(x), I_A(x), F_A(x)>), x ∈ U}
and B = {(x, <T_B(x), I_B(x), F_B(x)>), x ∈ U} be two interval-valued neutrosophic oversets / undersets / offsets.
T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x): U →P($[\Psi,\Omega]$),
where P($[\Psi,\Omega]$) means the set of all subsets of $[\Psi,\Omega]$,
and T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x) ⊆ $[\Psi,\Omega]$,
with $\Psi \le 0 < 1 \le \Omega$, and Ψ is called underlimit, while Ω is called overlimit.
We take the inequality sign ≤ instead of < on both extremes above, in order to comprise all three cases: overset {when Ψ = 0, and 1 < Ω }, underset {when Ψ < 0, and 1 = Ω }, and offset {when Ψ < 0, and 1 < Ω }.

8.1. Interval-Valued Neutrosophic Overset / Underset / Offset Union.

Then A∪B =
{(x, <[max{inf(T_A(x)), inf(T_B(x))}, max{sup(T_A(x)), sup(T_B(x))}],
[min{inf(I_A(x)), inf(I_B(x))}, min{sup(I_A(x)), sup(I_B(x))}],
[min{inf(F_A(x)), inf(F_B(x))}, min{sup(F_A(x)), sup(F_B(x))}]>, x ∈ U}.

8.2. Interval-Valued Neutrosophic Overset / Underset / Offset Intersection.

Then A∩B =
{(x, <[min{inf(T_A(x)), inf(T_B(x))}, min{sup(T_A(x)), sup(T_B(x))}],
[max{inf(I_A(x)), inf(I_B(x))}, max{sup(I_A(x)), sup(I_B(x))}],
[max{inf(F_A(x)), inf(F_B(x))}, max{sup(F_A(x)), sup(F_B(x))}]>, x ∈ U}.

8.3. Interval-Valued Neutrosophic Overset / Underset / Offset Complement.

The complement of the neutrosophic set A is
C(A) = {(x, <F_A(x), [$\Psi + \Omega$ - sup{I_A(x)}, $\Psi + \Omega$ - inf{I_A(x)}], T_A(x)>), x ∈ U}.

Conclusion

The membership degrees over 1 (overmembership), or below 0 (undermembership) are part of our real world, so they deserve more study in the future.

The neutrosophic overset / underset / offset together with neutrosophic overlogic / underlogic / offlogic and especially neutrosophic overprobability / underprobability / and offprobability have many applications in technology, social science, economics and so on that the readers may be interested in exploring.

After designing the neutrosophic operators for single-valued neutrosophic overset/underset/offset, we extended them to interval-valued neutrosophic overset/underset/offset operators. We also presented another example of membership above 1 and membership below 0.

Of course, in many real world problems the neutrosophic union, neutrosophic intersection, and neutrosophic complement for interval-valued neutrosophic overset/underset/offset can be used. Future research will be focused on practical applications.

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- neutrosophic overlogic, neutrosophic underlogic, neutrosophic offlogic;
- neutrosophic overmeasure, neutrosophic undermeasure, neutrosophic offmeasure;
- neutrosophic overprobability, neutrosophic underprobability, neutrosophic offprobability;
- neutrosophic overstatistics, neutrosophic understatistics, neutrosophic offstatistics; as follows:

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Assessing IT Projects Success with Extended Fuzzy Cognitive Maps & Neutrosophic Cognitive Maps in comparison to Fuzzy Cognitive Maps

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Abstract. IT projects hold a huge importance to economic growth. Today, half of the capital investments are in IT technology. IT systems and projects are extensive and time consuming; thus implying that its failure is not affordable, so proper feasibility study of assessing project success factors is required. A current methodology like Fuzzy Cognitive Maps has been experimented for identifying and evaluating the success factors in IT projects, but this technique has certain limitations. This paper discusses two new approaches to evaluate IT project success: Extended Fuzzy Cognitive Maps (E-FCM) & Neutrosophic Cognitive Maps (NCM).The limitations of FCM like non consideration for non-linear, conditional, time delay weights and indeterminate relations are targeted using E-FCM and NCM in this paper.

Keywords: IT project success factors, Fuzzy Cognitive Maps, Extended FCM, Neutrosophic Cognitive Maps.

1 Introduction

IT projects have become so essential that its applications can be seen in every domain of life [1] [2] [3]. The various success factors are time, budget, quality, owner satisfaction, cooperation, etc., among which the most accepted assessment criteria in measuring the IT projects success are: meeting the specification, delivery on time and within budget [4].

A project can be completed on time, within cost and satisfy the given specifications, but if it is not liked and used by the customers then IT project will be a failure[5]. The various causes of failure are poor methodology, over-optimism, complexity, weak ownership etc. [6].Therefore, there is a need to identify the important factors contributing to the success rate of IT projects. In 1986, Pinto and Slevin considered both the internal factors i.e. cost, time and technical specifications and external factors i.e. use, satisfaction and effectiveness, to be the success factors of IT projects [7].

Many researchers [8] [9] have used different techniques to evaluate IT project success factors. Soft computing techniques are equipped to handle uncertainties which are frequent in IT projects, so the authors have experimented with the newly proposed methodology by Vasantha and Smarandache (2003), i.e. Neutrosophic Cognitive Maps for evaluating IT projects success in this paper. A comparative study is conducted where it is shown that NCM methodology is preferred over Fuzzy Cognitive Maps (FCM) mainly because NCM facilitates the computation of indeterminate cause-effect relationships that FCM does not permit.

The NCM based technique of evaluating IT project success has been tested on a small case study: Mobile Payment System Project [10]. The same case study was discussed by Rodriguez-Repiso et al. [10] where they used FCM methodology to evaluate IT project success factors.

FCM methodology has certain drawbacks which are highlighted by researcher Hagiwara [11] .It is proposed in his research that the limitations can be overcome by Extended FCM. The authors used two techniques Extended FCM and NCM and compare their results with the work done by Rodriguez- Repiso et. al.

The remaining of the paper is organised as follows. Section 2 gives the literature review of project success and cognitive maps. Section 3 describes the case study of MPS (Mobile payment system). Section 4, 5 and 6 discusses the FCM, E-FCM and NCM methodology with its implementation on MPS project. Section 7 presents discussion of results. Section 8 outlines the conclusion & future work.

2 Literature Review

2.1 Project Success

There are various factors that determine the success of a project but a project is said to be successful if it meets the basic three criteria i.e. delivery on time, within budget and meeting the specification [12] [13].

Table 1 gives the compilation of the various prominent factors listed by different researchers that contribute towards the success of project [14] [15] [16] [17].

Success factor	Description
Time	Some respondents noted that the measure of estimated time should include extensions and/or reductions due to variations in the original scope of the works, rather than measuring against the original baseline.
Budget	Some respondents noted comparison should be made between agreed project costs, not necessarily the contracted price.
Quality/Specification	Respondents noted that success could be measured by determining "was the project completed to specifications" or whether the project demonstrated "fitness for purpose".
Owner Satisfaction/Meeting Owner's Needs.	Some respondents stated that owner satisfaction is ultimately all that matters and that all other success criteria are subordinate to this measure.
Cooperation	Cooperation includes smooth project team coordination, an efficient and harmonious project team, good relations with the owner, no unresolved disputes, and cooperation between stakeholders, authorities, vendors and purchasers.
Risks Managed	Respondents specifically looked for clear risk identification, allocation & management; risk mitigation; along with only identified risks occurring i.e. no unpleasant surprises or crises occurring.
Safety	Safety criteria included safety targets were met or exceeded, a safe project, no accidents, excellent safety record, no accidents or injuries during delivery, and achieving satisfactory safety.

Table 1: Factors for success of project

2.2 Cognitive Maps

The concept of Cognitive Maps was introduced and applied by a political scientist Axelrod in 1976 to rectify those desired states which are unclear [18].These states are called as ill-structured problems. He developed signed digraphs design to extract the casual assertions of person with respect to certain area and then used them in order to find out the facts of alternative.

It has only two basic types of elements. First are the concepts and second are casual beliefs. In simple term they are known as nodes & arcs. Nodes describe the behavior of system and can be represented as variables. On the other side arcs are the relationships among the concepts which are either positive or negative. The positive relation means that the effect variable undergoes change in the same direction and negative relation means that the effect variable undergoes change in the opposite direction with respect to the change in cause variable [19].

2.3 Fuzzy Cognitive Maps

Kosko introduced the concept of fuzzy cognitive maps (FCM) [20]. It is an extension of cognitive maps consisting of elements (concepts / nodes) which represent the important attributes of the mapped system. FCM is a very simple and effective tool that is used in lots of applications like business [21] [22], banking [23], medical field [24] [25] , sports [26] , robotics [27], expert systems [28], decision making [29] [30], risk assessment [31].

Fuzzy cognitive maps (FCMs) are more applicable when the data in the first place is an unsupervised one. The FCMs work on the opinion of experts. FCMs model the world as a collection of classes and causal relation between classes.

2.3.1 Basics of FCM

- Assume C_i and C_j denote two nodes of the FCM. The directed edge from C_i to C_j denote the causality of C_i on C_j called connections. Each edge in the FCM is weighted with a number {-1, 0, 1}. Assume a_{ij} is the weight of the directed edge C_i C_j, a_{ij} ∈ {-1, 0, 1}.
  a_{ij} = 0 if C_i does not have any effect on C_j
  a_{ij} = 1 if increase (or decrease) in C_i causes increase (or decrease) in C_j
  a_{ij} = -1 if increase (or decrease) in C_i causes decrease (or increase) in C_j
- Let C_1C_2 , C_3C_4 , $\cdots$ C_iC_j , be a cycle when C_i is switched on and if the causality flows through the edges of a cycle and if it again causes C_i , We say that the dynamical system goes round and round. This is true for any node C_i , for $i = 1, 2, \cdots, n$. The equilibrium state for this dynamical system is called the hidden pattern.
- If the equilibrium state of a dynamical system is a unique state vector, then it is called a fixed point. Consider a FCM with C_1, C_2 ,...., C_n as nodes. For example let us start the dynamical system by switching on C_1.
- Let us assume that the FCM settles down with C_1 and C_n on, i.e. the state vector remains as (1, 0, 0,…, 0, 1) this state vector (1, 0, 0, …, 0, 1) is called the fixed point.
- If the FCM settles down with a state vector repeating in the form $A_1 \to A_2 \to \cdots A_i \to A_1$. Then this equilibrium is called limit cycle.

3 Case study on Mobile Payment system

Nowadays people need to make dozens of payments every day. This requires the availability of cash or plastic cards any time and everywhere. Though it is not always easy to have cash available and if the price of the purchase does not exceed a certain minimum value, the plastic cards are not accepted.

The basic idea behind the MPS project is to allow mobile phone users to make small and medium payments using their mobile phones. The user will send SMS to the mobile phone number of the payment recipient. The SMS sent will contain the code given to the user by the system provider of Mobile Payment System; followed by the amount to be paid. This amount of money will be directly debited from the bank account of the user and credited to the bank account associated with the mobile phone number that receives the SMS[10].

Rodriguez-Repiso et.al [10] considered the MPS project and FCM methodology was used to check its feasibility. The authors conducted a survey from 40 individuals belonging to different continents to identify various factors and their degree of importance to MPS project success.

The authors identified following factors that contribute to the success of MPS project [10]:

C1. Ability to store money in your mobile

C2. Avoid using coins (you won't need coins in your pocket as you will use your mobile to
pay)

C3. Less to carry with you

C4. Independence of time and place (subject to the area covered by the network operator)

C5. Getting rid of plastic cards

C6. Convenience

C7. Security

C8. Comfort

C9. Able to make small payments (up to 40 GBP)

C10. Able to make medium sized payments (up to 300 GBP)

C11. Interface easy to use

C12. Direct debiting from account

C13. Ability to pay using a mobile phone in store

C14. Avoid using cash

C15. Possibility of multiple mobile cash accounts to divide own and company purse

C16. Flexibility

C17. Efficiency

C18. Economy

C19. Your phone is always with you

C20. Remote control for everyday things

C21. The cost of the payment for the user is the cost of 1 SMS.

C22. The user does not pay credit/debit card maintenance costs to the bank

C23. The cost of the payment for the shop is the cost of 1 SMS

C24. The bank receives a commission from the network operator processing each payment transaction.

The following steps were broadly executed by Rodriguez-Repiso et.al [10] to check the feasibility of MPS using FCM:-

1) The results of the survey is recorded in Initial matrix of success (IMS).The dimension of the IMS matrix is 24×40, where 24 is the number of success factors and 40 corresponds to the number of experts.

2) The values in the IMS matrix are fuzzified in the interval [0,1] which is recorded in Fuzzified matrix of success (FZMS).

3) The strength of relationship matrix of success (SRMS) was constructed which is a 24×24 matrix. The rows and columns of the matrix are the success factors and each element S_{ij} in the matrix indicates the relationship between factor "i" and factor "j". S_{ij} can accept values in the interval [-1, 1] .

4) Once the SRMS matrix is completed, some of the data contained in it could be misleading data. Not all success factors represented in the matrix are related, and not always there is a relationship of causality between them. An expert opinion is required to analyse the data and convert the SRMS matrix into the FMS (Final matrix of success) matrix, which contains only those numerical fuzzy components representing relationships of causality between the success factors.

Thus, the model developed in the paper [10] is easy to understand and can be used to evaluate, and test the effect of factors and predict the performance of the MPS system.

4 Working of FCM methodology in MPS

4.1 Computation of hidden pattern using FCM

The authors determine the hidden pattern for the FCM methodology used in the MPS project [10] and is described as:

Let A_1 be the initial state vector where C_1 and C_{19} are in ON state.

$A_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0)$

D =FCM matrix shown in Table 2 [10].

Success factors	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1		0.66			0.65				0.7	0.73				0.63										
2	0.66		0.72																					
3		0.72			0.7	0.79								0.74										
4						0.79										0.87			0.83	0.74				
5	0.65		0.7																					
6			0.79	0.79				0.79	0.7	0.75	0.76									0.76				
7												0.66												
8						0.79					0.85		0.78											
9						0.7			0.7															
10	0.73					0.75																		
11						0.76		0.85									0.89							
12							0.66									0.62	0.74							0.7
13								0.78									0.82							
14	0.63		0.74																					
15												0.62				0.65	0.61							
16				0.87											0.65		0.88		0.78					
17											0.89	0.74	0.82		0.61	0.88								
18																					0.72	0.83	0.65	-0.6
19				0.83												0.78				0.71				
20				0.74		0.76													0.71					
21																		0.72						
22																		0.83						
23																		0.65						
24												0.7						-0.6						

Table 2. FCM Matrix

$A_1D =$ (0 0.68 0 0.83 0.65 0 0 0 0.70 0.73 0 0 0 0.63 0 0.78 0 0 0 0.71 0 0 0 0) $\rightarrow$ (1 1 0 1 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 1 0 0 0 0) = A_2

$A_2D =$ (3.39 0.68 2.16 2.44 0.65 3 0 0 0.70 0.73 0 0 0 0.63 0.65 1.65 0.88 0 2.32 1.45 0 0 0 0) $\rightarrow$ (1 1 1 1 1 1 0 0 1 1 0 0 0 1 1 1 1 0 1 1 0 0 0 0) = A_3

$A_3D =$ (3.39 1.4 2.95 3.23 1.35 3.79 0 0.79 1.40 1.48 1.65 1.36 0.82 1.37 1.26 3.18 1.49 0 2.32 2.21 0 0 0 0) $\rightarrow$ (1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 0) = A_4

$A_4D =$ (3.39 1.4 2.95 3.23 1.35 5.34 0.66 2.42 1.40 1.48 2.5 1.36 1.6 1.37 1.88 3.18 3.94 0 2.32 2.21 0 0 0 0.70) $\rightarrow$ (1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 1) = A_5

$A_5D =$ (3.39 1.4 2.95 3.23 1.35 5.34 0.66 2.42 1.40 1.48 2.5 2.72 1.6 1.37 1.88 3.18 3.94 0 2.32 2.21 0 0 0 0.70) $\rightarrow$ (1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 1) = $A_6 = A_5$

As $A_6 = A_5$, so this is a fixed point.

This implies that the concepts C_1 and C_{19} does not have any effect on concepts C_{18}, C_{21}, C_{22} and C_{23}. This is a saturation point. By making further iterations, there is no change in the results.

4.2 Limitations of FCM

There are three important drawbacks in the conventional FCMs [11]:

1) Connections in FCMs are just numeric ones: relationship of two events should be linear.

In MPS [10], the relation $C_{19} \rightarrow C_4$ is considered as linear.

2) Lack of a concept of time; practically each causal has different time delay.

In MPS [10], the relation $C_{24} \rightarrow C_{12}$ tends to have some time delay, but is not considered.

3) They cannot deal with co-occurrence of multiple causes such as expressed by “and" conditions.

In MPS [10], the concepts C_1 and C_{19} can be combined to create an overall effect on C_9 ,but this idea is not considered and implemented in the system.

The drawbacks of FCM can be overcome by the proposed methodology E-FCM and NCM.

5 EXTENDED FUZZY COGNITIVE MAPS (E-FCM)

E-FCM has certain features [11]:

1) Weights have nonlinear membership functions.
2) Conditional weights
3) Time delay weights.

Authors have considered features of E-FCM: one non-linear weight, conditional weight and time delay weight for evaluating the success of MPS [10].

5.1 Non-linear membership functions

The relationship between the concepts is not always linear stating that change in concept C_i will not always lead to the change in C_j even if there exists a relation between them. The change occurs till certain limit and after that there will be no/inverse effect.

Consider the relation, C_{19} (Your phone is always with you) $\rightarrow$ C_4 (Independence of time and place)

This relationship used in MPS [10] always considered linear relationship. Consider a situation, if there is no proper network or the phone is switched off due to low battery, so FCM will not give realistic results. So this relation holds a non-linear relationship which can be represented in E-FCM.

Non-linear activation function i.e. sinusoidal function is used to show non linear relationship. In non linear relation, eq. (1) is used to get the saturation point [32].

$$v^{(k+1)} = f(\rho_1 v^{(k)} + \rho_2 w^T.v^{(k)}) \quad (1)$$

where

$v^{(k)} = [v_1^{(k)} v_2^{(k)} \cdots v_n^{(k)}]^T$ is the state vector

n = number of concepts

k= kth state vector used to derive the succeeding states.

Weight matrix $w = [w_{ij}]_{n\times n}$, $1 \le i, j \le n$,

$$f(v^{(k)}) = [f(v_1^{(k)} \cdots f(v_n^{(k)}]^{\mathrm{T}}. \quad (2)$$

Sinusoidal function is given as,

$$f(x) = 0.5(\sin(\beta x) + 1) \quad (3)$$

where,

$$\beta = \frac{1.5708}{(\rho_1 + \rho_2 \| w \|) n^{1/2} M} \quad (4)$$

β is calculated using $\rho_1 = 1, \rho_2 = 1, M = 1, n = 24$ (number of concepts).The domain of sinusoidal function is restricted within the range $[-\beta\pi/2, \beta\pi/2]$ so the value of $\pi/2$=1.5708 is used in calculation of β . Since classification or logistic regression aims to have 1 and 0 extremities, both sigmoid and sinusoidal functions achieve that. Instead sinusoidal does a better job in that its extremities are absolute instead of being asymptotic.

$\| w \|$=max Eigen value of $w^T w$.

β is calculated as 0.03.

Using eq. (3) in eq. (1) the saturation state is reached. Initially, the state of the concepts is taken as the average i.e.

$$C_j = \sum_{i=1}^{k} C_i / i \tag{5}$$

k =number of experts which is taken as 40 in the paper [10]. C_i is the value given to the concept C_j by i^{th} expert. (In the paper, the FZMS matrix gives the value of the concepts given by different experts).

Initially, all the values given to the concepts using eq. (5) are taken at time instant 0. In further iterations eq. (1) is used until saturation state is not reached (saturation means that the concept state vectors at time instant a, matching with the subsequent concept state vectors i.e. from time instant $a+1$,…). The saturation state using the non linear relation is found at instant 4 which is shown in Table 3. Linear membership function is used for all concepts except C_4 . As $C_{19} \rightarrow C_4$ holds a non-linear relationship so, non-linear membership is used to determine the saturation state of C_4 .

T	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
0	0.7	0.7	0.7	0.8	0.5	0.7	0.7	0.8	0.8	0.6	0.9	0.6
1	1	1	1	0.5473	1	1	1	1	1	1	1	1
2	1	1	1	0.5565	1	1	1	1	1	1	1	1
3	1	1	1	0.5567	1	1	1	1	1	1	1	1
4	1	1	1	0.5567	1	1	1	1	1	1	1	1
5	1	1	1	0.5567	1	1	1	1	1	1	1	1

T	C13	C14	C15	C16	C17	C18	C19	C20	C21	C22	C23	C24
0	0.7	0.7	0.5	0.8	0.9	0.8	0.8	0.6	0.6	0.9	0.6	0.5
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1

Table 3. Results using non-linear relation (α)

Hence, for handling non-linear relations (like $C_{19} \rightarrow C_4$), FCM would not suffice; rather E-FCM should be used.

5.2 Conditional weights

Sometimes, different concepts can affect a single concept, so the concepts can be combined to show the combined effect on the concept which is considered in E-FCM.

In this case AND function is used to represent the combined effect on the concepts, where two or more concepts can together create an overall effect on a particular concept.

In MPS [10], there is a direct connection between C_1 and C_9 (consider w_{19}) and C_{19} is not having a direct connection to C_9 , so authors wanted to show the combined effect of C_1 and C_{19} on C_9 . (5)

If C_1 (Ability to store money in your mobile) AND C_{19} (Your phone is always with you) then C_9 (Able to make small payments) means that if you have mobile with you and you have money too in your mobile then only you will be able to make payments. If any of the conditions in the antecedent part is not true then you will not be able to make payments.

So,

$$C_1, C_{19} \rightarrow C_9 \text{ with a proportion } w_{19} = 0.70/3$$

Ability to store money in your mobile (C_1) in the conventional FCM (Table 4) is saturated from time 3, that is why 3 is taken as a denominator.

T	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
0	0.7	0.7	0.7	0.8	0.5	0.7	0.7	0.8	0.8	0.6	0.9	0.6
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1	1	1

T	C13	C14	C15	C16	C17	C18	C19	C20	C21	C22	C23	C24
0	0.7	0.7	0.5	0.8	0.9	0.8	0.8	0.6	0.6	0.9	0.6	0.5
1	1	1	1	1	1	1	1	1	1	1	1	0
2	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1	1	1

Table 4. Conventional FCM

Now the weight w_{19} is updated to 0.233 in the weight matrix i.e. FCM Matrix. A change in the weight of the connection between the concepts C_1 and C_9 can be observed when two concepts are combined. The updated weight is shown in Figure 1 and rest of the weights are same as FCM. Authors have combined the results of conditional and non-linear relations which is shown in Table 5.

Using eq.(1),saturation point is calculated, where w is the newly constructed matrix after updating w_{19} to 0.233.The saturation point is found at instant 4 shown in Table 5, by taking initially the concept states using eq. (5).

T	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
0	0.7	0.7	0.7	0.8	0.5	0.7	0.7	0.8	0.8	0.6	0.9	0.6
1	1	1	1	0.5473	1	1	1	1	0.5267	1	1	1
2	1	1	1	0.5565	1	1	1	1	0.5289	1	1	1
3	1	1	1	0.5567	1	1	1	1	0.5289	1	1	1
4	1	1	1	0.5567	1	1	1	1	0.5289	1	1	1
5	1	1	1	0.5567	1	1	1	1	0.5289	1	1	1
T	C13	C14	C15	C16	C17	C18	C19	C20	C21	C22	C23	C24
0	0.7	0.7	0.5	0.8	0.9	0.8	0.8	0.6	0.6	0.9	0.6	0.5
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1

Table 5. Results using non-linear relation (α) and conditional weight (β)

Normally in real world problems both non-linear relationship and conditional weights are observed, so for such cases E-FCM is a better choice to find the hidden patterns as shown in Table 5.

5.3 Time delay weights

In E-FCMs, total input to node C_j at time t can be expressed as,

$$net_j = \sum_{i=1}^{k} w_{ij}(C_i(t - delay_{ij}))C_i(t - delay_{ij})$$

where, $C_j(t)$ is a causal concept at time t, $w_{ij}(.)$ is a weight function from concept $C_i(t)$ to concept $C_j(t)$, and $delay_{ij}$ is a time delay from causal concept $C_i(t)$ to concept $C_j(t)$ and k =number of concepts [11].

The relation between C_{24} and C_{12} has been discussed in MPS [10], but it is quite evident that this relation will incur some time delay, which was not considered in MPS [10].

The relation C_{24} (The bank receives a commission from the network operator processing each payment transaction) $\rightarrow C_{12}$ (Direct debiting from account) has some time delay, i.e. by the time the bank receives a commission from the operator, debiting takes place but the debiting and its changes to the account takes place after a certain amount of delay .

By using eq. (6) the state of concept C_{12} (since the time delay effect is on C_{12} and delay taken is 1 min) is calculated and the rest of the concept values are calculated using eq.(1) which is same as the values in Table 5. The Table 6 shows the saturation state using non-linear, conditional and time delay weights.

T	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
0	0.7	0.7	0.7	0.8	0.5	0.7	0.7	0.8	0.8	0.6	0.9	0.6
1	1	1	1	0.5473	1	1	1	1	0.5267	1	1	0.5
2	1	1	1	0.5565	1	1	1	1	0.5289	1	1	0.175
3	1	1	1	0.5567	1	1	1	1	0.5289	1	1	0.0214
4	1	1	1	0.5567	1	1	1	1	0.5289	1	1	0.0003
5	1	1	1	0.5567	1	1	1	1	0.5289	1	1	0
6	1	1	1	0.5567	1	1	1	1	0.5289	1	1	0
T	C13	C14	C15	C16	C17	C18	C19	C20	C21	C22	C23	C24
0	0.7	0.7	0.5	0.8	0.9	0.8	0.8	0.6	0.6	0.9	0.6	0.5
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1	1	1	1	1

Table 6. Results using non-linear relation (α) , conditional weight (β) and time delay weight (γ)

So, for handling relationships in which a time delay is observed between antecedent and consequent, again E-FCM emerges as a better option for modelling.

5.4 Computation of fixed point in E-FCM

Let, E be the E-FCM Matrix which is formulated using three factors: non-linear, conditional and time delay weights. (6)

This E matrix is different from D, with respect to the change recorded in conditional weight, as shown in Table 7.

Success factors	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1		0.66			0.65				0.23	0.73				0.63										
2	0.66		0.72																					
3		0.72			0.7	0.79								0.74										
4						0.79										0.87			0.83	0.74				
5	0.65		0.7																					
6			0.79	0.79				0.79	0.7	0.75	0.76									0.76				
7												0.66												
8						0.79					0.85		0.78											
9						0.7			0.7															
10	0.73					0.75																		
11						0.76		0.85									0.89							
12							0.66									0.62	0.74							0.7
13								0.78									0.82							
14	0.63		0.74																					
15												0.62				0.65	0.61							
16				0.87											0.65		0.88		0.78					
17											0.89	0.74	0.82		0.61	0.88								
18																					0.72	0.83	0.65	-0.6
19				0.83												0.78				0.71				
20				0.74		0.76													0.71					
21																		0.72						
22																		0.83						
23																		0.65						
24												0.7						-0.6						

Table 7. E-FCM Matrix

Now the hidden pattern using E-FCM can be calculated as,

If A_1 = (1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0)

$A_1 E =$ (0 0.68 0 0.83 0.65 0 0 0 0.233 0.73 0 0 0 0.63 0 0.78 0 0 0 0.71 0 0 0 0) $\rightarrow$(1 1 0 1 1 0 0 0 1 1 0 0 0 1 0 1 0 0 1 1 0 0 0 0) = A_2

$A_2 E =$ (3.39 0.68 2.16 2.44 0.65 3 0 0 0.23 0.73 0 0 0 0.63 0.65 1.65 0.88 0 2.32 1.45 0 0 0 0) $\rightarrow$(1 1 1 1 1 1 0 0 1 1 0 0 0 1 1 1 1 0 1 1 0 0 0 0) = A_3

$A_3 E =$ (3.39 1.4 2.95 3.23 1.35 3.79 0 0.79 0.93 1.48 1.65 1.36 0.82 1.37 1.26 3.18 1.49 0 2.32 2.21 0 0 0 0) $\rightarrow$ (1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 0) = A_4

$A_4 E =$ (3.39 1.4 2.95 3.23 1.35 5.34 0.66 2.42 0.93 1.48 2.5 1.36 1.6 1.37 1.88 3.18 3.94 0 2.32 2.21 0 0 0 0.70) $\rightarrow$ (1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 1) = A_5

$A_5 E =$ (3.39 1.4 2.95 3.23 1.35 5.34 0.66 2.42 0.93 1.48 2.5 2.72 1.6 1.37 1.88 3.18 3.94 0 2.32 2.21 0 0 0 0.70) $\rightarrow$ (1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 1) = $A_6 = A_5$

Hence, $A_6 = A_5$ we got a fixed point.

This implies that the concepts C_1 and C_{19} does not have any effect on the concepts C_{18}, C_{21}, C_{22} and C_{23}. Thus, this is a saturation point. By making further iterations, there is no change in the results. The results obtained by E-FCM are same as observed when FCM was used. But the discussion in this section indicates the suitability of deploying E-FCM when non-linearity, conditional and time delay is observed.

5.5 Limitations of E-FCM

The drawbacks of FCM are overcome by E-FCM considering non-linear weights, conditional and time delay weights. Though these three aspects are quite frequent and important in the relationships, but indeterminacy is also one of the prominent attribute of any relationship; Example: Consider unemployment and crime rate to be the two main causes of corruption, there may or may not be a possibility that due to unemployment the crime rate will increase, so this relation holds indeterminacy.

Capturing of indeterminacy is not done by FCM and E-FCM. This aspect of indeterminacy in the relationship is tackled using Neutrosophic Cognitive Maps (NCM), which is discussed next.

6 NEUTROSOPHIC COGNITIVE MAPS (NCM)

NCM is an extension of FCM where indeterminacy is included. The concept of fuzzy cognitive maps deal with the relationship between two nodes but it fails to deal with the indeterminate relation. Neutrosophic logic is the only tool, which deals with the notions of indeterminacy. NCM will certainly give a more appropriate result when we deal with unsupervised data, and no relation can be determined between two nodes. NCM applications can be found in medical field [33] [34], social issue [35] and other areas [36] [37].

6.1 Basics of NCM [35]

- Let $C_1, C_2, \ldots, C_n$ be n nodes, and we assume every node is a neutrosophic vector from neutrosophic vector space V. A node C_i will be represented by $(x_1, \cdots, x_n)$ where x_k is zero or one or *I* (*I* is the indeterminate). The concept's state x_k = 1 means the node C_k is in on state; x_k = 0 means the node is in off state and x_k = I means the node state is indeterminate at that time or in that situation.
- Assume C_i and C_j denote two nodes of the NCM. The directed edge from C_i to C_j denote the causality of C_i on C_j called connections. Each edge in the NCM is weighted with a number {-1, 0, 1, I }. Assume a_{ij} is the weight of the directed edge $C_i C_j$, $a_{ij} \in$ {-1, 0, 1, I }.
  a_{ij} = 0 if C_i does not have any effect on C_j
  a_{ij} = 1 if increase (or decrease) in C_i causes increase (or decrease) in C_j
  a_{ij} = -1 if increase (or decrease) in C_i causes decrease (or increase) in C_j
  a_{ij} = I if the relation or effect of C_i on C_j is an indeterminate.
- Let $C_1C_2, C_3C_4, \cdots C_iC_j$ be the edges of NCM and the edges form a directed cycle. An NCM is said to be cyclic if it has a directed cycle and acyclic if it does not have any directed cycle.
- If the NCM settles down with a unique neutrosophic state vector, then it is known as fixed point. Assume the NCM with C_1, C_2, ..., C_n as nodes. For example let us start by switching on C_1. Let us consider that the NCM settles down with C_1 and C_n on, i.e. The state vector remain as (1, 0, $\cdots$, 1) this neutrosophic state vector (1,0,$\cdots$, 0, 1) is known as the fixed point.
- If the NCM settles down with a neutrosophic state vector repeating in the form of $A_1 \rightarrow A_2 \rightarrow \cdots A_i \rightarrow A_1$, then this equilibrium is called as limit cycle of the NCM.

6.2 NCM Methodology

The MPS is based only on FCM where no indeterminacy relations are considered.

In the paper [10] following indeterminate relations are highlighted:

C_1(Ability to store money in your mobile) $\longrightarrow$ C_5(Getting rid of plastic cards)

If the user has money in his/her mobile then there may be a possibility that he carries credit card with him/her for making large payments since MPS is designed for small and medium payments.

C_{19}(Your phone is always with you) $\longrightarrow$ C_{13}(Ability to pay from mobile in store)

If the user has phone with him/her, still he/she may or may not be able to pay from the mobile because there can be a network problem or may be they don't have enough balance with them, preventing them in making payments.

C_{12}(Direct debiting from account) $\longrightarrow$ *C_7(Security)*

The direct debiting from the account may or may not affect security since the information related to risks analysis and management are missing which can lead to indeterminacy.

C_{22}(User does not pay credit/debit card maintenance costs to the bank) $\longrightarrow$ *C_8(Economy)*

User does not need credit/debit card for small and medium payments so no maintenance cost for the cards. If the range of payment exceeds medium payment then the user can pay through the card and there will be some maintenance cost to be paid by the user to the bank for card maintenance, hence this relationship is also indeterminate.

6.3 Working of NCM

The 'I' factor was introduced in the FCM matrix which is now relabelled as NCM matrix as shown in Table 8.The hidden pattern using NCM was calculated as,
N(E) = NCM Matrix shown in Table 8.

Success factors	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1		0.66			I				0.23	0.73				0.63										
2	0.66		0.72																					
3		0.72			0.7	0.79								0.74										
4						0.79										0.87			0.83	0.74				
5	0.65		0.7																					
6			0.79	0.79				0.79	0.7	0.75	0.76									0.76				
7												0.66												
8						0.79					0.85		0.78											
9						0.7			0.7															
10	0.73					0.75																		
11						0.76		0.85									0.89							
12							I									0.62	0.74							0.7
13								0.78									0.82							
14	0.63		0.74																					
15												0.62				0.65	0.61							
16				0.87											0.65		0.88		0.78					
17											0.89	0.74	0.82		0.61	0.88								
18																					0.72	0.83	0.65	-0.6
19				0.83									I			0.78				0.71				
20				0.74		0.76													0.71					
21																		0.72						
22																		I						
23																		0.65						
24												0.7						-0.6						

Table 8. NCM Matrix

$I =$ Indeterminacy

The NCM for the MPS project is shown in Figure 1. and its related FCM is shown in paper [10].

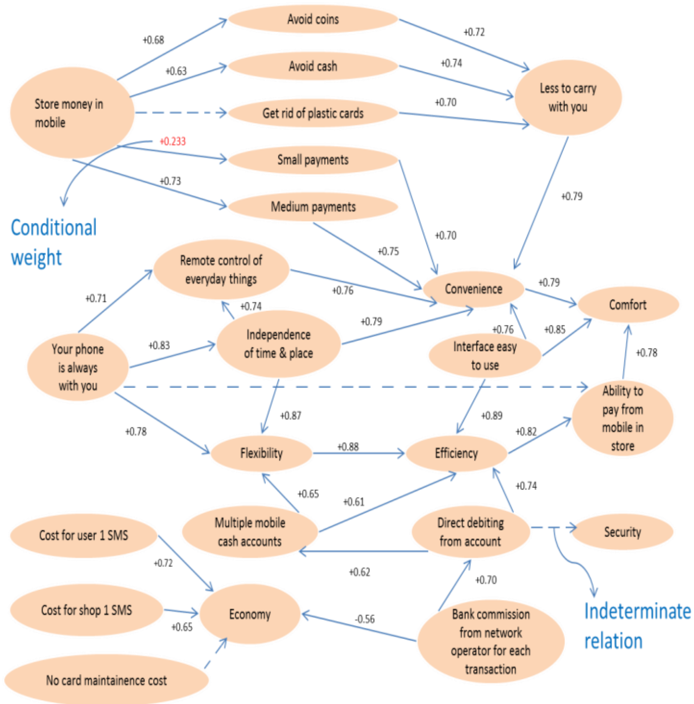

Figure 1. NCM for the MPS project

The hidden pattern for NCM is calculated as follows:
Initially C_1 and C_{19} are taken in ON state i.e.

If $A_1 =$ (1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0),

$A_1N(E) =$ (0 0.68 0 0.83 I 0 0 0 0.233 0.73 0 0 I 0.63 0 0.78 0 0 0 0.71 0 0 0 0) $\rightarrow$(1 1 0 1 I 0 0 0 1 1 0 0 I 1 0 1 0 0 1 1 0 0 0 0) $= A_2$

$A_2N(E) =$ (2.74+0.65I 0.68 1.46+0.70I 2.44 I 3 0 0.78I 0.233 0.73 0 0 I 0.63 0.65 1.65 0.88+0.82I 0 2.32 1.45 0 0 0 0)$\rightarrow$ (1 1 1 1 I 1 0 I 1 1 0 0 I 1 1 1 1 0 1 1 0 0 0 0) $= A_3$

$A_3N(E) =$ (2.74+0.65I 1.4 2.25+0.70I 3.23 0.70+I 3.79+0.79I 0 0.79+0.78I 0.933 1.48 1.65+0.85I 1.36 0.82+1.78I 1.37 1.26 3.18 1.49+0.82I 0 2.32 2.21 0 0 0 0)$\rightarrow$(1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 0) $= A_4$

$A_4N(E) =$ (3.39 1.4 2.95 3.23 0.70+I 5.34 I 2.42 0.933 1.48 2.5 1.36 1.6+I 1.37 1.88 3.18 3.94 0 2.32 2.21 0 0 0 0.70)$\rightarrow$(1 1 1 1 1 1 I 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 1) $= A_5$

$A_5N(E) =$ (3.39 1.4 2.95 3.23 0.70+I 5.34 I 2.42 0.933 1.48 2.5 2.06+0.66I 1.6+I 1.37 1.88 3.18 3.94 -0.56 2.32 2.21 0 0 0 0.70)$\rightarrow$(1 1 1 1 1 1 I 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 1) $= A_6 = A_5$

Here in state A_6 concept 7 is showing indeterminacy, i.e. by making the concepts C_1 and C_{19} on, concepts with state as 1 shows that they are affected by the factors C_1 and C_{19} , but C_7 is I that indicates that even if user has his phone with him and has money in mobile, security factor may or may not be affected, also it can have positive or negative impact.

With the availability of internet facility on mobile phones, there is a chance of some virus attack which may affect the performance of MPS software by making user's device slow or hang. Considering such situation if the user executes a transaction and does not get the confirmation, it may lead to another transaction. Since the previous one was under processing, of which the user was not aware; can make user pay for the same transaction twice. Thus, giving negative influence between C_1 and C_{19}.

Contrary to this the positive influence between C_1 and C_{19} can be recorded if the user joins MPS system, he is given a secret code, which he knows it personally and can use it for payments in a secure way.

So the relationship between C_1 and C_{19} can be either positive or negative; thus reflecting indeterminacy in it.

7 DISCUSSION OF RESULTS

To record the effect of factors C_1(*Ability to store money in your mobile*) and C_{19}(*Your phone is always with you*) initially the vector is taken as (1000000000000000100000). The results for different methodologies FCM, E-FCM and NCM and their comparison is shown in Table 9.

Methodology Used	Results	Summary
Fuzzy Cognitive Maps	The hidden pattern calculated was (1111111111111111110110001) which shows that there will always be effect on concept 7 i.e. security by the concepts C1 and C19	FCMs measure the existence of causal relation between two concepts and if no relation exists it is denoted by 0. It does not consider non-linear weights, conditional and time delay weights which are the drawbacks of FCM.
Extended-Fuzzy Cognitive Maps	The hidden pattern calculated was (1111111111111111110110001) which is same as FCM showing that there will always be effect on security when we hold C1 and C19. Though considering the conditional weight, change in weight [w1,9] is observed.	E-FCM is an extension of FCM that provides option for capturing non-linear, conditional, time delay weights. Though it overcomes the drawbacks of FCM but does not represents indeterminate relations.
Neutrosophic Cognitive Maps	The hidden pattern calculated was (111111*I* 11111111110110001). Results obtained clearly indicates that the effect on concept 7 i.e. security is indeterminate, means that if the user always has phone with him/her and even has money in his mobile, this may or may not lead to secure payments.	NCMs measure not only the existence or non-existence of causal relation between concepts but also allows the representation of indeterminacy in the relations. Hence, NCM model to map the indeterminate relationship which are frequent in real world, thus more realistic results are expected.

Table 9. Comparison of Results

CONCLUSION & FUTURE WORK

Compared to the results of FCM and E-FCM in the MPS project, the hidden pattern showed that security will always be affected as FCM and E-FCM can represent positive, negative or no effect. But, in NCM, security concept is '*I*' depicting that this factor may or may not be affected. NCM provided the option of handling the indeterminate relationship.

Neutrosophic Cognitive Map is an innovative research approach. The concept of NCM can be used in modelling of system success, since the concept of indeterminacy plays role while evaluating project success. This was authors' main aim to use NCMs in place of FCMs. When an indeterminate causality is present in an FCM we term it as an NCM.

As an extension of the presented work, authors project to study the following:

a) More number of parameters can be used to predict the results. Increment in sample size will also lead to give more accurate results.

b) Opinion of different experts can be combined & implemented using Linked FCM & Linked NCM.

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Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making

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Abstract. This paper is devoted to propose triangular fuzzy number neutrosophic sets by combining triangular fuzzy numbers with single valued neutrosophic set and define some of its operational rules. Then, triangular fuzzy number neutrosophic weighted arithmetic averaging operator and triangular fuzzy number neutrosophic weighted geometric averaging operator are defined to aggregate triangular fuzzy number neutrosophic sets. We have also established some of their properties of the proposed operators. The operators have been employed to multi attribute decision making problem to aggregate the triangular fuzzy neutrosophic numbers based rating values of each alternative over the attributes. The collective rating values of each alternative have been ordered with the help of score and accuracy values to find out the best alternative. Finally, an illustrative example has been provided to validate the proposed approach for multi attribute decision making problem.

Keywords: Triangular fuzzy number neutrosophic set, Score and accuracy function, Triangular fuzzy number neutrosophic weighted arithmetic averaging operator, Triangular fuzzy number neutrosophic weighted geometric averaging operator, Multi-attribute decision making problem.

1 Introduction

Zadeh [1] has been credited with having pioneered the development of the concept of fuzzy set in 1965. It is generally agreed that a major breakthrough in the evolution of the modern concept of uncertainty was achieved in defining fuzzy set, even though some ideas presented in the paper were envisioned in 1937 by Black [2]. In order to define fuzzy set, Zadeh [1] introduced the concept of membership function with a range covering the interval [0, 1] operating on the domain of all possible values. It should be noted that the concept of membership in a fuzzy set is not a matter of affirmation or denial, rather a matter of a degree. Zadeh's original ideas blossomed into a comprehensive corpus of methods and tools for dealing with gradual membership and non-probabilistic uncertainty. In essence, the basic concept of fuzzy set is a generalization of classical set or crisp set [3, 4]. The field has experienced an enormous development, and Zadeh's seminal concept of fuzzy set [1] has naturally evolved in different directions. Different sets have been derived in the literature such as L-fuzzy sets [5], flou sets [6], interval-valued fuzzy sets [7-10], intuitionistic fuzzy sets [11-13], two fold fuzzy sets [14], interval valued intuitionistic fuzzy set [15], intuitionistic L-fuzzy sets [16], etc. Interval-valued fuzzy sets are a special case of L-fuzzy sets in the sense of Goguen [5] and a special case of type 2 fuzzy set. Mathematical equivalence of intuitionistic fuzzy set (IFS) with interval-valued fuzzy sets was noticed by Atanassov [17], Atanassov and Gargov [15]. Wang and He [18] proved that the concepts of IFS [11-13] and intuitionistic L-fuzzy sets [5] and the concept of L-fuzzy sets [5] are equivalent. Kerre [19] provided a summary of the links that exist between fuzzy sets [1] and other mathematical models such as flou sets [6], two-fold fuzzy sets [14] and L-fuzzy sets [5]. Deschrijver and Kerre [20] established the relationships between IFS [11], L-fuzzy sets [5], interval-valued fuzzy sets [7], interval-valued IFS [15]. Dubois et al. [21] criticized the term IFSs in the sense of [11-13], and termed it "to be unjustified, misleading, and possibly offensive to people in intuitionistic mathematics and logic" as it clashes with the correct usage of intuitionistic fuzzy set proposed by Takeuti and Titani [22]. Dubois et al. [21] suggested changing the name of IFS as I-fuzzy set. Smarandache incorporated the degree of indeterminacy as independent component in IFS and defined neutrosophic set [23-24] as the generalization of IFSs. Georgiev [25] explored some properties of the neutrosophic logic and defined simplified neutrosophic set. A neutrosophic set is simplified [25] if its elements are comprised of singleton subsets of the real unit interval. Georgiev [25] concluded that the neutrosophic logic is not capable of maintaining modal operators, since there is no normalization rule for the components T, I and F. The author [25] claimed that the IFSs have the chance to become a consistent model of the modal logic, adopting all the necessary properties [26].However certain type of uncertain information such as indeterminate, incomplete and inconsistent information cannot be dealt with fuzzy sets as well as IFSs. Smarandache [27-28] re-established neutrosophic set as the generalization of IFS, which plays a key role to handle uncertain, inconsistent and indeterminacy information existing in real world. In this set [27-28] each element of the universe is characterized by the truth degree, indeterminacy degree and falsity degree lying in the nonstandard unit interval. The neutrosophic set [27-28] emerged as one of the research focus in many branches such as image processing [29-31], artificial intelligence [32], applied physics [33-34], topology [35] and social science [36]. Furthermore, single valued neutrosophic set[37], interval neutrosophic set[38],neutrosophic soft set[39], neutrosophic soft expert set [40], rough neutrosophic set [41], interval neutrosophic rough set, interval valued neutrosophic soft rough set [42], complex neutrosophic set[43], bipolar neutrosophic sets [44] and neutrosophic cube set[45] have been studied in the literature which are connected with neutrosophic set. However, in this study, we have applied single valued neutrosophic set [37] (SVNS), a subclass of NS, in which each element of universe is characterized by truth membership, indeterminacy membership and falsity membership degrees lying in the real unit interval. Recently, SVNS has caught attention to the researcher on various topics such as similarity measure [46-50], medical diagnosis [51] and multi criteria/ attribute decision making [52-58], etc

Aggregation of SVNS information becomes an important research topic for multi attribute decision making in which the rating values of alternatives are expressed in terms of SVNSs. Aggregation operators of SVNSs, usually taking the forms of mathematical functions, are common techniques to fuse all the input individual data that are typically interpreted as the truth, indeterminacy and the falsity membership degree in SVNS into a single one. Ye [59] proposed weighted arithmetic average operator and weighted geometric average operator for simplified neutrosophic sets. Peng et al.[60] developed some aggregation operators to aggregate single valued neutrosophic information, such as simplified neutrosophic number weighted averaging (SNNWA), simplified neutrosophic number weighted geometric (SNNWG), simplified neutrosophic number ordered weighted averaging (SNNOWA), simplified neutrosophic number ordered weighted geometric averaging (SNNOWG), simplified neutrosophic number hybrid ordered weighted averaging operator(SNNHOWA), simplified neutrosophic number hybrid ordered weighted geometric operator (SNNHOWG), generalised simplified neutrosophic number weighted averaging operator(GSNNWA) and generalised simplified neutrosophic number weighted geometric operator(GSNNGA) operators. Peng et al. [60] applied these aggregation operators in multi criteria group decision making problem to get an overall evaluation value for selecting the best alternative. Liu et al. [61] defined some generalized neutrosophic Hamacher aggregation operators and applied them to multi attribute group decision making problem. Liu and Wang [62] proposed a single valued neutrosophic normalized weighted Bonferroni mean operator for multi attribute decision making problem.

Application of SVNS has been extensively studied in multi-attribute decision making problem. However, in uncertain and complex situations, the truth membership, indeterminacy membership, and falsity membership degree of SVNS cannot be represented with exact real numbers or interval numbers. Moreover, triangular fuzzy number can handle effectively fuzzy data rather than interval number. Therefore, combination of triangular fuzzy number with SVNS can be used as an effective tool for handling incomplete, indeterminacy, and uncertain information existing in decision making problems. Recently, Ye [63] defined trapezoidal fuzzy neutrosophic set and developed trapezoidal fuzzy neutrosophic number weighted arithmetic averaging and trapezoidal fuzzy neutrosophic number weighted geometric averaging operators to solve multi attribute decision making problem.

Zhang and Liu [64] presented method for aggregating triangular fuzzy intuitionistic fuzzy information and its application to decision making. However, their approach cannot deal the decision making problems which involve indeterminacy. So new approach is essentially needed which can deal indeterminacy. Literature review reflects that this is the first time that aggregation operator of triangular fuzzy number neutrosophic values has been studied although this number can be used as an effective tool to deal with uncertain information. In this paper, we have first presented triangular fuzzy number neutrosophic sets (TFNNS), score function and accuracy function of TFNNS. Then we have extended the aggregation method of triangular fuzzy intuitionistic fuzzy information [64] to triangular fuzzy number neutrosophic weighted arithmetic averaging (TFNNWA) operator and triangular fuzzy number neutrosophic weighted geometric averaging (TFNNWG) operator to aggregate TFNNSs. The proposed TFNNWA and TFNNWG operators are more flexible and powerful than their fuzzy and intuitionistic fuzzy counterpart as they are capable of dealing with uncertainty and indeterminacy.

The objectives of the study include to:

- propose triangular fuzzy number neutrosophic sets (TFNNS), score function and accuracy function of TFNNS.
- propose two aggregation operators, namely, TFNNWA and TFNNWG.
- prove some properties of the proposed operators namely, TFNNWA and TFNNWG.
- establish a multi attribute decision making (MADM) approach based on TFNNWA and TFNNWG.
- provide an illustrative example of MADM problem.

The rest of the paper has been organized in the following way. In Section 2, a brief overview of IFS, SVNS have been presented. In Section 3, we have defined TFNNS, score function and accuracy function of TFNNS, and some operational rules of TFNNS. Section 4 has been devoted to propose two aggregation operators, namely, TFNNWA and TFNNWG operators to aggregate TFNNSs. In Section 5, applications of two proposed operators have been presented in multi attribute decision making problem. In Section 6, an illustrative example of MADM has been provided. Finally, conclusion and future direction of research have been presented in Section 7.

2 Preliminaries

In this section we recall some basic definitions of intuitionistic fuzzy sets, triangular fuzzy number intuitionistic fuzzy set (TFNIFS), score function and accuracy function of TFNIFS.

2.1 Intuitionistic fuzzy sets

Definition1. (Intuitionistic fuzzy set [13]) An intuitionistic fuzzy set A in finite universe of discourse $X=\{x_1, x_2, ..., x_n\}$ is given by

$$A=\{\langle x, \mu_A(\mathrm{x}), \nu_A(\mathrm{x})\rangle \mid x\in X\}, \qquad (1)$$

where $\mu_A : X\to[0,1]$ and $\nu_A : X\to[0,1]$ with the condition $0\le \mu_A(x)+\nu_A(x)\le 1$. The numbers $\mu_A(\mathrm{x})$ and $\nu_A(\mathrm{x})$ denote, respectively, the degree of membership degree and degree of non-membership of x in A. In addition $\pi_A(\mathrm{x})=1-\mu_A(\mathrm{x})-\nu_A(\mathrm{x})$ is called a hesitancy degree of $x\in X$ in A. For convenience, $A=(\mu_A(\mathrm{x}), \nu_A(\mathrm{x}))$ is considered as an intuitionistic fuzzy number (IFN).

Definition 2. (Operations rules of IFNs [65-67])

Let $A=(\mu_A(\mathrm{x}), \nu_A(\mathrm{x}))$ and $B=(\mu_B(\mathrm{x}), \nu_B(\mathrm{x}))$ be two IFNs, then the basic operations of IFNs are presented as follows:

1. $A\oplus B=(\mu_A(\mathrm{x})+\mu_B(\mathrm{x})-\mu_A(\mathrm{x})\mu_B(\mathrm{x}),\ \nu_A(\mathrm{x})\nu_B(\mathrm{x})),$ (2)
2. $A\otimes B=(\mu_A(\mathrm{x})\mu_B(\mathrm{x}),\ \nu_A(\mathrm{x})\nu_B(\mathrm{x})-\nu_A(\mathrm{x})\nu_B(\mathrm{x})),$ (3)
3. $\lambda A=\left(1-(1-\mu_A(\mathrm{x}))^{\lambda}, (\nu_A(x))^{\lambda}\right)$ for $\lambda>0,$ (4)
4. $A^{\lambda}=\left((\mu_A(x))^{\lambda}, 1-(1-\nu_A(\mathrm{x}))^{\lambda}\right)$ for $\lambda>0.$ (5)

Definition 3. [68] Let X be a finite universe of discourse and $F[0,1]$ be the set of all triangular fuzzy numbers on $[0,1]$. A triangular fuzzy number intuitionistic fuzzy set (TFNIFS) A in X is represented by

$$A=\{\langle x, \tilde{\mu}_A(x), \tilde{\nu}_A(x)\rangle \mid x\in X\},$$

where, $\tilde{\mu}_A(x): X\to F[0,1]$ and $\tilde{\nu}_A(x): X\to F[0,1]$.

The triangular fuzzy numbers $\tilde{\mu}_A(x)=\left(\mu_A^1(\mathrm{x}), \mu_A^2(\mathrm{x}), \mu_A^3(\mathrm{x})\right)$ and $\tilde{\nu}_A(x)=\left(\nu_A^1(\mathrm{x}), \nu_A^2(\mathrm{x}), \nu_A^3(\mathrm{x})\right)$, respectively, denote the membership degree and non-membership degree of x in A and for every $x\in X$:

$$0\le \mu_A^3(\mathrm{x})+\nu_A^3(\mathrm{x})\le 1.$$

For convenience, we consider $A=\langle(\mathrm{a,b,c}), (\mathrm{e,f,g})\rangle$ as the trapezoidal fuzzy number intuitionistic fuzzy values (TFNIFV) where, $\left(\mu_A^1(\mathrm{x}), \mu_A^2(\mathrm{x}), \mu_A^3(\mathrm{x})\right)=(a,b,c)$ and $\left(\nu_A^1(\mathrm{x}), \nu_A^2(\mathrm{x}), \nu_A^3(\mathrm{x})\right)=(e,f,g)$.

Definition 4. [69-70] Let $A_1=\langle(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1), (\mathrm{e}_1, \mathrm{f}_1, \mathrm{g}_1)\rangle$ and $A_2=\langle(\mathrm{a}_2, \mathrm{b}_2, \mathrm{c}_2), (\mathrm{e}_2, \mathrm{f}_2, \mathrm{g}_2)\rangle$ be two TFNIFVs, then the following operations are valid:

1. $A_1 \oplus A_2 = \left\langle \begin{matrix} \left(a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2, c_1 + c_2 - c_1c_2\right), \\ \left(e_1e_2, f_1f_2, g_1g_2\right) \end{matrix} \right\rangle;$ (6)

2. $A_1 \otimes A_2 = \left\langle \begin{matrix} \left(a_1a_2, b_1b_2, c_1c_2\right), \\ \left(e_1 + e_2 - e_1e_2, f_1 + f_2 - f_1f_2, g_1 + g_2 - g_1g_2\right) \end{matrix} \right\rangle;$ (7)

3. $\lambda A_1 = \left\langle \left(1-(1-a_1)^\lambda, 1-(1-b_1)^\lambda, 1-(1-c_1)^\lambda\right), (e_1^\lambda, f_1^\lambda, g_1^\lambda) \right\rangle$ for $\lambda > 0$, , and (8)

4. $A_1^\lambda = \left\langle \begin{matrix} (a_1^\lambda, b_1^\lambda, b_1^\lambda), \\ \left(1-(1-e_1)^\lambda, 1-(1-f_1)^\lambda, 1-(1-g_1)^\lambda\right) \end{matrix} \right\rangle$ for $\lambda > 0.$ (9)

Definition 5. [69-70]Let $A_1 = \langle (a_1, b_1, c_1), (e_1, f_1, g_1) \rangle$ be a TFNIFV, the score function $S(A_1)$ of A_1 is defined as follows:

$$S(A_1) = \frac{1}{4}\left[\left(a_1 + 2b_1 + c_1\right) - \left(e_1 + 2f_1 + g_1\right)\right],\ S(A_1) \in [-1,1] \quad (10)$$

The score function $S(A^+)=1$ for the TFNIFV $A^+ = \langle (1,1,1), (0,0,0) \rangle$ and $S(A^-) = -1$ for the TFNIFV $A^- = \langle (0,0,0), (1,1,1) \rangle$.

Definition 6. [69-70] Let $A_1 = \langle (a_1, b_1, c_1), (e_1, f_1, g_1) \rangle$ be a TFNIFV, the accuracy function $H(A_1)$ is of A_1 is defined as follows:

$$H(A_1) = \frac{1}{4}\left[\left(a_1 + 2b_1 + c_1\right) + \left(e_1 + 2f_1 + g_1\right)\right],\ H(A_1) \in [0,1]. \quad (11)$$

2.2 Single valued neutrosophic sets

In this section, some basic definitions of single valued neutrosophic sets are reviewed.

Definition 7. [37] Let X be a space of points (objects) with a generic element in X denoted by x. A single valued neutrosophic set A in X is characterized by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$ and is denoted by

$$\tilde{A} = \left\{x, \left\langle T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\right\rangle \mid x \in X\right\}.$$

Here $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$ are real subsets of [0, 1] that is $T_{\tilde{A}}(x): X \to [0,1]$, $I_{\tilde{A}}(x): X \to [0,1]$ and $F_{\tilde{A}}(x): X \to [0,1]$. The sum of $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$ lies in [0, 3] that is $0 \le \sup T_{\tilde{A}}(x) + \sup I_{\tilde{A}}(x) + \sup F_{\tilde{A}}(x) \le 3.$

For convenience, SVNS $\tilde{A}$ can be denoted by $\tilde{A} = \left\langle T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\right\rangle$ for all x in X.

Definition 8. [37] Assume that $\tilde{A} = \left\langle T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\right\rangle$ and $\tilde{B} = \left\langle T_{\tilde{B}}(x), I_{\tilde{B}}(x), F_{\tilde{B}}(x)\right\rangle$ be two SVNSs in a universe of discourse X. Then the following operations are defined as follows:

1. $\tilde{A} \oplus \tilde{B} = \left\langle \begin{matrix} T_{\tilde{A}}(x) + T_{\tilde{B}}(x) - T_{\tilde{A}}(x)T_{\tilde{B}}(x), \\ I_{\tilde{A}}(x)I_{\tilde{B}}(x), F_{\tilde{A}}(x)F_{\tilde{B}}(x) \end{matrix} \right\rangle;$ (12)

2. $\tilde{A} \otimes \tilde{B} = \left\langle \begin{matrix} T_{\tilde{A}}(x)T_{\tilde{B}}(x), I_{\tilde{A}}(x) + I_{\tilde{B}}(x) - I_{\tilde{A}}(x)I_{\tilde{B}}(x), \\ F_{\tilde{A}}(x) + F_{\tilde{B}}(x) - F_{\tilde{A}}(x)F_{\tilde{B}}(x) \end{matrix} \right\rangle;$ (13)

3. $\lambda\tilde{A} = \left\langle 1-\left(1-T_{\tilde{A}}(x)\right)^\lambda, \left(I_{\tilde{A}}(x)\right)^\lambda, \left(F_{\tilde{A}}(x)\right)^\lambda \right\rangle$ for $\lambda > 0$, and (14)

4. $(\tilde{A})^\lambda = \left\langle \left(T_{\tilde{A}}(x)\right)^\lambda, 1-\left(1-I_{\tilde{A}}(x)\right)^\lambda, 1-\left(1-F_{\tilde{A}}(x)\right)^\lambda \right\rangle$ for $\lambda > 0.$ (15)

3 Triangular fuzzy number neutrosophic set

SVNS can represent imprecise, incomplete and inconsistent type information existing in the real world problem. However, decision maker often expresses uncertain information with truth, indeterminacy and falsity membership functions that are represented with uncertain numeric values instead of exact real number values. These uncertain numeric values of truth, indeterminacy and falsity membership functions of SVNSs can be represented in terms of triangular fuzzy numbers.

In this section, we combine triangular fuzzy numbers (TFNs) with SVNSs to develop triangular fuzzy number neutrosophic set (TFNNS) in which, the truth, indeterminacy and falsity membership functions are expressed with triangular fuzzy numbers.

Definition 9. Assume that X be the finite universe of discourse and F [0, 1] be the set of all triangular fuzzy numbers on $[0,1]$. A triangular fuzzy number neutrosophic set (TFNNS) $\tilde{A}$ in X is represented by

$\tilde{A} = \left\{ \left\langle x, \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x) \right\rangle | x \in X \right\}$,

where, $\tilde{T}_A(x): X \to F[0,1]$, $\tilde{I}_A(x): X \to F[0,1]$, and $\tilde{F}_A(x): X \to F[0,1]$.

The triangular fuzzy numbers $\tilde{T}_{\tilde{A}}(x) = \left(T^1_{\tilde{A}}(\mathrm{x}), T^2_{\tilde{A}}(\mathrm{x}), T^3_{\tilde{A}}(\mathrm{x})\right)$, $\tilde{I}_{\tilde{A}}(x) = \left(I^1_{\tilde{A}}(\mathrm{x}), I^2_{\tilde{A}}(\mathrm{x}), I^3_{\tilde{A}}(\mathrm{x})\right)$, and $\tilde{F}_{\tilde{A}}(x) = \left(F^1_{\tilde{A}}(x), F^2_{\tilde{A}}(x), F^3_{\tilde{A}}(\mathrm{x})\right)$, respectively, denote the truth membership degree, indeterminacy degree, and falsity membership degree of x in $\tilde{A}$ and for every $x \in X$:

$$0 \le T^3_A(\mathrm{x}) + I^3_A(\mathrm{x}) + F^3_A(\mathrm{x}) \le 3 . \quad (16)$$

For notational convenience, we consider $\tilde{A} = \langle (a,b,c),(e,f,g),(r,s,t) \rangle$ as a trapezoidal fuzzy number neutrosophic values (TFNNV) where,

$\left(T^1_{\tilde{A}}(\mathrm{x}), T^2_{\tilde{A}}(\mathrm{x}), T^3_{\tilde{A}}(\mathrm{x})\right) = (a,b,c)$,

$\left(I^1_{\tilde{A}}(\mathrm{x}), I^2_{\tilde{A}}(\mathrm{x}), I^3_{\tilde{A}}(\mathrm{x})\right) = (e,f,g)$,

and $\left(F^1_{\tilde{A}}(\mathrm{x}), F^2_{\tilde{A}}(\mathrm{x}), F^3_{\tilde{A}}(\mathrm{x})\right) = (r,s,t)$.

Definition 10. Let $\tilde{A}_1 = \langle (a_1,b_1,c_1),(e_1,f_1,g_1),(r_1,s_1,t_1) \rangle$ and $\tilde{A}_2 = \langle (a_2,b_2,c_2),(e_2,f_2,g_2),(r_2,s_2,t_2) \rangle$ be two TFNNVs in the set of real numbers. Then the following operations are defined as follows:

$$1.\ \tilde{A}_1 \oplus \tilde{A}_2 = \left\langle \begin{matrix} \left(a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2, c_1 + c_2 - c_1c_2\right), \\ \left(e_1e_2, f_1f_2, g_1g_2\right), \left(r_1r_2, s_1s_2, t_1t_2\right) \end{matrix} \right\rangle; \quad (17)$$

$$2.\ \tilde{A}_1 \otimes \tilde{A}_2 = \left\langle \begin{matrix} \left(a_1a_2, b_1b_2, c_1c_2\right), \\ \left(e_1 + e_2 - e_1e_2, f_1 + f_2 - f_1f_2, g_1 + g_2 - g_1g_2\right), \\ \left(r_1 + r_2 - r_1r_2, s_1 + s_2 - s_1s_2, t_1 + t_2 - t_1t_2\right) \end{matrix} \right\rangle; \quad (18)$$

$$3.\ \lambda\tilde{A} = \left\langle \begin{matrix} \left(1-(1-\mathrm{a}_1)^\lambda, 1-(1-b_1)^\lambda, 1-(1-c_1)^\lambda\right), \\ \left(e_1^\lambda, f_1^\lambda, g_1^\lambda\right), \left(r_1^\lambda, s_1^\lambda, t_1^\lambda\right) \end{matrix} \right\rangle \text{ for } \lambda > 0 \text{ and} \quad (19)$$

$$4.\ \tilde{A}^\lambda = \left\langle \begin{matrix} \left(a_1^\lambda, b_1^\lambda, c_1^\lambda\right), \\ \left(1-(1-e_1)^\lambda, 1-(1-f_1)^\lambda, 1-(1-g_1)^\lambda\right), \\ \left(1-(1-r_1)^\lambda, 1-(1-s_1)^\lambda, 1-(1-t_1)^\lambda\right) \end{matrix} \right\rangle \text{ for } \lambda > 0 . \quad (20)$$

The operations defined in Definition 10 satisfy the following properties:

1. $\tilde{A}_1 \oplus \tilde{A}_2 = \tilde{A}_2 \oplus \tilde{A}_1$, $\tilde{A}_1 \otimes \tilde{A}_2 = \tilde{A}_2 \otimes \tilde{A}_1$;
2. $\lambda\left(\tilde{A}_1 \oplus \tilde{A}_2\right) = \lambda\tilde{A}_1 \oplus \lambda\tilde{A}_2$, $\left(\tilde{A}_1 \otimes \tilde{A}_2\right)^\lambda = \tilde{A}_1^\lambda \otimes \tilde{A}_2^\lambda$ for $\lambda > 0$, and
3. $\lambda_1\tilde{A}_1 \oplus \lambda_2\tilde{A}_1 = \left(\lambda_1 + \lambda_2\right)\tilde{A}_1$, $\tilde{A}_1^{\lambda_1} \oplus \tilde{A}_1^{\lambda_2} = \tilde{A}_1^{(\lambda_1+\lambda_2)}$ for $\lambda_1, \lambda_2 > 0$.

3.1 Score and accuracy function of TFNNV

In the following section, we define score function and accuracy function of TFNNV from Definition 5, Definition 6.

Definition 11. Assume that $\tilde{A}_1 = \langle (\mathrm{a}_1,\mathrm{b}_1,\mathrm{c}_1),(\mathrm{e}_1,\mathrm{f}_1,\mathrm{g}_1),(\mathrm{r}_1,\mathrm{s}_1,\mathrm{t}_1) \rangle$ be a *TFNNVs* in the set of real numbers, the score function $S(\tilde{A}_1)$ of $\tilde{A}_1$ is defined as follows:

$$S\left(\tilde{A}_1\right) = \frac{1}{12}\begin{bmatrix} 8 + (a_1 + 2b_1 + c_1) - (e_1 + 2f_1 + g_1) \\ -(r_1 + 2s_1 + t_1) \end{bmatrix}. \quad (21)$$

The value of score function of TFNNV $A^+ = \langle (1,1,1),(0,0,0),(0,0,0) \rangle$ is $S(\mathrm{A}^+) = 1$ and value of accuracy function of TFNNV $A^- = \langle (0,0,0),(1,1,1),(1,1,1) \rangle$ is $S(\mathrm{A}^-) = -1$.

Definition 12. Assume that $\tilde{A}_1 = \langle (\mathrm{a}_1,\mathrm{b}_1,\mathrm{c}_1),(\mathrm{e}_1,\mathrm{f}_1,\mathrm{g}_1),(\mathrm{r}_1,\mathrm{s}_1,\mathrm{t}_1) \rangle$ be a TFNNV in the set of real numbers, the accuracy function $H\left(\tilde{A}_1\right)$ of $\tilde{A}_1$ is defined as follows:

$$H\left(\tilde{A}_1\right) = \frac{1}{4}\left[(\mathrm{a}_1 + 2\,\mathrm{b}_1 + \mathrm{c}_1) - (\mathrm{r}_1 + 2\,\mathrm{s}_1 + \mathrm{t}_1)\right]. \quad (22)$$

The accuracy function $H(\tilde{\mathrm{A}}_1) \in [-1,1]$ determines the difference between truth and falsity. Larger the difference reflects the more affirmative of the TFNNV. The accuracy function $H(\tilde{\mathrm{A}}^+) = 1$ for $A^+ = \langle (1,1,1),(0,0,0),(0,0,0) \rangle$ and $H(\tilde{\mathrm{A}}^-) = -1$ for the TFNNV $A^- = \langle (0,0,0),(1,1,1),(1,1,1) \rangle$.

Based on Definition 11 and Definition 12, we present the order relations between two TFNNVs.

Definition 13. Assume that $\tilde{A}_1 = \langle (\mathrm{a}_1,\mathrm{b}_1,\mathrm{c}_1),(\mathrm{e}_1,\mathrm{f}_1,\mathrm{g}_1),(\mathrm{r}_1,\mathrm{s}_1,\mathrm{t}_1) \rangle$ and $\tilde{A}_2 = \langle (\mathrm{a}_2,\mathrm{b}_2,\mathrm{c}_2),(\mathrm{e}_2,\mathrm{f}_2,\mathrm{g}_2),(\mathrm{r}_2,\mathrm{s}_2,\mathrm{t}_2) \rangle$ be two TFNNVs in the set of real numbers. Suppose that $S(\tilde{\mathrm{A}}_i)$ and $H(\tilde{\mathrm{A}}_i)$ are the score and accuracy functions of TFNNS $\tilde{A}_i (i = 1,2)$, then the following order relations are defined as follows:

1. If $S(\tilde{A}_1) > S(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$ that is $\tilde{A}_1 \succ \tilde{A}_2$;
2. If $S(\tilde{A}_1) = S(\tilde{A}_2)$ and $H(\tilde{A}_1) \geq H(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1 \succ \tilde{A}_2$;
3. If $S(\tilde{A}_1) = S(\tilde{A}_2)$, $H(\tilde{A}_1) = H(\tilde{A}_2)$ then $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, i.e. $\tilde{A}_1 \approx \tilde{A}_2$.

Example 1. Consider two TFNNVs in the set of real numbers:

$\tilde{A}_1 = \langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$,

$\tilde{A}_2 = \langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$.

Then from Eqs.(21) and (22), we obtain the following results:

1. Score value of $S(\tilde{A}_1) = (8 + 3 - 0.8 - 0.6)/12 = 0.80$, and $S(\tilde{A}_2) = (8 + 1.8 - 1.8 - 1.6)/12 \approx 0.53$;
2. Accuracy value of $H(\tilde{A}_1) = (3 - 0.6)/4 = 0.60$, and $H(\tilde{A}_2) = (1.8 - 1.6)/4 = 0.05$.

   Therefore from Definition 13, we obtain $A_1 \succ A_2$.

Example 2. Consider two TFNNVs in the set of real numbers:

$\tilde{A}_1 = \langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$

$\tilde{A}_2 = \langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$.

Using Eqs. (21) and (22), we obtain the following results:

1. Score value of $S(\tilde{A}_1) = (8 + 2.2 - 1.2 - 1.0)/12 \approx 0.67$, and $S(\tilde{A}_2) = (8 + 1.8 - 1.8 - 1.6)/12 \approx 0.53$;
2. Accuracy value of $H(\tilde{A}_1) = (2.2 - 1.2)/4 = 0.25$, and $H(\tilde{A}_2) = (1.8 - 1.6)/4 = 0.05$.

Therefore from Definition 13, we have $\tilde{A}_1 \succ \tilde{A}_2$.

4 Aggregation of triangular fuzzy number neutrosophic sets

In this section, we first recall some basic definitions of aggregation operators for real numbers.

Definition 14. [72] Assume that $W : (\mathrm{Re})^n \to \mathrm{Re}$, and $a_j\,(j = 1, 2, ..., n)$ be a collection of real numbers. The weighted averaging operator WA_w is defined as

$$WA_w(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j a_j \tag{23}$$

where *Re* is the set of real numbers, $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of $a_j\,(j = 1, 2, ..., n)$ such that $w_j \in [0,1]$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 15. [73] Assume that $W : (\mathrm{Re})^n \to \mathrm{Re}$, and $a_j\,(j = 1, 2, ..., n)$ be a collection of real numbers. The weighted geometric operator WG_w is defined as follows:

$$WG_w(a_1, a_2, ..., a_n) = \prod_{j=1}^{n} a_j^{w_j}, \tag{24}$$

where *Re* is the set of real numbers, $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of $a_j\,(j = 1, 2, ..., n)$ with $w_j \in [0,1]$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$

Based on Definition 14 and Definition 15, we propose the following two aggregation operators of TFNNSs to be used in decision making.

4.1 Triangular fuzzy number neutrosophic arithmetic averaging operator

Definition 16. Assume that $\tilde{A}_j = \langle (a_j, b_j, c_j), (e_j, f_j, g_j), (r_j, s_j, t_j) \rangle$ $(j = 1, 2, ..., n)$ be a collection TFNNVs in the set of real numbers and let $TFNNWA : \Theta^n \to \Theta$. The triangular fuzzy number neutrosophic weighted averaging (TFNNWA) operator denoted by $TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ is defined as

$$TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = w_1 \tilde{A}_1 \oplus w_2 \tilde{A}_2 \oplus \cdots w_n \tilde{A}_n = \bigoplus_{j=1}^{n} (w_j \tilde{A}_j), \tag{25}$$

where $w_j \in [0,1]$ is the weight vector of $A_j\,(j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$.

In particular, if $w = (1/n, 1/n, ..., 1/n)^T$ then the $TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ operator reduces to triangular fuzzy number neutrosophic averaging (TFNNA) operator:

$$TFNNA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \frac{1}{n}\left(\tilde{A}_1 \oplus \tilde{A}_2 \oplus \cdots \oplus \tilde{A}_n\right) \tag{26}$$

We can now establish the following theorem by using the basic operations of TFNNVs defined in Definition 10.

Theorem 1.

Let $\tilde{A}_j = \langle (\mathrm{a}_j, \mathrm{b}_j, \mathrm{c}_j), (\mathrm{e}_j, \mathrm{f}_j, \mathrm{g}_j), (\mathrm{r}_j, \mathrm{s}_j, \mathrm{t}_j) \rangle$ $(\mathrm{j} = 1, 2, ..., \mathrm{n})$ be a collection TFNNVs in the set of real numbers. Then the aggregated value obtained by TFNNWA, is also a TFNNV, and

$$TFNNWA_w(\tilde{\mathrm{A}}_1, \tilde{\mathrm{A}}_2, ..., \tilde{\mathrm{A}}_n)$$

$$= w_1\tilde{A}_1 \oplus w_2\tilde{A}_2 \oplus \cdots \oplus w_n\tilde{A}_n = \bigoplus_{j=1}^{n} (\mathrm{w}_j \tilde{\mathrm{A}}_j)$$

$$= \left\langle \begin{array}{l} \left(1 - \prod_{j=1}^{n} (1 - \mathrm{a}_j)^{w_j}, 1 - \prod_{j=1}^{n} (1 - b_j)^{w_j}, 1 - \prod_{j=1}^{n} (1 - c_j)^{w_j}\right), \\ \left(\prod_{j=1}^{n} e_j^{w_j}, \prod_{j=1}^{n} f_j^{w_j}, \prod_{j=1}^{n} g_j^{w_j}\right), \left(\prod_{j=1}^{n} r_j^{w_j}, \prod_{j=1}^{n} s_j^{w_j}, \prod_{j=1}^{n} t_j^{w_j}\right) \end{array} \right\rangle, \tag{27}$$

where $w_j \in [0,1]$ is the weight vector of TFNNV A_j $(j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$.

Proof: We prove the theorem by mathematical induction.

1. When $n = 1$, it is a trivial case

When $n = 2$, we have $\bigoplus_{j=1}^{2} (\mathrm{w}_j \tilde{\mathrm{A}}_j) = w_1\tilde{A}_1 \oplus w_2\tilde{A}_2$

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2. $$= \left(\begin{array}{l} \left\langle \left(1 - (1 - \mathrm{a}_1)^{w_1}, 1 - (1 - b_1)^{w_1}, 1 - (1 - c_1)^{w_1}\right), \left(e_1^{w_1}, f_1^{w_1}, g_1^{w_1}\right), \left(r_1^{w_1}, s_1^{w_1}, t_1^{w_1}\right) \right\rangle \\ \oplus \left\langle \left(1 - (1 - \mathrm{a}_2)^{w_2}, 1 - (1 - b_2)^{w_2}, 1 - (1 - c_2)^{w_2}\right), \left(e_2^{w_2}, f_2^{w_2}, g_2^{w_2}\right), \left(r_2^{w_2}, s_2^{w_2}, t_2^{w_2}\right) \right\rangle \end{array} \right)$$

$$= \left\langle \begin{array}{l} \left(\begin{array}{l} \left(1 - (1 - \mathrm{a}_1)^{w_1}\right) + \left(1 - (1 - \mathrm{a}_2)^{w_2}\right) - \left(1 - (1 - \mathrm{a}_1)^{w_1}\right).\left(1 - (1 - \mathrm{a}_2)^{w_2}\right), \\ \left(1 - (1 - b_1)^{w_1}\right) + \left(1 - (1 - b_2)^{w_2}\right) - \left(1 - (1 - b_1)^{w_1}\right).\left(1 - (1 - b_2)^{w_2}\right), \\ \left(1 - (1 - c_1)^{w_1}\right) + \left(1 - (1 - c_2)^{w_2}\right) - \left(1 - (1 - c_1)^{w_1}\right).\left(1 - (1 - c_2)^{w_2}\right) \end{array} \right), \\ \left(e_1^{w_1} e_2^{w_2}, f_1^{w_1} f_2^{w_2}, g_1^{w_1} g_2^{w_2}\right), \left(r_1^{w_1} r_2^{w_2}, s_1^{w_1} s_2^{w_2}, t_1^{w_1} t_2^{w_2}\right) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left(1 - \prod_{j=1}^{2} (1 - \mathrm{a}_j)^{w_j}, 1 - \prod_{j=1}^{2} (1 - b_j)^{w_j}, 1 - \prod_{j=1}^{2} (1 - b_j)^{w_j}\right), \\ \left(\prod_{j=1}^{2} e_j^{w_j}, \prod_{j=1}^{2} f_j^{w_j}, \prod_{j=1}^{2} g_j^{w_j}\right), \left(\prod_{j=1}^{2} r_j^{w_j}, \prod_{j=1}^{2} s_j^{w_j}, \prod_{j=1}^{2} t_j^{w_j}\right) \end{array} \right\rangle. \tag{28}$$

   Thus the theorem is true for n = 2

3. When n = k, we assume that Eq.(27) is also true.

   Then, $TFNNWA(\tilde{\mathrm{A}}_1, \tilde{\mathrm{A}}_2, ..., \tilde{\mathrm{A}}_k) = w_1\tilde{A}_1 \oplus w_1\tilde{A}_1 \oplus \cdots \oplus w_n\tilde{A}_n = \bigoplus_{j=1}^{k} (\mathrm{w}_j \tilde{\mathrm{A}}_j)$

$$= \left\langle \begin{array}{l} \left(1 - \prod_{j=1}^{k} (1 - \mathrm{a}_j)^{w_j}, 1 - \prod_{j=1}^{k} (1 - b_j)^{w_j}, 1 - \prod_{j=1}^{k} (1 - b_j)^{w_j}\right), \\ \left(\prod_{j=1}^{k} e_j^{w_j}, \prod_{j=1}^{k} f_j^{w_j}, \prod_{j=1}^{n} g_j^{w_j}\right), \left(\prod_{j=1}^{k} r_j^{w_j}, \prod_{j=1}^{k} s_j^{w_j}, \prod_{j=1}^{k} t_j^{w_j}\right) \end{array} \right\rangle. \tag{29}$$

4. When n = k + 1, we have

$$TNFNWA(\tilde{\mathrm{A}}_1, \tilde{\mathrm{A}}_2, ..., \tilde{\mathrm{A}}_{k+1}) = \bigoplus_{j=1}^{k} (\mathrm{w}_j \tilde{\mathrm{A}}_j) \oplus (\mathrm{w}_{k+1} \tilde{\mathrm{A}}_{k+1})$$

$$=\left\langle \begin{array}{c} \begin{pmatrix} 1-\prod_{j=1}^{k}(1-a_j)^{w_j}+1-(1-a_{k+1})^{w_{k+1}}-1-\prod_{j=1}^{k}(1-a_j)^{w_j}1-(1-a_{k+1})^{w_{k+1}}, \\ 1-\prod_{j=1}^{k}(1-b_j)^{w_j}+1-(1-b_{k+1})^{w_{k+1}}-1-\prod_{j=1}^{k}(1-b_j)^{w_j}1-(1-b_{k+1})^{w_{k+1}}, \\ 1-\prod_{j=1}^{k}(1-c_j)^{w_j}+1-(1-c_{k+1})^{w_{k+1}}-1-\prod_{j=1}^{k}(1-c_j)^{w_j}1-(1-c_{k+1})^{w_{k+1}} \end{pmatrix}, \\ \left(\prod_{j=1}^{k}e_j^{w_j}.e_{k+1}^{w_{k+1}},\prod_{j=1}^{k}f_j^{w_j}.f_{k+1}^{w_{k+1}},\prod_{j=1}^{n}g_j^{w_j}.e_{k+1}^{w_{k+1}}\right),\left(\prod_{j=1}^{k}r_j^{w_j}.r_{k+1}^{w_{k+1}},\prod_{j=1}^{k}s_j^{w_j}.s_{k+1}^{w_{k+1}},\prod_{j=1}^{k}t_j^{w_j}.t_{k+1}^{w_{k+1}}\right) \end{array} \right\rangle$$

$$=\left\langle \begin{array}{c} \left(1-\prod_{j=1}^{k+1}(1-a_j)^{w_j},1-\prod_{j=1}^{k+1}(1-b_j)^{w_j},1-\prod_{j=1}^{k+1}(1-b_j)^{w_j}\right), \\ \left(\prod_{j=1}^{k+1}e_j^{w_j},\prod_{j=1}^{k+1}f_j^{w_j},\prod_{j=1}^{k+1}g_j^{w_j}\right),\left(\prod_{j=1}^{k+1}r_j^{w_j},\prod_{j=1}^{k+1}s_j^{w_j},\prod_{j=1}^{k+1}t_j^{w_j}\right) \end{array} \right\rangle. \tag{30}$$

We observe that the theorem is true for n = k + 1. Therefore, by mathematical induction, we can say that Eq. (27) holds for all values of *n*. As the components of all three membership functions of $\tilde{A}_j$ belong to [0, 1], the following relations are valid

$$0\le\left(1-\prod_{j=1}^{n}(1-c_j)^{w_j}\right)\le 1,\ \ 0\le\left(\prod_{j=1}^{n}g_j^{w_j}\right)\le 1,$$

$$0\le\left(\prod_{j=1}^{n}t_j^{w_j}\right)\le 1. \tag{31}$$

It follows that the relation

$0\le\left(1-\prod_{j=1}^{n}(1-c_j)^{w_j}+\prod_{j=1}^{n}g_j^{w_j}+\prod_{j=1}^{n}t_j^{w_j}\right)\le 3$ is also valid.

This completes the proof of the Theorem 1.

Now, we highlight some necessary properties of TFNNWA operator.

Property 1.(*Idempotency*): If all $\tilde{A}_j\,(\mathrm{j}=1,2,...,\mathrm{n})$ are equal i.e. $\tilde{A}_j=\tilde{A}=\langle(\mathrm{a,b,c}),(\mathrm{e,f,g}),(\mathrm{r,s,t})\rangle$, for all j, then $TFNNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_k)=\tilde{A}$.

Proof: From Eq.(27), we have

$$TFNNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)$$

$$=TFNNWA(\tilde{A},\tilde{A},...,\tilde{A})=\bigoplus_{j=1}^{n}(\mathrm{w}_j\tilde{A})$$

$$=\left\langle \begin{array}{c} \left(1-\prod_{j=1}^{n}(1-a)^{w_j},1-\prod_{j=1}^{n}(1-b)^{w_j},1-\prod_{j=1}^{n}(1-c)^{w_j}\right), \\ \left(\prod_{j=1}^{n}e^{w_j},\prod_{j=1}^{n}f^{w_j},\prod_{j=1}^{n}g^{w_j}\right),\left(\prod_{j=1}^{n}r^{w_j},\prod_{j=1}^{n}s^{w_j},\prod_{j=1}^{n}t^{w_j}\right) \end{array} \right\rangle$$

$$=\left\langle \begin{array}{c} \left(1-(1-a)^{\sum_{j=1}^{n}w_j},1-(1-b)^{\sum_{j=1}^{n}w_j},1-(1-c)^{\sum_{j=1}^{n}w_j}\right), \\ \left(e^{\sum_{j=1}^{n}w_j},f^{\sum_{j=1}^{n}w_j},g^{\sum_{j=1}^{n}w_j}\right), \\ \left(e^{\sum_{j=1}^{n}w_j},f^{\sum_{j=1}^{n}w_j},g^{\sum_{j=1}^{n}w_j}\right) \end{array} \right\rangle$$

$$=\langle(\mathrm{a,b,c}),(\mathrm{e,f,g}),(\mathrm{r,s,t})\rangle=\tilde{A}.$$

This completes the proof the Property 1.

Property 2. (*Boundedness*)

Let $\tilde{A}_j=\langle(\mathrm{a}_j,\mathrm{b}_j,\mathrm{c}_j),(\mathrm{e}_j,\mathrm{f}_j,\mathrm{g}_j),(\mathrm{r}_j,\mathrm{s}_j,\mathrm{t}_j)\rangle$ $(\mathrm{j}=1,2,...,\mathrm{n})$ be a collection TFNNVs in the set of real numbers.

Assume $\tilde{A}^{+} = \left\langle \begin{array}{c} \left(\max_j(a_j), \max_j(b_j), \max_j(c_j)\right), \\ \left(\min_j(e_j), \min_j(f_j), \min_j(g_j)\right), \\ \left(\min_j(r_j), \min_j(s_j), \min_j(t_j)\right) \end{array} \right\rangle$ and

$$\tilde{A}^{-} = \left\langle \begin{array}{c} \left(\min_j(a_j), \min_j(b_j), \min_j(c_j)\right), \\ \left(\max_j(e_j), \max_j(f_j), \max_j(g_j)\right), \\ \left(\max_j(r_j), \max_j(s_j), \max_j(t_j)\right) \end{array} \right\rangle \text{ for all}$$

$j = 1, 2, ..., n.$

Then $\tilde{A}^{-} \le TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) \le \tilde{A}^{+}$. (32)

Proof: We have

$$\min_j(c_j) \le c_j \le \max_j(c_j),\ \min_j(g_j) \le g_j \le \max_j(g_j),$$
$$\min_j(t_j) \le t_j \le \max_j(t_j) \text{ for } j = 1,2,...,n. \quad (33)$$

Then

$$1-\prod_{j=1}^{n}(1-\min_j(c_j))^{w_j} \le 1-\prod_{j=1}^{n}(1-c_j)^{w_j}$$
$$\le 1-\prod_{j=1}^{n}(1-\max_j(c_j))^{w_j}$$
$$= 1-\left((1-\min_j(c_j))\right)^{\sum_{j=1}^{n} w_j} \le 1-\prod_{j=1}^{n}(1-c_j)^{w_j}$$
$$\le 1-\left((1-\max_j(c_j))\right)^{\sum_{j=1}^{n} w_j}$$
$$= \min_j(c_j) \le 1-\prod_{j=1}^{n}(1-c_j)^{w_j} \le \max_j(c_j).$$

Again from Eq.(33), we have for j = 1, 2, …, n

$$\prod_{j=1}^{n}\left(\min_j(g_j)\right)^{w_j} \le \prod_{j=1}^{n} g^{w_j} \le \prod_{j=1}^{n}\left(\max_j(g_j)\right)^{w_j}$$
$$= \left(\min_j(g_j)\right)^{\sum_{j=1}^{n} w_j} \le \prod_{j=1}^{n} g^{w_j} \le \left(\max_j(g_j)\right)^{\sum_{j=1}^{n} w_j}$$
$$= \min_j(g_j) \le \prod_{j=1}^{n} e^{w_j} \le \max_j(g_j);$$

and $\prod_{j=1}^{n}\left(\min_j(t_j)\right)^{w_j} \le \prod_{j=1}^{n} t^{w_j} \le \prod_{j=1}^{n}\left(\max_j(t_j)\right)^{w_j} =$

$$\left(\min_j(t_j)\right)^{\sum_{j=1}^{n} w_j} \le \prod_{j=1}^{n} t^{w_j} \le \left(\max_j(t_j)\right)^{\sum_{j=1}^{n} w_j}$$
$$= \min_j(t_j) \le \prod_{j=1}^{n} t^{w_j} \le \max_j(t_j).$$

Similarly, we have

$$\min_j(a_j) \le 1-\prod_{j=1}^{n}(1-a_j)^{w_j} \le \max_j(a_j),$$
$$\min_j(b_j) \le 1-\prod_{j=1}^{n}(1-b_j)^{w_j} \le \max_j(b_j);$$
$$\min_j(e_j) \le 1-\prod_{j=1}^{n}(1-e_j)^{w_j} \le \max_j(e_j),$$
$$\min_j(f_j) \le 1-\prod_{j=1}^{n}(1-f_j)^{w_j} \le \max_j(f_j);$$
$$\min_j(r_j) \le 1-\prod_{j=1}^{n}(1-r_j)^{w_j} \le \max_j(r_j),$$
$$\min_j(s_j) \le 1-\prod_{j=1}^{n}(1-s_j)^{w_j} \le \max_j(s_j)$$

for $j = 1,2,...,n.$

Assume that

$TFNNWA_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \tilde{A} = \left\langle (a,b,c),(e,f,g),(r,s,t) \right\rangle$,

then the score function of $\tilde{A}$

$$S(\tilde{A}) = \frac{1}{12}\left[8+(a+2b+c)-(e+2f+g)-(r+2s+t)\right]$$
$$\le \frac{1}{12}\left[\begin{array}{l} 8+\left(\max_j(a_j)+\max_j(2b_j)+\max_j(c_j)\right) \\ -\left(\min_j(e_j)+2\min_j(f_j)+\min_j(g_j)\right) \\ -\left(\min_j(r_j)+2\min_j(s_j)+\min_j(t_j)\right) \end{array}\right], \quad (34)$$
$$= S\left(\tilde{A}^{+}\right).$$

Similarly, the score function of $\tilde{A}$

$$S(\tilde{A}) = \frac{1}{12}\left[8+(a+2b+c)-(e+2f+g)-(r+2s+t)\right];$$
$$\ge \frac{1}{12}\left[\begin{array}{l} 8+\left(\min_j(a_j)+\min_j(2b_j)+\min_j(c_j)\right) \\ -\left(\max_j(e_j)+\max_j(2f_j)+\max_j(g_j)\right) \\ -\left(\max_j(r_j)+\max_j(2s_j)+\max_j(t_j)\right) \end{array}\right]$$
$$= S\left(\tilde{A}^{-}\right).$$

Now, we consider the following cases:

1. If $S(\tilde{A}) < S(\tilde{A}^{+})$ and $S(\tilde{A}) > S(\tilde{A}^{-})$ then we have
$$\tilde{A}^{-} < TFNNWA(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) < \tilde{A}^{+}. \quad (35)$$

2. If $S(\tilde{A}) = S(\tilde{A}^{+})$, then we can take

$$\frac{1}{12}\left[8+(a+2b+c)-(e+2f+g)-(r+2s+t)\right]$$

$$=\frac{1}{12}\left[\begin{array}{l} 8+\left(\max_j(a_j)+2\max_j(b_j)+\max_j(c_j)\right) \\ -\left(\min_j(e_j)+2\min_j(f_j)+\min_j(g_j)\right) \\ -\left(\min_j(r_j)+2\min_j(s_j)+\min_j(t_j)\right) \end{array}\right].$$

3. It follows that

$$(a+2b+c)=\left(\max_j(a_j)+2\max_j(b_j)+\max_j(c_j)\right),$$

$$(e+2f+g)=\left(\min_j(e_j)+2\min_j(f_j)+\min_j(g_j)\right) \text{ and}$$

$$(r+2s+t)=\left(\min_j(r_j)+2\min_j(s_j)+\min_j(t_j)\right).$$

Therefore the accuracy function of $\tilde{A}$

$$H(\tilde{A}) = \frac{1}{4}\left[(a+2b+c)-(r+2s+t)\right]$$

$$=\frac{1}{4}\left[\begin{array}{l} \left(\max_j(a_j)+\max_j(2b_j)+\max_j(c_j)\right) \\ -\left(\min_j(r_j)+\min_j(2s_j)+\min_j(t_j)\right) \end{array}\right].$$

$$= H(\tilde{A}^{+}), \quad (36)$$

From (36),we have $TFNNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)=\tilde{A}^{+}$ (37)

Similarly, for $S(\tilde{A}) = S(\tilde{A}^{-})$, the accuracy function of $\tilde{A}$

$$H(\tilde{A}) = \frac{1}{4}\left[(a+2b+c)-(r+2s+t)\right]$$

$$=\frac{1}{4}\left[\begin{array}{l} \left(\min_j(a_j)+\min_j(2b_j)+\min_j(c_j)\right) \\ -\left(\max_j(r_j)+\max_j(2s_j)+\max_j(t_j)\right) \end{array}\right]$$

$$= H(\tilde{A}^{-}) \quad (38)$$

From (38), we have $TNFNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)=\tilde{A}^{-}$. (39)

Combining Eqs. (35), (37) and (39), we obtain the following result

$$\tilde{A}^{-} \le TFNNWA(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n) \le \tilde{A}^{+} \quad (40)$$

This proves the Property 2. □

Property 3. (*Monotonicity*) Suppose that $\tilde{A}_j^1 = \left\langle (a_j^1,b_j^1,c_j^1),(e_j^1,f_j^1,g_j^1),(r_j^1,s_j^1,t_j^1)\right\rangle$ and $\tilde{A}_j^2 = \left\langle (a_j^2,b_j^2,c_j^2),(e_j^2,f_j^2,g_j^2),(r_j^2,s_j^2,t_j^2)\right\rangle$ $(j=1,2,...,n)$ be a collection of two TFNNVs in the set of real numbers. If $\tilde{A}_j^1 \preccurlyeq \tilde{A}_j^2$ for $j=1,2,...,n$ then

$$TFNNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1) \preccurlyeq TFNNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2). \quad (41)$$

Proof: We first consider c_j^1, g_j^1, t_j^1 of $\tilde{A}_j^1$ and c_j^2, g_j^2, t_j^2 of $\tilde{A}_j^2$ to prove the property 3.

We can consider $c_j^1 \le c_j^2$, $g_j^1 \ge g_j^2$ and $t_j^1 \ge t_j^2$ for $\tilde{A}_j^1 \preccurlyeq \tilde{A}_j^2$ $(j=1,2,...,n)$.

Then we have

$$\left(1-c_j^1\right)^{w_j} \ge \left(1-c_j^2\right)^{w_j}, \left(g_j^1\right)^{w_j} \ge \left(g_j^2\right)^{w_j}, \left(t_j^1\right)^{w_j} \ge \left(t_j^2\right)^{w_j};$$

$$1-\prod_{j=1}^{n}\left(1-c_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-c_j^2\right)^{w_j}, \left(g_j^1\right)^{w_j} \ge \left(g_j^2\right)^{w_j} \text{ and } \left(t_j^1\right)^{w_j} \ge \left(t_j^2\right)^{w_j}.$$

Therefore,

$$1-\prod_{j=1}^{n}\left(1-c_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-c_j^2\right)^{w_j}; \prod_{j=1}^{n}\left(g_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(g_j^2\right)^{w_j},$$

$$\text{and} \prod_{j=1}^{n}\left(t_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(t_j^2\right)^{w_j}. \quad (42)$$

Similarly, we can show

$$1-\prod_{j=1}^{n}\left(1-a_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-a_j^2\right)^{w_j}; \prod_{j=1}^{n}\left(e_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(e_j^2\right)^{w_j},$$

$$\text{and} \prod_{j=1}^{n}\left(r_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(r_j^2\right)^{w_j};$$

$$1-\prod_{j=1}^{n}\left(1-b_j^1\right)^{w_j} \le 1-\prod_{j=1}^{n}\left(1-b_j^2\right)^{w_j}; \prod_{j=1}^{n}\left(f_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(f_j^2\right)^{w_j},$$

$$\text{and} \prod_{j=1}^{n}\left(s_j^1\right)^{w_j} \ge \prod_{j=1}^{n}\left(s_j^2\right)^{w_j}.$$

Assume that

$$\tilde{A}^1 = TFNNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)$$

$$=\left\langle (a^1,b^1,c^1),(e^1,f^1,g^1),(r^1,s^1,t^1)\right\rangle \text{ and}$$

$$\tilde{A}^2 = TFNNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2)$$

$$=\left\langle (a^2,b^2,c^2),(e^2,f^2,g^2),(r^2,s^2,t^2)\right\rangle, \text{ where}$$

$$a^s = 1-\prod_{j=1}^{n}\left(1-a_j^s\right)^{w_j}, b^s = 1-\prod_{j=1}^{n}\left(1-b_j^s\right)^{w_j},$$

$$c^s = 1-\prod_{j=1}^{n}\left(1-c_j^s\right)^{w_j};$$

$$e^s = \prod_{j=1}^{n}\left(e_j^s\right)^{w_j}, f^s = \prod_{j=1}^{n}\left(f_j^s\right)^{w_j}, g^s = \prod_{j=1}^{n}\left(g_j^s\right)^{w_j} \text{ and}$$

$$r^s = \prod_{j=1}^{n}\left(r_j^s\right)^{w_j}, s^s = \prod_{j=1}^{n}\left(s_j^s\right)^{w_j}, t^s = \prod_{j=1}^{n}\left(t_j^s\right)^{w_j} \text{ for } s=1, 2.$$

Now we consider the score function of $\tilde{A}_1$:

$$S\left(\tilde{A}^1\right)=\frac{1}{12}\begin{bmatrix}8+(a^1+2b^1+c^1)-(e^1+2f^1+g^1)\\ -(r^1+2s^1+t^1)\end{bmatrix}$$

$$\le\frac{1}{12}\left[8+(a^2+2b^2+c^2)-(e^2+2f^2+g^2)+(r^2+2s^2+t^2)\right]=S\left(\tilde{A}^2\right)$$

Now we consider the following two cases:

Case 1 . If $S\left(\tilde{A}^1\right)<S\left(\tilde{A}^2\right)$, from Definition-13, we have

$TNFNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)\prec TNFNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2)$. (43)

Case 2 . If $S\left(\tilde{A}^1\right)=S\left(\tilde{A}^2\right)$, then by Eq.(21) we can consider

$$\frac{1}{12}\begin{bmatrix}8+(a^1+2b^1+c^1)-(e^1+2f^1+g^1)\\ -(r^1+2s^1+t^1)\end{bmatrix}$$

$$=\frac{1}{12}\begin{bmatrix}8+(a^2+2b^2+c^2)-(e^2+2f^2+g^2)\\ -(r^2+2s^2+t^2)\end{bmatrix}.$$

Thus for $\tilde{A}_j^1\preccurlyeq\tilde{A}_j^2$ $(j=1,2,...,n)$ i.e., for $a_j^1\le a_j^2$, $b_j^1\le b_j^2$ $c_j^1\le c_j^2$; $e_j^1\ge e_j^2$ $f_j^1\ge f_j^2$, $g_j^1\ge g_j^2$ and $r_j^1\le r_j^2$, $s_j^1\ge s_j^2$, $t_j^1\ge t_j^2$ we have

$a^1=a^2$, $b^1=b^2$, $c^1=c^2$, $e^1=e^2$, $f^1=f^2$, $g^1=g^2$, $r^1=r^2$, $s^1=s^2$ and $t^1=t^2$.

Then, the accuracy function of $\tilde{A}^1$ yields

$$H(\tilde{A}^1)=\frac{1}{4}\left[(a^1+2b^1+c^1)-(r^1+2s^1+t^1)\right]$$

$$=\frac{1}{4}\left[(a^2+2b^2+c^2)-(r^2+2s^2+t^2)\right]$$

$$=H(\tilde{A}^2). \tag{44}$$

Thus from Definition-13, we have

$$TNFNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)=TNFNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2). \tag{45}$$

Finally, from Eqs. (43) and (45), we have the following result

$TFNNWA(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)\preccurlyeq TFNNWA(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2)$.

This completes the proof of Property 3. □

Example3. We consider the following four TFNNVs:

$\tilde{A}_1=\langle(0.80,0.85,0.90),\ (0.10,0.15,0.20),$ $(0.05,0.10,0.15)\rangle$; $\tilde{A}_2=\langle(0.70,0.75,0.80),$ $(0.15,0.20,0.25),\ (0.10,0.15,0.20)\rangle$;

$\tilde{A}_3=\langle(0.40,0.45,0.50),\ (0.40,0.45,0.50),$ $(0.35,0.40,0.45)\rangle$ and

$\tilde{A}_4=\langle(0.70,0.75,0.80),\ (0.15,0.20,0.25),$ $(0.10,0.15,0.20)\rangle$.

Using TFNNWA operator defined in Eq.(27), we can aggregate $\tilde{A}_1,\tilde{A}_2,\tilde{A}_3$, and $\tilde{A}_4$ with weight vector $w=(0.30,0.25,0.25,0.20)$ as:

$$\tilde{A}=TFNNWA(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,\tilde{A}_4)$$
$$=w_1\tilde{A}_1\oplus w_2\tilde{A}_2\oplus w_3\tilde{A}_3\oplus w_4\tilde{A}_4$$

$$=\left\langle\begin{pmatrix}\left(1-(1-0.80)^{0.30}(1-0.70)^{0.25}(1-0.40)^{0.25}(1-0.70)^{0.20}\right),\\ \left(1-(1-0.85)^{0.30}(1-0.75)^{0.25}(1-0.45)^{0.25}(1-0.75)^{0.20}\right),\\ \left(1-(1-0.90)^{0.30}(1-0.80)^{0.25}(1-0.50)^{0.25}(1-0.80)^{0.20}\right)\end{pmatrix},\begin{pmatrix}\left((0.10)^{0.30}(0.15)^{0.25}(0.40)^{0.25}(0.15)^{0.20}\right),\\ \left((0.15)^{0.30}(0.20)^{0.25}(0.45)^{0.25}(0.20)^{0.20}\right),\\ \left((0.20)^{0.30}(0.25)^{0.25}(0.50)^{0.25}(0.25)^{0.20}\right)\end{pmatrix},\begin{pmatrix}\left((0.05)^{0.30}(0.10)^{0.25}(0.35)^{0.25}(0.10)^{0.20}\right),\\ \left((0.10)^{0.30}(0.15)^{0.25}(0.40)^{0.25}(0.15)^{0.20}\right),\\ \left((0.15)^{0.30}(0.20)^{0.25}(0.45)^{0.25}(0.20)^{0.20}\right)\end{pmatrix}\right\rangle \tag{46}$$

$$=\left\langle\begin{matrix}\left((1-0.617\times0.740\times0.880\times0.786),(1-0.566\times0.707\times0.861\times0.756),(1-0.501\times0.669\times0.841\times0.725)\right),\\ \left((0.501\times0.622\times0.795\times0.684),(0.566\times0.667\times0.819\times0.725),(0.617\times0.707\times0.841\times0.887)\right),\\ \left((0.407\times0.562\times0.769\times0.631),(0.501\times0.622\times0.795\times0.684),(0.566\times0.669\times0.819\times0.725)\right)\end{matrix}\right\rangle$$

$=\left\langle (0.6842, 0.7395, 0.7956), (0.1804, 0.2605, 0.3254), (0.1110, 0.1694, 0.2249) \right\rangle$.

4.2 Triangular fuzzy number neutrosophic geometric averaging operator

Definition 17. Suppose that

that $\tilde{A}_j = \left\langle (a_j, b_j, c_j), (e_j, f_j, g_j), (r_j, s_j, t_j) \right\rangle$ $(j = 1, 2, ..., n)$ be a collection TFNNVs in the set of real numbers and $TFNNWG: \Theta^n \to \Theta$. The triangular fuzzy number neutrosophic weighted geometric (TFNNWG) operator denoted by $TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ is defined as follows:

$$TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2} \otimes \cdots \otimes \tilde{A}_n^{w_n}$$

$$= \bigotimes_{j=1}^{n} (\tilde{A}_j^{w_j}) \tag{47}$$

where $w_j \in [0,1]$ is the exponential weight vector of $\tilde{A}_j$ $(j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$. In particular, if $w = (1/n, 1/n, ..., 1/n)^T$ then the $TFNNWG(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ operator reduces to triangular fuzzy neutrosophic geometric(TNFG) operator:

$$TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = \left(\tilde{A}_1 \otimes \tilde{A}_2 \otimes \cdots \otimes \tilde{A}_n \right)^{\frac{1}{n}}. \tag{48}$$

We now establish the following theorem with the basic operations of TFNNV defined in Definition 10.

Theorem 2. Assume that

$\tilde{A}_j = \left\langle (a_j, b_j, c_j), (e_j, f_j, g_j), (r_j, s_j, t_j) \right\rangle$ $(j = 1, 2, ..., n)$ be a collection TFNNVs in the set of real numbers. Then the aggregated value obtained from TFNNWG, is also a TFNNV, and then we have

$$TFNNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$$

$$= \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2} \otimes \cdots \otimes \tilde{A}_n^{w_n}$$

$$= \bigotimes_{j=1}^{n} (\tilde{A}_j^{w_j})$$

$$= \left\langle \begin{array}{l} \left(\prod_{j=1}^{n} a_j^{w_j}, \prod_{j=1}^{n} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j} \right), \\ \left(1 - \prod_{j=1}^{n} (1-e_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-f_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-g_j)^{w_j} \right), \\ \left(1 - \prod_{j=1}^{n} (1-r_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-s_j)^{w_j}, 1 - \prod_{j=1}^{n} (1-t_j)^{w_j} \right) \end{array} \right\rangle \tag{49}$$

where $w_j \in [0,1]$ is the weight vector of TFNNV $\tilde{A}_j$ $(j = 1, 2, ..., n)$ such that $\sum_{j=1}^{n} w_j = 1$.

Similar to arithmetic averaging operator, we can also prove the theorem by mathematical induction.

1. When n = 1, the theorem is true.
2. When n = 2, we have

$$\bigotimes_{j=1}^{2} \left(\tilde{A}_j \right)^{w_j} = \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2}$$

$$= \left(\begin{array}{l} \left\langle \left(a_1^{w_1}, b_1^{w_1}, c_1^{w_1} \right), \left(1-(1-e_1)^{w_1}, 1-(1-f_1)^{w_1}, 1-(1-g_1)^{w_1} \right), \left(1-(1-r_1)^{w_1}, 1-(1-s_1)^{w_1}, 1-(1-t_1)^{w_1} \right) \right\rangle \\ \otimes \left\langle \left(a_2^{w_1}, b_2^{w_1}, c_2^{w_1} \right), \left(1-(1-e_2)^{w_1}, 1-(1-f_2)^{w_1}, 1-(1-g_2)^{w_1} \right), \left(1-(1-r_2)^{w_1}, 1-(1-s_2)^{w_1}, 1-(1-t_2)^{w_1} \right) \right\rangle \end{array} \right)$$

$$
=\left\langle \begin{array}{c}
\left(a_1^{w_1}a_2^{w_2}, b_1^{w_1}b_2^{w_2}, c_1^{w_1}c_2^{w_2}\right), \begin{pmatrix}
\left(1-(1-e_1)^{w_1}\right)+\left(1-(1-e_2)^{w_2}\right)-\left(1-(1-e_1)^{w_1}\right).\left(1-(1-e_2)^{w_2}\right), \\
\left(1-(1-f_1)^{w_1}\right)+\left(1-(1-f_2)^{w_2}\right)-\left(1-(1-f_1)^{w_1}\right).\left(1-(1-f_2)^{w_2}\right), \\
\left(1-(1-c_1)^{w_1}\right)+\left(1-(1-c_2)^{w_2}\right)-\left(1-(1-c_1)^{w_1}\right).\left(1-(1-c_2)^{w_2}\right)
\end{pmatrix} \\
\begin{pmatrix}
\left(1-(1-r_1)^{w_1}\right)+\left(1-(1-r_2)^{w_2}\right)-\left(1-(1-r_1)^{w_1}\right).\left(1-(1-r_2)^{w_2}\right), \\
\left(1-(1-s_1)^{w_1}\right)+\left(1-(1-s_2)^{w_2}\right)-\left(1-(1-s_1)^{w_1}\right).\left(1-(1-s_2)^{w_2}\right), \\
\left(1-(1-t_1)^{w_1}\right)+\left(1-(1-t_2)^{w_2}\right)-\left(1-(1-t_1)^{w_1}\right).\left(1-(1-t_2)^{w_2}\right)
\end{pmatrix}
\end{array}\right\rangle
$$

$$
=\left\langle \begin{array}{c}
\left(\prod_{j=1}^{2} a_j^{w_j}, \prod_{j=1}^{2} b_j^{w_j}, \prod_{j=1}^{2} c_j^{w_j}\right), \left(1-\prod_{j=1}^{2}(1-e_j)^{w_j}, 1-\prod_{j=1}^{2}(1-f_j)^{w_j}, 1-\prod_{j=1}^{2}(1-g_j)^{w_j}\right), \\
\left(1-\prod_{j=1}^{2}(1-e_j)^{w_j}, 1-\prod_{j=1}^{2}(1-f_j)^{w_j}, 1-\prod_{j=1}^{2}(1-g_j)^{w_j}\right)
\end{array}\right\rangle \tag{50}
$$

3. When n = k, we assume that Eq.(49) is true then,

$$
TNFNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_k) = \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2} \otimes \cdots \otimes \tilde{A}_k^{w_k}
$$

$$
=\left\langle \begin{array}{c}
\left(\prod_{j=1}^{k} a_j^{w_j}, \prod_{j=1}^{k} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j}\right), \left(1-\prod_{j=1}^{k}(1-e_j)^{w_j}, 1-\prod_{j=1}^{k}(1-f_j)^{w_j}, 1-\prod_{j=1}^{k}(1-g_j)^{w_j}\right), \\
\left(1-\prod_{j=1}^{k}(1-r_j)^{w_j}, 1-\prod_{j=1}^{k}(1-s_j)^{w_j}, 1-\prod_{j=1}^{k}(1-t_j)^{w_j}\right)
\end{array}\right\rangle \tag{51}
$$

4. When n = k+1, we can consider the following expression:

$$
TNFNWG_w(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_{k+1}) = \bigotimes_{j=1}^{k}\left(\tilde{A}_j\right)^{w_j} \otimes \left(\tilde{A}_{k+1}\right)^{w_{k+1}}
$$

$$
=\left\langle\left(\prod_{j=1}^{k} a_j^{w_j}.a_{k+1}^{w_{k+1}}, \prod_{j=1}^{k} b_j^{w_j}.b_{k+1}^{w_{k+1}}, \prod_{j=1}^{n} c_j^{w_j}.c_{k+1}^{w_{k+1}}\right),\right.
$$

$$
\begin{pmatrix}
\left(1-\prod_{j=1}^{k}(1-e_j)^{w_j}\right)+\left(1-(1-e_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-e_j)^{w_j}\right).\left(1-(1-e_{k+1})^{w_{k+1}}\right), \\
\left(1-\prod_{j=1}^{k}(1-f_j)^{w_j}\right)+\left(1-(1-f_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-f_j)^{w_j}\right).\left(1-(1-f_{k+1})^{w_{k+1}}\right), \\
\left(1-\prod_{j=1}^{k}(1-g_j)^{w_j}\right)+\left(1-(1-g_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-g_j)^{w_j}\right).\left(1-(1-g_{k+1})^{w_{k+1}}\right)
\end{pmatrix},
$$

$$\left\langle\begin{pmatrix}\left(1-\prod_{j=1}^{k}(1-r_j)^{w_j}\right)+\left(1-(1-r_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-r_j)^{w_j}\right).\left(1-(1-r_{k+1})^{w_{k+1}}\right),\\ \left(1-\prod_{j=1}^{k}(1-s_j)^{w_j}\right)+\left(1-(1-s_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-s_j)^{w_j}\right).\left(1-(1-s_{k+1})^{w_{k+1}}\right),\\ \left(1-\prod_{j=1}^{k}(1-t_j)^{w_j}\right)+\left(1-(1-t_{k+1})^{w_{k+1}}\right)-\left(1-\prod_{j=1}^{k}(1-t_j)^{w_j}\right).\left(1-(1-t_{k+1})^{w_{k+1}}\right)\end{pmatrix}\right\rangle \tag{52}$$

$$=\left\langle\begin{matrix}\left(\prod_{j=1}^{k+1}a_j^{w_j},\prod_{j=1}^{k+1}b_j^{w_j},\prod_{j=1}^{k+1}c_j^{w_j}\right),\left(1-\prod_{j=1}^{k+1}(1-e_j)^{w_j},1-\prod_{j=1}^{k+1}(1-f_j)^{w_j},1-\prod_{j=1}^{k+1}(1-g_j)^{w_j}\right),\\ \left(1-\prod_{j=1}^{k+1}(1-r_j)^{w_j},1-\prod_{j=1}^{k+1}(1-s_j)^{w_j},1-\prod_{j=1}^{k+1}(1-t_j)^{w_j}\right)\end{matrix}\right\rangle \tag{53}$$

We observe that the theorem is also true for n = k+1.

Therefore, by mathematical induction, Eq. (49) holds for all values of *n*.

Since the components of all three membership functions of $\tilde{A}_j(j=1,2,...,n)$ belong to [0, 1] the following relations are valid

$0\le\left(\prod_{j=1}^{n}c_j^{w_j}\right)\le 1$, $0\le\left(1-\prod_{j=1}^{n}(1-g_j)^{w_j}\right)\le 1$, ,

and $0\le\left(1-\prod_{j=1}^{n}(1-t_j)^{w_j}\right)\le 1$. (54)

It follows that

$$0\le\left(\prod_{j=1}^{n}c_j^{w_j}+1-\prod_{j=1}^{n}(1-g_j)^{w_j}+1-\prod_{j=1}^{n}(1-t_j)^{w_j}\right)\le 3.$$

This completes the proof of Theorem 2.

Now, we discuss some essential properties of TFNNWG operator for TFNNs.

Property 4.(*Idempotency*): If all $\tilde{A}_j\,(j=1,2,...,n)$ are equal that is $\tilde{A}_j=\langle(a,b,c),(e,f,g),(r,s,t)\rangle$, for all *j* , then $TFNNWG_w(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_k)=\tilde{A}$.

Proof: From Eq.(49), we have

$TFNNWG_w(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)$

$$=TFNNWG_g(\tilde{A},\tilde{A},...,\tilde{A})=\bigotimes_{j=1}^{n}\left(\tilde{A}_j\right)^{w_j}$$

$$=\left\langle\begin{matrix}\left(\prod_{j=1}^{n}a^{w_j},\prod_{j=1}^{n}b^{w_j},\prod_{j=1}^{n}c^{w_j}\right),\\ \left(1-\prod_{j=1}^{n}(1-e)^{w_j},1-\prod_{j=1}^{n}(1-f)^{w_j},1-\prod_{j=1}^{n}(1-g)^{w_j}\right),\\ \left(1-\prod_{j=1}^{n}(1-r)^{w_j},1-\prod_{j=1}^{n}(1-s)^{w_j},1-\prod_{j=1}^{n}(1-t)^{w_j}\right)\end{matrix}\right\rangle$$

$$=\left\langle\begin{matrix}\left(a^{\sum_{j=1}^{n}w_j},b^{\sum_{j=1}^{n}w_j},c^{\sum_{j=1}^{n}w_j}\right),\\ \left(1-(1-e)^{\sum_{j=1}^{n}w_j},1-(1-f)^{\sum_{j=1}^{n}w_j},1-(1-g)^{\sum_{j=1}^{n}w_j}\right),\\ \left(1-(1-e)^{\sum_{j=1}^{n}w_j},1-(1-f)^{\sum_{j=1}^{n}w_j},1-(1-g)^{\sum_{j=1}^{n}w_j}\right)\end{matrix}\right\rangle$$

$=\langle(a,b,c),(e,f,g),(r,s,t)\rangle=\tilde{A}.$

This completes the Property 4. □

Property 5. (*Boundedness*).

Let $\tilde{A}_j=\langle(a_j,b_j,c_j),(e_j,f_j,g_j),(r_j,s_j,t_j)\rangle$ $(j=1,2,...,n)$ be a collection TFNNs in the set of real numbers. Assume

$$\tilde{A}^{+}=\left\langle\begin{matrix}\left(\max_j a_j,\max_j b_j,\max_j c_j\right),\left(\min_j e_j,\min_j f_j,\min_j g_j\right),\\ \left(\min_j r_j,\min_j s_j,\min_j t_j\right)\end{matrix}\right\rangle$$

and

$$\tilde{A}^{-}=\left\langle \left(\min_j a_j,\min_j b_j,\min_j c_j\right),\left(\max_j e_j,\max_j f_j,\max_j g_j\right),\left(\max_j r_j,\max_j s_j,\max_j t_j\right)\right\rangle$$

for all $j=1,2,...,n.$. Then

$$\tilde{A}^{-}\le TNFNWG_w(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)\le\tilde{A}^{+}. \quad (55)$$

Proof: The proof of the Property 5 is similar to Property 2.

Property 6. (*Monotonicity*).
Let $\tilde{A}_j^1=\left\langle(a_j^1,b_j^1,c_j^1),(e_j^1,f_j^1,g_j^1),(r_j^1,s_j^1,t_j^1)\right\rangle$ and $\tilde{A}_j^2=\left\langle(a_j^2,b_j^2,c_j^2),(e_j^2,f_j^2,g_j^2),(r_j^2,s_j^2,t_j^2)\right\rangle$ $(j=1,2,...,n)$ be a collection of two TFNNVs in the set of real numbers. If $\tilde{A}_j^1\preccurlyeq\tilde{A}_j^2$ for $j=1,2,...,n$ then

$$TFNNWG_w(\tilde{A}_1^1,\tilde{A}_2^1,...,\tilde{A}_n^1)\preccurlyeq TFNNWG_w(\tilde{A}_1^2,\tilde{A}_2^2,...,\tilde{A}_n^2). \quad (56)$$

Proof: Property 6 can be proved by a similar argument of Property 3. Therefore, we do not discuss again to avoid repetition. □

Example 4. Assume that $\tilde{A}_1=\langle(0.80,0.85,0.90),(0.10,0.15,0.20),(0.05,0.10,0.15)\rangle$; $\tilde{A}_2=\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$; $\tilde{A}_3=\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$ and $\tilde{A}_4=\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$ are four TFNNVs. Then using TFNNWG operator defined in Eq.(49), we can aggregate $\tilde{A}_1,\tilde{A}_2,\tilde{A}_3$, and $\tilde{A}_4$ with the considered weight vector $w=(0.30,0.25,0.25,0.20)$ as:

$$\tilde{A}=TFNNWG_w(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,\tilde{A}_4)$$
$$=w_1\tilde{A}_1\otimes w_2\tilde{A}_2\otimes w_3\tilde{A}_3\otimes w_4\tilde{A}_4$$

$$=\left\langle\begin{pmatrix}\left((0.80)^{0.30}(0.70)^{0.25}(0.40)^{0.25}(0.70)^{0.20}\right),\\ \left((0.85)^{0.30}(0.75)^{0.25}(0.45)^{0.25}(0.75)^{0.20}\right),\\ \left((0.90)^{0.30}(0.80)^{0.25}(0.50)^{0.25}(0.80)^{0.20}\right)\end{pmatrix},\begin{pmatrix}\left(1-(1-0.10)^{0.30}(1-0.15)^{0.25}(1-0.40)^{0.25}(1-0.15)^{0.20}\right),\\ \left(1-(1-0.15)^{0.30}(1-0.20)^{0.25}(1-0.45)^{0.25}(1-0.20)^{0.20}\right),\\ \left(1-(1-0.20)^{0.30}(1-0.25)^{0.25}(1-0.50)^{0.25}(1-0.25)^{0.20}\right)\end{pmatrix},\begin{pmatrix}\left(1-(1-0.05)^{0.30}(1-0.10)^{0.25}(1-0.35)^{0.25}(1-0.10)^{0.20}\right),\\ \left(1-(1-0.10)^{0.30}(1-0.15)^{0.25}(1-0.40)^{0.25}(1-0.15)^{0.20}\right),\\ \left(1-(1-0.15)^{0.30}(1-0.20)^{0.25}(1-0.45)^{0.25}(1-0.20)^{0.20}\right)\end{pmatrix}\right\rangle$$

$$=\left\langle\begin{matrix}\left((0.935\times0.915\times0.795\times0.931),(0.952\times0.930\times0.819\times0.944),(0.969\times0.946\times0.841\times0.956)\right),\\ \left((1-0.969\times0.960\times0.880\times0.968),(1-0.952\times0.946\times0.861\times0.956),(1-0.935\times0.930\times0.841\times0.944)\right),\\ \left((1-0.985\times0.974\times0.898\times0.979),(1-0.969\times0.960\times0.880\times0.968),(1-0.952\times0.946\times0.861\times0.956)\right)\end{matrix}\right\rangle$$

$$=\left\langle(0.6332,0.6845,0.7370),(0.2076,0.2587,0.3097),(0.1565,0.2075,0.2587)\right\rangle.$$

---

5 Application of TFNNWA and TFNNWG operators to multi attribute decision making

Consider a multi attribute decision making problem in which $Y=\{Y_1,Y_2,...,Y_m\}$ be the set of n feasible alternatives and $C=\{C_1,C_2,...,C_n\}$ be the set of attributes. Assume that $w=(w_1,w_2,...,w_n)^T$ be the weight vector of the attributes, where w_j denotes the importance degree of the attribute C_j such that $w_j\ge0$ and $\sum_{j=1}^n w_j=1$ for $j=1,2,...,n$.

The ratings of all alternatives $Y_i\,(i=1,2,...,m)$ with respect to the attributes $C_j\,(j=1,2,...,n)$ have been presented in a TFNNV based decision matrix $U=(u_{ij})_{m\times n}$ (see the Table 1). Furthermore, in the decision matrix $U=(u_{ij})_{m\times n}$, the rating $u_{ij}=\left\langle(a_{ij},b_{ij},c_{ij}),(e_{ij},f_{ij},g_{ij}),(r_{ij},s_{ij},t_{ij})\right\rangle$ represents a

TFNNV, where the fuzzy number (a_{ij}, b_{ij}, c_{ij}) represents the degree that the alternative $Y_i\,(i = 1, 2, ..., m)$ satisfies the attribute $C_j\,(j = 1, 2, ..., n)$, the fuzzy number (e_{ij}, f_{ij}, g_{ij}) represents the degree that the alternative Y_i is uncertain about the attribute C_j and fuzzy number (r_{ij}, s_{ij}, t_{ij}) indicates the degree that the alternative Y_i does not satisfy the attribute C_j such that

$0 \le c_{ij} + g_{ij} + t_{ij} \le 3$, for *i = 1, 2,..., m* and *j = 1, 2, ..., n.*

Based on the TFNNWA and TFNNWG operators, we develop a practical approach for solving MADM problems, in which the ratings of the alternatives over the attributes are expressed with TFNNVs. The schematic diagram of the proposed approach for MADM is depicted in the Figure-1.

Table 1. Triangular fuzzy number neutrosophic value based decision matrix

	C_1	C_2	...	C_n
Y_1	$\left\langle (a_{11},b_{11},c_{11}), (e_{11},f_{11},g_{11}), (r_{11},s_{11},t_{11}) \right\rangle$	$\left\langle (a_{12},b_{12},c_{12}), (e_{12},f_{12},g_{12}), (r_{12},s_{12},t_{12}) \right\rangle$	...	$\left\langle (a_{1n},b_{1n},c_{1n}), (e_{1n},f_{1n},g_{1n}), (r_{1n},s_{1n},t_{1n}) \right\rangle$
Y_2	$\left\langle (a_{21},b_{21},c_{21}), (e_{21},f_{21},g_{21}), (r_{21},s_{21},t_{21}) \right\rangle$	$\left\langle (a_{22},b_{22},c_{22}), (e_{22},f_{22},g_{22}), (r_{22},s_{22},t_{22}) \right\rangle$	...	$\left\langle (a_{2n},b_{2n},c_{2n}), (e_{2n},f_{2n},g_{2n}), (r_{2n},s_{2n},t_{2n}) \right\rangle$
....	....	....	....	...
Y_m	$\left\langle (a_{m1},b_{m1},c_{m1}), (e_{m1},f_{m1},g_{m1}), (r_{m1},s_{m1},t_{m1}) \right\rangle$	$\left\langle (a_{m2},b_{m2},c_{m2}), (e_{m2},f_{m2},g_{m2}), (r_{m2},s_{m2},t_{m2}) \right\rangle$	...	$\left\langle (a_{mn},b_{mn},c_{mn}), (e_{mn},f_{mn},g_{mn}), (r_{mn},s_{mn},t_{mn}) \right\rangle$

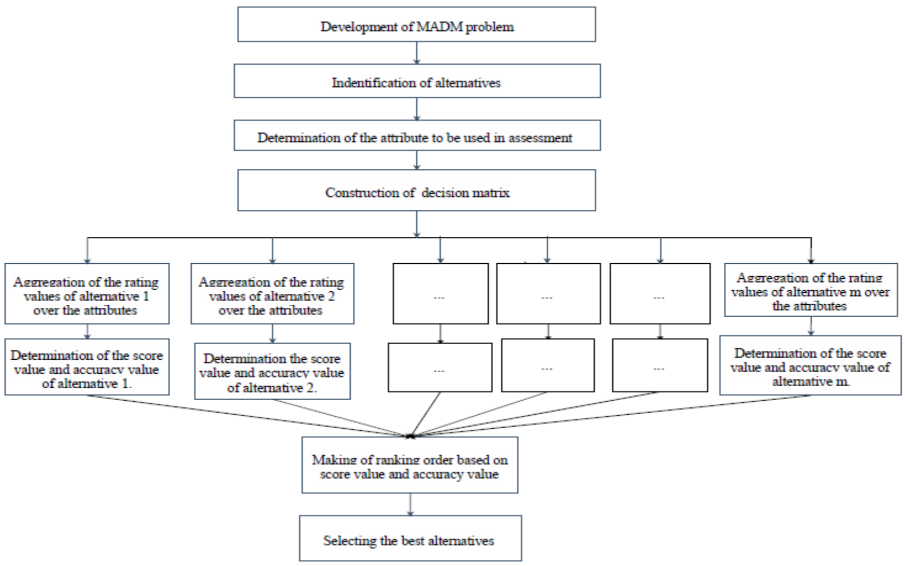

Figure-1. Framework for the proposed MADM method

Therefore, we design the proposed approach in the following steps:

Step 1: First aggregate all rating values p_{ij} $(j = 1, 2, ..., n)$ of the i -th row of the decision matrix $(p_{ij})_{m\times n}$ defined in Table 1.

Step 2: Determine the aggregation value u_i corresponding to the alternative Y_i obtained from TFNNWA operator:

$$u_i = \langle (a_i, b_i, c), (e_i, f_i, g_i), (r_i, s_i, t_i) \rangle = TFNNWA_w(p_{i1}, p_{i2}, ..., p_{in}) \quad (57)$$

or by the TFNNWG operator as

$$u_i = \langle (a_i, b_i, c), (e_i, f_i, g_i), (r_i, s_i, t_i) \rangle = TFNNWG_w(p_{i1}, p_{i2}, ..., p_{in}) \quad (58)$$

Step 3: For each alternative $A_i (i = 1, 2, ..., m)$, calculate the score values $S(u_i)$ and accuracy values $A(u_i)$ of the aggregated rating values obtained by TFNNWA or TFNNWG operators that are in Eqs. (21) and (22).

Step 4: Using Definition 11 to Definition 13, determine the ranking order of aggregated values obtained in Step 3.

Step 5: Select the best alternative in accordance with the ranking order obtained in Step 4.

6 An illustrative example of multi attribute decision making

In this section, we consider an illustrative example of medical representative selection problem to demonstrate and applicability of the proposed approach to multi attribute decision making problem.

Assume that a pharmacy company wants to recruit a medical representative. After initial scrutiny four candidates $Y_i (i = 1, 2, 3, 4)$ have been considered for further evaluation with respect to the five attributes $C_j (j = 1, 2, 3, 4, 5)$ namely,

1. oral communication skill (C_1);
2. past experience (C_2);
3. general aptitude (C_3);
4. willingness (C_4) and
5. self confidence (C_5).

The ratings of the alternatives $Y_i (i = 1, 2, 3, 4)$ with respect to the attributes $C_j (j = 1, 2, 3, 4, 5)$ are expressed with TFNNVs shown in the decision matrix $P = (p_{ij})_{4\times 5}$ (see Table 2.). Assume w = (0.10, 0.25, 0.25, 0.15, 0.25)^T be the relative weight vector of all attributes $C_j (j = 1, 2, 3, 4, 5)$.

Table 2. Triangular fuzzy number neutrosophic value based rating values

	C_1	C_2	C_3	C_4	C_5
Y_1	$\langle (0.80,0.85,0.90), (0.10,0.15,0.20), (0.05,0.10,0.15) \rangle$	$\langle (0.50,0.55,0.60), (0.25,0.30,0.35), (0.20,0.25,0.30) \rangle$	$\langle (0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20) \rangle$	$\langle (0.80,0.85,0.90), (0.10,0.15,0.20), (0.05,0.10,0.15) \rangle$	$\langle (0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20) \rangle$
Y_2	$\langle (0.50,0.55,0.60), (0.25,0.30,0.35), (0.20,0.25,0.30) \rangle$	$\langle (0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20) \rangle$	$\langle (0.80,0.85,0.90), (0.10,0.15,0.20), (0.05,0.10,0.15) \rangle$	$\langle (0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20) \rangle$	$\langle (0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20) \rangle$
Y_3	$\langle (0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45) \rangle$	$\langle (0.50,0.55,0.60), (0.25,0.30,0.35), (0.20,0.25,0.30) \rangle$	$\langle (0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45) \rangle$	$\langle (0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45) \rangle$	$\langle (0.50,0.55,0.60), (0.25,0.30,0.35), (0.20,0.25,0.30) \rangle$
Y_4	$\langle (0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45) \rangle$	$\langle (0.50,0.55,0.60), (0.25,0.30,0.35), (0.20,0.25,0.30) \rangle$	$\langle (0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45) \rangle$	$\langle (0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20) \rangle$	$\langle (0.70,0.75,0.80), (0.15,0.20,0.25), (0.10,0.15,0.20) \rangle$

Here, we apply two proposed aggregation operators TFNNWA and TFNNWG to solve the medical representative selection problem by using the following steps.

6.1 Utilization of TFNNWA operator:

Step 1: Aggregate the rating values of the alternative Y_i (i= 1, 2, 3, 4) defined in the i -th row of decision matrix $P=(\mathrm{p}_{ij})_{4\times 5}$ (see Table 2.) with TFNNWA operator.

Step 2: The aggregated rating values u_i corresponding to the alternative Y_i are determined by Eq.(27) and the values are shown in Table 3.

Table 3. Aggregated TFNNV based ratings

	Aggregated ratings
u_1	$\langle(0.6920, 0.7451, 0.8000), (0.1540, 0.2026, 0.2572), (0.1000, 0.1540, 0.2060)\rangle$
u_2	$\langle(0.7147, 0.7667, 0.8197), (0.1426, 0.1938, 0.2445), (0.0901, 0.1426, 0.1938)\rangle$
u_3	$\langle(0.4523, 0.5025, 0.5528), (0.3162, 0.3674, 0.4183), (0.2646, 0.3162, 0.3674)\rangle$
u_4	$\langle(0.5655, 0.6184, 0.6722), (0.2402, 0.2940, 0.3466), (0.1844, 0.2402, 0.2940)\rangle$

Step 3: The score and accuracy values of alternatives Y_i (i= 1, 2, 3, 4) are determined by Eq.(21) and Eq.(22) in Table 4.

Table 4. Score and accuracy values of aggregated rating values

Alternative	Score values $S(u_i)$	Accuracy values $A(u_i)$
Y_1	0.7960	0.5921
Y_2	0.8103	0.6247
Y_3	0.6464	0.1864
Y_4	0.6951	0.3789

Step 4: The order of the alternatives Y_i (i= 1, 2, 3, 4) is determined according to the descending order of the score and accuracy values shown in Table 4. Thus the ranking order of the alternatives is presented as follows:

$$Y_2 \succ Y_1 \succ Y_4 \succ Y_3.$$

Step 5: The ranking order in Step 4 reflects that, Y_2 is the best medical representative.

6.2 Utilization of TFNNWG operator:

Step 1: Using Eq.(49), we aggregate all the rating values of the alternative Y_i (i= 1, 2, 3, 4) for the i- throw of the decision matrix $P=(\mathrm{p}_{ij})_{4\times 5}$ (see Table 2.).

Step 2: The aggregated rating values u_i corresponding to the alternative Y_i are shown in the Table 5.

Table 5. Aggregated TFNN based rating values

	Aggregated rating values
u_1	$\langle(0.6654, 0.7161, 0.7667), (0.1643, 0.2144, 0.2646), (0.1142, 0.1643, 0.2144)\rangle$
u_2	$\langle(0.6998, 0.7502, 0.8002), (0.1485, 0.1986, 0.2486), (0.0984, 0.1485, 0.1986)\rangle$
u_3	$\langle(0.4472, 0.4975, 0.5477), (0.3292, 0.3795, 0.4299), (0.2789, 0.3292, 0.3795)\rangle$
u_4	$\langle(0.5291, 0.5804, 0.6316), (0.2707, 0.3214, 0.3721), (0.2202, 0.2707, 0.3214)\rangle$

Step 3: The score and accuracy values of alternatives Y_i (i= 1, 2, 3, 4) are determined by Eqs.(21) and (22) and the results are shown in the Table 6.

Table 6. Score and accuracy values of rating values

Alternative	Score values $S(u_i)$	Accuracy values $A(u_i)$
Y_1	0.7791	0.5518
Y_2	0.8010	0.6016
Y_3	0.5962	0.1683
Y_4	0.6627	0.3096

Step 4: The order of alternatives Y_i (i = 1, 2, 3, 4) has been determined according to the descending order of score and accuracy values shown in Table 4.

Thus the ranking order of the alternative is presented as follows:

$Y_2 \succ Y_1 \succ Y_4 \succ Y_3$.

Step 5: The ranking order in Step 4 reflects that Y_2 is the best medical representative.

7 Conclusions

MADM problems generally takes place in a complex environment and usually connected with imprecise data and uncertainty. The triangular neutrosophic fuzzy numbers are an effective tool for dealing with impreciseness and incompleteness of the decision maker's assessments over alternative with respect to attributes. We have first introduced TFNNs and defined some of its operational rules. Then we have proposed two aggregation operators called TFNNWAA and TFNNWGA operators and score function and applied them to solve multi attribute decision making problem under neutrosophic environment. Finally, the effectiveness and applicability of the proposed approach have been illustrated with medical representative selection problem. We hope that the proposed approach can be also applied in other decision making problems such as pattern recognition, personnel selection, medical diagnosis, etc.

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Similarity Measure of Refined Single-Valued Neutrosophic Sets and Its Multicriteria Decision Making Method

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Abstract. This paper introduces a refined single-valued neutrosophic set (RSVNS) and presents a similarity measure of RSVNSs. Then a multicriteria decision-making method with RSVNS information is developed based on the similarity measure of RSVNSs. By the similarity measure between each alternative and the ideal solution (ideal alternative), all the alternatives can be ranked and the best one can be selected as well. Finally, an actual example on the selecting problems of construction projects demonstrates the application and effectiveness of the proposed method.

Keywords: Refined single-valued neutrosophic set, similarity measure, decision making.

1 Introduction

To deal with indeterminate and inconsistent information, Smarandache [1] proposed a neutrosophic set, which is composed of the neutrosophic components of truth, indeterminacy, and falsity denoted by *T*, *I*, *F*. Then, Wang et al. [2] constrained the neutrosophic set to a single-valued neutrosophic set (SVNS) as a subclass of the neutrosophic set for convenient actual applications. Further, Smarandache [3] extended the classical neutrosophic logic to n-valued refined neutrosophic logic, in which neutrosophic components *T*, *I*, *F* are refined (splitted) into $T_1, T_2, ..., T_p$ and $I_1, I_2, ..., I_r$, and $F_1, F_2, ..., F_t$, respectively, and constructed as a *n*-valued refined neutrosophic set. In existing literature [4-7], neutrosophic refined sets were studied and applied to medical diagnosis and decision making. However, the existing neutrosophic refined set is also a single-valued neutrosophic multiset [6] in the concept. In this paper, we present a refined single-valued neutrosophic set (RSVNS), then its concept is different from the concept of single-valued neutrosophic multisets (neutrosophic refined sets) [4-7]. In fact, RSVNSs are scarcely studied and applied in science and engineering fields. Therefore, it is necessary to propose a similarity measure between RSVNSs and its decision making method in this paper.

The rest of the paper is constructed as follows. Section 2 reviews basic concepts of a SVNS and a neutrosophic refined set (single-valued neutrosophic multiset). Section 3 introduces a RSVNS and a similarity measure of RSVNSs. Section 4 presents a multicriteria decision-making method based on the similarity method under a RSVNS environment. In section 5, an actual example is provided for the decision-making problem of selecting construction projects to illustrate the application of the proposed method. Section 6 contains conclusions and future research.

2 Preliminaries

Definition 1 [2]. Let *U* be a universe of discourse, then a SVNS *A* in *U* is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, such that $T_A(x), I_A(x), F_A(x) \in [0, 1]$ and $0 \le T_A(x) + T_A(x) + T_A(x) \le 3$. Thus, a SVNS *A* can be expressed as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in U\}$.

Let $U = \{x_1, x_2, \ldots, x_n\}$ be the universe of discourse, and *A* and *B* be two (non-refined) single-valued neutrosophic sets, $A = \{\langle x_i, T_A(x_i), I_A(x_i), F_A(x_i)\rangle \mid x_i \in U\}$ and $B = \{\langle x_i, T_B(x_i), I_B(x_i), F_B(x_i)\rangle \mid x_i \in U\}$. Majumdar and Samanta's similarity method of two (non-refined) single-valued neutrosophic sets *A* and *B* is:

$$M_{MS}(A,B) = \frac{\sum_{i=1}^{n} \left\{ \begin{array}{l} \min(T_{iA}(x_i), T_{iB}(x_i)) \\ + \min(I_{iA}(x_i), I_{iB}(x_i)) \\ + \min(F_{iA}(x_i), F_{iB}(x_i)) \end{array} \right\}}{\sum_{i=1}^{n} \left\{ \begin{array}{l} \max(T_{iA}(x_i), T_{iB}(x_i)) \\ + \max(I_{iA}(x_i), I_{iB}(x_i)) \\ + \max(F_{iA}(x_i), F_{iB}(x_i)) \end{array} \right\}}. \quad (1)$$

Based on *n*-valued refined neutrosophic sets [3], Ye and Ye [6] introduced a single-valued neutrosophic multisets (also called a single-valued neutrosophic refined set (SVNRS) [4, 5, 7]) and defined it below.

Definition 2. Let U be a universe of discourse, then a SVNRS R in U can be defined as follows:

$$R = \left\{ \left\langle \begin{array}{l} x, (T_{1R}(x), T_{2R}(x), ..., T_{pR}(x)), (I_{1R}(x), I_{2R}(x), \\ ..., I_{pR}(x)), (F_{1R}(x), F_{2R}(x), ..., F_{pR}(x)) \end{array} \right\rangle | \, x \in U \right\},$$

where p is a positive integer, $T_{1R}(x), T_{2R}(x), ..., T_{pR}(x)$ $: U \to [0,1]$, $I_{1R}(x), I_{2R}(x), ..., I_{pR}(x)$ $: U \to [0,1]$, and $F_{1R}(x), F_{2R}(x), ..., F_{pR}(x)$ $: U \to [0,1]$, and there are $0 \le T_{jR}(x) + I_{jR}(x) + F_{jR}(x) \le 3$ for $j = 1, 2, …, p$.

Definition 3. Let two SVNRS R and S in U be:

$$R = \left\{ \left\langle \begin{array}{l} x, (T_{1R}(x), T_{2R}(x), ..., T_{pR}(x)), (I_{1R}(x), I_{2R}(x), \\ ..., I_{pR}(x)), (F_{1R}(x), F_{2R}(x), ..., F_{pR}(x)) \end{array} \right\rangle | \, x \in U \right\},$$

$$S = \left\{ \left\langle \begin{array}{l} x, (T_{1S}(x), T_{2S}(x), ..., T_{pS}(x)), (I_{1S}(x), I_{2S}(x), \\ ..., I_{pS}(x)), (F_{1S}(x), F_{2S}(x), ..., F_{pS}(x)) \end{array} \right\rangle | \, x \in U \right\}.$$

Then there are the following relations of R and S:

(1) Containment:

$R \subseteq S$, if and only if $T_{jR}(x) \le T_{jS}(x)$, $I_{jR}(x) \ge I_{jS}(x)$, $F_{jR}(x) \ge F_{jS}(x)$ for $j = 1, 2, …, p$;

(2) Equality:

$R = S$, if and only if $T_{jR}(x) = T_{jS}(x)$, $I_{jR}(x) = I_{jS}(x)$, $F_{jR}(x) = F_{jS}(x)$ for $j = 1, 2, …, p$;

(3) Union:

$$R \cup S = \left\{ \left\langle \begin{array}{l} x, (T_{1R}(x) \vee T_{1S}(x), T_{2R}(x) \vee T_{2S}(x), ..., T_{pR}(x) \vee T_{pS}(x)), \\ (I_{1R}(x) \wedge I_{1S}(x), I_{2R}(x) \wedge I_{2S}(x), ..., I_{pR}(x) \wedge I_{pS}(x)), \\ (F_{1R}(x) \wedge F_{1S}(x), F_{2R}(x) \wedge F_{2S}(x), ..., F_{pR}(x) \wedge F_{pS}(x)) \end{array} \right\rangle | \, x \in U \right\};$$

(4) Intersection:

$$R \cap S = \left\{ \left\langle \begin{array}{l} x, (T_{1R}(x) \wedge T_{1S}(x), T_{2R}(x) \wedge T_{2S}(x), \\ ..., T_{pR}(x) \wedge T_{pS}(x)), (I_{1R}(x) \vee I_{1S}(x), \\ I_{2R}(x) \vee I_{2S}(x), ..., I_{pR}(x) \vee I_{pS}(x)), \\ (F_{1R}(x) \vee F_{1S}(x), F_{2R}(x) \vee F_{2S}(x), \\ ..., F_{pR}(x) \vee F_{pS}(x)) \end{array} \right\rangle | \, x \in U \right\}.$$

3 Similarity Methods of RSVNSs

In this section, we introduce a RSVNS and propose a similarity method between RSVNSs based on the extension of Majumdar and Samanta's similarity method of two (non-refined) single-valued neutrosophic sets [8].

Definition 4. Let R and S in the universe of discourse $U = \{x_1, x_2, …, x_n\}$ be two refined single-valued neutrosophic sets, which are defined as

$$R = \left\{ \left\langle \begin{array}{l} x_i, (T_{1R}(x_i), T_{2R}(x_i), ..., T_{p_iR}(x_i)), \\ (I_{1R}(x_i), I_{2R}(x_i), ..., I_{p_iR}(x_i)), \\ (F_{1R}(x_i), F_{2R}(x_i), ..., F_{p_iR}(x_i)) \end{array} \right\rangle | \, x_i \in U \right\},$$

$$S = \left\{ \left\langle \begin{array}{l} x_i, (T_{1S}(x_i), T_{2S}(x_i), ..., T_{p_iS}(x_i)), \\ (I_{1S}(x_i), I_{2S}(x_i), ..., I_{p_iS}(x_i)), \\ (F_{1S}(x_i), F_{2S}(x_i), ..., F_{p_iS}(x_i)) \end{array} \right\rangle | \, x_i \in U \right\},$$

where p_i is a positive integer, and all $T_{jR}(x_i)$, $I_{jR}(x_i)$, $F_{jR}(x_i)$ and $T_{jS}(x_i)$, $I_{jS}(x_i)$, $F_{jS}(x_i)$ ($i = 1, 2, …, n$; $j = 1, 2, …, p_i$) belong to [0, 1].

As an extension of Majumdar and Samanta's similarity method of SVNSs [8], we present a similarity method between two RSVNSs R and S as follows:

$$M(R,S) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\{ \begin{array}{l} \min(T_{jR}(x_i), T_{jS}(x_i)) \\ + \min(I_{jR}(x_i), I_{jS}(x_i)) \\ + \min(F_{jR}(x_i), F_{jS}(x_i)) \end{array} \right\}}{\sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\{ \begin{array}{l} \max(T_{jR}(x_i), T_{jS}(x_i)) \\ + \max(I_{jR}(x_i), I_{jS}(x_i)) \\ + \max(F_{jR}(x_i), F_{jS}(x_i)) \end{array} \right\}}. \quad (2)$$

Obviously, the above similarity measure $M(R, S)$ satisfies the following properties:

(1) $0 \le M(R, S) \le 1$;

(2) $M(R, S) = M(S, R)$

(3) $M(R, S) = 1$ if and only if $R = S$.

In general, we usually consider the weights of criteria. Assume that the weight of each criterion x_i is w_i ($i = 1, 2, …, n$), with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, we can introduce the weighted similarity formula:

$$M_w(R,S)=\frac{\sum_{i=1}^{n} w_i \sum_{j=1}^{p_i}\left\{\begin{array}{l}\min(T_{jR}(x_i),T_{jS}(x_i))\\ +\min(I_{jR}(x_i),I_{jS}(x_i))\\ +\min(F_{jR}(x_i),F_{jS}(x_i))\end{array}\right\}}{\sum_{i=1}^{n} w_i \sum_{j=1}^{p_i}\left\{\begin{array}{l}\max(T_{jR}(x_i),T_{jS}(x_i))\\ +\max(I_{jR}(x_i),I_{jS}(x_i))\\ +\max(F_{jR}(x_i),F_{jS}(x_i))\end{array}\right\}}. \quad (3)$$

4 Decision-making method using the similarity measure

In a decision making problem, there is a set of alternatives $R = \{R_1, R_2, \ldots, R_m\}$, which needs to satisfies a set of criteria $C = \{C_1, C_2, \ldots, C_n\}$, where C_i ($i = 1, 2, \ldots, n$) may be splitted into some sub-criteria C_{ij} ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, p_i$). If the decision maker provides the suitability evaluation values of the criteria for C_i (i =1,2,..., n) on the alternative R_k (k = 1, 2,…, m) by using a RSVNS:

$$R_k=\left\{\left\langle\begin{array}{l}C_i,(T_{1R_k}(C_i),T_{2R_k}(C_i),\ldots,T_{p_iR_k}(C_i)),\\ (I_{1R_k}(C_i),I_{2R_k}(C_i),\ldots,I_{p_iR_k}(C_i)),\\ (F_{1R_k}(C_i),F_{2R_k}(C_i),\ldots,F_{p_iR_k}(C_i))\end{array}\right\rangle \mid C_i\in C\right\}.$$

Then for convenience, each basic element in the RSVNS R_k is represented by the refined single-valued neutrosophic number (RSVNN):

$$\left\langle (T_{1R_k},T_{2R_k},\ldots,T_{p_iR_k}),(I_{1R_k},I_{2R_k},\ldots,I_{p_iR_k}),(F_{1R_k},F_{2R_k},\ldots,F_{p_iR_k})\right\rangle$$

for $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$. Hence, we can construct the refined single-valued neutrosophic decision matrix D, as shown in Table 1.

When the weights of criteria are considered as the different importance of each criterion C_i ($i = 1, 2, \ldots, n$), the weight vector of the three criteria is given by $W = (w_1, w_2, \ldots, w_n)$ with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. Thus, the decision-making steps are described as follows:

Step 1: Based on the refined single-valued neutrosophic decision matrix D, we can determine the ideal solution (ideal RSVNN) by

$$s_i^*=\left\langle\begin{array}{l}(\max_k(T_{1R_k}),\max_k(T_{2R_k}),\ldots,\max_k(T_{p_iR_k})),\\ (\min_k(I_{1R_k}),\min_k(I_{2R_k}),\ldots,\min_k(I_{p_iR_k})),\\ (\min_k(F_{1R_k}),\min_k(F_{2R_k}),\ldots,\min_k(F_{p_iR_k}))\end{array}\right\rangle, \quad (4)$$

which is constructed as the ideal alternative $S^* = \{s_1^*, s_2^*, \ldots, s_n^*\}$.

Step 2: The similarity measure between each alternative R_k ($k = 1, 2, \ldots, m$) and the ideal alternative S^* can be calculated according to Eq. (3) and the values of $M_w(R_k, S^*)$ for $k = 1, 2, \ldots, m$ can be obtained.

Step 3: The alternatives are ranked in a descending order according to the values of $M_w(R_k, S^*)$ for $k = 1, 2, \ldots, m$. The greater value of $M_w(R_k, S^*)$ means the better alternative R_k.

Step 4: End.

5 Actual example on the selection of construction projects

In this section, we give the application of the decision making method for the selection of construction projects.

A construction company needs to determine the selecting problem of construction projects. Decision makers provide four construction projects as a set of four alternatives $R = \{R_1, R_2, R_3, R_4\}$. Then, the selection of these construction projects is dependent on three main criteria and seven sub-criteria: (1) Financial state (C_1): budget control (C_{11}) and risk/return ratio (C_{12}); (2) Environmental protection (C_2): public relation (C_{21}), geographical location (C_{22}), and health and safety (C_{23}); (3) Technology (C_3): technical knowhow (C_{31}), technological capability (C_{32}).

Experts or decision makers are required to evaluate the four possible alternatives under the above three criteria (seven sub-criteria) by suitability judgments, which are represented by RSVNSs. Thus we can construct the following refined single-valued neutrosophic decision matrix D, as shown in Table 2.

Table 1. The refined single-valued neutrosophic decision matrix D

	$C_1(C_{11},C_{12},\ldots,C_{1p_1})$	…	$C_n(C_{n1},C_{n2},\ldots,C_{np_n})$
R_1	$\langle(T_{1R_1},T_{2R_1},\ldots,T_{p_1R_1}),(I_{1R_1},I_{2R_1},\ldots,I_{p_1R_1}),(F_{1R_1},F_{2R_1},\ldots,F_{p_1R_1})\rangle$	…	$\langle(T_{1R_1},T_{2R_1},\ldots,T_{p_nR_1}),(I_{1R_1},I_{2R_1},\ldots,I_{p_nR_1}),(F_{1R_1},F_{2R_1},\ldots,F_{p_nR_1})\rangle$
R_2	$\langle(T_{1R_2},T_{2R_2},\ldots,T_{p_1R_2}),(I_{1R_2},I_{2R_2},\ldots,I_{p_1R_2}),(F_{1R_2},F_{2R_2},\ldots,F_{p_1R_2})\rangle$	...	$\langle(T_{1R_2},T_{2R_2},\ldots,T_{p_nR_2}),(I_{1R_2},I_{2R_2},\ldots,I_{p_nR_2}),(F_{1R_2},F_{2R_2},\ldots,F_{p_nR_2})\rangle$
…	…	...	…
R_m	$\langle(T_{1R_m},T_{2R_m},\ldots,T_{p_1R_m}),(I_{1R_m},I_{2R_m},\ldots,I_{p_1R_m}),(F_{1R_m},F_{2R_m},\ldots,F_{p_1R_m})\rangle$	...	$\langle(T_{1R_m},T_{2R_m},\ldots,T_{p_nR_m}),(I_{1R_m},I_{2R_m},\ldots,I_{p_nR_m}),(F_{1R_m},F_{2R_m},\ldots,F_{p_nR_m})\rangle$

Table 2. Defined single-valued neutrosophic decision matrix D for the four alternatives on three criteria (seven sub-criteria)

	C_1 (C_{11}, C_{12})	C_2 (C_{21}, C_{22}, C_{23})	C_3 (C_{31}, C_{32})
R_1	<(0.6, 0.7), (0.2, 0.1), (0.2, 0.3)>	<(0.9, 0.7, 0.8), (0.1, 0.3, 0.2), (0.2, 0.2, 0.1)>	<(0.6, 0.8), (0.3, 0.2), (0.3, 0.4)>
R_2	<(0.8, 0.7), (0.1, 0.2), (0.3, 0.2)>	<(0.7, 0.8, 0.7), (0.2, 0.4, 0.3), (0.1, 0.2, 0.1)>	<(0.8, 0.8), (0.1, 0.2), (0.1, 0.2)>
R_3	<(0.6, 0.8), (0.1, 0.3), (0.3, 0.4)>	<(0.8, 0.6, 0.7), (0.3, 0.1, 0.1), (0.2, 0.1, 0.2)>	<(0.8, 0.7), (0.4, 0.3), (0.2, 0.1)>
R_4	<(0.7, 0.6), (0.1, 0.2), (0.2, 0.3)>	<(0.7, 0.8, 0.7), (0.2, 0.2, 0.1), (0.1, 0.2, 0.2)>	<(0.7, 0.7), (0.2, 0.3), (0.2, 0.3)>

Then, the weight vector of the three criteria is given by W = (0.4, 0.3, 0.3). Thus, the proposed decision making method is applied to the selecting problem of the construction projects. Consequently, the decision-making steps are described as follows:

Step 1: By Eq. (4), the ideal solution (ideal RSVNS) can be determined as the following ideal alternative:

S^* ={<(0.8, 0.8), (0.1, 0.1), (0.2, 0.2)>, <(0.9, 0.8, 0.8), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1)>, <(0.8, 0.8), (0.1, 0.2), (0.1, 0.1)>}.

Step 2: According to Eq. (3), the weighted similarity measure values between each alternative R_k (k = 1, 2, 3, 4) and the ideal alternative S^* can be obtained as follows:

$M_w(R_1, S^*) = 0.7743$, $M_w(R_2, S^*) = 0.8370$, $M_w(R_3, S^*) = 0.7595$, and $M_w(R_4, S^*) = 0.7778$.

Step 3: Since the measure values are $M_w(R_2, S^*) > M_w(R_4, S^*) > M_w(R_1, S^*) > M_w(R_3, S^*)$, the ranking order of the four alternatives is $R_2 > R_4 > R_1 > R_3$. Hence, the alternative R_2 is the best choice among all the construction projects.

6 Conclusion

This paper introduced RSVNSs and presented the similarity measure of RSVNSs. Then, we proposed a similarity measure-based multicriteria decision-making method under a RSVNS environment. In the decision-making process, through the similarity measure between each alternative and the ideal alternative, the ranking order of all alternatives can be determined and the best alternative can be selected as well. Finally, an actual example on the selecting problem of construction projects demonstrated the application of the proposed method. The main advantage of the proposed approach is easy evaluation and more suitable for actual applications in decision-making problems with RSVNS information. In the future, we shall extend the proposed decision-making method to medical diagnosis and fault diagnosis.

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Restricted Interval Valued Neutrosophic Sets and Restricted Interval Valued Neutrosophic Topological Spaces

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Abstract: In this paper we introduce the concept of restricted interval valued neutrosophic sets (RIVNS in short). Some basic operations and properties of RIVNS are discussed. The concept of restricted interval valued neutrosophic topology is also introduced together with restricted interval valued neutrosophic finer and restricted interval valued neutrosophic coarser topology. We also define restricted interval valued neutrosophic interior and closer of a restricted interval valued neutrosophic set. Some theorems and examples are cites. Restricted interval valued neutrosophic subspace topology is also studied.

Keywords: Neutrosophic Set, Interval Valued Nuetrosophic Set, Restricted Interval Valued Neutrosophic Set, Restricted Interval Valued Neutrosophic Topological Space.

AMS subject classification:03E72

1 Introduction

In 1999, Molodtsov [10] introduced the concept of soft set theory which is completely new approach for modeling uncertainty. In this paper [10] Molodtsov established the fundamental results of this new theory and successfully applied the soft set theory into several directions. Maji et al. [8] defined and studied several basic notions of soft set theory in 2003. Pie and Miao [14], Aktas and Cagman [1] and Ali et al. [2] improved the work of Maji et al. [9]. The intuitionistic fuzzy set is introduced by Atanaasov [4] as a generalization of fuzzy set [19] where he added degree of non-membership with degree of membership. Neutrosophic set introduced by F. Smarandache in 1995 [16]. Smarandache [17] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistant data. Maji [9] combined neutrosophic set and soft set and established some operations on these sets. Wang et al. [18] introduced interval neutrosophic sets. Deli [7] introduced the concept of interval-valued neutrosophic soft sets.

In this paper we introduce the concept of restricted interval valued neutrosophic sets (RIVNS in short). Some basic operations and properties of RIVNS are discussed. The concept of restricted interval valued neutrosophic topology is also introduced together with restricted interval valued neutrosophic finer and restricted interval valued neutrosophic coarser topology. We also define restricted interval valued neutrosophic interior and closer of a restricted interval valued neutrosophic set. Some theorems and examples are cited. Restricted interval valued neutrosophic subspace topology is also studied. We establish some properties of restricted interval valued neutrosophic soft topological space with supporting proofs and examples.

2 Preliminaries

Definition 2.1[17] A neutrosophic set A on the universe of discourse U is defined as

$A=\left\{\left\langle x,\mu_A(x),\gamma_A(x),\delta_A(x)\right\rangle : x\in U\right\}$, where

$\mu_A,\gamma_A,\delta_A : U \to]^-0,1^+[$ are functions such that the condition:

$\forall x \in U$, $^{-}0 \le \mu_A(x)+\gamma_A(x)+\delta_A(x) \le 3^{+}$ is satisfied.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0,1^{+}[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^{-}0,1^{+}[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0,1]$.

Definition 2.2 [6] An interval valued neutrosophicset A on the universe of discourse U is defined as $A=\{\langle x,\mu_A(x),\gamma_A(x),\delta_A(x)\rangle : x \in U\}$, where $\mu_A,\gamma_A,\delta_A : U \to Int]^{-}0,1^{+}[$ are functions such that the condition: $\forall x \in U$, $^{-}0 \le sup\mu_A(x)+sup\gamma_A(x)+sup\delta_A(x) \le 3^{+}$ is satisfied.

In real life applications it is difficult to use interval valued neutrosophic set with interval-value from real standard or non-standard subset of $Int(]^{-}0,1^{+}[)$. Hence we consider the interval-valued neutrosophic set which takes the interval-value from the subset of $Int([0,1])$ (where $Int([0,1])$ denotes the set of all closed sub intervals of $[0,1]$).

Definition 2.3 [15] Let *X* be a non-empty fixed set. A generalized neutrosophic set (*GNS* in short) *A* is an object having the form $A=\{\langle x,\mu_A(x),\sigma_A(x),\gamma_A(x)\rangle : x \in X\}$ Where $\mu_A(x),\sigma_A(x)$ and $\gamma_A(x)$ which represent the degree of member ship function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set *A*where the functions satisfy the condition $\mu_A(x)\wedge\sigma_A(x)\wedge\gamma_A(x) \le 0.5$.

We call this generalizedneutrosophic set**[15]** as **restricted neutrosophic set**.

Definition 2.4 [15] Let *A* and *B* be two *RNS*s on *X* defined by $A=\{\langle x,\mu_A(x),\sigma_A(x),\gamma_A(x)\rangle : x \in X\}$ and $B=\{\langle x,\mu_B(x),\sigma_B(x),\gamma_B(x)\rangle : x \in X\}$. Then union, intersection, subset and complement may be defined as

(i) The union of *A* and *B* is denoted by $A \cup B$ and is defined as

$$A \cup B=\{\langle x,\mu_A(x)\vee\mu_B(x),\sigma_A(x)\wedge\sigma_B(x),\gamma_A(x)\wedge\gamma_B(x)\rangle : x \in X\}$$

or

$$A \cup B=\{\langle x,\mu_A(x)\vee\mu_B(x),\sigma_A(x)\vee\sigma_B(x),\gamma_A(x)\wedge\gamma_B(x)\rangle : x \in X\}.$$

(ii) The intersection of *A* and *B* is denoted by $A \cap B$ and is defined as

$$A \cap B=\{\langle x,\mu_A(x)\wedge\mu_B(x),\sigma_A(x)\vee\sigma_B(x),\gamma_A(x)\vee\gamma_B(x)\rangle : x \in X\}$$

or

$$A \cap B=\{\langle x,\mu_A(x)\wedge\mu_B(x),\sigma_A(x)\wedge\sigma_B(x),\gamma_A(x)\vee\gamma_B(x)\rangle : x \in X\}.$$

or

$$A \cap B=\{\langle x,\mu_A(x).\mu_B(x),\sigma_A(x).\sigma_B(x),\gamma_A(x).\gamma_B(x)\rangle : x \in X\}$$

(iii) *A* is called subset of *B*, denoted by $A \subseteq B$ if and only if

$\mu_A(x) \le \mu_B(x)$, $\sigma_A(x) \ge \sigma_B(x)$, $\gamma_A(x) \ge \gamma_B(x)$

or

$\mu_A(x) \le \mu_B(x)$, $\sigma_A(x) \le \sigma_B(x)$,
$\gamma_A(x) \ge \gamma_B(x)$.

(iv) The complement of *A* is denoted by A^c and is defined as

$$A = \{\langle x, \gamma_A(x), 1-\sigma_A(x), \mu_A(x)\rangle : x \in X\}$$

or

$$A = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$$

or

$$A = \{\langle x, 1-\mu_A(x), \sigma_A(x), 1-\gamma_A(x)\rangle : x \in X\}$$

Definition 2.5: [15] A restricted neutrosophic topology (*RN*-topology in short) on a non empty set *X* is a family of restricted neutrosophic subsets in *X* satisfying the following axioms

(i) $0_N, 1_N \in \tau$

(ii) $\bigcup_i G_i \in \tau, \forall \{G_i : i \in J\} \subseteq \tau$

(iii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.

The pair (X, τ) is called restricted neutrosophic topological space (*RN*-topological space in short). The members of τ are called restricted neutrosophic open sets. A *RNS F* is closed if and only if F^c is *RN* open set.

3 Restricted Interval Valued Neutrosophic Set

In this section we introduce the concept of restricted interval valued neutrosophic set along with some examples, operators and results.

Definition 3.1 Let *X* be a non empty set. A restricted interval valued neutrosophic set (*RIVNS* in short) *A*is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x), \delta_A(x)\rangle : x \in X\}$, where $\mu_A(x), \gamma_A(x), \delta_A(x) : X \to Int\,]^-0, 1^+[$ are functions such that the condition: $\forall x \in X$, $sup\mu_A(x) \wedge sup\gamma_A(x) \wedge sup\delta_A(x) \le 0.5$ is satisfied.

Here $\mu_A(x), \gamma_A(x)$ and $\delta_A(x)$ represent truth-membership interval, indeterminacy-membership interval and falsity- membership interval respectively of the element $x \in X$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \gamma_A, \delta_A \rangle$ for the *RIVNS* $A = \{\langle x, \mu_A(x), \gamma_A(x), \delta_A(x)\rangle : x \in X\}$.

Example 3.2 Let $X = \{x_1, x_2, x_3\}$, then the *RIVNS* $A = \{\langle x, \mu_A(x), \gamma_A(x), \delta_A(x)\rangle : x \in X\}$ can be represent by the following table

X	$\mu_A(x)$	$\gamma_A(x)$	$\delta_A(x)$	$sup\mu_A(x) \wedge sup\gamma_A(x) \wedge sup\delta_A(x)$
x_1	$[.2,.3]$	$[0,.1]$	$[.4,.5]$	.1
x_2	$[.3,.5]$	$[.1,.4]$	$[.5,.6]$	.4
x_3	$[.4,.7]$	$[.2,.4]$	$[.6,.8]$	.4

The *RIVNS*s $\tilde{0}$ and $\tilde{1}$ are defined as
$\tilde{0} = \{\langle x, [0,0], [1,1], [1,1]\rangle : x \in X\}$ and
$\tilde{1} = \{\langle x, [1,1], [0,0], [0,0]\rangle : x \in X\}$.

Definition 3.3 Let $J_1 = [inf\ J_1, sup\ J_1]$ and $J_2 = [inf\ J_2, sup\ J_2]$ be two intervals then

(i) $J_1 \le J_2$ iff $inf\ J_1 \le inf\ J_2$ and $sup\ J_1 \le sup\ J_2$.

(ii) $J_1 \vee J_2 = [max(inf\ J_1, inf\ J_2), max(sup\ J_1, sup\ J_2)]$.

(iii) $J_1 \wedge J_2 = [min(inf\ J_1, inf\ J_2), min(sup\ J_1, sup\ J_2)]$.

Definition 3.4 Let *A* and *B* be two *RIVNS*s on *X* defined by
$A = \{\langle x, \mu_A(x), \gamma_A(x), \delta_A(x)\rangle : x \in X\}$ and

$B=\{\langle x,\mu_B(x),\gamma_B(x),\delta_B(x)\rangle : x\in X\}$. Then we can define union, intersection, subset and complement in several ways.

(i) The *RIVN*union of *A* and *B* is denoted by $A\cup B$ and is defined as

$$A\cup B=\{\langle x,\mu_A(x)\vee\mu_B(x),\gamma_A(x)\wedge\gamma_B(x),\delta_A(x)\wedge\delta_B(x)\rangle : x\in X\}$$

or

$$A\cup B=\{\langle x,\mu_A(x)\vee\mu_B(x),\gamma_A(x)\vee\gamma_B(x),\delta_A(x)\wedge\delta_B(x)\rangle : x\in X\}$$

We take first definition throughout the paper.

(ii) The*RIVN*intersection of *A* and *B* is denoted by $A\cap B$ and is defined as

$$A\cap B=\{\langle x,\mu_A(x)\wedge\mu_B(x),\gamma_A(x)\vee\gamma_B(x),\delta_A(x)\vee\delta_B(x)\rangle : x\in X\}$$

or

$$A\cap B=\{\langle x,\mu_A(x)\wedge\mu_B(x),\gamma_A(x)\wedge\gamma_B(x),\delta_A(x)\vee\delta_B(x)\rangle : x\in X\}$$

We take first definition throughout the paper.

(iii) *A* is called *RIVN* subset of *B*, denoted by $A\subseteq B$ if and only if

$\mu_A(x)\le\mu_B(x)$, $\gamma_A(x)\ge\gamma_B(x)$, $\delta_A(x)\ge\delta_B(x)$

or

$\mu_A(x)\le\mu_B(x)$, $\gamma_A(x)\le\gamma_B(x)$, $\delta_A(x)\ge\delta_B(x)$.

We take first definition throughout the paper.

(iv) The *RIVN* complement of *A* is denoted by A^c and is defined as

$$A=\{\langle x,\delta_A(x),[1-sup\gamma_A(x),1-inf\,\gamma_A(x)],\mu_A(x)\rangle : x\in X\}$$

or

$$A=\{\langle x,\delta_A(x),\gamma_A(x),\mu_A(x)\rangle : x\in X\}$$

We take first definition throughout the paper.

Definition 3.5 Let $\{A_i : i\in J\}$ be an arbitrary family of *RIVNS*s in *X*, then $\bigcup A_i$ and $\bigcap A_i$ can be respectively defined as

$$\bigcup A_i=\{\langle x,\vee_{i\in J}\mu_{A_i}(x),\wedge_{i\in J}\gamma_{A_i}(x),\wedge_{i\in J}\delta_{A_i}(x)\rangle : x\in X\}$$

or

$$\bigcup A_i=\{\langle x,\vee_{i\in J}\mu_{A_i}(x),\vee_{i\in J}\gamma_{A_i}(x),\wedge_{i\in J}\delta_{A_i}(x)\rangle : x\in X\}$$

$$\bigcap A_i=\{\langle x,\wedge_{i\in J}\mu_{A_i}(x),\vee_{i\in J}\gamma_{A_i}(x),\vee_{i\in J}\delta_{A_i}(x)\rangle : x\in X\}$$

or

$$\bigcap A_i=\{\langle x,\wedge_{i\in J}\mu_{A_i}(x),\wedge_{i\in J}\gamma_{A_i}(x),\vee_{i\in J}\delta_{A_i}(x)\rangle : x\in X\}$$

.

Theorem 3.6 Let*A*, *B* and *C*be three *RIVNS*s then

(1) $A\cup A=A$
(2) $A\cap A=A$
(3) $A\cup B=B\cup A$
(4) $A\cap B=B\cap A$
(5) $(A\cup B)^c=A^c\cap B^c$
(6) $(A\cap B)^c=A^c\cup B^c$
(7) $(A\cup B)\cup C=A\cup(B\cup C)$
(8) $(A\cap B)\cap C=A\cap(B\cap C)$
(9) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
(10) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Proof: Let $A=\langle x,[a_1,a_2],[a_3,a_4],[a_5,a_6]\rangle$, $B=\langle x,[b_1,b_2],[b_3,b_4],[b_5,b_6]\rangle$ and $C=\langle x,[c_1,c_2],[c_3,c_4],[c_5,c_6]\rangle$

(1) - (4) Straight forward.

(5) $A\cup B=\langle x,[max(a_1,b_1),max(a_2,b_2)],[min(a_3,b_3),min(a_4,b_4)],[min(a_5,b_5),min(a_6,b_6)]\rangle$

$$(A\cup B)^c=\langle x,[min(a_5,b_5),min(a_6,b_6)],[1-min(a_4,b_4),1-min(a_3,b_3)],[max(a_1,b_1),max(a_2,b_2)]\rangle$$

Now

$$A^c = \langle x, [a_5, a_6], [1-a_4, 1-a_3], [a_1, a_2] \rangle$$

$$B^c = \langle x, [b_5, b_6], [1-b_4, 1-b_3], [b_1, b_2] \rangle$$

$$A^c \cap B^c = \langle x, [min(a_5, b_5), min(a_6, b_6)], [max(1-a_4, 1-b_4), max(1-a_3, 1-b_3)], [max(a_1, b_1), max(a_2, b_2)] \rangle$$

$$= \langle x, [min(a_5, b_5), min(a_6, b_6)], [1-min(a_4, b_4), 1-min(a_3, b_3)], [max(a_1, b_1), max(a_2, b_2)] \rangle$$

(6) Same as **(5).**

(7) $A \cup B = \langle x, [max(a_1, b_1), max(a_2, b_2)], [min(a_3, b_3), min(a_4, b_4)], [min(a_5, b_5), min(a_6, b_6)] \rangle$

$$(A \cup B) \cup C = \langle x, [max(max(a_1, b_1), c_1), max(max(a_2, b_2), c_2)], [min(min(a_3, b_3), c_3), min(min(a_4, b_4), c_4)], [min(min(a_5, b_5), c_5), min(min(a_6, b_6), c_6)] \rangle$$

$$= \langle x, [max(a_1, b_1, c_1), max(a_2, b_2, c_2)], [min(a_3, b_3, c_3), min(a_4, b_4, c_4)], [min(a_5, b_5, c_5), min(a_6, b_6, c_6)] \rangle$$

$$B \cup C = \langle x, [max(b_1, c_1), max(b_2, c_2)], [min(b_3, c_3), min(b_4, c_4)], [min(b_5, c_5), min(b_6, c_6)] \rangle$$

$$A \cup (B \cup C) = \langle x, [max(a_1, max(b_1, c_1)), max(a_2, max(b_2, c_2))], [min(a_3, min(b_3, c_3)), min(a_4, min(b_4, c_4))], [min(a_5, min(b_5, c_5)), min(a_6, min(b_6, c_6))] \rangle$$

$$= \langle x, [max(a_1, b_1, c_1), max(a_2, b_2, c_2)], [min(a_3, b_3, c_3), min(a_4, b_4, c_4)], [min(a_5, b_5, c_5), min(a_6, b_6, c_6)] \rangle$$

(8) Same as **(7)**.

(9) $B \cap C = \langle x, [min(b_1, c_1), min(b_2, c_2)], [max(b_3, c_3), max(b_4, c_4)], [max(b_5, c_5), max(b_6, c_6)] \rangle$

$$A \cup (B \cap C) = \langle x, [max(a_1, min(b_1, c_1)), max(a_2, min(b_2, c_2))], [min(a_3, max(b_3, c_3)), min(a_4, max(b_4, c_4))], [min(a_5, max(b_5, c_5)), min(a_6, max(b_6, c_6))] \rangle$$

$$A \cup B = \langle x, [max(a_1, b_1), max(a_2, b_2)], [min(a_3, b_3), min(a_4, b_4)], [min(a_5, b_5), min(a_6, b_6)] \rangle$$

$$A \cup C = \langle x, [max(a_1, c_1), max(a_2, c_2)], [min(a_3, b_3), min(a_4, c_4)], [min(a_5, c_5), min(a_6, c_6)] \rangle$$

$$(A \cup B) \cap (A \cup C) = \langle x, [min(max(a_1, b_1), max(a_1, c_1)), min(max(a_2, b_2), max(a_2, c_2))], [max(min(a_3, b_3), min(a_3, c_3)), max(min(a_4, b_4), min(a_4, c_4))], [max(min(a_5, b_5), min(a_5, c_5)), max(min(a_6, b_6), min(a_6, c_6))] \rangle$$

Now let us consider a_1, b_1 and c_1, six cases may arise as

$a_1 \geq b_1 \geq c_1$, for this

$$max(a_1, min(b_1, c_1)) = min(max(a_1, b_1), max(a_1, c_1)) = a_1$$

$a_1 \geq c_1 \geq b_1$, for this

$$max(a_1, min(b_1, c_1)) = min(max(a_1, b_1), max(a_1, c_1)) = a_1$$

$b_1 \geq a_1 \geq c_1$, for this

$max\left(a_1, min\left(b_1, c_1\right)\right) =$

$min\left(max\left(a_1, b_1\right), max\left(a_1, c_1\right)\right) = a_1$

$b_1 \geq c_1 \geq a_1$, for this

$max\left(a_1, min\left(b_1, c_1\right)\right) =$

$min\left(max\left(a_1, b_1\right), max\left(a_1, c_1\right)\right) = c_1$

$c_1 \geq a_1 \geq b_1$, for this

$max\left(a_1, min\left(b_1, c_1\right)\right) =$

$min\left(max\left(a_1, b_1\right), max\left(a_1, c_1\right)\right) = a_1$

$c_1 \geq b_1 \geq a_1$, for this

$max\left(a_1, min\left(b_1, c_1\right)\right) =$

$min\left(max\left(a_1, b_1\right), max\left(a_1, c_1\right)\right) = b_1$.

Similarly it can be shown that other results are true for a_2, b_2, c_2 ; a_3, b_3, c_3 ; a_4, b_4, c_4 ; a_5, b_5, c_5 ; a_6, b_6, c_6 . Hence

$$A \cup \left(B \cap C\right) = \left(A \cup B\right) \cap \left(A \cup C\right) .$$

(10) Same as **(9)**.

4. Restricted Interval Valued neutrosophic Topological Spaces

In this section we give the definition of restricted interval valued Neutrosophic topological spaces with some examples and results.

Definition 4.1A restricted interval valued neutrosophic topology (*RIVN*-topology in short) on a non empty set *X* is a family of restricted interval valued neutrosophic subsets in *X* satisfying the following axioms

(iv) $\tilde{0}, \tilde{1} \in \tau$

(v) $\bigcup_i G_i \in \tau, \forall \left\{G_i : i \in J\right\} \subseteq \tau$

(vi) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.

The pair $\left(X, \tau\right)$ is called restricted interval valued neutrosophic topological space (*RIVN*-topological space in short). The members of τ are called restricted interval valued neutrosophic open sets. A *RIVNSF* is closed if and only if F^c is *RIVN* open set.

Example 4.2 Let*X* be a non-empty set. Let us consider the following *RIVNS*s

$G_1 = \left\{\left\langle x, [.5,.8], [.2,.3], [.2,.5]\right\rangle : x \in X\right\}$,

$G_2 = \left\{\left\langle x, [.6,.7], [.5,.6], [.3,.4]\right\rangle : x \in X\right\}$,

$G_3 = G_1 \cup G_2 = \left\{\left\langle x, [.6,.8], [.2,.3], [.2,.4]\right\rangle : x \in X\right\}$,

$G_4 = G_1 \cap G_2 = \left\{\left\langle x, [.5,.7], [.5,.6], [.3,.5]\right\rangle : x \in X\right\}$.

The family $\tau_1 = \left\{\tilde{0}, \tilde{1}, G_1, G_2, G_3, G_4\right\}$ is a *RIVN*-topology in *X* and $\left(X, \tau_1\right)$ is called a *RIVN*-topological space. But $\tau_2 = \left\{\tilde{0}, \tilde{1}, G_1, G_2\right\}$ is not a *RIVN*-topology as $G_1 \cup G_2 = G_3 \notin \tau_2$.

Definition 4.3 The two *RIVN* subsets $\tilde{0}, \tilde{1}$ constitute a *RIVN*-topology on *X*, called indiscrete *RIVN*-topology. The family of all *RIVN* subsets of *X* constitutes a *RIVN*-topology on*X*, such topology is called discrete *RIVN*-topology.

Theorem 4.4 Let $\left\{\tau_j : j \in J\right\}$ be a collection of *RIVN*-topologies on*X*. Then their intersection $\bigcap_{j \in J} \tau_j$ is also a*RIVS*-topology on *X*.

Proof: (i) Since $\tilde{0}, \tilde{1} \in \tau_j$ for each $j \in J$. Hence $\tilde{0}, \tilde{1} \in \bigcap_{j \in J} \tau_j$.

(ii) Let $\left\{G_k : k \in K\right\}$ be an arbitrary family *RIVNS*s where $G_k \in \bigcap_{j \in J} \tau_j$ for each $k \in K$. Then for each $j \in J$, $G_k \in \tau_j$ for $k \in K$ and since for each $j \in J$, τ_j ia a *RIVN*-topology, therefore $\bigcup_{k \in K} G_k \in \tau_j$ for each $j \in J$. Hence $\bigcup_{k \in K} G_k \in \bigcap_{j \in J} \tau_j$.

(iii) Let $G_1, G_2 \in \bigcap_{j \in J} \tau_j$, then $G_1, G_2 \in \tau_j$ for each $j \in J$. Since for each $j \in J$, τ_j is an *RIVN*-topology, therefore $G_1, G_2 \in \tau_j$ for each $j \in J$. Hence $G_1 \cap G_2 \in \bigcap_{j \in J} \tau_j$.

Thus $\bigcap_{j\in J}\tau_j$ forms a *RIVN*-topology as it satisfies all the axioms of RIVN-topology. But union of R*IVN*-topologies need not be a R*IVN*-topology.

Let us show this with the following example.

Example 4.5 In example 4.2, let us consider two *RIVN*- topologies τ_3 and τ_4 on *X* as $\tau_3=\{\tilde{0},\tilde{1},G_1\}$ and $\tau_4=\{\tilde{0},\tilde{1},G_2\}$. Here their union $\tau_3\cup\tau_4=\{\tilde{0},\tilde{1},G_1,G_2\}=\tau_2$ is not a *RIVN*-topology on*X*.

Definition 4.6 Let (X,τ) be an R*IVN*-topological space over X . A R*IVN* subset G of X is called restricted intervalvalued neutrosophicclosed set (in short R*IVN*-closed set) if its complement G^c is a member of τ .

Definition 4.7 Let (X,τ_1) and (X,τ_2) be two R*IVN*-topological spaces over X . If each $G\in\tau_2$ implies $G\in\tau_1$, then τ_1 is called restrictedinterval valued neutrosophic finer topology than τ_2 and τ_2 is called restricted interval valued neutrosophic coarser topology than τ_1 .

Example 4.8 In example 4.2 and 4.5, τ_1 is restricted interval valued neutrosophic finer topology than τ_3 and τ_3 is called restricted interval valued neutrosophic coarser topology than τ_1 .

Definition 4.9 Let τ be a *RIVN*-topological space on X and ß be a subfamily of τ . If every element of τ can be express as the arbitrary restrictedinterval valued neutrosophic union of some elements of ß, then ß is called restricted interval valued neutrosophic basis for the R*IVN*-topology τ .

5 Some Properties of Restricted Interval Valued Neutrosophic Soft Topological Spaces

In this section some properties of *RIVN*-topological spaces are introduced. Some results on *RIVNInt* and *RIVNCl* are also introduced.Restricted interval valued neutrosophic subspace topology is also studied.

Definition 5.1 Let (X,τ) be a *RIVN*-topological space and *A* be a *RIVNS* in *X*. The restrictedinterval valued neutrosophic interior and restrictedinterval valued neutrosophiccloser of*A* is denoted by *RIVNInt(A)* and *RIVNCl(A)* are defined as

$$RIVNInt(A)=\bigcup\{G:G \text{ is an } RIVN \text{ open set and } G\subseteq A\}$$

and

$$RIVNCl(A)=\bigcap\{F:F \text{ is an } RIVN \text{ closed set and } F\supseteq A\}$$

respectively.

Theorem 5.2 Let (X,τ) be a *RIVN*-topological space and G and H be two *RIVNS*s then the following properties hold

(1) $RIVNInt(G)\subseteq G$

(2) $G\subseteq H\Rightarrow RIVNInt(G)\subseteq RIVNInt(H)$

(3) $RIVNInt(G)\in\tau$

(4) $G\in\tau\Leftrightarrow RIVNInt(G)=G$

(5) $RIVNInt(RIVNInt(G))=RIVNInt(G)$

(6) $RIVNInt(\tilde{0})=\tilde{0}$, $RIVNInt(\tilde{1})=\tilde{1}$

Proof:

(1) Straight forward.

(2) Let $G\subseteq H$, then all the R*IVN*-open sets Contained in G also contained in H .

i.e. $\{G^*\in\tau:G^*\subseteq G\}\subseteq\{H^*\in\tau:H^*\subseteq H\}$

i.e. $\bigcup\{G^*\in\tau:G^*\subseteq G\}\subseteq\bigcup\{H^*\in\tau:H^*\subseteq H\}$

i.e. $RIVNInt(G)\subseteq RIVNInt(H)$

(3) $RIVNInt(G)=\bigcup\{G^*\in\tau:G^*\subseteq G\}$

Now clearly $\bigcup\{G^*\in\tau:G^*\subseteq G\}\in\tau$

$\therefore RIVNInt(G)\in\tau$.

(4) Let $G\in\tau$, then by (1) $RIVNInt(G)\subseteq G$.

Now since $G\in\tau$ and $G\subseteq G$, therefore $G\subseteq\bigcup\{G^*\in\tau:G^*\subseteq G\}=RIVNInt(G)$

i.e, $G\subseteq RIVNInt(G)$

Thus $RIVNInt(G)=G$

Conversely, let $RIVNInt(G)=G$

Since by (3) $RIVNInt(G)\in\tau$

Therefore $G\in\tau$

(5) By (3) $RIVNInt(G)\in\tau$

$\therefore$ By (4)

$RIVNInt\left(RIVNInt\left(f_A, E\right)\right) = RIVNInt\left(f_A, E\right)$

(6) We know that $\tilde{0}, \tilde{1} \in \tau$

$\therefore$ By (4)

$RIVNInt\left(\tilde{0}\right) = \tilde{0},\ RIVNInt\left(\tilde{1}\right) = \tilde{1}$

Theorem 5.3 Let $\left(X, \tau\right)$ be a *RIVN*-topological space and G and H are two *RIVNS*s then the following properties hold

(1) $G \subseteq RIVNCl\left(G\right)$

(2) $G \subseteq H \Rightarrow RIVNCl\left(G\right) \subseteq RIVNCl\left(H\right)$

(3) $\left(RIVNCl\left(G\right)\right)^c \in \tau$

(4) $G^c \in \tau \Leftrightarrow RIVNCl\left(G\right) = G$

(5) $RIVNCl\left(RIVNCl\left(G\right)\right) = RIVNCl\left(G\right)$

(6) $RIVNCl\left(\tilde{0}\right) = \tilde{0},\ RIVNCl\left(\tilde{1}\right) = \tilde{1}$

Proof: straight forward.

Theorem 5.4 Let $\left(X, \tau\right)$ be an *RIVN*-topological space on X and A be a *RIVNS* of*X*and let $\tau_A = \left\{A \cap U : U \in \tau\right\}$. Then τ_A forms a *RIVN*-topology on A.

Proof:

(i) Clearly $\tilde{0} = A \cap \tilde{0} \in \tau_A$ and $\tilde{1} = A \cap \tilde{1} \in \tau_A$.

(ii) Let $G_j \in \tau_A, \forall j \in J$, then $G_j = A \cap U_j$ where $U_j \in \tau$ for each $j \in J$.

Now $\bigcup_{j \in J} G_j = \bigcup_{j \in J}\left(A \cap U_j\right) = A \cap \left(\bigcup_{j \in J} U_j\right) \in \tau_A$ (since $\bigcup_{j \in J} U_j \in \tau$ as each $U_j \in \tau$).

(iii) Let $G, H \in \tau_A$ then $G = A \cap U$ and $H = A \cap V$ where $U, V \in \tau$.

Now

$G \cap H = \left(A \cap U\right) \cap \left(A \cap V\right) = A \cap \left(U \cap V\right) \in \tau_A$

(since $U \cap V \in \tau$ as $U, V \in \tau$).

Definition 5.5 Let $\left(X, \tau\right)$ be an *RIVN*-topological space on X and A be a *RIVNS* of*X*. Then $\tau_A = \left\{A \cap U : U \in \tau\right\}$ is called restricted interval valued neutrosophic subspace topology and $\left(A, \tau_A\right)$ is called restricted interval valued neutrosophic subspace of *RIVN*-topological space $\left(X, \tau\right)$.

Conclusion: In this paper we introduce the concept of restricted interval valued neutrosophic set which is the generalization of restrictedneutrosophic set. We define some operators on *RIVNS*. We also introduce a topological structure based on this. *RIVN* interior and *RIVN* closer of a restricted interval valued neutrosophic set are also defined. Restricted interval valued neutrosophic subspace topology is also studied. In future combining the ideas presented in this paper with concept of soft set one can define a new concept named restricted interval valued Neutrosophic soft set and can define a topological structure too.

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Some Studies in Neutrosophic Graphs

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Abstract. The main purpose of this paper is to discuss the notion of neutrosophic graphs, weak isomorphisms, co-weak isomorphisms and isomorphisms between two neutrosophic graphs. It is proved that the isomorphism between the two neutrosophic graphs is an equivalence relation. Some other properties of morphisms are also discussed in this paper.

Keywords: Neutrosophic graphs, Weak isomorphisms, Co-weak isomorphisms, Equivalence relation and Isomorphisms.

1 Introduction

Graph theory has its origins in a 1736 paper by the celebrated mathematician Leonhard Euler (10), known as the father of graph theory, when he settled a famous unsolved problem known as Ko¨ nigsburg Bridge problem. Graph theory is considered as a part of combinatorial mathematics. The theory has greatly contributed to our understanding of communication theory, programming, civil engineering, switching circuits, architecture, operational research, economics linguistic, psychology and anthropology. A graph is also used to create a relationship between a given set of objects. Each object can be represented by a vertex and the relationship between them can be represented by an edge.

In 1965, L.A. Zadeh (22) published the first paper on his new theory of fuzzy sets and systems. A fuzzy set is an extension of classical set theory. His work proved to be a mathematical tool for explaining the concept of uncertainty in real life problems. A fuzzy set is characterized by a membership function which assigns to each object a grade of membership ranging between zero and one. Azriel Rosenfeld (18) introduced the of notion of fuzzy graphs in 1975 and proposed another definitions including paths, cycles. connectedness etc. Mordeson and Peng (15) studied operations on fuzzy graphs. Many researchers contributed a lot and gave some more generalized forms of fuzzy graphs which have been studied in (6) and (8). These contributions show a new dimension of graph theory.

F. Smarandache (20) introduced the notion of neutrosophic set which is useful for dealing real life problems having imprecise, indeterminacy and inconsistant data. The theory is generalization of classical sets and fuzzy sets and is applied in decision making problems, control theory, medicines, topology and in many more real life problems. The notion of neutrosphic soft graph is introduced by N. Shah and A. Hussain (19). In the present paper neutrosophic graphs, their types, different operations like union intersection complement are definend. Furthermore different morphisms such as weak isomorphisms, co-weak isomorphism and isomorphisms are defined. Some theorems on morphisms are also proven here. This paper has been arranged as the following; In section 2, some basic concepts about graphs and neutrosophic sets are presented which will be employed in later sections. In section 3, concept of neutrosophic graphs is given and some of their fundamental properties have been studied. In section 4, concept of strong neutrosophic graphs and its complement is studied. Section 5 is devoted for the study of morphisms of neutrosophic graphs. Conclusions are also given at the end of Section 5.

2 PRELIMINARIES

In this section, some definitions about graphs and neutrosophic sets are given. These will be helpful in later sections.

2.1 Definition (21) A graph G^* consists of set of finite objects $V = \{v_1, v_2, v_3 ...., v_n\}$ called vertices (also called points or nodes) and other set $E = \{e_1, e_2, e_3 ...., e_n\}$ whose element are called edges (also called lines or arcs).

Usually a graph is denoted as $G^* = (V, E)$. Let G^* be a graph and e $= \{u, v\}$ be an edge of G^*. Since $\{u, v\}$ is 2-element set, we may write $\{u, v\}$ instead of $\{v, u\}$. It is often more convenient to represent this edge by uv or vu.

2.2 Definition (21) An edge of a graph that joins a vertex to itself is called loop.

2.3 Definition (21) In a multigraph no loops are allowed but more than one edge can join two vertices, these edges are called multiple edges or parallel edges and a graph is called multigraph.

2.4 Definition (21) A Graph which has neither loops nor multiple edges is called a simple graph.

2.5 Definition (21) A sub graph H^* of G^* is a graph having all of its vertices and edges in G^*.

2.6 Definition (21) Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs. A function $f : V_1 \to V_2$ is called Isomorphism if

i) f is one to one and onto.

ii) for all $a, b \in V_1, \{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$ when such a function exists, G_1^* and G_2^* are called isomorphic graphs and is written as $G_1^* \cong G_2^*$.

2.7 Definition (21) The union of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The union of G_1^* and G_2^* is denoted by $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2) = (V, E)$.

2.8 Definition (21) The join of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup E'$, where E' is the set of all edges joining the nodes of V_1 and V_2 and assume that $V_1 \cap V_2 \neq \theta$. The join of G_1^* and G_2^* is denoted by $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$.

2.9 Definition (21) The intersection of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V = V_1 \cap V_2$ and edge set $E = E_1 \cap E_2$. The intersection of G_1^* and G_2^* is denoted by $G^* = G_1^* \cap G_2^* = (V_1 \cap V_2, E_1 \cap E_2) = (V, E)$.

2.10 Definition (20) A neutrosophic set A on the universe of discourse X is defined as $A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$, where $T, I, F : X \to]^-0, 1^+[$ and $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.. From philosophical point of view, the neutrosophic set takes the value from real standard or non standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$.

3 NEUTROSOPHIC GRAPHS

3.1 Definition Let $G^* = (V, E)$ be a simple graph and $E \subseteq V \times V$. Let $T_f, I_f, F_f : V \to [0, 1]$ denote the truth-membership, indeterminacy- membership and falsity-membership of an element $x \in V$ and $T_g, I_g, F_g : E \to [0, 1]$ denote the truth-membership, indeterminacy-membership and falsity- membership of an element$(x, y) \in E$. By a neutrosophic graphs, we mean a 3-tuple $G = (G^*, f, g)$ such that

$$T_g(x, y) \leq \min\{T_f(x), T_f(y)\}$$
$$I_g(x, y) \leq \min\{I_f(x), I_f(y)\}$$
$$F_g(x, y) \geq \max\{F_f(x), F_f(y)\}$$

For all $x, y \in V$.

3.2 Example Let $G^* = (V, E)$ be a simple graph with $V = \{x_1, x_2, x_3\}$ and $E = \{(x_1, x_2)\ (x_2,, x_3)\ , (x_1, x_3)\}$. A neutrosophic graph G is given in table 1 below and $T(x_i, x_j) = 0$, $I(x_i, x_j) = 0$ and $F(x_i, x_j) = 1$ for all $(x_i, x_j) \in E \setminus \{(x_1, x_2), (x_2, x_3), (x_1, x_3)\}$.

$Table1$

f	x_1	x_2	x_3
T_f	0.3	0.1	0.1
I_f	0.4	0.3	0.3
F_f	0.5	0.4	0.6

g	(x_1, x_2)	(x_2, x_3)	(x_1, x_3)
T_g	0.1	0.1	0.1
I_g	0.1	0.2	0.2
F_g	0.9	0.8	0.7

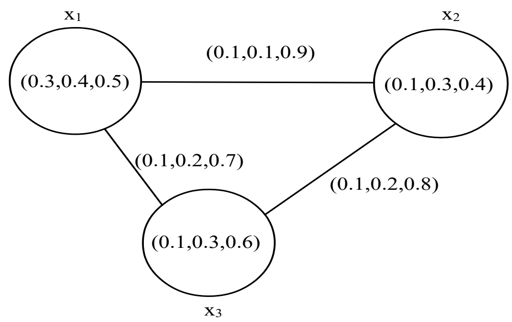

Figure 1

3.3 Definition A neutrosophic graph $G = (G^*, f_1, g_1)$ is called a neutrosophic subgraph of $G = (G^*, f, g)$ if

(i) $T_{f^1}(x) \leq T_f(x), I_{f^1}(x) \leq I_f(x), F_{f^1}(x) \geq F_f(x),$

(ii) $T_{g^1}(x, y) \leq T_g(x, y), I_{g^1}(x, y) \leq I_g(x, y), F_{g^1}(x, y) \geq F_g(x, y).$

for all $x, y \in V$.

A neutrosophic subgraph of example 3.2 is given in table 2 below and $T_g(x_i, x_j) = 0$,$I_g(x_i, x_j) = 0$ and $F_g(x_i, x_j) = 1$ for all $(x_i, x_j) \in E \setminus \{(x_1, x_2), (x_2, x_3), (x_1, x_3)\}$.

Table 2

f^1	x_1	x_2	x_3
T_{f^1}	0.1	0.1	0.1
I_{f^1}	0.1	0.2	0.2
F_{f^1}	0.5	0.4	0.6

g^1	(x_1,x_2)	(x_2,x_3)	(x_1,x_3)
T_{g^1}	0.1	0.1	0.1
I_{g^1}	0.1	0.1	0.1
F_{g^1}	0.9	0.8	0.7

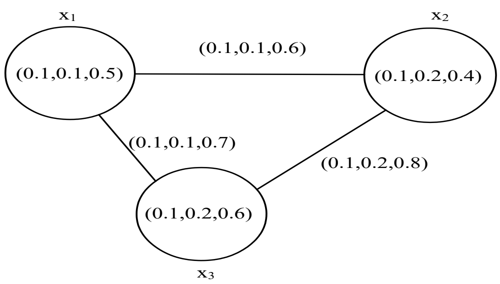

Figure 2

3.4 Definition A neutrosophic graph $G=(G^*, f^1, g^1)$ is said to be spanning neutrosophic subgraph of $G=(G^*, f, g)$ if $T_f(x)=T_f^1(x),\ I_f(x)=I_f^1(x),\ F_f(x)=F_f^1(x)$ for all $x \in V$.

3.5 Definition Let $G_1^*=(V_1,E_1)$ and $G_2^*=(V_2,E_2)$ be two simple graphs. The union of two neutrosophic graphs $G_1=(G_1^*, f^1, g^1)$ and $G_2=(G_2^*, f^2, g^2)$ is denoted by $G=(G^*, f, g)$, *where* $G^*=G_1^*\cup G_2^*$, $f=f^1\cup f^2$ $g=g^1\cup g^2$ where the truth-membership, indeterminacy-membership and falsity- membership of union are as follows

$$T_f(x)=\begin{cases} T_f^1(x) & \text{if } x\in V_1-V_2 \\ T_f^2(x) & \text{if } x\in V_2-V_1 \\ \max\{T_f^1(x),T_f^2(x)\} & \text{if } x\in V_1\cap V_2 \end{cases}$$

$$I_f(x)=\begin{cases} I_f^1(x) & \text{if } x\in V_1-V_2 \\ I_f^2(x) & \text{if } x\in V_2-V_1 \\ \max\{I_f^1(x),I_f^2(x)\} & \text{if } x\in V_1\cap V_2 \end{cases}$$

$$F(x)=\begin{cases} F_f^1(x) & \text{if } x\in V_1-V_2 \\ F_f^2(x) & \text{if } x\in V_2-V_1 \\ \min\{F_f^1(x),F_f^2(x)\} & \text{if } x\in V_1\cap V_2 \end{cases}$$

Also

$$T_g(x,y)=\begin{cases} T_{g^1}(x,y) & \text{if } (x,y)\in E_1-E_2 \\ T_{g^2}(x,y) & \text{if } (x,y)\in E_2-E_1 \\ \max\{T_{g^1}(x,y),T_{g^2}(x,y)\} & \text{if } (x,y)\in E_1\cap E_2 \end{cases}$$

$$I_g(x,y)=\begin{cases} I_{g^1}(x,y) & \text{if } (x,y)\in E_1-E_2 \\ I_{g^2}(x,y) & \text{if } (x,y)\in E_2-E_1 \\ \max\{I_{g^1}(x,y),I_{g^2}(x,y)\} & \text{if } (x,y)\in E_1\cap E_2 \end{cases}$$

$$F_g(x,y)=\begin{cases} F_{g^1}(x,y) & \text{if } (x,y)\in E_1-E_2 \\ F_{g^2}(x,y) & \text{if } (x,y)\in E_2-E_1 \\ \min\{F_{g^1}(x,y),F_{g^2}(x,y)\} & \text{if } (x,y)\in E_1\cap E_2 \end{cases}$$

3.6 Example Let $G_1^*=(V_1,E_1)$ be a simple graph with $V_1=\{x_1, x_3, x_4\}$ & $E_1=\{(x_1,x_4),(x_3,,x_4),(x_1,x_3)\}$. A neutrosophic graph G_1 is given in table 3 below and $T_g(x_i,x_j)=0$, $I_g(x_i,x_j)=0$ and $F_g(x_i,x_j)=1$ for all $(x_i,x_j)\in E_1 \setminus \{(x_1,x_4),(x_3,x_4),(x_1,x_3)\}$.

Table 3

f^1	x_1	x_3	x_4
T_{f^1}	0.1	0.2	0.2
I_{f^1}	0.2	0.4	0.5
F_{f^1}	0.3	0.5	0.7

g^1	(x_1,x_4)	(x_3,x_4)	(x_1,x_3)
T_{g^1}	0.1	0.1	0.1
I_{g^1}	0.2	0.3	0.2
F_{g^1}	0.7	0.8	0.5

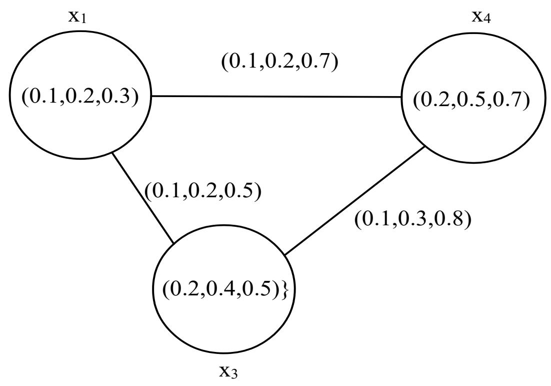

Figure 3

A neutrosophic graph $G_2 = (G_2^*, f^2, g^2)$ is given in table 4 below where $G_2^* = (V_2, E_2)$, $V_2 = \{x_2,\ x_3,\ x_4,\ x_5\}$ and $E_2 = \{(x_2,\ x_3)\ ,\ (x_3,\ x_4)\ ,\ (x_4,\ x_5)\ ,\ (x_2,\ x_5)\}$ and $T_g(x_i,\ x_j) = 0,\ I_g\ (x_i,\ x_j) = 0$ and $F_g\ (x_i,\ x_j) = 1$ for all $(x_i,\ x_j) \in E_2 \setminus \{(x_2,\ x_3)\ ,(x_2,x_4),(x_3,\ x_4)\ ,\ (x_4,\ x_5)\ ,\ (x_2,\ x_5)\}$.

Table 4

f^2	x_2	x_3	x_4	x_5
T_{f^2}	0.1	0.2	0.4	0.2
I_{f^2}	0.2	0.3	0.6	0.1
F_{f^2}	0.4	0.4	0.7	0.6

g^2	(x_2,x_3)	(x_3,x_4)	(x_2,x_5)	(x_4,x_5)	(x_2,x_4)
T_{g^2}	0.1	0.2	0.1	0.2	0.1
I_{g^2}	0.2	0.3	0.1	0.1	0.2
F_{g^2}	0.8	0.9	0.9	0.8	0.7

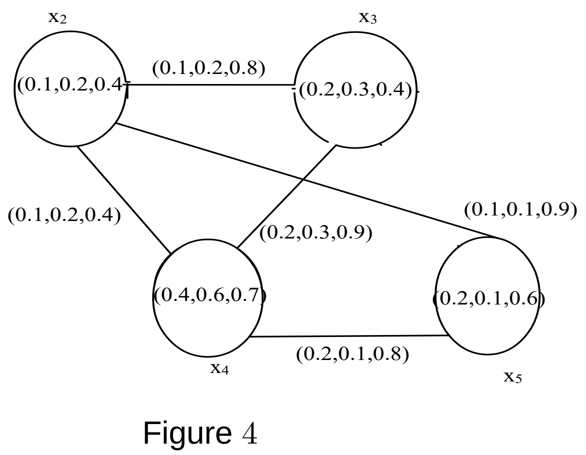

Figure 4

The union $G = (G^*,\ f,\ g)$ is given in table 5 below and $T_g\ (x_i,\ x_j) = 0,\ I_g\ (x_i,\ x_j) = 0$ and $F_g\ (x_i,\ x_j) = 1$ for all $(x_i,\ x_j) \in V \times V \setminus\ \{(x_1,\ x_4),\ (x_3,\ x_4),\ (x_1,\ x_3),$ $,(x_2,x_4\ ),(x_2,\ x_3),\ (x_4,\ x_5),\ (x_2,\ x_5)\}$.

Table 5

f	x_1	x_2	x_3	x_4	x_5
T_f	0.1	0.1	0.2	0.4	0.2
I_f	0.2	0.2	0.4	0.6	0.1
F_f	0.3	0.4	0.4	0.7	0.6

g	(x_1,x_4)	(x_3,x_4)	(x_1,x_3)	(x_2,x_3)	(x_4,x_5)	(x_2,x_4)	(x_2,x_5)
T_g	0.1	0.2	0.1	0.1	0.2	0.1	0.1
I_g	0.2	0.3	0.2	0.2	0.1	0.2	0.1
F_g	0.7	0.8	0.5	0.8	0.8	0.7	0.9

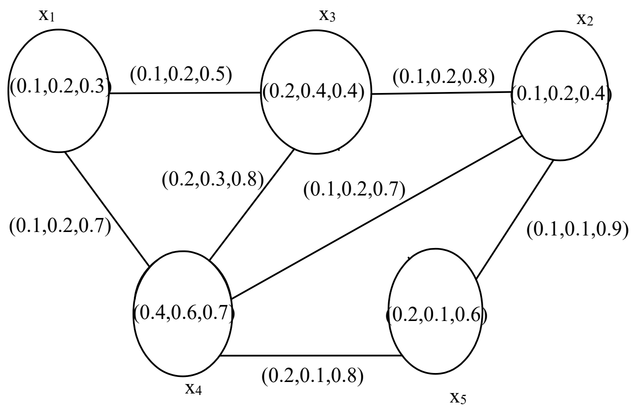

Figure 5

3.7 Proposition The union $G=(G^*, f, g)$ of two neutrosophic graphs $G_1=(G_1^*, f^1, g^1)$ and $G_2=(G_2^*, f^2, g^2)$ is a neutrosophic graph.

Proof

Case i) $\quad If\,(x,y)\in E_1-E_2$ *then*

$$T_g(x,y)=T_{g^1}(x,y)\le \min\{T_{f^1}(x),T_{f^2}(y)\}=\min\{T_f(x),T_f(y)\}$$

so $\quad T_g(x,y)\le \min\{T_f(x),T_f(y)\}$

Also $\quad I_g(x,y)=I_{g^1}(x,y)\le \min\{I_{f^1}(x),I_{f^2}(y)\}=\min\{I_f(x),I_f(y)\}$

so $\quad I_g(x,y)\le \min\{I_f(x),I_f(y)\}$

Now $\quad F_g(x,y)=F_{g^1}(x,y)\ge \max\{F_{f^1}(x),F_{f^2}(y)\}=\max\{F_f(x),F_f(y)\}$

Similarly $\quad If\,(x,y)\in E_2-E_1$ *then we can show the same as done above.*

Case ii) $\quad If\,(x,y)\in E_1\cap E_2,$ *then*

$$T_g(x,y)=\max\{T_{g^1}(x,y),T_{g^2}(x,y)\}$$
$$\le \max\{\min\{T_{f^1}(x),T_{f^1}(y)\},\min\{T_{f^2}(x),T_{f^2}(y)\}$$
$$\le \min\{\max\{T_{f^1}(x),T_{f^2}(x)\},\max\{T_{f^1}(y),T_{f^2}(y))=\min\{T_f(x),T_f(y)\}$$

Also $\quad I_g(x,y)=\max\{I_{g^1}(x,y),I_{g^2}(x,y)\}$
$$\le \max\{\min\{I_{f^1}(x),I_{f^1}(y)\},\min\{I_{f^2}(x),I_{f^2}(y)\}$$
$$\le \min\{\max\{I_{f^1}(x),I_{f^2}(x)\},\max\{I_{f^1}(y),I_{f^2}(y))=\min\{I_f(x),I_f(y)\}$$

Now $\quad F_g(x,y)=\min\{F_{g^1}(x,y),F_{g^2}(x,y)\}$
$$\ge \min\{\max\{F_{f^1}(x),F_{f^1}(y)\},\max\{F_{f^2}(x),F_{f^2}(y)\}$$
$$\ge \max\{\min\{F_{f^1}(x),F_{f^2}(x)\},\min\{F_{f^1}(y),F_{f^2}(y))$$
$$=\max\{F_f(x),F_f(y)\}$$

Hence the union $G=G_1\cup G_2$ is a neutrosophic graph.

3.8 Definition The intersection of two neutrosophic graphs $G_1=(G_1^*, f^1, g^1)$ and $G_2=(G_2^*, f^2, g^2)$ is denoted by $G=(G^*, f, g)$, *where* $G=G_1^*\cap G_2^*$, $f=f_1^1\cap f_1^1$, $g=g^1\cap g^2$, $V=V_1\cap V_2$ and the truth-membership, indeterminacy-membership and falsity- membership of intersection are as follows

$$T_f(x)=\min\{T_{f^1}(x),T_{f^2}(x)\},\quad I_f(x)=\min\{I_{f^1}(x),I_{f^2}(x)\},$$
$$F_f(x)=\max\{F_{f^1}(x),F_{f^2}(x)\}$$
$$T_g(x,y)=\min\{T_{g^1}(x,y),T_{g^2}(x,y)\},\quad I_g(x,y)=\min\{I_{g^1}(x,y),I_{g^2}(x,y)\}$$
$$F_g(x,y)=\max\{F_{g^1}(x,y),F_{g^2}(x,y)\}$$

for all $x,\ y\in V.$

3.9 Example Let $G_1^*=(V_1,E_1)$ be a simple graph with $V_1=\{x_1,\ x_2,\ x_5\}$ & $E_1=\{(x_1,\ x_5),\ (x_1,\ x_2),\ (x_2,\ x_5)\}$.

A neutrosophic graph $G_1=(G_1^*, f^1, g^1)$ is given in table 6 below and $T_g(x_i,\ x_j)=0,\ I_g(x_i,\ x_j)=0$ and $F_g(x_i,\ x_j)=1$ for all $(x_i,\ x_j)\in E_1\setminus\{(x_1,\ x_5),\ (x_1,\ x_2),\ (x_2,\ x_5)\}$.

Table 6

f^1	x_1	x_2	x_5
T_{f^1}	0.2	0.4	0.3
I_{f^1}	0.3	0.6	0.4
F_{f^1}	0.7	0.7	0.6

g^1	(x_1,x_2)	(x_2,x_5)	(x_1,x_5)
T_{g^1}	0.2	0.3	0.2
I_{g^1}	0.3	0.4	0.3
F_{g^1}	0.7	0.8	0.7

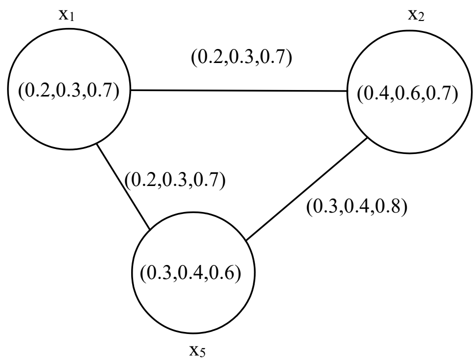

Figure 6

Let $G_2^*=(V_2,E_2)$ be a simple graph with $V_2=\{x_2,\ x_3,\ x_5\}$ and $E_2=\{(x_2,\ x_3),\ (x_3,\ x_5)(x_2,\ x_5)\}$.

A neutrosophic graph $G_2=(G_2^*, f^2, g^2)$ is given in table 7 below and $T_g(x_i,\ x_j)=0,\ I_g(x_i,\ x_j)=0$ and $F_g(x_i,\ x_j)=1$ for all $(x_i,\ x_j)\in E_2\setminus\{(x_2,\ x_3),\ (x_3,\ x_5)(x_2,\ x_5)\}$.

Table 7

f^2	x_2	x_3	x_5
T_{f^2}	0.3	0.2	0.4
I_{f^2}	0.5	0.4	0.5
F_{f^2}	0.6	0.6	0.9

g^2	(x_2, x_3)	(x_3, x_5)	(x_2, x_5)
T_{g^2}	0.1	0.2	0.2
I_{g^2}	0.3	0.4	0.4
F_{g^2}	0.7	0.9	0.9

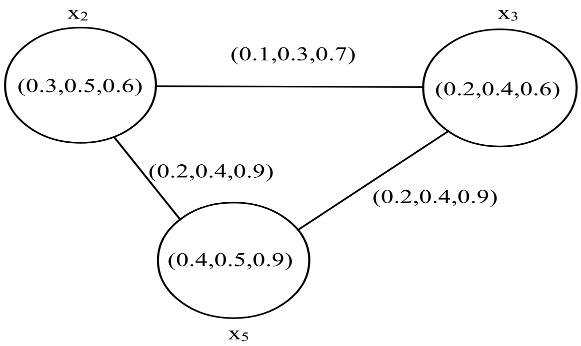

Figure 7

Let $V = V_1 \cap V_2 = \{x_2, x_5\}$, $E = E_1 \cap E_2 = \{(x_2, x_5)\}$. The intersection of the above two graphs G_1 and G_2 is given in the table 8 below and figure 8.

Table 8

f	x_2	x_5	g	(x_2, x_5)
T_f	0.3	0.3	T_g	0.2
I_f	0.5	0.4	I_g	0.4
F_f	0.7	0.9	F_g	0.9

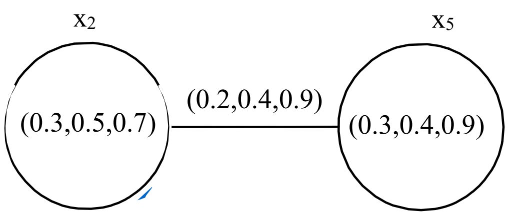

Figure 8

3.10 Proposition The intersection $G = (G^*, f, g)$ of two neutrosophic graphs $G_1 = (G_1^*, f^1, g^1)$ and $G_2 = (G_2^*, f^2, g^2)$ is a neutrosophic graph where $V = V_1 \cap V_2$.

Proof

Let $x, y \in V = V_1 \cap V_2$ and $(x, y) \in E = E_1 \cap E_2$,

then $T_g(x,y) = \min\{T_{g^1}(x,y), T_{g^2}(x,y)$

$\leq \min\{\min\{T_{f^1}(x), T_{f^1}(y)\}, \min\{T_{f^2}(x), T_{f^2}(y)\}$

$\leq \min\{\min\{T_{f^1}(x), T_{f^2}(x)\}, \min\{T_{f^1}(y), T_{f^2}(y))$

$= \min\{T_f(x), T_f(y)\}$

Also $I_g(x,y) = \min\{I_{g^1}(x,y), I_{g^2}(x,y)\}$

$\leq \min\{\min\{I_{f^1}(x), I_{f^1}(y)\}, \min\{I_{f^2}(x), I_{f^2}(y)\}$

$\leq \min\{\min\{I_{f^1}(x), I_{f^2}(x)\}, \min\{I_{f^1}(y), I_{f^2}(y)) = \min\{I_f(x), I_f(y)\}$

Now $F_g(x,y) = \max\{F_{g^1}(x,y), F_{g^2}(x,y)\}$

$\geq \max\{\max\{F_{f^1}(x), F_{f^1}(y)\}, \max\{F_{f^2}(x), F_{f^2}(y)\}\}$

$\geq \max\{\max\{F_{f^1}(x), F_{f^2}(x)\}, \max\{F_{f^1}(y), F_{f^2}(y)\}\} = \max\{F_f(x), F_f(y)\}$

Hence the intersection $G = G_1 \cap G_2$ is a neutrosophic graph.

4 Strong Neutrosophic Graphs

In this section we will study the notion of strong neutrosophic graphs and complement of such graphs.

4.1 Definition A neutrosophic graph $G = (G^*, f, g)$ is called strong if

$T_g(x, y) = \min\{T_f(x), T_f(y)\}$

$I_g(x, y) = \min\{I_f(x), I_f(y)\}$

$F_g(x, y) = \max\{F_f(x), F_f(y)\}$

for all $(x, y) \in E$.

4.2 Example Let $V = \{x_1, x_2, x_3\}$ and $E = \{(x_1, x_2), (x_2, x_3), (x_2, x_3)\}$. A strong neutrosophic graph $G = (G^*, f, g)$ where $G^* = (V, E)$ is simple graph, is given in table 9 below and $T_g(x_i, x_j) = 0$, $I_g(x_i, x_j) = 0$ and $F_g(x_i, x_j) = 1$ for all $(x_i, x_j) \in V \times V \setminus \{(x_1, x_2), (x_2, x_3), (x_2, x_3)\}$.

Table 9

f	x_1	x_2	x_3
T_f	0.1	0.2	0.3
I_f	0.2	0.3	0.4
F_f	0.4	0.5	0.7

g	(x_1,x_2)	(x_2,x_3)	(x_1,x_3)
T_g	0.1	0.2	0
I_g	0.2	0.3	0
F_g	0.5	0.7	1

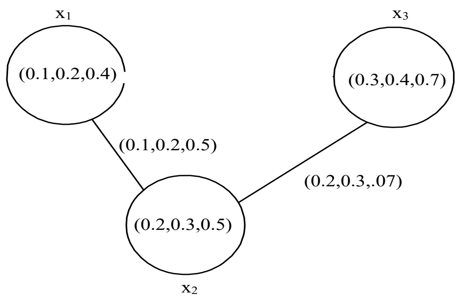

Figure 9

4.3 Definition Let $G=(G^*,f,g)$ be a strong neutrosophic graph. The complement $\overline{G}=(G^*,\overline{f},\overline{g})$ of strong neutrosophic graph $G=(G^*,f,g)$ is neutrosophic graph where

(i) $T_f(x)=\overline{T}_f(x), I_f(x)=\overline{I}_f(x), F_f(x)=\overline{F}_f(x),$ *for* all $x\in V$.

and

$$(ii)\quad \overline{T}_g(x,y)=\begin{cases}\min\{T_f(x),T_f(y)\} & \text{if } T_g(x,y)=0\\ 0 & \textit{otherwise}\end{cases}$$

$$\overline{I}_g(x,y)=\begin{cases}\min\{I_f(x),I_f(y)\} & \text{if } I_g(x,y)=0\\ 0 & \textit{otherwise}\end{cases}$$

$$\overline{F}_g(x,y)=\begin{cases}\max\{F_f(x),F_f(y)\} & \text{if } F_g(x,y)=1\\ 1 & \textit{otherwise}\end{cases}$$

4.4 Example For the strong neutrosophic graph in previous example, i.e. The complement of

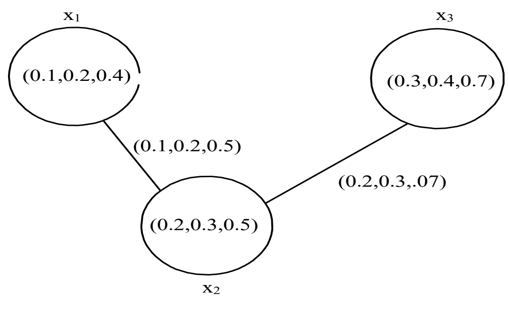

Figure 10

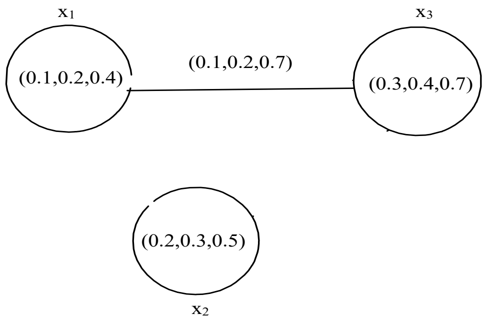

Figure 11

Similarly the complement of neutrosophic graph

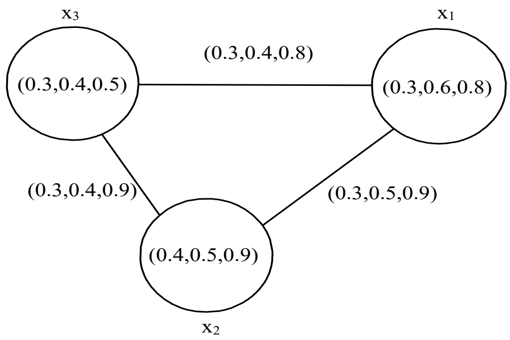

Figure 12

is given by

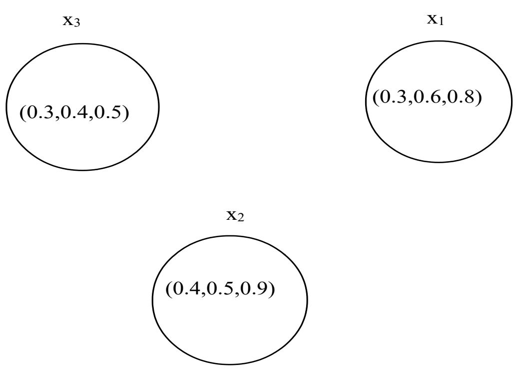

Figure 13

5 Homomorphism Of Neutrosophic Graphs

In this section we introduced and discussed the notion of homomorphisms of neutrosophic graphs. We have also discussed weak isomorphism, co- weak isomorphism and isomorphism here.

5.1 Definition A Homomorphism $h: G_1 \to G_2$ between two neutrosophic graphs $G_1 = (G_1^*, f^1, g^1)$ and $G_2 = (G_2^*, f^2, g^2)$ is a mapping $h: V_1 \to V_2$ which satisfies

(*i*) $T_{f^1}(x) \le T_{f^2}(h(x)),\ I_{f^1}(x) \le I_{f^2}(h(x)),$

$F_{f^1}(x) \ge F_{f^2}(h(x)),$ for all $x \in V_1$.

(*ii*) $T_{g^1}(x,y) \le T_{g^2}(h(x),h(y)),\ I_{g^1}(x,y) \le I_{g^2}(h(x),h(y)),$

$F_{g^1}(x,y) \ge F_{g^2}(h(x),h(y)),$ for all $x, y \in V_1$.

5.2 Definition A weak isomorphism $h: G_1 \to G_2$ between two neutrosophic graphs $G_1 = (G_1^*, f^1, g^1)$ and $G_2 = (G_2^*, f^2, g^2)$ is a mapping $h: V_1 \to V_2$ which is a bijective homomorphism such that $T_{f^1}(x) = T_{f^2}(h(x))$, $I_{f^1}(x) = I_{f^2}(h(x))$, $F_{f^1}(x) = F_{f^2}(h(x))$, for all $x, y \in V$.

5.3 Example Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs with $V_1 = \{x_1, x_2, x_3\}$, $E_1 = \{$

$\{(x_1,x_2),\ (x_2,x_3),\ (x_3,x_1)\}$

$V_2 = \{x_1', x_2', x_3'\}$. $E_2 = \{(x_1', x_2'), (x_2', x_3'), (x_1', x_3')\}$. Two neutrosophic graphs $G_1 = (G_1^*, f^1, g^1)$ and $G_2 = (G_2^*, f^2, g^2)$ are given in table 10 and Table 11 below and $T_g(x_i,x_j) = 0, I_g(x_i,x_j) = 0$ and $F_g(x_i,x_j) = 1$ for all $(x_i,x_j) \in V \times V \setminus \{\{(x_1,x_2),\ (x_2,x_3),\ (x_1,x_3)\}$.

Also

$T_g(x_i', x_j') = 0, I_g(x_i', x_j') = 0$ and $F_g(x_i', x_j') = 1$

for all $(x_i', x_j') \in V \times V \setminus \{(x_1', x_2'), (x_2', x_3'), (x_1', x_3')\}$.

Table 10

f^1	x_1	x_2	x_3
T_{f^1}	0.2	0.1	0.1
I_{f^1}	0.3	0.2	0.5
F_{f^1}	0.5	0.4	0.7

g^1	(x_1,x_2)	(x_2,x_3)	(x_1,x_3)
T_{g^1}	0.1	0.1	0.1
I_{g^1}	0.1	0.2	0.3
F_{g^1}	0.9	0.7	0.9

Table 11

f^2	x_1'	x_2'	x_3'
T_{f^2}	0.2	0.1	0.1
I_{f^2}	0.3	0.2	0.5
F_{f^2}	0.5	0.4	0.7

g^2	(x_1',x_2')	(x_2',x_3')	(x_1',x_3')
T_{g^2}	0.1	0.1	0.1
I_{g^2}	0.2	0.2	0.3
F_{g^2}	0.8	0.7	0.8

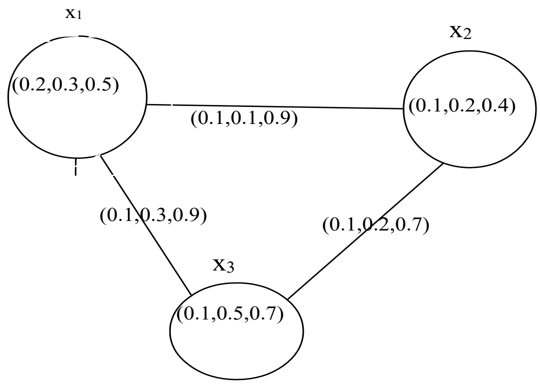

Figure 14

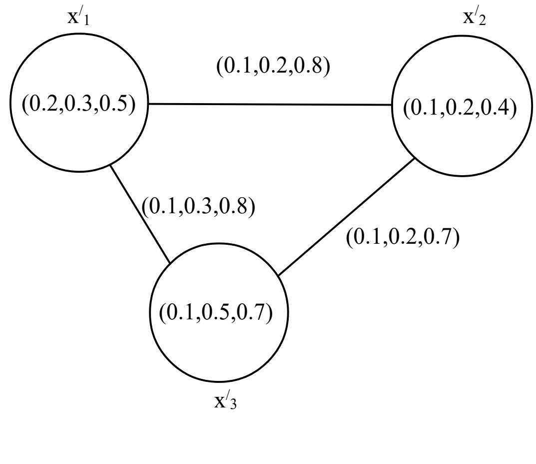

Figure 15

Now we define $h: V_1 \to V_2$ *by* $h(x_1) = x_1'$, $h(x_2) = x_2'$, $h(x_3) = x_3'$, *then* $T_{f^1}(x) = T_{f^2}(h(x))$, $I_{f^1}(x) = I_{f^2}(h(x))$, $F_{f^1}(x) = F_{f^2}(h(x))$, for all $x \in V_1$ ▷

By easy calculation, it can be seen that h is a weak isomorphism.

5.4 Proposition

Weak isomorphism between neutrosophic graphs satisfies the partial order relation.

Proof

Let $G_1 = (G_1^*, f^1, g^1)$, $G_2 = (G_2^*, f^2, g^2)$ and $G_3 = (G_3^*, f^3, g^3)$ be three neutrosophic graphs with sets of vertices V_1, V_2 and V_3 respectively. Then

1) The relation is reflexive. Let $h: V_1 \to V_2$ be an identity mapping such that $h(x_1) = x_1$, for all $x \in V_1$. That is $Tf1\ (x) = Tf2\ (h(x))$, $If1\ (x) = If2\ (h(x))$, $Ff1\ (x) = Ff2\ (h(x))$, for all $x \in V_1$ and $Tg1\ (x, y) \le Tg2\ (h(x), h(y))$, $Ig1\ (x, y) \le Ig2\ (h(x), h(y))$, $F_{g1}(x, y) \ge F_{g2}(h(x), h(y))$,for all $x, y \in V1$. So $h: V1 \to V1$ is a weak isomorphism of the neutrosophic graph $G1$ onto itself.

2) The relation is anti-symmetric. Let h be a weak isomorphism between the neutrosophic graphs $G1$ and $G2$, that is $h: V1 \to V2$ is a bijective mapping. Therefore $h(x_1) = x2$, for all $x_1 \in V_1$ satisfying $T_{f1}(x_2) = T_{f2}(h(x_1))$, $I_{f1}(x_2) = I_{f2}(h(x_1))$, $F_{f1}(x_2) = F_{f2}(h(x_1))$, for all $x_1 \in V_1$ and $T_{g1}(x_1, y_1) \le T_{g2}(h(x_1), h(y_1))$, $I_{g1}(x_1, y_1) \le I_{g2}(h(x_1), h(y_1)))$, $F_{g1}(x_1, y_1) \ge F_{g2}(h(x_1), h(y_1))$....(1) , for all $x_1, y_1 \in V_1$.

Let k be a weak isomorphism between the neutrosophic graph $G2$ and $G1$ so the relation is anti-symmetric that is $k: V2 \to V1$ is a bijective map with $T_{f2}(x_2) = T_{f1}(k(x_2))$, $I_{f2}(x_2) = I_{f1}(k(x_2))$,.$F_{f2}(x_2) = F_{f1}(k(x_2))$ for all $x_2 \in V2$ and $T_{g2}(x_2, y2) \le T_{g1}(k(x_2), k(y_2))$, $I_{g2}(x_2, y_2) \le I_{g1}(k(x_2), k(y_2))$, $F_{g2}(x_2, y_2) \le F_{g1}(k(x_2), k(y_2))$ for all $(x_2, y_2) \in (V2 \times V2)$....(2), for all $x_2, y_2 \in V2$. The subset relation (1) and (2) hold good on the finite sets $V1$, $V2$ when the neutrosophic graphs $G1$ and $G2$ have the same no. of edges and the corresponding edges are identical. Hence $G1$ and $G2$ are identical.

3) The relation is transitive. Let $h: V1 \to V2$ and $k: V2 \to V3$ be weak isomorphisms of the neutrosophic graphs $G_1 = (G_1^*, f^1, g^1)$ onto $G_2 = (G_2^*, f^2, g^2)$ and $G_2 = (G_2^*, f^2, g^2)$ onto $G_3 = (G_3^*, f^3, g^3)$ respectively. Then kof is a bijective mapping from $V1$ to $V3$ and defined as $(koh)(x_1) = k[h(x_1)]$, for all $x_1 \in V_1$. As h is a weak isomorphism, so $h(x_1) = x_2$, for all $x_1 \in V_1$ and $T_{f^1}(x) = T_{f^2}(h(x), I_{f^1}(x) = I_{f^2}(h(x), F_{f^1}(x) = F_{f^2}(h(x)$ for all $x_1 \in V_1$. Also $T_{g1}(x_1, y_1) \le T_{g2}(h(x_1), h(y_1))$, $I_{g1}(x_1, y_1) \le I_{g2}(h(x_1), h(y_1)))$, $F_{g1}(x_1, y_1) \ge F_{g2}(h(x_1), h(y_1))$, for all $x_1, y_1 \in V_1$.

As k is a weak isomorphism, so $k(x_2) = x_3$, for all $x_2 \in V2$ and $Tf2\ (x_2) = T_{f3}(k(x_2))$, $I_{f2}(x_2) = I_{f3}(k(x_2))$, $F_{f2}(x_2) = F_{f3}(k(x_2))$ and $T_{g2}(x_2, y_2) \le T_{g3}(k(x_2), k(y_2))$, $I_{g2}(x_2, y_2) \le I_{g3}(k(x_2), k(y_2))$, $F_{g2}(x_2, y_2) \le F_{g3}(k(x_2), k(y_2))$, for all $x_2, y_2 \in V2$.

As $T_{f1}(x) = T_{f2}(h(x))$, $I_{f1}(x_1) = I_{f2}(h(x_1))$, $F_{f1}(x_1) = F_{f2}(h(x_1))$, for all $x_1 \in V1$ and $T_{f2}(x_2) = T_{f3}(k(x_2))$, $I_{f2}(x_2) = I_{f3}(k(x_2))$, $F_{f2}(x_2) = F_{f3}(k(x_2))$ for all $x2 \in V2$. so $T_{f^1}(x_2) = T_{f^3}(k((h(x_1)))$, $I_f(x_2) = I_{f_e^3}(k((h(x_1)))$, $F_{f_e^1}(x_2) = F_{f_e^3}(k((h(x_1)))$ for all $x_1 \in V1$. As $T_{g1}(x_1, y_1) \le T_{g2}(h(x_1), h(y_1)) = T_{g2}(x_2, y_2)$, $I_{g1}(x_1, y_1) \le I_{g2}(h(x_1), h(y_1)) = I_{g2}(x_2, y_2)$, $F_{g1}(x_1, y_1) \ge F_{g2}(h(x_1), h(y1)) = F_{g2}(x_2, y_2)$ for all $x_1, y_1 \in V1$, so $T_{g1}(x_1, y_1) \le T_{g2}(x_2, y_2)$, $I_{g1}(x_1, y1) \le I_{g2}(x_2, y_2)$, $F_{g1}(x_1, y_1) \ge F_{g2}(x_2, y_2)$ for all $x_1, y_1 \in V1$.

But $T_{g2}(x_2, y_2) \le T_{g3}(k(x_2), k(y_2))$, $I_{g2}(x_2, y_2) \le I_{g3}(k(x_2), k(y_2))$, $F_{g2}(x_2, y_2) \ge F_{g3}(k(x_2), k(y_2))$,

Therefore

$T_{g1}(x1, y_1) \le T_{g3}(k(x_2), k(y_2))$, $I_{g1}(x_1, y_1) \le I_{g3}(k(x_2), k(y_2))$, $F_{g1}(x_1, y_1) \ge F_{g3}(k(x_2), k(y_2))$ for all $x_1, y_1 \in V1$.

So koh is a weak isomorphism between $G1$ and $G3$. that is, the relation is transitive. Hence the theorem.

5.5 Definition A co-weak isomorphism $h: G_1 \to G_2$ is a map $h: V1 \to V2$ between two neutrosophic graphs

$G_1 = (G_1^*, f^1, g^1)$ and $G_2 = (G_2^*, f^2, g^2)$ which is a bijective homomorphism that satisfies the condition
$T_{g1}(x_1, y_1) = T_{g2}(h(x_1), h(y_1))$,
$I_{g1}(x_1, y_1) = I_{g2}(h(x_1), h(y_1))$,
$F_{g1}(x_1, y_1) = F_{g2}(h(x_1), h(y_1))$
for all $x, y \in V_1$.

5.6 Definition An isomorphism $h : G_1 \to G_2$ is a mapping $h : V_1 \to V_2$ which is bijective that satisfies the following conditions

$$(i)\ T_{f^2}(x) = T_{f^3}(h(x)),\ \ I_{f^1}(x) = I_{f^2}(h(x)),$$
$$F_{f^1}(x) = F_{f^2}(h(x)),\ \ \text{for all } x \in V_1$$
$$(ii)\ T_{g^1}(x, y) = T_{g^2}(h(x), h(y)),$$
$$I_{g^1}(x, y) = I_{g^2}(h(x), h(y)),$$
$$F_{g^1}(x, y) = F_{g^2}(h(x), h(y)),\quad \text{for all } x, y \in V_1.$$

If such h exists then we say G_1 is isomorphic to G_2 and we write $G_1 \cong G_2$.

5.7 Proposition

The isomorphism between neutrosophic graphs is an equivalence relation.

Proof

Let $G_1 = (G_1^*, f^1, g^1)$, $G_2 = (G_2^*, f^2, g^2)$ and $G_3 = (G_3^*, f^3, g^3)$ be three neutrosophic graphs with sets of vertices V_1, V_2 and V_3 respectively then

i) The relation is reflexive. Consider the identity mapping $h : V_1 \to V_1$ such that $h(x_1) = x_1$, for all $x_1 \in V_1$. Then h is a bijective mapping satisfying
,

(i) $T_{f1}(x) = T_{f2}(h(x))$, $I_{f1}(x) = I_{f2}(h(x))$, $F_{f1}(x) = F_{f2}(h(x))$, for all $x \in V_1$.

ii) $T_{g1}(x, y) = T_{g2}(h(x), h(y))$,
$I_{g1}(x, y) = I_{g2}(h(x), h(y))$,
$F_{g1}(x, y) = F_{g2}(h(x), h(y))$, for all $x, y \in V_1$

showing that h is an isomorphism of the neutrosophic graph G_1 on to itself, that is $G_1 \cong G_1$.

i) The relation is symmetric. Let $h : V_1 \to V_2$ be an isomorphism of G_1 onto G_2 then h is bijective function. Therefore $h(x_1) = x_2$, for all $x_1 \in V$.

Also $T_{f1}(x) = T_{f2}(h(x))$, $I_{f1}(x) = I_{f2}(h(x))$, $F_{f1}(x) = F_{f2}(h(x))$, for all $x \in V_1$ and

$T_{g1}(x, y) = T_{g2}(h(x), h(y))$,
$I_g(x, y) = I_{g2}(h(x), h(y))$,
$F_{g1}(x, y) = F_{g2}(h(x), h(y))$, for all $x, y \in V_1$.

Since h is bijective, so it is invertible, that is, $h^{-1} : G_2 \to G_1$ will exist and $h^{-1}(x_2) = x_1$, for all $x_2 \in V_2$.

Since $T_{f1}(x_2) = T_{f2}(h(x_1))$,
$I_{f1}(x_2) = I_{f2}(h(x_1))$, $F_{f1}(x_2) = F_f(h(x_1))$ so
$T_{f1}(h^{-1}(x_2)) = T_{f2}(x_2)$ or
$T_{f2}(x_2) = T_{f1}(h^{-1}(x_2))$,
$I_{f2}(x_2) = I_{f1}(h^{-1}(x_2))$ *and*
$F_{f1}(h^{-1}(x_2)) = F_{f1}(h^{-1}(x_2))$ for all $x_2 \in V_2$. Also
$T_{g1}(x_1, y_2) = T_{g2}(h(x_1), h(y_1))$ so
$T_{g1}(h^{-1}(x_2), (h^{-1}(y_2))) = T_{g2}(x_2, y_2)$ or
$T_{g2}(x_2, y_2) = T_{g1}(h^{-1}(x_2), (h^{-1}(y_2)))$.
Similarly $I_{g1}(x_1, y_1) = I_{g2}(h(x_1), h(y_1))$ so
$I_{g1}(h^{-1}(x_2), (h^{-1}(y_2))) = I_{g2}(x_2, y_2)$ or
$I_{g2}(x_2, y_2) = I_{g1}(h^{-1}(x_2), (h^{-1}(y_2)))$, and
$F_{g1}(x_1, y_1) = F_{g2}(h(x_1), h(y_1))$ implies
$F_{g1}(h^{-1}(x_2), (h^{-1}(y_2))) = F_{g2}(x_2, y_2)$ or
$F_{g2}(x_2, y_2) = F_{g1}(h^{-1}(x_2), (h^{-1}(y_2)))$.

Hence $h^{-1} : G_2 \to G_1$ or $h^{-1} : V_2 \to V_1$ (Both one to one & onto) is an isomorphism from G_2 to G_1, that is $G_2 \cong G_1$. So $G_1 \cong G_2 \Rightarrow G_2 \cong G_1$.

iii) The relation is transitive.

Let $h : V_1 \to V_2$ and $k : V_2 \to V_3$ be the isomorphism of the neutrosophic graphs G_1 onto G_2 and G_2 onto G_3 respectively. Then $koh : V_1 \to V_3$ is also a bijective mapping from $V1$ to $V3$ defined as

$(koh)(x_1) = k[h(x_1)]$, for all $x_1 \in V_1$. Since $h : V_1 \to V_2$ is an isomorphism therefore $h(x_1) = x_2$, for all $x_1 \in V_1$. Also $T_{f1}(x_1) = T_{f2}(h(x_1))$,

$I_{f1}(x_1) = I_{f2}(h(x_1))$, $F_{f1}(x_1) = F_{f2}(h(x_1))$, for all $x_1 \in V_1$ and $T_{g1}(x_1, y_1) = T_{g2}(h(x_1), h(y_1))$,

$I_{g1}(x_1, y_1) = I_{g2}(h(x_1), h(y_1))$,
$F_{g1}(x_1, y_1) = F_{g2}(h(x_1), h(y_1))$, for all $x_1, y_1 \in V_1$.

Since $k : V_2 \to V_3$ is an isomorphism so

$k(x_2) = x_3$, $T_{f2}(x_2) = T_{f3}(k(x_2))$,
$I_{f2}(x_2) = I_{f3}(k(x_2))$, $F_{f2}(x_2) = F_{f3}(k(x_2))$ and
$T_{g2}(x_2, y_2) = T_{g3}(k(x_2), k(y_2))$, $I_{g2}(x_2, y_2) = I_{g3}(k(x_2), k(y_2))$, $F_{g2}(x_2, y_2) = F_{g3}(k(x_2), k(y_2))$, for all $x_2, y_2 \in V_2$. As $T_{f1}(x_1) = T_{f2}(h(x_1))$ and

$T_{f2}(x_2) = T_{f3}(k(x_2))$ so $T_{f1}(x_1) = T_{f2}(h(x_1)) =$

$T_{f_2}(x_2) = T_{f_3}(k(x_2)) = T_{f_3}(k(h(x_1)))$, for all $x_1 \in V_1$ which shows $T_{f_1}(x_1) = T_{f_3}(k(h(x_1)))$, for all $x_1 \in V_1$.

Similarly we can show $I_{f_1}(x_1) = I_{f_3}(k(h(x_1)))$, $F_{f_1}(x_1) = F_{f_3}(k(h(x_1)))$. Furthermore $T_{g_1}(x_1, y_1) = T_{g_2}(h(x_1), h(y_1))$ and $T_{g_2}(x_2, y_2) = T_{g_3}(k(x_2), k(y_2))$ so $T_{g_1}(x_1, y_1) = T_{g_2}(h(x_1), h(y_1)) = T_{g_2}(x_2, y_2) = T_{g_3}(k(x_2), k(y_2)) = T_{g_3}[(k(h(x_1)), (k(h(y_1))]$, so $T_{g_1}(x_1, y_1) = T_{g_3}[(k(h(x_1)), (k(h(y_1))]$ for all $x_1, y_1 \in V_1$.

Similarly we can show
$I_{g_1}(x_1, y_1) = I_{g_3}[(k(h(x_1)), (k(h(y_1))]$,
$F_{g_1}(x_1, y_1) = F_{g_3}[(k(h(x_1)), (k(h(y_1))]$.

So gof is isomorphism between G_1 and G_3.
Hence isomorphism between the neutrosophic graphs is an equivalence relation.

5.8 Remarks

1. If $G = G_1 = G_2$ then the homomorphism is called an endomorphism and the isomorphism is called an automorphism.
2. If $G_1 = G_2 = G$ then the co-weak and weak isomorphism become isomorphism.
3. A weak isomorphism preserves the equality of the of vertices but not necessarily the equality of edges.
4. A co-weak isomorphism preserves the equality of the edges but not necessarily the equality of vertices.
5. An isomorphism preserves the equality of edges and the equality of vertices.

Conclusion

In this paper we have described the neutrosophic graphs with the help of neutrosophic sets. Some operations on neutrosophic graphs are also presented in our work. We have proved that the isomorphism between neutrosophic graphs is an equivalence relation and weak isomorphism between neutrosophic graphs satisfies the partial order relation.

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Smooth Neutrosophic Topological Spaces

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Abstract. As a new branch of philosophy, the neutrosophy was presented by Smarandache in 1980. It was presented as the study of origin, nature, and scope of neutralities; as well as their interactions with different ideational spectra. The aim in this paper is to introduce the concepts of smooth neutrosophic topological space, smooth neutrosophic cotopological space, smooth neutrosophic closure, and smooth neutrosophic interior. Furthermore, some properties of these concepts will be investigated.

Keywords: Fuzzy Sets, Neutrosophic Sets, Smooth Neutrosophic Topology, Smooth Neutrosophic Cotopology, Smooth Neutrosophic Closure, Smooth Neutrosophic Interior.

1 Introduction

In 1986, Badard [1] introduced the concept of a smooth topological space as a generalization of the classical topological spaces as well as the Chang fuzzy topology [2]. The smooth topological space was rediscovered by Ramadan [3], and El-Gayyar et al. [4]. In [5], the authors introduced the notions of smooth interior and smooth closure. In 1983 the intuitionistic fuzzy set was introduced by Atanassov [[6], [7], [8]], as a generalization of fuzzy sets in Zadeh's sense [9], where besides the degree of membership of each element there was considered a degree of non-membership. Smarandache [[10], [11], [12]], defined the notion of neutrosophic set, which is a generalization of Zadeh's fuzzy sets and Atanassov's intuitionistic fuzzy set. The words "neutrosophy" and "neutrosophic" were invented by F. Smarandache in his 1998 book. Etymologically, "neutro-sophy" (noun) [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] means knowledge of neutral thought.

While "neutrosophic" (adjective), means having the nature of, or having the characteristic of Neutrosophy.

Neutrosophic sets have been investigated by Salama et al. [[13], [14], [15]]. The purpose of this paper is to introduce the concepts of smooth neutrosophic topological space, smooth neutrosophic cotopological space, smooth neutrosophic closure, and smooth neutrosophic interior. We also investigate some of their properties.

2 PRELIMINARIES

In this section we use X to denote a nonempty set, I to denote the closed unit interval $[0,1]$, I_o to denote the interval $(0,1]$, I_1 to denote the interval $[0,1)$, and I^X to be the set of all fuzzy subsets defined on X. By $\underline{0}$ and $\underline{1}$ we denote the characteristic functions of ϕ and X, respectively. The family of all neutrosophic sets in X will be denoted by $\aleph(X)$.

2.1 Definition [11], [12], [15]

A neutrosophic set A (NS for short) on a nonempty set X is defined as:
$A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$, $x \in X$
where $T, I, F: X \to [0,1]$, and
$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ representing the degree of membership (namely $T_A(x)$), the degree of indeterminacy (namely $I_A(x)$), and the degree of non-membership (namely $F_A(x)$); for each element $x \in X$ to the set A.

2.2Definition [13], [14], [15]

The Null (empty) neutrosophic set 0_N and the absolute (universe) neutrosophic set 1_N are defined as follows:

$$\text{TypeI} \;:\; 0_N = \langle x,0,0,1\rangle,\; x \in X$$
$$\text{TypeII} \;:\; 0_N = \langle x,0,1,1\rangle,\; x \in X$$
$$\text{TypeI} \;:\; 1_N = \langle x,1,1,0\rangle,\; x \in X$$
$$\text{TypeII} \;:\; 1_N = \langle x,1,0,0\rangle,\; x \in X$$

2.3 Definition [13], [14], [15]

A neutrosophic set A is a subset of a neutrosophic set B, ($A \subseteq B$), may be defined as:

$$\text{TypeI} \ : \ A \subseteq B \Leftrightarrow T_A(x) \le T_B(x), \quad I_A(x) \le I_B(x), F_A(x) \ge F_B(x), \quad \forall x \in X$$

$$\text{TypeII} \ : \ A \subseteq B \Leftrightarrow T_A(x) \le T_B(x), \quad I_A(x) \ge I_B(x), F_A(x) \ge F_B(x), \quad \forall x \in X$$

2.4 Definition [13], [14], [15]

The Complement of a neutrosophic set A, denoted by coA, is defined as:

$$\text{TypeI} \ : \ coA = \langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle$$

$$\text{TypeII} : \ coA = \langle x, 1 - T_A(x), 1 - I_A(x), 1 - F_A(x) \rangle$$

2.5 Definition [13], [14], [15]

Let $A, B \in \aleph(X)$ then:

$$\text{TypeI} \ : \ A \cup B = \langle x, \max(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \min(F_A(x), F_B(x)) \rangle$$

$$\text{TypeII} \ : \ A \cup B = \langle x, \max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_B(x)) \rangle$$

$$\text{TypeI} \ : \ A \cap B = \langle x, \min(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \max(F_A(x), F_B(x)) \rangle$$

$$\text{TypeII} \ : \ A \cap B = \langle x, \min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_B(x)) \rangle$$

$$[\ ]A = \langle x, T_A(x), I_A(x), 1 - T_A(x) \rangle$$

$$\langle\ \rangle A = \langle x, 1 - F_A(x), I_A(x), F_A(x) \rangle$$

2.6 Definition [13], [14], [15]

Let $\{A_i\}, i \in J$ be an arbitrary family of neutrosophic sets, then:

$$\text{TypeI} \ : \ \bigcup_{i \in J} A_i = \left\langle x, \sup_{i \in j} T_{A_i}(x), \sup_{i \in j} I_{A_i}(x), \inf_{i \in j} F_{A_i}(x) \right\rangle$$

$$\text{TypeII} \ : \ \bigcup_{i \in J} A_i = \left\langle x, \sup_{i \in j} T_{A_i}(x), \inf_{i \in j} I_{A_i}(x), \inf_{i \in j} F_{A_i}(x) \right\rangle$$

$$\text{TypeI} \ : \ \bigcap_{i \in J} A_i = \left\langle x, \inf_{i \in j} T_{A_i}(x), \inf_{i \in j} I_{A_i}(x), \sup_{i \in j} F_{A_i}(x) \right\rangle$$

$$\text{TypeII} \ : \ \bigcap_{i \in J} A_i = \left\langle x, \inf_{i \in j} T_{A_i}(x), \sup_{i \in j} I_{A_i}(x), \sup_{i \in j} F_{A_i}(x) \right\rangle$$

2.7 Definition [13], [14], [15]

The difference between two neutrosophic sets A and B defined as $A \setminus B = A \cap coB$.

2.8 Definition [13], [14]

Every intuitionistic set A on X is NS having the form $A = \langle x, T_A(x), 1 - (T_A(x) + F_A(x)), F_A(x) \rangle$, and every fuzzy set A on X is NS having the form $A = \langle x, T_A(x), 0, 1 - T_A(x) \rangle$, $x \in X$.

2.9 Definition [5]

Let Y be a subset of X and $A \in I^X$; the restriction of A on Y is denoted by $A_{/Y}$. For each $B \in I^Y$, the extension of B on X, denoted by B_X, is defined by:

$$B_X = \begin{cases} B(x) & \text{if } x \in A \\ 0.5 & \text{if } X - Y \end{cases}$$

2.10 Definition [1],[3]

A smooth topological space (STS, for short) is an ordered pair (X, τ) where X is a nonempty set and $\tau : I^X \to I$ is a mapping satisfying the following properties:

(O1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$

(O2) $\forall A_1, A_2 \in I^X, \ \tau(A_1 \cap A_2) \ge \tau(A_1) \wedge \tau(A_2)$

(O3) $\forall A_i, i \in J, \ \tau(\bigcup_{i \in J} A_i) \ge \bigwedge_{i \in J} \tau(A_i)$

2.11 Definition [1],[3]

A smooth cotopology is defined as a mapping $\Im : I^X \to I$ which satisfies:

(C1) $\Im(\underline{0})=\Im(\underline{1})=1$

(C2) $\forall B_1,B_2 \in I^X,\ \Im(B_1 \cup B_2) \ge \Im(B_1) \wedge \Im(B_2)$

(C3) $\forall A_i\ , i \in J,\ \Im(\bigcap_{i\in J} B_i) \ge \bigwedge_{i\in J} \Im(B_i)$

3.Smooth Neutrosophic Topological spaces

we will define two types of smooth neutrosophic topological spaces, a smooth neutrosophic topological space (SNTS, for short) take the form (X,τ^T,τ^I,τ^F) and the mappings $\tau^T,\tau^I,\tau^F : I^X \to I$ represent the degree of openness, the degree of indeterminacy, and the degree of non-openness respectively.

3.1 Smooth Neutrosophic Topological spaces of type I

In this part we will consider the definitions of typeI.

3.1.1Definition

A smooth neutrosophic topology (τ^T,τ^I,τ^F) of typeI satisfying the following axioms:

$(SNOI_1)$ $\tau^T(\underline{0})=\tau^I(\underline{0})=\tau^T(\underline{1})=\tau^I(\underline{1})=1$,
and $\tau^F(\underline{0})=\tau^F(\underline{1})=0$

$(SNOI_2)$ $\forall A_1,A_2 \in I^X$,
$\tau^T(A_1 \cap A_2) \ge \tau^T(A_1) \wedge \tau^T(A_2)$,
$\tau^I(A_1 \cap A_2) \ge \tau^I(A_1) \wedge \tau^I(A_2)$, and
$\tau^F(A_1 \cap A_2) \le \tau^F(A_1) \vee \tau^F(A_2)$

$(SNOI_3)$ $\forall A_i \in I^X, i \in J,\ \tau^T(\bigcup_{i\in J} A_i) \ge \bigwedge_{i\in J} \tau^T(A_i)$,
$\tau^I(\bigcup_{i\in J} A_i) \ge \bigwedge_{i\in J} \tau^I(A_i)$, and
$\tau^F(\bigcup_{i\in J} A_i) \le \bigvee_{i\in J} \tau^F(A_i)$

3.1.2Definition

Let $\Im^T,\Im^I,\Im^F : I^X \to I$ be mappings satisfying the following axioms:

$(SNCI_1)$ $\Im^T(\underline{0})=\Im^I(\underline{0})=\Im^T(\underline{1})=\Im^I(\underline{1})=1$,
and $\Im^F(\underline{0})=\Im^F(\underline{1})=0$

$(SNCI_2)$ $\forall B_1,B_2 \in I^X$,
$\Im^T(B_1 \cup B_2) \ge \Im^T(B_1) \wedge \Im^T(B_2)$,
$\Im^I(B_1 \cup B_2) \ge \Im^I(B_1) \wedge \Im^I(B_2)$, and
$\Im^F(B_1 \cup B_2) \le \Im^F(B_1) \vee \Im^F(B_2)$

$(SNCI_3)$ $\forall B_i \in I^X, i \in J,\ \Im^T(\bigcap_{i\in J} B_i) \ge \bigwedge_{i\in J} \Im^T(B_i)$,
$\Im^I(\bigcap_{i\in J} B_i) \ge \bigwedge_{i\in J} \Im^I(B_i)$, and
$\Im^F(\bigcap_{i\in J} B_i) \le \bigvee_{i\in J} \Im^F(B_i)$

The triple $(\Im^T,\Im^I,\Im^F)$ is a smooth neutrosophic cotopology of typeI, $\Im^T,\Im^I,\Im^F$ represent the degree of closedness, the degree of indeterminacy, and the degree of non-closedness respectively.

3.1.3Example

Let $X=\{a,b\}$ .Define the mappings $\tau^T,\tau^I,\tau^F : I^X \to I$ as:

$$\tau^T(A)=\begin{cases}1 & \text{if } A=\underline{0}\\ 1 & \text{if } A=\underline{1}\\ \min(A(a),A(b)) & \text{if } A \text{ is neither } \underline{0} \text{ nor } \underline{1}\end{cases}$$

$$\tau^I(A)=\begin{cases}1 & \text{if } A=\underline{0}\\ 1 & \text{if } A=\underline{1}\\ 0.5 & \text{if } A \text{ is neither } \underline{0} \text{ nor } \underline{1}\end{cases}$$

$$\tau^F(A)=\begin{cases}0 & \text{if } A=\underline{0}\\ 0 & \text{if } A=\underline{1}\\ \max(A(a),A(b)) & \text{if } A \text{ is neither } \underline{0} \text{ nor } \underline{1}\end{cases}$$

Then (X,τ^T,τ^I,τ^F) is a smooth neutrosophic topological space on X.

3.1.4Proposition

Let (τ^T,τ^I,τ^F) and $(\Im^T,\Im^I,\Im^F)$ be a smooth neutrosophic topology and a smooth neutrosophic cotopology, respectively, and let $A \in I^X$,

$\tau^T_{\Im^T}(A) = \Im^T(coA)$, $\tau^I_{\Im^I}(A) = \Im^I(coA)$,

$\tau^F_{\Im^F}(A) = \Im^F(coA), \Im^T_{\tau^T}(A) = \tau^T(coA)$,

$\Im^I_{\tau^I}(A) = \tau^I(coA)$, and $\Im^F_{\tau^F}(A) = \tau^F(coA)$, then

(1) $(\tau^T_{\Im^T}, \tau^I_{\Im^I}, \tau^F_{\Im^F})$ and $(\Im^T_{\tau^T}, \Im^I_{\tau^I}, \Im^F_{\tau^F})$ are a smooth neutrosophic topology and a smooth neutrosophic cotopology, respectively.

(2) $\tau^T_{\Im^T_{\tau^T}} = \tau^T, \tau^I_{\Im^I_{\tau^I}} = \tau^I, \tau^F_{\Im^F_{\tau^F}} = \tau^F$,

$\Im^T_{\tau^T_{\Im^T}} = \Im^T, \Im^I_{\tau^I_{\Im^I}} = \Im^I, \Im^F_{\tau^F_{\Im^F}} = \Im^F$,

Proof

(1) (a) $\tau^T_{\Im^T}(\underline{0}) = \tau^T_{\Im^T}(\underline{1}) = \tau^I_{\Im^I}(\underline{0}) = \tau^I_{\Im^T}(\underline{1}) = 1$, and

$\tau^F_{\Im^F}(\underline{0}) = \tau^F_{\Im^F}(\underline{1}) = 0$

(b) $\forall A_1, A_2 \in I^X$, $\tau^T_{\Im^T}(A_1 \cap A_2) =$

$\Im^T(co(A_1 \cap A_2) = \Im^T(coA_1 \cup coA_2) \geq$

$\Im^T(coA_1) \wedge \Im^T(coA_2) = \tau^T_{\Im^T}(A_1) \wedge \tau^T_{\Im^T}(A_2)$

,similarly, $\forall A_1, A_2 \in I^X$,

$\tau^I_{\Im^I}(A_1 \cap A_2) \geq \tau^I_{\Im^I}(A_1) \wedge \tau^I_{\Im^I}(A_2)$,and

$\tau^F_{\Im^F}(A_1 \cap A_2) \leq \tau^F_{\Im^F}(A_1) \vee \tau^F_{\Im^F}(A_2)$

(c) $\forall A_i \in I^X, i \in J, \tau^T_{\Im^T}(\underset{i\in J}{\cup} A_i) = \Im^T(co \underset{i\in J}{\cup} A_i)$

$= \Im^T(\underset{i\in J}{\cap} coA_i) \geq \underset{i\in J}{\wedge} \Im^T(coA_i) = \underset{i\in J}{\wedge} \tau^T_{\Im^T}(A_i)$

,similarly, $\forall A_i \in I^X, i \in J$,

$\tau^I_{\Im^I}(\underset{i\in J}{\cup} A_i) \geq \underset{i\in J}{\wedge} \tau^I_{\Im^I}(A_i)$,and

$\tau^F_{\Im^F}(\underset{i\in J}{\cup} A_i) \leq \underset{i\in J}{\vee} \tau^F_{\Im^F}(A_i)$ .Hence, $(\tau^T_{\Im^T}, \tau^I_{\Im^I}, \tau^F_{\Im^F})$ is a smooth neutrosophic topology.Similarly, we can prove that $(\Im^T_{\tau^T}, \Im^I_{\tau^I}, \Im^F_{\tau^F})$ is a smooth neutrosophic cotopology.

(2) the proof is straightforward.

3.1.5Proposition

Let $\{(\tau^T_i, \tau^I_i, \tau^F_i)\}_{i\in J}$ be a family of smooth neutrosophic topologies on X .Then their intersection $\underset{i\in J}{\cap}(\tau^T_i, \tau^I_i, \tau^F_i)$ is a a smooth neutrosophic topology.

Proof

The proof is a straightforward result of both definition(2.6) and difintion (3.1.1).

3.1.6Definition

Let (τ^T, τ^I, τ^F) be a smooth neutrosophic topology of type I, and $A \in I^X$. Then the smooth neutrosophic closure of A , denoted by $\overline{A}$ is defined by:

$$\overline{A} = \begin{cases} A & ,(\Im^T_{\tau^T}(A), \Im^I_{\tau^I}(A), \Im^F_{\tau^F}(A)) = (1,1,0) \\ \cap\{H : H \in I^X, A \subseteq H, \Im^T_{\tau^T}(H) > \Im^T_{\tau^T}(A), \\ \Im^I_{\tau^I}(H) > \Im^I_{\tau^I}(A), \Im^F_{\tau^F}(H) < \Im^F_{\tau^F}(A)\} , \\ \qquad (\Im^T_{\tau^T}(A), \Im^I_{\tau^I}(A), \Im^F_{\tau^F}(A)) \neq (1,1,0) \end{cases}$$

3.1.7Proposition

Let (τ^T, τ^I, τ^F) be a smooth neutrosophic topology on X, and $A, B \in I^X$. Then

(1) $\underline{0} = \overline{\underline{0}}$, $\underline{1} = \overline{\underline{1}}$

(2) $A \subseteq \overline{A}$

(3) $\Im^T_{\tau^T}(\overline{A}) \geq \Im^T_{\tau^T}(A), \Im^I_{\tau^I}(\overline{A}) \geq \Im^I_{\tau^I}(A)$, and

$\Im^F_{\tau^F}(\overline{A}) \leq \Im^F_{\tau^F}(A)\}$, $\forall A \in I^X$

(4) $B \subseteq A, \Im^T_{\tau^T}(A) \geq \Im^T_{\tau^T}(B), \Im^I_{\tau^I}(A) \geq \Im^I_{\tau^I}(B)$

and $\Im^F_{\tau^F}(A) \leq \Im^F_{\tau^F}(B) \Rightarrow \overline{B} \subseteq \overline{A}$, $\forall A, B \in I^X$

(5) $\overline{A} \subseteq \overline{\overline{A}}$

(6) $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof

(1) Obvious

(2) Directly from definition (3.1.6)

(3) (a) if $A = \overline{A}$, the proof is straightforward .

(b) if $A \neq \overline{A}$, we have from the definition (3.1.2) and the

definition (3.1.6):

$$\begin{aligned}
&\Im^{T}_{\tau_T}(\overline{A}) = \Im^{T}_{\tau_T}(\cap\{H : H \in I^X, A \subseteq H,\\
&\Im^{T}_{\tau_T}(H) > \Im^{T}_{\tau_T}(A), \Im^{I}_{\tau_I}(H) > \Im^{I}_{\tau_I}(A),\\
&\Im^{F}_{\tau_F}(H) < \Im^{F}_{\tau_F}(A)\}) \geq \wedge\{\Im^{T}_{\tau_T}(H) : H \in I^X, A \subseteq H, \text{ we}\\
&\Im^{T}_{\tau_T}(H) > \Im^{T}_{\tau_T}(A), \Im^{I}_{\tau_I}(H) > \Im^{I}_{\tau_I}(A),\\
&\Im^{F}_{\tau_F}(H) < \Im^{F}_{\tau_F}(A)\} \geq \Im^{T}_{\tau_T}(A)
\end{aligned}$$

can prove that $\Im^{I}_{\tau_I}(\overline{A}) \geq \Im^{I}_{\tau_I}(A)$ in a similar way.

$$\begin{aligned}
&\Im^{F}_{\tau_F}(\overline{A}) = \Im^{F}_{\tau_F}(\cap\{H : H \in I^X, A \subseteq H,\\
&\Im^{T}_{\tau_T}(H) > \Im^{T}_{\tau_T}(A), \Im^{I}_{\tau_I}(H) > \Im^{I}_{\tau_I}(A),\\
&\Im^{F}_{\tau_F}(H) < \Im^{F}_{\tau_F}(A)\}) \leq \vee\{\Im^{F}_{\tau_F}(H) : H \in I^X, A \subseteq H,\\
&\Im^{T}_{\tau_T}(H) > \Im^{T}_{\tau_T}(A), \Im^{I}_{\tau_I}(H) > \Im^{I}_{\tau_I}(A),\\
&\Im^{F}_{\tau_F}(H) < \Im^{F}_{\tau_F}(A)\} \leq \Im^{F}_{\tau_F}(A)
\end{aligned}$$

(4) (a) if $B = \overline{B}$, then $A = \overline{A}$ and $\overline{B} \subseteq \overline{A}$.

(b) if $B \neq \overline{B}$, and $A = \overline{A}$

$$\overline{B} = \cap\{H : H \in I^X, B \subseteq H, \Im^{T}_{\tau_T}(H) > \Im^{T}_{\tau_T}(B),$$
$$\Im^{I}_{\tau_I}(H) > \Im^{I}_{\tau_I}(B), \Im^{F}_{\tau_F}(H) < \Im^{F}_{\tau_F}(B)\},$$

this family contains A, hence, $\overline{B} \subseteq A = \overline{A}$

(c) if $B \neq \overline{B}$, and $A \neq \overline{A}$

From definition (3.1.6) every element in the family $\overline{A}$ will be an element in the family $\overline{B}$, hence $\overline{B} \subseteq \overline{A}$.

(5) From (2) , (3) and the definition (3.1.6) we have

$\overline{A} \subseteq \overline{\overline{A}}$.

(6) (a) if $A = \overline{A}$, and $B = \overline{B}$, then

$$\overline{A \cup B} = A \cup B \supseteq A \cap B = \overline{A} \cap \overline{B}$$

(b) if $A = \overline{A}$, $B \neq \overline{B}$, and $\overline{A \cup B} \neq A \cup B$,

from (4) $\overline{B} \subseteq \overline{A \cup B}$, hence $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

(c) if $A = \overline{A}$, $B \neq \overline{B}$, and $\overline{A \cup B} = A \cup B$,

then $\overline{A} \subseteq \overline{A \cup B}$, hence $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

(d) if $A \neq \overline{A}$, $B = \overline{B}$, and $\overline{A \cup B} \neq A \cup B$,

similar to(6b)

(e) if $A \neq \overline{A}$, $B = \overline{B}$, and $\overline{A \cup B} = A \cup B$,

similar to(6c)

(f) if $A \neq \overline{A}$, $B \neq \overline{B}$, and $\overline{A \cup B} = A \cup B$, it follows

from(4)that $\overline{A} \subseteq \overline{A \cup B}$, hence $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

(g) if $A \neq \overline{A}$, $B \neq \overline{B}$, and $\overline{A \cup B} \neq A \cup B$

$$\begin{aligned}
\overline{A \cup B} &= \cap\{H : H \in I^X, A \cup B \subseteq H, \\
&\mathfrak{I}^T_{\tau^T}(H) > \mathfrak{I}^T_{\tau^T}(A \cup B), \mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(A \cup B), \\
&\mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(A \cup B)\} \\
&\supseteq \cap\{H : H \in I^X, A \cup B \subseteq H, \mathfrak{I}^T_{\tau^T}(H) > \\
&\mathfrak{I}^T_{\tau^T}(A) \wedge \mathfrak{I}^T_{\tau^T}(B), \mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(A) \wedge \mathfrak{I}^I_{\tau^I}(B), \\
&\mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(A) \vee \mathfrak{I}^F_{\tau^F}(B)\} \\
&= \cap\{H : H \in I^X, A \subseteq H, B \subseteq H, \mathfrak{I}^T_{\tau^T}(H) > \mathfrak{I}^T_{\tau^T}(A) \\
&\text{or } \mathfrak{I}^T_{\tau^T}(H) > \mathfrak{I}^T_{\tau^T}(B), \mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(A) \text{ or} \\
&\mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(B), \mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(A) \text{ or} \\
&\mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(B)\} \\
&\supseteq \cap [\ \{H : H \in I^X, A \subseteq H, \mathfrak{I}^T_{\tau^T}(H) > \mathfrak{I}^T_{\tau^T}(A), \\
&\mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(A), \mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(A)\} \cup \\
&\{H : H \in I^X, B \subseteq H, \mathfrak{I}^T_{\tau^T}(H) > \mathfrak{I}^T_{\tau^T}(B), \\
&\mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(B), \mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(B)\}\ ] \\
&= [\ \cap\{H : H \in I^X, A \subseteq H, \mathfrak{I}^T_{\tau^T}(H) > \mathfrak{I}^T_{\tau^T}(A), \\
&\mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(A), \mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(A)\}\ ] \cap \\
&[\ \cap\{H : H \in I^X, B \subseteq H, \mathfrak{I}^T_{\tau^T}(H) > \mathfrak{I}^T_{\tau^T}(B), \\
&\mathfrak{I}^I_{\tau^I}(H) > \mathfrak{I}^I_{\tau^I}(B), \mathfrak{I}^F_{\tau^F}(H) < \mathfrak{I}^F_{\tau^F}(B)\}\ ] \\
&= \overline{A} \cap \overline{B}
\end{aligned}$$

3.1.8Definition

Let (τ^T, τ^I, τ^F) be a smooth neutrosophic topology of type I, and $A \in I^X$. Then the smooth neutrosophic interior of A, denoted by A^o is defined by:

$$A^o = \begin{cases} A & ,(\tau^T(A), \tau^I(A), \tau^F(A)) = (1,1,0) \\ \cup\{H : H \in I^X, H \subseteq A, \tau^T(H) > \tau^T(A), \\ \tau^I(H) > \tau^I(A), \tau^F(H) < \tau^F(A)\}, & (\tau^T(A), \tau^I(A), \tau^F(A)) \neq (1,1,0) \end{cases}$$

3.1.9Proposition

Let (τ^T, τ^I, τ^F) be a smooth neutrosophic topology on X, and $A, B \in I^X$. Then

(1) $\underline{0} = \underline{0}^o$, $\underline{1} = \underline{1}^o$

(2) $A^o \subseteq A$

(3) $\tau^T(A^o) \geq \tau^T(A), \tau^I(A^o) \geq \tau^I(A)$, and

$\tau^F(A^o) \leq \tau^F(A)\}$, $\forall A \in I^X$

(4) $B \subseteq A, \tau^T(B) \geq \tau^T(A), \tau^I(B) \geq \tau^I(A)$

and $\tau^F(B) \leq \tau^F(A) \Rightarrow B^o \subseteq A^o$, $\forall A, B \in I^X$

(5) $(A^o)^o \subseteq A^o$

(6) $(A \cap B)^o \subseteq A^o \cup B^o$

Proof

Similar to the procedure used to prove Proposition (3.1.7)

3.2. Smooth Neutrosophic Topological spaces of type II

In this part we will consider the definitions of typeII. In a similar way as in typeI, we can state the following definitions and propositions. The proofs of the propositions of typeII, will be similar to the proofs of the propositions in typeI.

3.2.1 Definition

A smooth neutrosophic topology (τ^T, τ^I, τ^F) of typeII satisfying the following axioms:

$(SNOII_1)$ $\tau^T(\underline{0}) = \tau^T(\underline{1}) = 1$, and

$\tau^I(\underline{0}) = \tau^I(\underline{1}) = \tau^F(\underline{0}) = \tau^F(\underline{1}) = 0$

$(SNOII_2)$ $\forall A_1, A_2 \in I^X$,

$\tau^T(A_1 \cap A_2) \geq \tau^T(A_1) \wedge \tau^T(A_2)$,

$\tau^I(A_1 \cap A_2) \leq \tau^I(A_1) \vee \tau^I(A_2)$, and

$\tau^F(A_1 \cap A_2) \leq \tau^F(A_1) \vee \tau^F(A_2)$

$(SNOII_3)$ $\forall A_i \in I^X, i \in J, \tau^T(\bigcup_{i\in J} A_i) \geq \bigwedge_{i\in J} \tau^T(A_i)$,

$\tau^I(\bigcup_{i\in J} A_i) \leq \bigvee_{i\in J} \tau^I(A_i)$, and

$\tau^F(\bigcup_{i\in J} A_i) \leq \bigvee_{i\in J} \tau^F(A_i)$

3.2.2 Definition

Let $\Im^T, \Im^I, \Im^F : I^X \to I$ be mappings satisfying the following axioms:

$(SNCII_1)$ $\Im^T(\underline{0}) = \Im^T(\underline{1}) = 1$, and

$\Im^I(\underline{0}) = \Im^I(\underline{1}) = \Im^F(\underline{0}) = \Im^F(\underline{1}) = 0$

$(SNCII_2)$ $\forall B_1, B_2 \in I^X$,

$\Im^T(B_1 \cup B_2) \geq \Im^T(B_1) \wedge \Im^T(B_2)$,

$\Im^I(B_1 \cup B_2) \leq \Im^I(B_1) \vee \Im^I(B_2)$, and

$\Im^F(B_1 \cup B_2) \leq \Im^F(B_1) \vee \Im^F(B_2)$

$(SNCII_3)$ $\forall B_i \in I^X, i \in J, \Im^T(\bigcap_{i\in J} B_i) \geq \bigwedge_{i\in J} \Im^T(B_i)$,

$\Im^I(\bigcap_{i\in J} B_i) \leq \bigvee_{i\in J} \Im^I(B_i)$, and

$\Im^F(\bigcap_{i\in J} B_i) \leq \bigvee_{i\in J} \Im^F(B_i)$

The triple $(\Im^T, \Im^I, \Im^F)$ is a smooth neutrosophic cotopology of typeII, $\Im^T, \Im^I, \Im^F$ represent the degree of closedness, the degree of indeterminacy, and the degree of non-closedness respectively.

3.2.3 Example

Let $X = \{a, b\}$. Define the mappings $\tau^T, \tau^I, \tau^F : I^X \to I$ as:

$$\tau^T(A) = \begin{cases} 1 & \text{if } A = \underline{0} \\ 1 & \text{if } A = \underline{1} \\ \min(A(a), A(b)) & \text{if } A \text{ is neither } \underline{0} \text{ nor } \underline{1} \end{cases}$$

$$\tau^I(A) = \begin{cases} 0 & \text{if } A = \underline{0} \\ 0 & \text{if } A = \underline{1} \\ 0.5 & \text{if } A \text{ is neither } \underline{0} \text{ nor } \underline{1} \end{cases}$$

$$\tau^F(A) = \begin{cases} 0 & \text{if } A = \underline{0} \\ 0 & \text{if } A = \underline{1} \\ \max(A(a), A(b)) & \text{if } A \text{ is neither } \underline{0} \text{ nor } \underline{1} \end{cases}$$

Then $(X, \tau^T, \tau^I, \tau^F)$ is a smooth neutrosophic topological space on X.

Note that: the Propositions (3.1.4) and (3.1.5) are satisfied for typeII.

3.2.4 Definition

Let (τ^T, τ^I, τ^F) be a smooth neutrosophic topology of type II, and $A \in I^X$. Then the smooth neutrosophic closure of A, denoted by $\overline{A}$ is defined by:

$$\overline{A} = \begin{cases} A & ,(\Im^T_{\tau^T}(A), \Im^I_{\tau^I}(A), \Im^F_{\tau^F}(A)) = (1,1,0) \\ \cap\{H : H \in I^X, A \subseteq H, \Im^T_{\tau^T}(H) > \Im^T_{\tau^T}(A), & \\ \Im^I_{\tau^I}(H) < \Im^I_{\tau^I}(A), \Im^F_{\tau^F}(H) < \Im^F_{\tau^F}(A)\}, & \\ & (\Im^T_{\tau^T}(A), \Im^I_{\tau^I}(A), \Im^F_{\tau^F}(A)) \neq (1,1,0) \end{cases}$$

Also, the smooth neutrosophic interior of A, denoted by A^o is defined by:

$$A^o = \begin{cases} A & ,(\tau^T(A), \tau^I(A), \tau^F(A)) = (1,1,0) \\ \cup\{H : H \in I^X, H \subseteq A, \tau^T(H) > \tau^T(A), & \\ \tau^I(H) < \tau^I(A), \tau^F(H) < \tau^F(A)\}, & \\ & (\tau^T(A), \tau^I(A), \tau^F(A)) \neq (1,1,0) \end{cases}$$

Note That: the Propositions (3.1.7) and (3.1.9) are satisfied for typeII.

4. Conclusion and Future Work

In this paper, the concepts of smooth neutrosophic topological structures were introduced. In two different types we've presented the concepts of smooth neutrosophic topological space, smooth neutrosophic cotopological space, smooth neutrosophic closure, and smooth neutrosophic interior. Due to unawareness of the behaviour of the degree of indeterminacy, we've chosen for τ^I to act like τ^T in the first type, while in the second type we preferred that τ^T behaves like τ^F. Therefore, the definitions given above can also be modified in several ways depending on the behaviour of τ^I. Moreover, as a consequence of our choices of the performance of τ^I, one can see that: In typeI, booth τ^T and τ^I defined in (3.1.1) with their conditions are smooth topologies; while in typeII, only τ^T defined in (3.2.1) with its conditions is a smooth topology.

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The Neutrosophic Statistical Distribution, More Problems, More Solutions

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Abstract: In this paper , the authors explore neutrosophic statistics, that was initiated by Florentin Smarandache in 1998 and developed in 2014, by presenting various examples of several statistical distributions, from the work **[1]**. The paper is presented with more case studies, by means of which this neutrosophic version of statistical distribution becomes more pronounced.

Key words: Neutrosophy, Binomial & Normal distributions, Neutrosophic logic etc.

I.Introduction: Neutrosophy was first proposed by Prof. Florentin Smarandache in 1995 . It is a new branch of philosophy , where one can study origin , nature and scope of neutralities . According to Prof. Dr.Huang, this gives advantages to break the mechanical understanding of human culture. For example, according to mechanical theory ,existence and non-existence couldn't be simultaneously , due to some indeterminacy [2].

This theory considers every notion or idea <A> together with its opposite or negation <Anti-A>. The <neut-A> and <Anti-A> ideas together called as a <non-A>. Neutrosophic logic is a general framework for unification of many existing logics, intutionstic logic, paraconsistent logic etc. The focal objective of neutrosophic logic is to characterize each logical statements in a 3D-neutrosophic space, where each dimension of space represents respectively the truth(T) , falsehood(F) and indeterminacies of the statements under consideration . Where T,I,F are standard or non-standard real subset of (-0,1+) without necessary connection between them. [3]

The classical distribution is extended neutrosophically. That means that there is some indeterminacy related to the probabilistic experiment. Each experimental observation of each trial can result in an outcome of each trial can result in an outcome labelled failure (F) or some indeterminacy(I).Neutrosophic statistics is an extended form of classical statistics, dealing with crisp values. In this paper, we will discuss about one discrete random distribution such as Binomial distribution and a continuousone by approaching neutrosophically. Before focusing the light on this context, we should familiar with the following notions.

Neutrosophic statistical number 'N' has the form

$$N = d + I;$$

Where, d: Determinate part

I: Indeterminate part of N.

For example, a = 5 + I ; where I ∈ [0 , 0.4] is equivalent to a∈ [5 , 5.4]. So for sure a ≥ 5, where I ∈[O , O.4] .

I.A. Preliminaries: In this context, we are going to discuss about the classical distributions[4] .

A). Binomial distribution,

B). Normal distribution.

I. A. a). Binomial distribution:

I. A.a.i. Definition: A random variable X is said to follow Binomial distribution, if it assumes only non-negative values and its probability mass function is given by,

$$p(X=x)=p(x)=\frac{n!}{x!(n-x)!}p^x q^{n-x};\ x=0,1,2\ldots n \text{ and } q=1-p$$

otherwise equal to zero .

I.A.a.ii. Physical conditions: We get Binomial distribution under the following conditions–

1. Each trials results in two exhaustive and mutually disjoint outcomes termed as success and failure.

2. The number of trials ‘n’ is finite.

3. The trials are independent on each other.

4. The probability of success ‘p’ is constant for each trial.

I.A.b. Normal Distribution:

I.A.b.i. Definition: A random variable is said to have a normal distribution with parameters μ and σ^2 , ifitsp.d.f is given by the probability law ,

$$f(x\,;\mu\,;\,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x-\mu)^2}{2\sigma_x^2}}$$

$-\infty < x < \infty$ and $-\infty<\mu<\infty, \sigma_X > 0.$

A.I.b.ii. Chief characteristics of Normal Distribution and normal probability curve:

The normal probability curve is given by the equation

$$f(x)=\frac{1}{\sigma_X\sqrt{2\pi}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} \quad ;-\infty < x < \infty$$

I.A.b.iii.Properties:

1.The point of inflexion of the curve are:

$$x=\mu_X \pm \sigma_X \ ,\ \text{f(x)} = \frac{1}{\sigma_X\sqrt{2\pi}}e^{-1/2}$$

2. The curve is symmetrical and bell shaped about the line x = μ.

3. Mean, Median, Mode of distribution coincide.

4. X-axis is an asymptote to the curve.

5. Quartiles, $Q_1 = \mu - 0.6745\sigma$

$Q_3 = \mu + 0.6745\sigma$.

6. Q.D : M.D : S.D :: ⅔ : ⅘ : 1 that implies Q.D : M.D: S.D :: 10 :12 : 15

II. Neutrosophic Statistical Distribution:

II.i. Neutrosophic Binomial Distribution: The neutrosophic binomial random variable ‘x’ is then defined as the number of success when we perform the experiment n ≥ 1 times. The neutrosophic probability distribution of ‘x’ is also called neutrosophic binomial probability distribution.

II.i.a.Definitions:

1. **Neutrosophic Binomial Random Variable**: It is defined as the number of success when we perform the experiment n ≥ 1 times, and is denoted as ‘x’.
2. **Neutrosophic Binomial Probability Distribution:** The neutrosophic probability distribution of ‘x’ is called n.p.d.
3. **Indeterminacy:** It is not confined to experimental results (either success or failures).
4. **Indeterminacy Threshold**: It is the number of trials whose outcome is indeterminate. Where

th∈{0,1,2…n}

Let P(S) = The chance of a particular trial results in a success.

P(F) = The chance of a particular trial results in a failure , for both S and f different from indeterminacy .

P(I) = The chance of a particular trial results in an indeterminacy .

For example: for x∈ {0,1,2,…,n} , NP =(TX,IX,FX) with

TX : Chances of 'x' success and (n-x) failures and indeterminacy but such that the no. of indeterminacy is less than or equal to indeterminacy threshold.

FX : Chances of 'y' success , with y≠x and (n-y) failures and indeterminacy is less than the indeterminacy threshold.

IX : Chances of 'z' indeterminacy , where 'z' is strictly greater than thee indeterminacy threshold.

$$TX + FX + IX = (P(S) + P(I) + P(F))^n$$

For complete probability, P(S) + P(I) + P(F) = 1 ;

For incomplete probability ,

$$0 \le P(S) + P(I) + P(F) < 1 \;;$$

For paraconsistent probability ,

$$1 < P(S) + P(I) + P(F) \le 3 \;.$$

Now ,

$$Tx = \frac{n!}{x!(n-x)!}[P(S)^x \sum_{k=0}^{th} \frac{k!}{(n-x)!(k-n+x)!} P(I)^K P(F)^{n-x-k}$$

$$= \frac{n!}{x!(n-x)!} P(S)^x \sum_{k=0}^{th} \frac{(n-x)!}{(n-x-k)!} P(I)^k P(F)^{n-x-k}$$

$$= \frac{n!}{x!} P(S)^x . \sum_{k=0}^{th} \frac{P(I)^k . P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$Fx = \sum_{y=0}^{n} T_y = \sum_{y=0, y\neq x} \frac{n!}{y!} P(S)^y \sum_{k=0}^{th} \frac{P(S)^k . P(F)^{n-y-k}}{k!(n-y-k)!}$$

$$Ix = \sum_{z=th+1}^{n} \frac{n!}{z!(n-z)!} P(I)^z . \sum_{k=0}^{n-z} \frac{(n-z)!}{(n-z)!(n-z-k)!} P(S)^k . P(F)^{n-z-k}$$

$$= \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k . P(F)^{n-z-k}}{k!(n-z-k)!}$$

Where ,

Tx , Ix , Fx ,P(S) ,P(I) , P(F) have their usual meaning. Now we are going to discuss several cases.

II.i.b.1. Case studies :

1. Two friends Asish and Rajesh are going to throw 5 coins simultaneously. There are 60% of chance to get head and 30% of chance to get tail. Independent on the view of Asish ,Rajes said that the probability of the result that are neither Head nor Tail is 20% . Then find the probability of getting 3 Heads when indeterminacy threshold is 2.

Solution:

$$Tx = \frac{5!}{3!(5-3)!}[(0.6)^3 \sum_{k=0}^{2} \frac{k!}{2!(k-2)!}(0.2)^k (0.3)^{2-k}]$$

$$= \frac{5!}{3!(5-3)!}[(0.6)^3 \{\frac{2!}{2!}(0.2)^2\}]$$

$$= 10[(0.6)^3 \{(0.2)^2\}] = 0.0864$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k . P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$\therefore \; I_3 = \sum_{z=3}^{5} \frac{5!}{3!}(0.2)^z \sum_{k=0}^{2} \frac{(0.6)^k (0.3)^{2-k}}{k!(2-k)!}$$

$$= \sum_{z=3}^{5} \frac{5!}{3!}(0.2)^z \{\frac{(0.3)^2}{2!} + (0.6)(0.3) + \frac{(0.6)^2}{2!}\}$$

$$= \sum_{z=3}^{5} \frac{5!}{3!}(0.2)^z \{\frac{(0.3)^2}{2!} + (0.6)(0.3) + \frac{(0.6)^2}{2!}\}$$

$$= \{0.324 + 0.072 + 0.1008\} = 0.496$$

$$F_x = (P(S)+P(I)+P(F))^n - Tx - Fx$$

$$\therefore F_3 = (0.6 + 0.3 + 0.2)^5 - 0.0864 - 0.496$$

$$= 1.02811$$

2. Five coins are thrown simultaneously , the probability of success is 1/3 and the indeterminacy (the surface is very rough , so the coins may stand up) is 1/3 . Then find the probability of getting 3 Heads when the indeterminacy threshold is 2.

Solution:

Let x be no. of chances of getting heads in 5 trials .

$$T_x = \frac{n!}{x!} P(S)^x \sum_{k=0}^{th} \frac{P(I)^k P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{5!}{3!}(0.33)^3 \sum_{k=0}^{2} \frac{(0.33)^k (0.33)^{2-k}}{k!(2-k)!}$$

$$= \frac{5!}{3!}(0.33)^3 \{\frac{(0.33)^2}{2!} + (0.33)^2 + \frac{(0.33)^2}{2!}\}$$

$$=40.(0.33)^5 = 0.15654$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(2-k)!}$$

$$\therefore I_3 = \sum_{z=3}^{5} \frac{5!}{3!}(0.33)^3 . \sum_{k=0}^{2} \frac{(0.33)^k (0.33)^{2-k}}{k!(2-k)!}$$

$$=(0.33)^5[40 + (7.5)(0.33) + 1] = 0.17014$$

$$F_x = (P(S) + P(I) + P(F))^n - T_x - I_x$$

$$\therefore F_3 = (0.33 + 0.33 + 0.33)^5 - T_x - I_x$$

$$so, (T_x, I_x, F_x) = (0.15654, 0.17014, 0.67332)$$

3. Two friends Liza and Laxmi play a game in which their chance of winning is 2:3 . The chances of dismissing game is 30% . Then find the probability of Liza's chances of winning at least 3 games out of 5 games played when the indeterminacy threshold is 2.

solution:

x *is* the no. of chances of winning the game

Let th = 2

$$T_X = \frac{n!}{x!} P(S)^X . \sum_{k=0}^{th} \frac{P(I)^k . P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{5!}{3!}(0.4)^3 . \sum_{k=0}^{th} \frac{(0.3)^k (0.6)^{2-k}}{k!(2-k)!}$$

$$=20(0.405)^4 = 0.53808$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^Z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$I_3 = \sum_{Z=3}^{5} \frac{5!}{z!}(0.3)^z . \sum_{k=0}^{n-z} \frac{(0.4)^k (0.6)^{5-z-k}}{k!(5-z-k)!}$$

$$=20(0.3)^3(0.5) + 5(0.3)^4(0.42)$$

$$=0.28701$$

and $F_3 = 2.88785$, *this is* a paraconsistent probability which is ≤ 3 .

4. In a precision bombing attack there is a 50% chance that any one bomb will strike the target . Two direct hits are required to destroy the target . If the chance of failure of mission is 30% , then find how many bombs are required to give a 99% chance with th=2 .

Solution:

Let x is the no. of chances of hitting bomb

$$T_x = \frac{n!}{x!} P(S)^x . \sum_{k=0}^{th} \frac{P(I)^k P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{5!}{3!}(0.5)^3 . \sum_{k=0}^{2} \frac{(0.3)^k (0.3)^{2-k}}{k!(2-k)!}$$

$$=40(0.5)^5 = 0.0972$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^Z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$\therefore I_3 = \sum_{z=3}^{5} \frac{5!}{z!}(0.3)^z . \sum_{k=0}^{n-z} \frac{(0.5)^k (0.3)^{5-z-k}}{k!(5-z-k)!}$$

$$=0.1728+0.0078975=0.18069$$

therefore $F_3 = (P(S) + P(F) + P(I))^5 - T_3 - I_3$

$$=(1.1)^5 - 0.27789 = 1.33262$$

so $(T_{x=3}, F_{x=3}, I_{x=3}) = (0.0972, 1.3326, 0.1806)$.

It is an example of paraconsistent probability.

5. It is decided that a cricket player , Jagadiswar has to appear 4 times for a physical test . If the possibility of passing the test is 2/3 ; and one referee guess that the chance of dismiss of game is 30% , then what is the probability of that the player passes the test at least 3 times, provided th=2?

Solution:

Let x *is* no. of chances that the player passes the test

$$T_x = \frac{n!}{x!} P(S)^x . \sum_{k=0}^{th} \frac{P(I)^k P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{4!}{3!}(0.66)^3 . \sum_{k=0}^{2} \frac{(0.3)^k (0.33)^{1-k}}{k!(1-k)!}$$

$$=8(0.66)^3(0.33) = 0.7589$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$\therefore I_3 = \sum_{z=3}^{4} 4(0.33)^z . \sum_{k=0}^{1} \{(0.66)^k (0.33)^{1-k}\} / \{k!(1-k)!\}$$

$$=(0.33)^3[3.96 + 0.33] = 0.15416$$

therefore $F_3 = (0.66 + 0.33 + 0.3)^4 - T_3 - I_3$

$$=2.76922 - 0.91306 = 1.85616$$

SO $(T_3, I_3, F_3) = (0.7589, 0.1541, 1.8561)$.

II.i.b.2.Exercises:

1. In a B.Sccourse , suppose that a student has to pass a minimum of 4 tests out of 8 conducted tests during the year to get promoted to next academic year . One student, Sarmistha says that his chance of winning is 80% , another student, Baisakhi

says that his chance of winning is 0.3 . Then find the probability of the promotion of Sarmistha , when the indeterminacy (either illegal paper correction or system error) is 20% , provided th=2.

2. If a car agency sells 40% of its inventory of a certain foreign cars equipped with air bags , the asst. manager says that the cars which are neither equipped with air bags nor a general one is 30% , then find theprobability distribution of the 2 cars with airbags among the next 4 cars sold with th=2?

3. A question paper contain 5 questions and a candidate will be declared to have passed the exam. If he/she answered at least one question correctly, considering the uncertainty as 33% (may be improper paper correction or system error etc.). What is the probability that the candidate passes the examination?

II.ii.Neutrosophic Normal Distribution:

Neutrosophic normal distribution of a continuous variable X is a classical normal distribution of X, but such that its mean μ or its standard deviation σ or variance σ^2 or both are imprecise. For example , μ or σ or both can be set with two or more elements . The most common such distribution are when μ, σ or both are intervals .

$$X_N \sim N_N(\mu_N,\sigma_N) = \frac{1}{\sigma_N\sqrt{2\pi}} e^{\left(\frac{(X-\mu_N)^2}{2\sigma_N^2}\right)}$$

N_N : *Normal* distribution may be neutrosophic

X_N : X may be neutrosophic

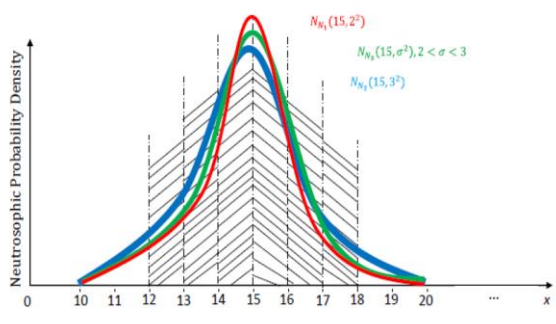

FIG-1: CREDIT TO FLORENTIN SMARANDACHE IN NEUTROSOPHIC STATISTICS

II.ii.a. Case studies:

1. In a college examination of a particular year, 60% of the Student failed when the mean of marks was 50% and the standard deviation is 5% with uncertainty I∈ [0,0.4] .The college decided to relax the condition of passing by lowering the passing marks to show its result as 80% passed , find the minimum marks to be kept for passing when marks are distributed normally .

Solution : Let μ = 50 , σ = 5 with indeterminacy I∈[0,0.4] ,so σ = 5 + [0,0.4] = [5,5.4] . therefore,μ±σ =50 ± [5,5.4] = [50-5.4,50+5] = [44.6 ,55] . Thus, 66.04 % of values lies in [44.6,55] .

$$0.8 = P(X_N \geq a_N)$$
$$=1\text{-}P(X_N \leq \alpha_N)$$
$$=1\text{-}P\left(\frac{X_N-\mu_N}{\sigma_N} \leq \frac{\alpha_N-\mu_N}{\sigma_N}\right) = 1 - P\left(Z_N \leq \frac{\alpha_N-50}{[5,5.4]}\right)$$
$$\therefore P\left(Z_N \leq \frac{\alpha\ \ -50}{[5,5.4]}\right) = 0.2, clearly \frac{a_N-50}{[5,5.4]} < 0, Let\ so, P(Z_N \leq Z_{0.2}) = 0.8$$
$$Z_{O.2} = -\left(\frac{\alpha_N-50}{[5,5.4]}\right) = 48.45\%\ approx.$$

2. If the monthly machine repair and maintenance cost X in a certain factory is known to be neutrosophically normal with mean 1000 and standard deviation 10000 , find the followings-

μ±σ,μ±2σ,when I∈[0,0.3].

Solution: Let μ=10000 ,σ=1000+[0,0.3],then μ±σ=10000±[1000,1000.03] . Thus 66.06% of values lies in [9000.03,11000]. And μ±2σ=10000±2[1000,1000.03]=[7999.97,12000]. Thus 75.04% of values lies in [7999.97,12000].

II.ii.b. Exercises:

1. A machine fills boxes weighting B kg with A kg of salt , where A and B are neutrosophically normal with mean 200kg and 10kg respectively and standard deviation of 2kg and 1kg respectively , what percentage of filled boxes weighting between 110kg an 120kg are to be expected when I∈[0,0.5].

2. The average life of a bulb is 2000 hours and the standard deviation is 400 hours .If X_N is the life period of a bulb which is distributed

normally in a neutrosophic plane. Find the probability that a randomly picked bulb will lasts ≤600 hrs. , considering the distribution is neutrosophically normal with indeterminacy I∈[0,0.2].

Till now, we have discussed various types of practical cases in statistical approach. Now we review the general formula for fusioning classical subjective probability provided by 2 sources.

The principle of redistributing the conflicting chances for ex. t and I are same as in PCR5 rule for the DSmT used in information fusion if 2 sources of information S_1 *and* S_2 give the subjective probability P_1 and P_2 about 't' to combining by PCR5 rule **,[5]**

$$(P_1 \wedge P_2)E = P_1(E)P_2(E) + \sum_{x \in E \wedge x \cap E = \phi} [\frac{P_1(E)^2 P_2(X)}{P_1(E)+P_2(X)} + \frac{P_2(E)^2 P_1(X)}{P_2(E)+P_2(X)}]$$

It helps to the generalization of classical probability theory, fuzzy set, fuzzy logic to their respective domains. They are useful in artificial intelligence, neutrosophic dynamic system, quantum mechanics **[6].**

This theory can be used for topical communication study **[7].** It may also be applied to neutrosophic cognitive map study **[8].**

Thus we have presented our discussion with certain essential area of neutrosophy in a synchronized manner. Now we are going to explore some open challenges as follows.

Which are designed for inquiring minds.

Open Problems:

1. Can this Neutrosophic Statistics be applied to Industrial Management study?
2. Can we apply it with the study of Digital Signal Processing?
3. Can we merge the Representation theory [9]with Neutrosophy for a new theory ?
4. Is the uncertain theory, K-theory [10]solve the recent intriguing statistical problems by the power of this Neutrosophic logic ?
5. Can we construct a special master-space by the fusion of manifold concepts [11], soft topology [12], Ergodic theory [13],with Neutrosophic distribution ?
6. Is it possible for the construction of Neutrosophic manifold?
7. Is it possible for the construction of neutrosophic algebraic geometry[14] ?

III. **Conclusion:**

The actual motto of this short paper is to present the theory of Neutrosophic probability distribution in a more lucid and clear-cut way .The author presents various solved and unsolved problems, which are existed in reference to Neutrosophic 3D- space .Various practical situations are described and were tried to solve by Neutrosophic logic. The spectra of this theory may be applied to Quantum physics [15]and M-theory [16]. It may be said that it can also be applied to Human psychology as well as Behavioral study. I hope that the more extended version (with large no. of case studies) with the area of application of this theory will see the light of the day in recent future. Here we limited our discussion of problem analysis to some extent due to limited scope of presentation. And lastly but important that if some unmatched/contradicted idea will occur in this paper, then it is surely unintentional. Finally I hope that the idea on the advanced version of this theory,which is already raised in my brain, will change their abstractskeleton into a paper, in coming future.

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Standard Neutrosophic Soft Theory: Some First Results

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Abstract. The traditional soft set is a mapping from a parameter set to family of all crisp subsets of a universe. Molodtsov introduced the soft set as a generalized tool for modelling complex systems involving uncertain or not clearly defined objects. In this paper, the notion of neutrosophic soft set is reanalysed. The novel theory is a combination of neutrosophic set theory and soft set theory. The complement, "and", "or", intersection and union operations are defined on the neutrosophic soft sets. The neutrosophic soft relations accompanied with their compositions are also defined. The basic properties of the neutrosophic soft sets, neutrosophic soft relations and neutrosophic soft compositions are also discussed.

Keywords: Soft sets, Fuzzy soft sets, Intuitionistic fuzzy soft sets, Neutrosophic soft sets, Neutrosophic soft relations

1 Introduction

Uncertain data modelling is a complex problem appearing in many areas such as economics, engineering, environmental science, sociology and medical science. Some mathematical theories such as probability, fuzzy set [1], [2], intuitionistic fuzzy set [3], [4], rough set [5], [6], and the interval mathematics [7], [8] are useful approaches to describing uncertainty. However each of these theories has its inherent difficulties as mentioned by Molodtsov [9]. Soft set theory developed by Molodtsov [9] has become a new useful approach for handling vagueness and uncertainty.

Later, Maji et al. [10] introduced several basic operations of soft set theory and proved some related propositions on soft set operations. Ali et al. [11] analysed the incorrectness of some theorems in [10]. Then they proposed some new soft set operations and proved that De Morgan's laws hold with these new definitions. Maji et al. also [12] gave an application of soft set theory in a decision making problem.

Above works are based on classical soft set. However, in practice, the objects may not precisely satisfy the problems' parameters, thus Maji et al. [13] put forward the concept of fuzzy soft set by combining the fuzzy set and the soft set, then they [14] presented a theoretical approach of the fuzzy soft set in decision making problem. In [15], they considered the concept of intuitionistic fuzzy soft set. By combining the interval-valued fuzzy set and soft set, Yang et al. [16] proposed the interval-valued fuzzy soft set and then analyzed a decision making problem in the interval-valued fuzzy soft set. Yang et al [17] presented the concept of interval-valued intuitionistic fuzzy soft sets which is an interval-valued fuzzy extension of the intuitionistic fuzzy soft set theory.

From philosophical point of view, Smarandache's neutrosophic set [26] generalizes fuzzy set and intuitionistic fuzzy set. However, it is difficult to apply it to the real applications and needs to be specified. Wang et al. [27] proposed interval neutrosophic sets and some operators of then. Wang et al. [28] proposed a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [29] defined the concept of simplified neutrosophic sets, which can be described by three real numbers in the real unit interval $[0,1]$, and some operational laws for simplified neutrosophic sets and to propose two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. In 2013 [18], we presented the definition of picture fuzzy sets, which is a generalization of the Zadeh's fuzzy sets and Atanassov's intuitionistic fuzzy sets, and some basic operations on picture fuzzy sets. In [18] we also discussed some properties of these operations, then the definition of the Cartesian product of picture fuzzy sets and the definition of picture fuzzy relations were given. Our picture fuzzy set turns out a special case of neutrosophic set. Thus, from now on, we also regard picture fuzzy set as standard neutrosophic set.

The purpose of this paper is to combine the standard neutrosophic sets and soft models, from which we can obtain neutrosophic soft sets. Intuitively, the neutrosophic soft set presented in this paper is an extension of the intuitionistic fuzzy soft sets [13][15].

The rest of this paper is organized as follows. Section 2 briefly reviews some background on soft sets, fuzzy soft sets, intuitionistic soft sets as well as neutrosophic set. In Section 3, we recall the concept of the standard neutrosophic sets (SNSs) with some operations on SNSs, then we present the concept of neutrosophic soft sets (NSSs) with some operations. Some properties of these operations are discussed in the Sub-section 3.3. Sub-section 3.4 is devoted to the Cartesian product of NSSs. The neutrosophic soft relations are presented in Section 4. Finally, in Section 5, we draw the conclusion and present some topics for future research.

2 Preliminaries

In this section, we briefly recall the notions of soft sets, fuzzy soft sets, intuitionistic fuzzy soft sets as well as neutrosophic sets. See especially [9][10][13][15] for further details and background.

2.1 Soft sets and some extensions

Molodtsov [8] defined the soft set in the following way. Let U be an initial universe of objects and E be the set of related parameters of objects in U. Parameters are often attributes, characteristics, or properties of objects. Let $P(U)$ denotes the power set of U and $A \subseteq E$.

Definition 2.1. [8] A pair (F, A) is called a soft set over U**,** where F is a mapping given by $F : A \to P(U)$.

In other words, the soft set is not a kind of set, but a parameterized family of subsets of U [9][10][16]. For any parameter $e \in E$, $F(e) \subseteq U$ is considered as the set of e-approximate elements of the soft set (F, A).

Maji et al. [13] initiated the study on hybrid structures involving both fuzzy sets and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of (crisp) soft set.

Definition 2.2 [13] Let $\mathcal{F}(U)$ be the set of all fuzzy subsets of U, E be the set of parameters and $A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F : A \to \mathcal{F}(U)$.

It is easy to see that every (crisp) soft set can be considered as a fuzzy soft set. Generally speaking, for any parameter $e \in E$, $F(e)$ is a fuzzy subset of U and it is called fuzzy value set of parameter e. If for any parameter $e \in A$, $F(e)$ is a subset of U, then (F, A) is degenerated to the standard soft set. For all $x \in U$ and $e \in E$, let us denote by $\mu_{F(e)}(x)$ the membership degree that the object x holds parameter e. So then $F(e)$ can be written as

$$F(e) = \left\{ \left\langle x, \mu_{F(e)}(x) \right\rangle \middle| x \in U \right\}.$$

Before introduce the notion of the intuitionistic fuzzy soft set, let us recall the concept of intuitionistic fuzzy set [3], [4].

Let X be a fixed set. An intuitionistic fuzzy set (IFS) in X is an object having the form

$$A = \left\{ \left\langle x, \mu_A(x), \nu_A(x) \right\rangle \middle| x \in X \right\},$$

where $\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$ respectively define the degree of membership and the degree of non-membership of the element x to the set A such that $\mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$. The set of all IFSs on X is denoted by $IFS(X)$.

In [15] Maji et al. proposed the concept of intuitionistic fuzzy soft set as follows.

Definition 2.3 [15] Let E the set of parameters and $A \subseteq E$. A pair (F, A) is called a intuitionistic fuzzy soft set over U, where F is a mapping $F : A \to IFS(U)$.

Clearly, for any parameter $e \in E$, $F(e)$ is an IFS

$$F(e) = \left\{ \left\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x) \right\rangle \middle| x \in U \right\},$$

where $\mu_{F(e)}$ and $\nu_{F(e)}$ are the membership and non-membership functions, respectively. If for any parameter $e \in A$, $\nu_{F(e)}(x) = 1 - \mu_{F(e)}(x)$, then $F(e)$ is a fuzzy set and (F, A) is reduced to a fuzzy soft set.

2.2 Neutrosophic sets

Definition 2.4 [26] A neutrosophic set A in a on a universe X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. For each $x \in X$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, that is T_A, I_A and F_A : $X \to]0^-, 1^+[$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$, for all $x \in X$.

Definition 2.5 [26] The complement of a neutrosophic set A is denoted by A^c and is defined as $T_{A^c}(x) = \{1^+\} \ominus T_A(x)$, $I_{A^c}(x) = \{1^+\} \ominus I_A(x)$, and $F_{A^c}(x) = \{1^+\} \ominus F_A(x)$ for every x in X.

Definition 2.6 [26] A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$ if and only if

$\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$,

$\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$,

$\inf F_A(x) \ge \inf F_B(x)$, and $\sup F_A(x) \ge \sup F_B(x)$ for every x in X.

Definition 2.7 [26] The union of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy membership and false-membership functions are related to those of A and B by

$T_C(x) = T_A(x) \oplus T_B(x) \ominus T_A(x) \odot T_B(x)$,

$I_C(x) = I_A(x) \oplus I_B(x) \ominus I_A(x) \odot I_B(x)$, and

$F_C(x) = F_A(x) \oplus F_B(x) \ominus F_A(x) \odot F_B(x)$ for any x in X.

Definition 2.8 [1] The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by $T_C(x) = T_A(x) \odot T_B(x)$, $I_C(x) = I_A(x) \odot I_B(x)$, and $F_C(x) = F_A(x) \odot F_B(x)$ for any x in X.

Definition 2.9 [29] Consider a neutrosophic set A in X characterized by a truth-membership function T_A, a indeterminacy-membership function I_A and a falsity - membership function F_A. If $T_A(x)$, $I_A(x)$ and $F_A(x)$ are singleton values in the real standard $[0,1]$ for every x in X, that is T_A, I_A and F_A : $X \to [0,1]$. Then, a simplification of the neutrosophic set A is denoted by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

which is called a simplified neutrosophic set.

3 Neutrosophic soft sets

In this section, first we recall the definition of the standard neutrosophic sets (SNSs), some basic operations with their properties, then we will present the neutrosophic soft set theory which is a combination of neutrosophic set theory and a soft set theory.

3.1 Standard neutrosophic sets

Intuitionistic fuzzy sets introduced by Atanassov in 1983 constitute a generalization of fuzzy sets (FS) [3]. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give a degree of membership and a degree of non-membership of an element in a given set.

A generalization of fuzzy sets and intuitionistic fuzzy sets are the following notion of standard neutrosophic set (SNS).

Definition 3.1 [18] A SNS A on a universe X is an object of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called the "degree of positive membership of x in A", $\eta_A(x) \in [0,1]$ is called the "degree of neutral membership of x in A" and $\nu_A(x) \in [0,1]$ is called the "degree of negative membership of x in A", and μ_A, η_A and ν_A satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \le 1, \ \forall x \in X.$$

The expression $(1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)))$ is termed as "degree of refusal membership" of x in A.

Basically, SNSs based models may be adequate in situations when we face human opinions involving more answers of type: *yes*, *abstain*, *no* and *refusal*. Voting can be a good example of such a situation as the voters are divided into four groups: *vote for*, *abstain*, *vote against* and *refusal of the voting*.

Let $SNS(X)$ denote the set of all the standard neutrosophic set SNSs on a universe X.

Definition 3.2 [18] For A, $B \in SNS(X)$, the union, intersection and complement are defined as follows:

- $A \subseteq B \Leftrightarrow \begin{cases} \mu_A(x) \le \mu_B(x) \\ \eta_A(x) \le \eta_B(x) \\ \nu_A(x) \ge \nu_B(x) \end{cases}, \ \forall x \in X$;
- $A = B \Leftrightarrow \begin{cases} A \subseteq B \\ B \subseteq A \end{cases}$;
- $A \cup B \in SNS(X)$ with

  $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$,

  $\eta_{A\cup B}(x) = \min(\eta_A(x), \eta_B(x))$, and

  $\nu_{A\cup B}(x) = \min(\nu_A(x), \nu_B(x))$, $\forall x \in X$;
- $A \cap B \in SNS(X)$ with

  $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$,

  $\eta_{A\cap B}(x) = \min(\eta_A(x), \eta_B(x))$, and

  $\nu_{A\cap B}(x) = \max(\nu_A(x), \nu_B(x))$, $\forall x \in X$;
- $CoA = A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) | x \in X\}$.

In this paper, we denote $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$, for every a, $b \in \mathbb{R}$.

Definition 3.3 [18] Let X, Y be two universes and $A \in SNS(X)$, $B \in SNS(Y)$. We define the Cartesian product of these two SNSs by $A \times B \in SNS(X \times Y)$ such that

$\mu_{A\times B}(x, y) = \mu_A(x) \wedge \mu_B(y)$,

$\eta_{A\times B}(x, y) = \eta_A(x) \wedge \eta_B(y)$, and

$\nu_{A\times B}(x,y)=\nu_A(x)\vee\nu_B(y)$, $\forall(x,y)\in X\times Y$.

The validation of Definition 3.3 was shown in [18]. Now we consider some properties of the defined operations on SNSs.

Proposition 3.4 [18] For every A, B, $C\in SNS(X)$:

(a) If $A\subseteq B$ and $B\subseteq C$, then $A\subseteq C$;

(b) $\left(A^c\right)^c=A$;

(c) Operations $\cap$ and $\cup$ are commutative, associative and distributive;

(d) Operations $\cap$, *Co* and $\cup$ satisfy the law of De Morgan.

Proof. See [19][20] for detail proof. □

Convex combination is an important operation in mathematics, which is a useful tool on convex analysis, linear spaces and convex optimization. In this sub-section convex combination firstly is defined with some simple propositions.

Definition 3.5 [18] Let A, $B\in SNS(X)$. For each $\theta\in[0,1]$, the convex combination of A and B is defined as follows:

$$C_\theta(A,B)=\left\{\left(x,\mu_{C_\theta}(x),\eta_{C_\theta}(x),\nu_{C_\theta}(x)\right)\middle|\,x\in X\right\},$$

where

$\mu_{C_\theta}(x)=\theta\mu_A(x)+(1-\theta)\mu_B(x)$,

$\eta_{C_\theta}(x)=\theta\eta_A(x)+(1-\theta)\eta_B(x)$, and

$\nu_{C_\theta}(x)=\theta\nu_A(x)+(1-\theta)\nu_B(x)$, $\forall x\in X$.

Proposition 3.6 [18] Let A, $B\in SNS(X)$ and θ, θ_1, $\theta_2\in[0,1]$, then

- If $\theta=1$, then $C_\theta(A,B)=A$; and if $\theta=0$, then $C_\theta(A,B)=B$;
- If $A\subseteq B$, then $A\subseteq C_\theta(A,B)\subseteq B$;
- If $B\subseteq A$ and $\theta_1\le\theta_2$, then $C_{\theta_1}(A,B)\subseteq C_{\theta_2}(A,B)$.

3.2 Neutrosophic soft sets

Definition 3.7 Let $SNS(U)$ be the set of all standard neutrosophic sets of U, E be the set of parameters and $A\subseteq E$. A pair (F,A) is called a standard neutrosophic soft set (or neutrosophic soft set for short) over U, where F is a mapping given by $F:A\to SNS(U)$.

Clearly, for any parameter $e\in E$, $F(e)$ is a SNS:

$$F(e)=\left\{\left(x,\mu_{F(e)}(x),\eta_{F(e)}(x),\nu_{F(e)}(x)\right)\middle|\,x\in U\right\},$$

where $\mu_{F(e)}$, $\eta_{F(e)}$ and $\nu_{F(e)}$ are positive membership, neutral membership and negative membership functions respectively. If for all parameter $e\in A$ and for all $x\in U$, $\eta_{F(e)}(x)=0$, then $F(e)$ will degenerated to be an intuitionistic fuzzy set and then (F,A) is degenerated to an intuitionistic fuzzy soft set.

We denote the set of all standard neutrosophic soft sets over U by $SNS(U)$.

Example 1. We consider the situation which involves four economic projects evaluated by a decision committee according to five parameters: *good finance indicator* (e_1), *average finance indicator* (e_2), *good social contribution* (e_3), *average social contribution* (e_4) and *good environment indicator* (e_5). The set of economic projects and the set of parameters are denoted $U=\{p_1,p_2,p_3,p_4\}$ and $A=\{e_1,e_2,e_3,e_4,e_5\}$, respectively. So, the attractiveness of the projects to the decision committee can be represented by a SNS (F,A):

$$F(e_1)=\begin{Bmatrix}(p_1,0.8,0.12,0.05),(p_2,0.6,0.18,0.16),\\(p_3,0.55,0.20,0.21),(p_4,0.50,0.20,0.24)\end{Bmatrix},$$

$$F(e_2)=\begin{Bmatrix}(p_1,0.82,0.05,0.10),(p_2,0.7,0.12,0.10),\\(p_3,0.60,0.14,0.10),(p_4,0.51,0.10,0.24)\end{Bmatrix},$$

$$F(e_3)=\begin{Bmatrix}(p_1,0.60,0.14,0.16),(p_2,0.55,0.20,0.16),\\(p_3,0.70,0.15,0.11),(p_4,0.63,0.12,0.18)\end{Bmatrix},$$

$$F(e_4)=\begin{Bmatrix}(p_1,0.7,0.12,0.07),(p_2,0.75,0.05,0.16),\\(p_3,0.60,0.17,0.18),(p_4,0.55,0.10,0.22)\end{Bmatrix},$$

$$F(e_5)=\begin{Bmatrix}(p_1,0.60,0.12,0.07),(p_2,0.62,0.14,0.16),\\(p_3,0.55,0.10,0.21),(p_4,0.70,0.20,0.05)\end{Bmatrix}.$$

The standard neutrosophic soft set (F,A) is a parameterized family $\left\{F(e_i)\middle|\,i=1,\ldots,5\right\}$ of standard neutrosophic sets over U.

Definition 3.8 1) For (F,A), $(G,B)\in SNS(U)$ over a common universe U, we say that (F,A) is a subset of (G,B), $(F,A)\subseteq(G,B)$, if the following conditions are satisfied:

(a) $A\subseteq B$;

(b) For all $e\in A$, $F(e)$ and $G(e)$ are identical approximations.

2) (F,A) is termed as a superset of (G,B), $(F,A)\supseteq(G,B)$, if (G,B) is a subset of (F,A).

3) (F,A) and (G,B) are called to be equal, $(F,A)=(G,B)$, if $(F,A)\subseteq(G,B)$ and $(G,B)\subseteq(F,A)$.

It is easy to show that $(F,A)=(G,B)$ iff $A=B$ and $F(e)=G(e)$ for all $e\in A$.

3.3 Some operations and properties

Now we define some operations on standard neutrosophic soft sets and present some properties.

Definition 3.9 The complement of a NSS (F,A), $(F,A)^c$, is defined by $(F,A)^c=(F^c,A)$, where $F^c:A\to P(U)$ is a mapping given by $F^c(e)=(F(e))^c$, for all $e\in A$.

Definition 3.10 If (F,A), $(G,B)\in NSS(U)$, then "$(F,A)\,and\,(G,B)$" is a NSS denoted by $(F,A)\wedge(G,B)$ and defined by $(F,A)\wedge(G,B)=(H,A\times B)$, where $H(\alpha,\beta)=F(\alpha)\cap G(\beta)$ for all $(\alpha,\beta)\in A\times B$, that is

$$\mu_{H(\alpha,\beta)}(x)=\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),$$

$$\eta_{H(\alpha,\beta)}(x)=\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right), \text{ and}$$

$$\nu_{H(\alpha,\beta)}(x)=\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\ \forall x\in U.$$

Definition 3.11 If (F,A), $(G,B)\in NSS(U)$, then "$(F,A)\,or\,(G,B)$" is a NSS denoted by $(F,A)\vee(G,B)$ and defined by $(F,A)\vee(G,B)=(H,A\times B)$, where $H(\alpha,\beta)=F(\alpha)\cup G(\beta)$ for all $(\alpha,\beta)\in A\times B$, that is

$$\mu_{H(\alpha,\beta)}(x)=\max\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),$$

$$\eta_{H(\alpha,\beta)}(x)=\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right), \text{ and}$$

$$\nu_{H(\alpha,\beta)}(x)=\min\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\ \forall x\in U.$$

Theorem 3.1 Let (F,A), $(G,B)\in NSS(U)$, then we have the following properties:

(1) $\left((F,A)\wedge(G,B)\right)^c=(F,A)^c\vee(G,B)^c$;

(2) $\left((F,A)\vee(G,B)\right)^c=(F,A)^c\wedge(G,B)^c$.

Proof. (1) Assume that $(F,A)\wedge(G,B)=(H,A\times B)$. Then

$$\left((F,A)\wedge(G,B)\right)^c=(H,A\times B)^c=(H^c,A\times B).$$

For any $(\alpha,\beta)\in A\times B$, $x\in U$, we have

$$\begin{aligned} H(\alpha,\beta)(x) \;=\; &\Big(\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),\\ &\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right)\\ &\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right)\Big),\end{aligned}$$

which implies

$$\begin{aligned} H^c(\alpha,\beta)(x) \;=\; &\Big(\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\\ &\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right)\\ &\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right)\Big).\end{aligned} \qquad (1)$$

On the other hand,

$$(F,A)^c\vee(G,B)^c=(F^c,A)\vee(G^c,B).$$

Let us assume that $(F^c,A)\vee(G^c,B)=(K,A\times B)$. We obtain

$$\begin{aligned} K(\alpha,\beta)(x) \;=\; &\Big(\max\left(\mu_{F^c(\alpha)}(x),\mu_{G^c(\beta)}(x)\right),\\ &\min\left(\eta_{F^c(\alpha)}(x),\eta_{G^c(\beta)}(x)\right)\\ &\min\left(\nu_{F^c(\alpha)}(x),\nu_{G^c(\beta)}(x)\right)\Big).\end{aligned}$$

Since $\mu_{F^c(\alpha)}=\nu_{F(\alpha)}$, $\eta_{F^c(\alpha)}=\eta_{F(\alpha)}$, $\nu_{F^c(\alpha)}=\mu_{F(\alpha)}$, $\mu_{G^c(\beta)}=\nu_{G(\beta)}$, $\eta_{G^c(\beta)}=\eta_{G(\beta)}$, $\nu_{G^c(\beta)}=\mu_{G(\beta)}$,

$$\begin{aligned} K(\alpha,\beta)(x) \;=\; &\Big(\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\\ &\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right)\\ &\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right)\Big).\end{aligned} \qquad (2)$$

Combining (1) and (2), the proof is completed.

(2) The proof is similar to (1). □

Theorem 3.2 Let (F,A), (G,B), $(H,C)\in NSS(U)$, then we have the following properties:

a) $(F,A)\wedge((G,B)\wedge(H,C))=((F,A)\wedge(G,B))\wedge(H,C)$;

b) $(F,A)\vee((G,B)\vee(H,C))=((F,A)\vee(G,B))\vee(H,C)$.

Proof. (1) Assume that

$$(G,B)\wedge(H,C)=(I,B\times C),$$

We have

$$\begin{aligned} I(\beta,\gamma)(x) &= \Big(\min\big(\mu_{G(\beta)}(x),\mu_{H(\gamma)}(x)\big),\\ &\quad \min\big(\eta_{G(\beta)}(x),\eta_{H(\gamma)}(x)\big)\\ &\quad \max\big(\nu_{G(\beta)}(x),\nu_{H(\gamma)}(x)\big)\Big), \end{aligned}$$

$\forall(\beta,\gamma)\in B\times C$, $x\in U$.

We assume that

$$(F,A)\wedge((G,B)\wedge(H,C))=(K,A\times B\times C).$$

In other words,

$$(K,A\times B\times C)=(F,A)\wedge(I,B\times C).$$

By definition of $\wedge$ operator for two NSSs,

$$\begin{aligned} K(\alpha,\beta,\gamma)(x) &= \Big(\min\big(\mu_{F(\alpha)}(x),\min\big(\mu_{G(\beta)}(x),\mu_{H(\gamma)}(x)\big)\big),\\ &\quad \min\big(\eta_{F(\alpha)}(x),\min\big(\eta_{G(\beta)}(x),\eta_{H(\gamma)}(x)\big)\big)\\ &\quad \max\big(\nu_{F(\alpha)}(x),\max\big(\nu_{G(\beta)}(x),\nu_{H(\gamma)}(x)\big)\big)\Big), \end{aligned}$$

or

$$\begin{aligned} K(\alpha,\beta,\gamma)(x) &= \Big(\min\big(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x),\mu_{H(\gamma)}(x)\big),\\ &\quad \min\big(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x),\eta_{H(\gamma)}(x)\big)\\ &\quad \max\big(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x),\nu_{H(\gamma)}(x)\big)\Big). \end{aligned}$$

By a similar argument, we get

$$((F,A)\wedge(G,B))\wedge(H,C)=(K,A\times B\times C).$$

This concludes the proof of a).

The proof of b) is analogous. □

Definition 3.12 The intersection of two NSSs (F,A), $(G,B)\in NSS(U)$, denoted by $(F,A)\cap(G,B)$, is a NSSs (H,C), where $C=A\cup B$ and for all $e\in C$,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B \\ G(e) & \text{if } e\in B\setminus A \\ F(e)\cap G(e) & \text{if } e\in A\cap B \end{cases}. \tag{3}$$

Definition 3.13 The union of two NSSs (F,A), $(G,B)\in NSS(U)$, denoted by $(F,A)\cup(G,B)$, is a NSSs (H,C), where $C=A\cup B$ and for all $e\in C$,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B \\ G(e) & \text{if } e\in B\setminus A \\ F(e)\cup G(e) & \text{if } e\in A\cap B \end{cases}. \tag{3}$$

Theorem 3.3. Let (F,A), $(G,B)\in NSS(U)$, then we have the following properties:

a) $((F,A)\cap(G,B))^c=(F,A)^c\cup(G,B)^c$;

b) $((F,A)\cup(G,B))^c=(F,A)^c\cap(G,B)^c$.

Proof. a) Assume that $(F,A)\cap(G,B)=(H,C)$, with $C=A\cup B$, then

$$((F,A)\cap(G,B))^c=(H,C)^c=(H^c,C).$$

By Definition 3.13,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B \\ G(e) & \text{if } e\in B\setminus A \\ F(e)\cap G(e) & \text{if } e\in A\cap B \end{cases}.$$

It implies

$$H^c(e)=\begin{cases} F^c(e) & \text{if } e\in A\setminus B \\ G^c(e) & \text{if } e\in B\setminus A \\ F^c(e)\cup G^c(e) & \text{if } e\in A\cap B \end{cases}. \tag{5}$$

Similarly, we denote $(F,A)^c\cup(G,B)^c=(K,C)$ with $C=A\cup B$. Since $(K,C)=(F^c,A)\cup(G^c,B)$,

$$K(e)=\begin{cases} F^c(e) & \text{if } e\in A\setminus B \\ G^c(e) & \text{if } e\in B\setminus A \\ F^c(e)\cup G^c(e) & \text{if } e\in A\cap B \end{cases}. \tag{6}$$

From (5) and (6), we get $H^c=K$. Hence,

$$\left((F,A)\cap(G,B)\right)^c=(F,A)^c\cup(G,B)^c.$$

b) Similarly, we have b). □

3.4 Cartesian product of neutrosophic soft sets

Definition 3.14 Let $O_1\in SNS(X_1)$ and $O_2\in SNS(X_2)$. The Cartesian product of these two NSSs is $O_1\times O_2\in SNS(X_1\times X_2)$ defined as

$$\mu_{O_1\times O_2}(x,y)=\mu_{O_1}(x)\wedge\mu_{O_2}(y),$$

$$\eta_{O_1\times O_2}(x,y)=\eta_{O_1}(x)\wedge\eta_{O_2}(y),\text{ and}$$

$$\nu_{O_1\times O_2}(x,y)=\nu_{O_1}(x)\vee\nu_{O_2}(y),\ \forall(x,y)\in X_1\times X_2.$$

It is easy to check the validation of Definition 3.15.

Theorem 3.4 For O_1, $O_2\in SNS(X_1)$, $O_3\in SNS(X_2)$, and $O_4\in SNS(X_3)$:

a) $O_1\times O_3=O_3\times O_1$;

b) $(O_1\times O_3)\times O_4=O_1\times(O_3\times O_4)$;

c) $(O_1\cup O_2)\times O_3=(O_1\times O_3)\cup(O_2\times O_3)$;

d) $(O_1\cap O_2)\times O_3=(O_1\times O_3)\cap(O_2\times O_3)$.

Proof. a) and b) are straightforward. We consider c) and d).

c) We have

$$\mu_{O_1\cup O_2}(x)=\mu_{O_1}(x)\vee\mu_{O_2}(x),$$

$$\eta_{O_1\cup O_2}(x)=\eta_{O_1}(x)\wedge\eta_{O_2}(x),\text{ and}$$

$$\nu_{O_1\cup O_2}(x)=\nu_{O_1}(x)\wedge\nu_{O_2}(x),\ \forall x\in X_1.$$

Thus,

$$\mu_{(O_1\cup O_2)\times O_3}(x,y)=\left(\mu_{O_1}(x)\vee\mu_{O_2}(x)\right)\wedge\mu_{O_3}(y),$$

$$\eta_{(O_1\cup O_2)\times O_3}(x,y)=\left(\eta_{O_1}(x)\wedge\eta_{O_2}(x)\right)\wedge\eta_{O_3}(y),\text{ and}$$

$$\nu_{(O_1\cup O_2)\times O_3}(x,y)=\left(\nu_{O_1}(x)\wedge\nu_{O_2}(x)\right)\vee\nu_{O_3}(y),$$

$$\forall(x,y)\in X_1\times X_2.$$

Using the properties of the operations $\wedge$ and $\vee$ we obtain

$$\begin{aligned}\mu_{(O_1\cup O_2)\times O_3}(x,y)&=\left(\mu_{O_1}(x)\wedge\mu_{O_3}(y)\right)\vee\left(\mu_{O_2}(x)\wedge\mu_{O_3}(y)\right)\\&=\mu_{O_1\times O_3}(x,y)\vee\mu_{O_2\times O_3}(x,y)\\&=\mu_{(O_1\times O_3)\cup(O_2\times O_3)}(x,y),\\ \eta_{(O_1\cup O_2)\times O_3}(x,y)&=\left(\eta_{O_1}(x)\wedge\eta_{O_3}(y)\right)\wedge\left(\eta_{O_2}(x)\wedge\eta_{O_3}(y)\right)\\&=\eta_{O_1\times O_3}(x,y)\wedge\eta_{O_2\times O_3}(x,y)\\&=\eta_{(O_1\times O_3)\cup(O_2\times O_3)}(x,y),\\ \nu_{(O_1\cup O_2)\times O_3}(x,y)&=\left(\nu_{O_1}(x)\vee\nu_{O_3}(y)\right)\wedge\left(\nu_{O_2}(x)\vee\nu_{O_3}(y)\right)\\&=\nu_{O_1\times O_3}(x,y)\wedge\nu_{O_2\times O_3}(x,y)\\&=\nu_{(O_1\times O_3)\cup(O_2\times O_3)}(x,y),\forall(x,y)\in X_1\times X_2.\end{aligned}$$

The proof is given.

d) The proof of d) is analogous. □

Now we give the definition of the Cartesian product of neutrosophic soft sets.

Definition 3.15 Let X_1, X_2 be two universes, E be the set of parameters, A, $B\subseteq E$. Then the Cartesian product of $\langle F,A\rangle\in NSS(X_1)$ and $\langle G,B\rangle\in NSS(X_2)$ is denoted by $\langle F,A\rangle\times\langle G,B\rangle$ and defined by $\langle H,A\times B\rangle$, where

$$\begin{aligned}H(\alpha,\beta)(x,y)=&\Big(\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(y)\right),\\&\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(y)\right),\\&\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(y)\right)\Big),\end{aligned}$$

$\forall(\alpha,\beta)\in A\times B$, $\forall(x,y)\in X_1\times X_2$.

Theorem 3.5 Let X_1, X_2, X_3 be three universes, E be the set of parameters, A_1, A_2, B, $D\subseteq E$. For $\langle F_1,A_1\rangle$, $\langle F_2,A_2\rangle\in NSS(X_1)$, $\langle G,B\rangle\in NSS(X_2)$ and $\langle H,D\rangle\in NSS(X_3)$, we have:

a) $\langle F_1,A_1\rangle\times\langle G,B\rangle=\langle G,B\rangle\times\langle F_1,A_1\rangle$;

b) $\left(\langle F_1,A_1\rangle\times\langle G,B\rangle\right)\times\langle H,D\rangle$

$=\langle F_1,A_1\rangle\times\left(\langle G,B\rangle\times\langle H,D\rangle\right)$;

c) $\left(\langle F_1,A_1\rangle\cup\langle F_2,A_2\rangle\right)\times\langle G,B\rangle$

$=\left(\langle F_1,A_1\rangle\times\langle G,B\rangle\right)\cup\left(\langle F_2,A_2\rangle\times\langle G,B\rangle\right)$;

d) $\left(\langle F_1,A_1\rangle\cap\langle F_2,A_2\rangle\right)\times\langle G,B\rangle$

$$=\left(\langle F_1,A_1\rangle\times\langle G,B\rangle\right)\cap\left(\langle F_2,A_2\rangle\times\langle G,B\rangle\right).$$

Proof. The proof of a) and b) is omitted.

c) Use Definition 3.14, if $\langle F',A_1\cup A_2\rangle=\langle F_1,A_1\rangle\cup\langle F_2,A_2\rangle$, then for all $\alpha\in A_1\cup A_2$:

$$H'(\alpha)=\begin{cases}F_1(\alpha) & \text{if } \alpha\in A_1\setminus A_2\\ F_2(\alpha) & \text{if } \alpha\in A_2\setminus A_1\\ F_1(\alpha)\cup F_2(\alpha) & \text{if } \alpha\in A_1\cap A_2\end{cases}.$$

Let assume that $\langle K,(A_1\cup A_2)\times B\rangle=\langle F',A_1\cup A_2\rangle\times\langle G,B\rangle$. For all $(x,y)\in X_1\times X_2$, there are following three cases :

* Case 1: $(\alpha,\beta)\in(A_1\setminus A_2)\times B$.

$$\mu_{K(\alpha,\beta)}(x,y)=\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\beta)}(y)\right),$$

$$\eta_{K(\alpha,\beta)}(x,y)=\min\left(\eta_{F_1(\alpha)}(x),\eta_{G(\beta)}(y)\right), \text{ and}$$

$$\nu_{K(\alpha,\beta)}(x,y)=\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\beta)}(y)\right).$$

* Case 2: $(\alpha,\beta)\in(A_2\setminus A_1)\times B$.

$$\mu_{K(\alpha,\beta)}(x,y)=\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\beta)}(y)\right),$$

$$\eta_{K(\alpha,\beta)}(x,y)=\min\left(\eta_{F_2(\alpha)}(x),\eta_{G(\beta)}(y)\right), \text{ and}$$

$$\nu_{K(\alpha,\beta)}(x,y)=\max\left(\nu_{F_2(\alpha)}(x),\nu_{G(\beta)}(y)\right).$$

* Case 3: $(\alpha,\beta)\in(A_2\cap A_1)\times B$.

$$\mu_{K(\alpha,\beta)}(x,y)=\min\left(\mu_{F_1(\alpha)\cup F_2(\alpha)}(x),\mu_{G(\beta)}(y)\right)$$

$$=\min\left(\max\left(\mu_{F_1(\alpha)}(x),\mu_{F_2(\alpha)}(x)\right),\mu_{G(\beta)}(y)\right)$$

$$=\max\left(\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\beta)}(y)\right),\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\beta)}(y)\right)\right),$$

$$\eta_{K(\alpha,\beta)}(x,y)=\min\left(\eta_{F_1(\alpha)\cup F_2(\alpha)}(x),\eta_{G(\beta)}(y)\right)$$

$$=\min\left(\min\left(\eta_{F_1(\alpha)}(x),\eta_{F_2(\alpha)}(x)\right),\eta_{G(\beta)}(y)\right)$$

$$=\min\left(\eta_{F_1(\alpha)}(x),\eta_{F_2(\alpha)}(x),\eta_{G(\beta)}(y)\right), \text{ and}$$

$$\nu_{K(\alpha,\beta)}(x,y)=\max\left(\nu_{F_1(\alpha)\cup F_2(\alpha)}(x),\nu_{G(\beta)}(y)\right)$$

$$=\max\left(\min\left(\nu_{F_1(\alpha)}(x),\nu_{F_2(\alpha)}(x)\right),\nu_{G(\beta)}(y)\right)$$

$$=\min\left(\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\beta)}(y)\right),\max\left(\nu_{F_2(\alpha)}(x),\nu_{G(\beta)}(y)\right)\right).$$

Let us denote $\langle H_1,A_1\times B\rangle=\langle F_1,A_1\rangle\times\langle G,B\rangle$ and $\langle H_2,A_2\times B\rangle=\langle F_2,A_2\rangle\times\langle G,B\rangle$. We have:

$$\mu_{H_1(\alpha,\beta)}(x,y)=\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\alpha)}(y)\right),$$

$$\eta_{H_1(\alpha,\beta)}(x,y)=\min\left(\eta_{F_1(\alpha)}(x),\eta_{G(\alpha)}(y)\right),$$

$$\nu_{H_1(\alpha,\beta)}(x,y)=\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\alpha)}(y)\right),$$

$$\forall(\alpha,\beta)\in A_1\times B \text{ and } (x,y)\in X_1\times X_2;$$

$$\mu_{H_2(\alpha,\beta)}(x,y)=\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\alpha)}(y)\right),$$

$$\eta_{H_2(\alpha,\beta)}(x,y)=\min\left(\eta_{F_2(\alpha)}(x),\eta_{G(\alpha)}(y)\right),$$

$$\nu_{H_2(\alpha,\beta)}(x,y)=\max\left(\nu_{F_2(\alpha)}(x),\nu_{G(\alpha)}(y)\right),$$

$$\forall(\alpha,\beta)\in A_2\times B \text{ and } (x,y)\in X_1\times X_2.$$

We consider,

$$\langle K',(A_1\times B)\cup(A_2\times B)\rangle=\langle H_1,A_1\times B\rangle\cup\langle H_1,A_1\times B\rangle.$$

Again, we have following three cases:

* Case 1: $(\alpha,\beta)\in(A_1\times B)\setminus(A_2\times B)=(A_1\setminus A_2)\times B$. We have:

$$\mu_{K'(\alpha,\beta)}(x,y)=\mu_{H_1(\alpha,\beta)}(x,y)$$

$$=\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\alpha)}(y)\right);$$

$$\eta_{K'(\alpha,\beta)}(x,y)=\eta_{H_1(\alpha,\beta)}(x,y)$$

$$=\min\left(\eta_{F_1(\alpha)}(x),\eta_{G(\alpha)}(y)\right);$$

$$\nu_{K'(\alpha,\beta)}(x,y)=\nu_{H_1(\alpha,\beta)}(x,y)$$

$$=\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\alpha)}(y)\right).$$

* Case 2: $(\alpha,\beta)\in(A_2\times B)\setminus(A_1\times B)=(A_2\setminus A_1)\times B$.

$$\mu_{K'(\alpha,\beta)}(x,y)=\mu_{H_2(\alpha,\beta)}(x,y)$$

$$=\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\alpha)}(y)\right);$$

$$\eta_{K'(\alpha,\beta)}(x,y)=\eta_{H_2(\alpha,\beta)}(x,y)$$

$$= \min\left(\eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)\right);$$

$$\nu_{K'(\alpha,\beta)}(x,y) = \nu_{H_2(\alpha,\beta)}(x,y)$$

$$= \max\left(\nu_{F_2(\alpha)}(x), \nu_{G(\alpha)}(y)\right).$$

* Case 3: $(\alpha,\beta) \in (A_1 \times B) \cap (A_2 \times B) = (A_2 \cap A_1) \times B$.

$$\mu_{K'(\alpha,\beta)}(x,y) = \mu_{(H_1(\alpha,\beta))\cup(H_2(\alpha,\beta))}(x,y)$$

$$= \max\left(\mu_{H_1(\alpha,\beta)}(x,y), \mu_{H_2(\alpha,\beta)}(x,y)\right)$$

$$= \max\left(\min\left(\mu_{F_1(\alpha)}(x), \mu_{G(\alpha)}(y)\right), \min\left(\mu_{F_2(\alpha)}(x), \mu_{G(\alpha)}(y)\right)\right);$$

$$\eta_{K'(\alpha,\beta)}(x,y) = \eta_{(H_1(\alpha,\beta))\cup(H_2(\alpha,\beta))}(x,y)$$

$$= \min\left(\eta_{H_1(\alpha,\beta)}(x,y), \eta_{H_2(\alpha,\beta)}(x,y)\right)$$

$$= \min\left(\min\left(\eta_{F_1(\alpha)}(x), \eta_{G(\alpha)}(y)\right), \min\left(\eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)\right)\right)$$

$$= \min\left(\eta_{F_1(\alpha)}(x), \eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)\right); \text{ and}$$

$$\nu_{K'(\alpha,\beta)}(x,y) = \nu_{(H_1(\alpha,\beta))\cup(H_2(\alpha,\beta))}(x,y)$$

$$= \min\left(\nu_{H_1(\alpha,\beta)}(x,y), \nu_{H_2(\alpha,\beta)}(x,y)\right)$$

$$= \min\left(\max\left(\nu_{F_1(\alpha)}(x), \nu_{G(\alpha)}(y)\right), \max\left(\nu_{F_2(\alpha)}(x), \nu_{G(\alpha)}(y)\right)\right).$$

We then obtain $K = K'$ which completes the proof of c). The proof of d) is analogous. □

4 Standard neutrosophic soft relations

4.1 Standard neutrosophic relations

Fuzzy relations are one of the most important notions of fuzzy set theory and fuzzy system theory. The Zadeh's composition rule of inference [2] is a well-known method in approximation theory and inference methods in fuzzy control theory. Intuitionistic fuzzy relations were received many results [21][22]. Xu [24] defined some new intuitionistic preference relations, such as the consistent intuitionistic preference relation, incomplete intuitionistic preference relation and studied their properties. Thus, it is necessary to develop new approaches to issues, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation. In this section we shall present some preliminary results on standard neutrosophic relations.

4.1.1 Standard neutrosophic relations

Let X, Y and Z be ordinary non-empty sets. A standard neutrosophic relation is defined as follows.

Definition 4.1 [18] A standard neutrosophic relation (SNR) R between X and Y is a SNS on $X \times Y$, i.e.

$$R = \left\{\left((x,y), \mu_R(x,y), \eta_R(x,y), \nu_R(x,y)\right) \middle| (x,y) \in X \times Y\right\},$$

where μ_R, η_R, $\nu_R : X \times Y \to [0,1]$ satisfy the condition

$$\mu_R(x,y) + \eta_R(x,y) + \nu_R(x,y) \le 1, \ (x,y) \in X \times Y.$$

We will denote by $SNR(X \times Y)$ the set of all SNRs between X and Y.

Definition 4.2 [18] Let $R \in SNR(X \times Y)$, the inverse relation R^{-1} of R is a SNR between Y and X defined as

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y), \ \eta_{R^{-1}}(y,x) = \eta_R(x,y), \text{ and}$$

$$\nu_{R^{-1}}(y,x) = \nu_R(x,y), \forall (y,x) \in Y \times X.$$

Now we will consider some simple properties of SNRs.

Definition 4.3 [18] Let R, $P \in SNR(X \times Y)$, for every, we define**:**

a) $$R \le P \Leftrightarrow \begin{cases} \mu_R(x,y) \le \mu_P(x,y) \\ \eta_R(x,y) \le \eta_P(x,y) \\ \nu_R(x,y) \ge \nu_P(x,y) \end{cases};$$

b) $$R \vee P = \left\{\left((x,y), \mu_R(x,y) \vee \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y)\right) \middle| (x,y) \in X \times Y\right\};$$

c) $$R \wedge P = \left\{\left((x,y), \mu_R(x,y) \wedge \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \vee \nu_P(x,y)\right) \middle| (x,y) \in X \times Y\right\};$$

d) $$R^c = \left\{\left((x,y), \nu_R(x,y), \eta_R(x,y), \mu_R(x,y)\right) \middle| (x,y) \in X \times Y\right\}$$

Proposition 4.1 [18] Let R, P, $Q \in SNS(X \times Y)$. Then

a) $\left(R^{-1}\right)^{-1} = R$;

b) $R \le P \Rightarrow R^{-1} \le P^{-1}$;

c1) $(R \vee P)^{-1} = R^{-1} \vee P^{-1}$; c2) $(R \wedge P)^{-1} = R^{-1} \wedge P^{-1}$;

d1) $R \wedge (P \vee Q) = (R \wedge P) \vee (R \wedge Q)$;

d2) $R \vee (P \wedge Q) = (R \vee P) \wedge (R \vee Q)$;

e) $R \wedge P \le R$, $R \wedge P \le P$;

f1) If $R \ge P$ and $R \ge Q$ then $R \ge P \vee Q$;

f2) If $R \le P$ and $R \le Q$ then $R \le P \wedge Q$.

Proof. For the detail proof of this proposition, see [20].

4.1.2 Composition of standard neutrosophic relations

In this sub-section we present some compositions of SNRs.

Definition 4.4 [20] Let $R \in SNR(X \times Y)$ and $P \in SNR(Y \times Z)$. We will call max - min composed relation $P \circ_1 R \in SNR(X \times Z)$ to the one defined by

$$\mu_{P \circ_1 R}(x,z) = \vee_y \{\mu_R(x,y) \wedge \mu_P(y,z)\},$$

$$\eta_{P \circ_1 R}(x,z) = \wedge_y \{\eta_R(x,y) \wedge \eta_P(y,z)\}, \text{ and}$$

$$\nu_{P \circ_1 R}(x,z) = \wedge_y \{\nu_R(x,y) \vee \nu_P(y,z)\}, \ \forall (x,z) \in X \times Z.$$

Definition 4.5 [20] Let $R \in SNR(X \times Y)$ and $P \in SNR(Y \times Z)$. We will call max - prod composed relation $P \circ_2 R \in SNR(X \times Z)$ to the one defined by

$$\mu_{P \circ_2 R}(x,z) = \vee_y \{\mu_R(x,y) \cdot \mu_P(y,z)\},$$

$$\eta_{P \circ_2 R}(x,z) = \wedge_y [\eta_R(x,y) \cdot \eta_P(y,z)], \text{ and}$$

$$\nu_{P \circ_2 R}(x,z) = \wedge_y \{\nu_R(x,y) + \nu_P(y,z) - \nu_R(x,y) \cdot \nu_P(y,z)\},$$

$\forall (x,z) \in X \times Z$.

Definition 4.6 [20] Let β be a t-norm, ρ be a t-conorm, $R \in SNR(X \times Y)$ and $P \in SNR(Y \times Z)$. We will call max - t composed relation $R \circ_3 P \in PFR(X \times Z)$ to the one defined by

$$\mu_{R \circ_3 P}(x,z) = \vee_y \left(\beta\left(\mu_R(x,y), \mu_P(y,z)\right)\right),$$

$$\eta_{R \circ_3 P}(x,z) = \wedge_y \left\{\beta\left(\eta_R(x,y), \eta_P(y,z)\right)\right\}, \text{ and}$$

$$\nu_{R \circ_3 P}(x,z) = \wedge_y \left\{\rho\left(\nu_R(x,y), \nu_P(y,z)\right)\right\},$$

$\forall (x,z) \in X \times Z$.

The validation of Definitions 4.5-4.7 were given in [30].

4.2 Neutrosophic soft relations

4.2.1 Some operations on neutrosophic soft relations

In this sub-section, we give the definition of standard neutrosophic soft relation (SNSR) as a generalization of fuzzy soft relation and intuitionistic fuzzy soft relation. The novel concept is actually a parameterized family of standard neutrosophic relations (SNRs).

In following definitions, X, Y are ordinary non-empty sets and E is a set of parameters.

Definition 4.7 Let $A \subseteq E$. A pair (R, A) is called a *standard neutrosophic soft relation* (SNSR) over $X \times Y$ if R assigns to each parameter e in E a SNR $R(e)$ in $SNR(X \times Y)$, that is

$$R : A \to SNR(X \times Y).$$

The set of all SNSRs between X and Y is denoted by $SNSR(X \times Y)$.

Definition 4.8 Let A, $B \subseteq E$. The *intersection* of two SNSRs (R_1, A) and (R_2, B) over $X \times Y$ is a SNSR (R_3, C) over $X \times Y$ such that $C = A \cup B$ and for all $e \in C$,

$$R_3(e) = \begin{cases} R_1(e) & \text{if} \quad e \in A \setminus B, \\ R_2(e) & \text{if} \quad e \in B \setminus A, \\ R_1(e) \wedge R_2(e) & \text{if} \quad e \in A \cap B. \end{cases}$$

This relation is denoted by $(R_1, A) \cap (R_1, B)$.

Definition 4.9 Let $A, B \subseteq E$ .The *union* of two SNSRs (R_1, A) and (R_2, B) over $X \times Y$ is a SNSR (R_3, C) over $X \times Y$, where $C = A \cup B$ and for all $e \in C$,

$$R_3(e) = \begin{cases} R_1(e) & \text{if} \quad e \in A \setminus B, \\ R_2(e) & \text{if} \quad e \in B \setminus A, \\ R_1(e) \vee R_2(e) & \text{if} \quad e \in A \cap B. \end{cases}$$

This relation is denoted by $(R_1, A) \cup (R_2, B)$.

4.2.2 Composition of neutrosophic soft relations

We denote by $SNSR_{E_1}(X \times Y)$ the set of all SNSRs on $X \times Y$ with the corresponding parameter set E_1. Similarly, $SNSR_{E_2}(Y \times Z)$ denotes the set of all SNSRs on $Y \times Z$ with the corresponding parameter set E_2.

Definition 4.10 Let $R \in SNSR_{E_1}(X \times Y)$ and $P \in SNSR_{E_2}(Y \times Z)$. We will call max - min composed relation $P \bullet_1 R \in SNSR_{E_1 \times E_2}(X \times Z)$ to the one defined by

$$P \bullet_1 R(e_1,e_2) = \Big\{(x,z), \mu_{P\bullet_1 R}(x,y)(e_1,e_2), \eta_{P\bullet_1 R}(x,z)(e_1,e_2) \nu_{P\bullet_1 R}(x,z)(e_1,e_2) \Big| (x,z) \in X \times Z\Big\},$$

$\forall (e_1,e_2) \in A_1 \times A_2$. Where

$$\mu_{P\bullet_1 R}(x,z)(e_1,e_2) = \vee_y \left\{\mu_{R(e_1)}(x,y) \wedge \mu_{P(e_2)}(y,z)\right\},$$

$$\eta_{P\bullet_1 R}(x,z)(e_1,e_2) = \wedge_y \left\{\eta_{R(e_1)}(x,y) \wedge \eta_{P(e_2)}(y,z)\right\},$$

$$\nu_{P\bullet_1 R}(x,z)(e_1,e_2) = \wedge_y \left\{\nu_{R(e_1)}(x,y) \vee \nu_{P(e_2)}(y,z)\right\},$$

for all $(x,z) \in X \times Z$, $(e_1,e_2) \in A_1 \times A_2$.

Definition 4.11 Let $R \in SNSR_{E_1}(X \times Y)$ and $P \in SNSR_{E_2}(Y \times Z)$. We will call max - prod composed relation $P \bullet_2 R \in SNSR_{E_1 \times E_2}(X \times Z)$ to the one defined by

$$P \bullet_2 R(e_1,e_2) = \Big\{(x,z), \mu_{P\bullet_2 R}(x,y)(e_1,e_2), \eta_{P\bullet_2 R}(x,z)(e_1,e_2) \nu_{P\bullet_2 R}(x,z)(e_1,e_2) \Big| (x,z) \in X \times Z\Big\},$$

$\forall (e_1,e_2) \in A_1 \times A_2$. Where

$$\mu_{P\bullet_2 R}(x,z)(e_1,e_2) = \vee_y \left\{\mu_{R(e_1)}(x,y) \cdot \mu_{P(e_2)}(y,z)\right\},$$

$$\eta_{P\bullet_2 R}(x,z)(e_1,e_2) = \wedge_y \left\{\eta_{R(e_1)}(x,y) \cdot \eta_{P(e_2)}(y,z)\right\},$$

$$\nu_{P\bullet_2 R}(x,z)(e_1,e_2) = \wedge_y \left\{\nu_{R(e_1)}(x,y) + \nu_{P(e_2)}(y,z) - \nu_{R(e_1)}(x,y) \cdot \nu_{P(e_2)}(y,z)\right\},$$

for all $(x,z) \in X \times Z$, $(e_1,e_2) \in A_1 \times A_2$.

Definition 4.12 Let $R \in SNSR_{E_1}(X \times Y)$, $P \in SNSR_{E_2}(Y \times Z)$, β is a t-norm and ρ is a t-conorm. We will call max - t composed relation $P \bullet_3 R \in SNSR_{E_1 \times E_2}(X \times Z)$ to the one defined by

$$P \bullet_3 R(e_1,e_2) = \Big\{(x,z), \mu_{P\bullet_3 R}(x,y)(e_1,e_2), \eta_{P\bullet_3 R}(x,z)(e_1,e_2) \nu_{P\bullet_3 R}(x,z)(e_1,e_2) \Big| (x,z) \in X \times Z\Big\},$$

$\forall (e_1,e_2) \in A_1 \times A_2$. Where

$$\mu_{P\bullet_3 R}(x,z)(e_1,e_2) = \vee_y \left\{\beta\left(\mu_{R(e_1)}(x,y), \mu_{P(e_2)}(y,z)\right)\right\},$$

$$\eta_{P\bullet_3 R}(x,z)(e_1,e_2) = \wedge_y \left\{\beta\left(\eta_{R(e_1)}(x,y), \eta_{P(e_2)}(y,z)\right)\right\},$$

$$\nu_{P\bullet_2 R}(x,z)(e_1,e_2) = \wedge_y \left\{\rho\left(\nu_{R(e_1)}(x,y), \nu_{P(e_2)}(y,z)\right)\right\},$$

for all $(x,z) \in X \times Z$, $(e_1,e_2) \in A_1 \times A_2$.

The validation of Definitions 4.11-4.13 is trivial by following arguments. For each pair $(e_1,e_2) \in A_1 \times A_2$, $P \bullet_1 R(e_1,e_2)$ is max - min composition of two SNRs $R(e_1)$ and $P(e_2)$, i.e.

$$P \bullet_1 R(e_1,e_2) = P(e_2) \circ_1 P(e_2).$$

By the validation of $\circ_1$, $P \bullet_1 R(e_1,e_2) \in SNR(X \times Z)$ which yields $P \bullet_1 R \in SNSR_{E_1 \times E_2}(X \times Z)$. The validation of $\bullet_2$ and $\bullet_3$ are also obtained by analogous calculations.

Conclusion

In 2013, the new notion of picture fuzzy sets was introduced. The novel concept, which is also termed as standard neutrosophic set (SNS), constitutes an importance case of neutrosophic set. Our neutrosophic soft set (NSS) theory is a combination of the standard neutrosophic theory and the soft set theory. In other words, neutrosophic soft set theory is a neutrosophic extension of the intuitionistic fuzzy soft set theory. The complement, "and", "or", union and intersection operations are defined on the NSSs. The standard neutrosophic soft relations (SNSR) are also considered. The basic properties of the NSSs and the SNSRs are also discussed. Some future work may be concerned interval- valued neutrosophic soft sets and interval-valued neutrosophic relations should be considered.

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Neutrosophic Crisp α-Topological Spaces

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Abstract.
In this paper, a generalization of the neutrosophic topological space is presented. The basic definitions of the neutrosophic crisp α-topological space and the neutrosophic crisp α-compact space with some of their characterizations are deduced. Furthermore, we aim to construct a netrosophic crisp α-continuous function, with a study of a number its properties.

Keywords: Neutrosophic Crisp Set, Neutrosophic Crisp Topological space, Neutrosophic Crisp Open Set.

1 Introduction

In 1965, Zadeh introduced the degree of membership5 and defined the concept of fuzzy set [15]. A degree of non-membership was added by Atanassov [2], to give another dimension for Zadah's fuzzy set. Afterwards in late 1990's, Smarandache introduced a new degree of indeterminacy or neutrality as an independent third component to define the neutrosophic set as a triple structure [14]. Since then, laid the foundation for a whole family of new mathematical theories to generalize both the crisp and the fuzzy counterparts [4-10]. In this paper, we generalize the neutrosophic topological space to the concept of neutrosophic crisp α-topological space. Moreover, we present the netrosophic crisp α-continuous function as well as a study of several properties and some characterization of the neutrosophic crisp α-compact space.

2 Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [12,13,14], and Salama et al. [4, 5,6,7,8,9,10,11]. Smarandache introduced the neutrosophic components T, I, F which respectively represent the membership, indeterminacy, and non-membership characteristic mappings of the space X into the non-standard unit interval $]^-0, 1^+[$.

Hanafy and Salama et al. [3,10] considered some possible definitions for basic concepts of the neutrosophic crisp set and its operations. We now improve some results by the following.

2.1 Neutrosophic Crisp Sets

2.1.1 Definition

For any non-empty fixed set X, a neutrosophic crisp set A (NCS for short), is an object having the form $A = \langle A_1, A_2, A_3 \rangle$ where A_1, A_2 and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$.

2.1.2 Remark

Every crisp set A formed by three disjoint subsets of a non-empty set X is obviously a NCS having the form $A = \langle A_1, A_2, A_3 \rangle$.

Several relations and operations between NCSs were defined in [11].

For the purpose of constructing the tools for developing neutrosophic crisp sets, different types of NCSs ϕ_N, X_N, A^c in X were introduced in [9] to be as follows:

2.1.3 Definition

ϕ_N may be defined in many ways as a NCS, as follows:

i) $\phi_N = \langle \phi, \phi, X \rangle$, or
ii) $\phi_N = \langle \phi, X, X \rangle$, or
iii) $\phi_N = \langle \phi, X, \phi \rangle$, or
iv) $\phi_N = \langle \phi, \phi, \phi \rangle$.

2.1.4 Definition

X_N may also be defined in many ways as a NCS:

i) $X_N = \langle X, \phi, \phi \rangle$, or
ii) $X_N = \langle X, X, \phi \rangle$, or
iii) $X_N = \langle X, \phi, X \rangle$, or
iv) $X_N = \langle X, X, X \rangle$.

2.1.5 Definition

Let $A=\langle A_1, A_2, A_3\rangle$ a NCS on X, then the complement of the set A, (A^c for short) may be defined in three different ways:

(C_1) $A^c=\langle A_1{}^c, A_2{}^c, A_3{}^c\rangle$,

(C_2) $A^c=\langle A_3, A_2, A_1\rangle$

(C_3) $A^c=\langle A_3, A_2{}^c, A_1\rangle$

Several relations and operations between NCSs were introduced in [9] as follows:

2.1.6 Definition

Let X be a non-empty set, and the NCSs A and B in the form $A=\langle A_1,A_2,A_3\rangle$, $B=\langle B_1,B_2,B_3\rangle$, then we may consider two possible definitions for subsets $(A\subseteq B)$

$(A\subseteq B)$ may be defined in two ways:

1) $A\subseteq B \Leftrightarrow A_1\subseteq B_1, A_2\subseteq B_2$ and $A_3\supseteq B_3$ or
2) $A\subseteq B \Leftrightarrow A_1\subseteq B_1, A_2\supseteq B_2$ and $A_3\supseteq B_3$

2.1.7 Proposition

For any neutrosophic crisp set A, and the suitable choice of ϕ_N, X_N, the following are hold:

i) $\phi_N \subseteq A$, $\phi_N \subseteq \phi_N$.

ii) $A\subseteq X_N$, $X_N \subseteq X_N$.

2.1.8 Definition

Let X is a non-empty set, and the NCSs A and B in the form $A=\langle A_1,A_2,A_3\rangle$, $B=\langle B_1,B_2,B_3\rangle$. Then:

1) $A\bigcap B$ may be defined in two ways:
   i) $A\cap B=\langle A_1\cap B_1, A_2\cap B_2, A_3\cup B_3\rangle$ or
   ii) $A\cap B=\langle A_1\cap B_1, A_2\cup B_2, A_3\cup B_3\rangle$
2) $A\bigcup B$ may also be defined in two ways:
   i) $A\cup B=\langle A_1\cup B_1, A_2\cap B_2, A_3\cap B_3\rangle$ or
   ii) $A\cup B=\langle A_1\cup B_1, A_2\cup B_2, A_3\cap B_3\rangle$

2.1.9 Proposition

For any two neutrosophic crisp sets A and B on X, then the followings are true:

1) $(A\cap B)^c = A^c \cup B^c$.
2) $(A\cup B)^c = A^c \cap B^c$.

The generalization of the operations of intersection and union given in definition 2.1.8, to arbitrary family of neutrosophic crisp subsets are as follows:

2.1.10 Proposition

Let $\{A_j : j\in J\}$ be arbitrary family of neutrosophic crisp subsets in X, then

1) $\underset{j}{\cap} A_j$ may be defined as the following types :
   i) $\underset{j}{\cap} A_j=\langle \cap A_{j1}, \cap A_{j_2}, \cup A_{j_3}\rangle$, or
   ii) $\underset{j}{\cap} A_j=\langle \cap A_{j1}, \cup A_{j_2}, \cup A_{j_3}\rangle$.
2) $\underset{j}{\cup} A_j$ may be defined as the following types :
   i) $\underset{j}{\cup} A_j=\langle \cup A_{j1}, \cup A_{j_2}, \cap A_{j_3}\rangle$ or
   ii) $\underset{j}{\cup} A_j=\langle \cup A_{j1}, \cap A_{j_2}, \cap A_{j_3}\rangle$.

2.1.11 Definition

The Cartesian product of two neutrosophic crisp sets A and B is a neutrosophic crisp set $A\times B$ given by $A\times B=\langle A_1\times B_1, A_2\times B_2, A_3\times B_3\rangle$.

2.1.12 Definition

Let (X,Γ) be $NCTS$ and $A=\langle A_1,A_2,A_3\rangle$ be a NCS in X. Then the neutrosophic crisp closure of A ($NCcl(A)$ for short) and neutrosophic crisp interior ($NCint(A)$ for short) of A are defined by

$NCcl(A)=\cap\{K: K$ is a $NCCS$ in X and $A\subseteq K\}$

$NCint(A)=\cup\{G: G$ is a $NCOS$ in X and $G\subseteq A)$,

Where NCS is a neutrosophic crisp set and $NCOS$ is a neutrosophic crisp open set. It can be also shown that $NCcl(A)$ is a $NCCS$ (neutrosophic crisp closed set) and $NCint(A)$ is a $NCOS$ (neutrosophic crisp open set) in X.

3 Neutrosophic Crisp α-Topological Spaces

We introduce and study the concepts of neutrosophic crisp α-topological space

3.1 Definition

Let (X,Γ) be a neutrosophic crisp topological space (NCTS) and $A=\langle A_1,A_2,A_3\rangle$ be a NCS in X, then A is said to be neutrosophic crisp α-open set of X if and only if the following is true: $A\subseteq NCint(NCcl(NCint(A)))$.

3.2 Definition

A neutrosophic crisp α-topology (NCαT for short) on a non-empty set X is a family Γ^α of neutrosophic crisp subsets of X satisfying the following axioms

i) $\phi_N, X_N \in \Gamma^\alpha$.

ii) $A_1\cap A_2\in\Gamma^\alpha$ for any A_1 and $A_2\in\Gamma^\alpha$.

iii) $\underset{j}{\cup} A_j \in \Gamma^\alpha \quad \forall \{A_j : j\in J\}\subseteq \Gamma^\alpha$.

In this case the pair (X,Γ^α) is called a neutrosophic crisp α-topological space ($NC\alpha TS$ for short) in X. The elements in Γ^α are called neutrosophic crisp α-open sets (NCαOSs for short) in X. A neutrosophic crisp set F is α-closed if and only if its complement F^C is an α-open neutrosophic crisp set.

3.3 Remark

Neutrosophic crisp α-topological spaces are very natural generalizations of neutrosophic crisp topological spaces, as one can prof that every open set in a NCTS is an α-open set in a NCαTS

3.4 Example

Let $X=\{a,b,c,d\}$, ϕ_N, X_N be any types of the universal and empty sets on X, and A, B are two neutrosophic crisp sets on X defined by $A=\langle\{a\},\{b,d\},\{c\}\rangle$,

$B = \langle \{a\},\{b\},\{c\} \rangle$, $\Gamma = \{\phi_N, X_N, A\}$ then the family $\Gamma^\alpha = \{\phi_N, X_N, A, B\}$ is a neutrosophic crisp α-topology on X.

3.5 Definition

Let $(X, \Gamma_1^\alpha), (X, \Gamma_2^\alpha)$ be two neutrosophic crisp α-topological spaces on X . Then Γ_1^α is said be contained in Γ_2^α (in symbols $\Gamma_1^\alpha \subseteq \Gamma_2^\alpha$) if $G \in \Gamma_2^\alpha$ for each $G \in \Gamma_1^\alpha$. In this case, we also say that Γ_1^α is coarser than Γ_2^α .

3.6 Proposition

Let $\{ \Gamma_j^\alpha : j \in J \}$ be a family of NCαTS on X . Then $\cap \Gamma_j^\alpha$ is a neutrosophic crisp α-topology on X . Furthermore, $\cup \Gamma_j^\alpha$ is the coarsest NCαT on X containing all α-topologies.

Proof

Obvious.

Now, we can define the neutrosophic crisp α-closure and neutrosophic crisp α-interior operations on neutrosophic crisp α-topological spaces:

3.7 Definition

Let (X, Γ^α) be NCαTS and $A = \langle A_1, A_2, A_3 \rangle$ be a NCS in X, then the neutrosophic crisp α- closure of A (NCαCl(A) for short) and neutrosophic crisp α-interior crisp (NCαInt (A) for short) of A are defined by

$$NC\alpha Cl(A) = \cap\{K : K \text{ is an NCS in X and A} \subseteq \text{K}\}$$
$$NC\alpha Int\,(A) = \cup\{G : G \text{ is an NCOS in X and G} \subseteq \text{A}\}$$

where NCS is a neutrosophic crisp set, and NCOS is a neutrosophic crisp open set.

It can be also shown that $NC\alpha Cl\ (A)$ is a NCαCS (neutrosophic crisp α-closed set) and $NC\alpha Int(A)$ is a NCαOS (neutrosophic crisp α-open set) in X .

a) A is a NCα-closed in X if and only if $A = NC\alpha Cl(A)$.

b) A is a NCα-open in X if and only if $A = NC\alpha Int(A)$.

3.8 Proposition

For any neutrosophic crisp α-open set A in (X, Γ^α) we have

(a) $NC\alpha Cl(A^c) = (NC\alpha Int(A))^c$,

(b) $NC\alpha Int(A^c) = (NC\alpha Cl(A))^c$.

Proof

a) Let $A = \langle A_1, A_2, A_3 \rangle$ and suppose that the family of all neutrosophic crisp subsets contained in A are indexed by the family $\{A_j\}_{j \in J} = \{< A_{j_1}, A_{j_2}, A_{j_3} >: j \in J\}$. Then we see that we have two types defined as follows:

Type1: $NC\alpha Int(A) = < \cup A_{j_1}, \cup A_{j_2}, \cap A_{j_3} >$

$(NC\alpha Int(A))^c = < \cap A_{j_1}^c, \cap A_{j_2}^c, \cup A_{j3}^c >$

Hence $NC\alpha Cl(A^c) = (NC\alpha Int(A))^c$

Type 2: $NC\alpha Int(A) = < \cup A_{j_1}, \cap A_{j_2}, \cap A_{j_3} >$

$(NC\alpha Int(A))^c = < \cap A_{j_1}^c, \cup A_{j_2}^c, \cup A_{j3}^c >$.

Hence $NC\alpha Cl(A^c) = (NC\alpha Int(A))^c$

b) Similar to the proof of part (a).

3.9 Proposition

Let (X, Γ^α) be a NCαTS and A, B are two neutrosophic crisp α-open sets in X . Then the following properties hold:

(a) $NC\alpha Int(A) \subseteq A$,

(b) $A \subseteq NC\alpha Cl(A)$,

(c) $A \subseteq B \Rightarrow NC\alpha Int(A) \subseteq NC\alpha Int(B)$,

(d) $A \subseteq B \Rightarrow NC\alpha Cl(A) \subseteq NC\alpha Cl(B)$,

(e) $NC\alpha Int\ (A \cap B) = NC\alpha Int(A) \cap NC\alpha Int(B)$,

(f) $NC\alpha Cl(A \cup B) = NC\alpha Cl(A) \cup NC\alpha Cl(B)$,

(g) $NC\alpha Int(X_N) = X_N$,

(h) $NC\alpha Cl(\phi_N) = \phi_N$.

Proof. Obvious

4 Neutrosophic Crisp α-Continuity

In this section, we consider f:$X \longrightarrow Y$ to be a map between any two fixed sets X and Y.

4.1 Definition

(a) If $A = \langle A_1, A_2, A_3 \rangle$ is a NCS in X, then the neutrosophic crisp image of A under f, denoted by $f(A)$, is the a NCS in Y defined by $f(A) = \langle f(A_1), f(A_2), f(A_3) \rangle$.

(b) If f is a bijective map then f^{-1}: $Y \longrightarrow X$ is a map defined such that: for any NCS $B = \langle B_1, B_2, B_3 \rangle$ in Y, the neutrosophic crisp preimage of B, denoted by $f^{-1}(B)$, is a NCS in X defined by $f^{-1}(B) = \langle f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3) \rangle$.

Here we introduce the properties of neutrosophic images and neutrosophic crisp preimages, some of which we shall frequently use in the following sections.

4.2 Corollary

Let A = $\{A_i : i \in J\}$, be NCαOSs in X, and B = $\{B_j : j \in K\}$ be NCαOSs in Y, and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$, $B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(b) $A \subseteq f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$.

(c) $f^{-1}(f(B)) \subseteq B$ and if f is surjective, then $f^{-1}(f(B)) = B$.

(d) $f^{-1}(\cup B_i) = \cup f^{-1}(B_i)$, $f^{-1}(\cap B_i)) \subseteq \cap f^{-1}(B_i)$,

(e) $f(\cup A_i) = \cup f(A_i)$, $f(\cap A_i) \subseteq \cap f(A_i)$.

(f) $f^{-1}(Y_N) = X_N$, $f^{-1}(\phi_N) = \phi_N$.

(g) $f(\phi_N) = \phi_N$, $f(X_N) = Y_N$, if f is subjective.

Proof

Obvious.

4.3 Definition

Let (X, Γ_1^α) and (Y, Γ_2^α) be two NCαTSs, and let $f: X \to Y$ be a function. Then f is said to be α-continuous iff the neutrosophic crisp preimage of each NCS in Γ_2^α is a NCS in Γ_1^α.

4.4 Definition

Let (X, Γ_1^α) and (Y, Γ_2^α) be two NCαTSs and let $f: X \to Y$ be a function. Then f is said to be open iff the neutrosophic crisp image of each NCS in Γ_1^α is a NCS in Γ_2^α.

4.5 Proposition

Let (X, Γ_o^α) and (Y, ψ_o^α) be two NCαTSs.

If $f: X \to Y$ is α-continuous in the usual sense, then in this case, f is α-continuous in the sense of Definition 4.3 too.

Proof

Here we consider the NCαTs on X and Y, respectively, as follows : $\Gamma_1^\alpha = \{\langle G, \phi, G^c \rangle : G \in \Gamma_o^\alpha\}$ and $\Gamma_2^\alpha = \{\langle H, \phi, H^c \rangle : H \in \Psi_o^\alpha\}$,

In this case we have, for each $\langle H, \phi, H^c \rangle \in \Gamma_2^\alpha$, $H \in \Psi_o^\alpha$,

$f^{-1}\langle H, \phi, H^c \rangle = \langle f^{-1}(H), f^{-1}(\phi), f^{-1}(H^c) \rangle$

$= \langle f^{-1}H, \phi, (f^{-1}(H))^c \rangle \in \Gamma_1^\alpha$.

4.6 Proposition

Let $f: (X, \Gamma_1^\alpha) \to (Y, \Gamma_2^\alpha)$.

f is continuous iff the neutrosophic crisp preimage of each CNαCS (crisp neutrosophic α-closed set) in Γ_1^α is a CNαCS in Γ_2^α.

Proof

Similar to the proof of Proposition 4.5.

4.7 Proposition

The following are equivalent to each other:

(a) $f: (X, \Gamma_1^\alpha) \to (Y, \Gamma_2^\alpha)$ is continuous .

(b) $f^{-1}(CN\alpha Int(B) \subseteq CN\alpha Int(f^{-1}(B))$ for each CNS B in Y.

(c) $CN\alpha Cl(f^{-1}(B)) \subseteq f^{-1}(CN\alpha Cl(B))$ for each CNC B in Y.

4.8 Corollary

Consider (X, Γ_1^α) and (Y, Γ_2^α) to be two NCαTSs, and let $f: X \to Y$ be a function.

if $\Gamma_1^\alpha = \{f^{-1}(H) : H \in \Gamma_2^\alpha\}$. Then Γ_1^α will be the coarsest NCαT on X which makes the function $f: X \to Y$ α-continuous. One may call it the initial neutrosophic crisp α-topology with respect to f.

5 Neutrosophic Crisp α-Compact Space

First we present the basic concepts:

5.1 Definition

Let (X, Γ^α) be an NCαTS.

(a) If a family $\{\langle G_{i_1}, G_{i_2}, G_{i_3} \rangle : i \in J\}$ of NCαOSs in X satisfies the condition $\cup \{\langle G_{i_1}, G_{i_2}, G_{i_3} \rangle : i \in J\} = X_N$, then it is called an neutrosophic α-open cover of X.

(b) A finite subfamily of an α-open cover $\{\langle G_{i_1}, G_{i_2}, G_{i_3} \rangle : i \in J\}$ on X, which is also a neutrosophic α-open cover of X , is called a neutrosophic crisp finite α-open subcover.

5.2 Definition

A neutrosophic crisp set $A = \langle A_1, A_2, A_3 \rangle$ in a NCαTS (X, Γ^α) is called neutrosophic crisp α-compact iff every neutrosophic crisp open cover of A has a finite neutrosophic crisp open subcover.

5.3 Definition

A family $\{\langle K_{i_1}, K_{i_2}, K_{i_3} \rangle : i \in J\}$ of neutrosophic crisp α-compact sets in X satisfies the finite intersection property (FIP for short) iff every finite subfamily $\{\langle K_{i_1}, K_{i_2}, K_{i_3} \rangle : i = 1,2,...,n\}$ of the family satisfies the condition $\cap \{\langle K_{i_1}, K_{i_2}, K_{i_3} \rangle : i = 1,2,...,n\} \neq \phi_N$.

5.4 Definition

A NCαTS (X, Γ^α) is called neutrosophic crisp α-compact iff each neutrosophic crisp α-open cover of X has a finite α-open subcover.

5.5 Corollary

A NCαTS (X, Γ^α) is a neutrosophic crisp α-compact iff every family $\{\langle G_{i_1}, G_{i_2}, G_{i_3} \rangle : i \in J\}$ of neutrosophic crisp α-compact sets in X having the the finite intersection properties has nonempty intersection.

5.6 Corollary

Let (X, Γ_1^α), (Y, Γ_2^α) be NCαTSs and $f: X \to Y$ be a continuous surjection. If (X, Γ_1^α) is a neutrosophic crisp α-compact, then so is (Y, Γ_2^α).

5.7 Definition

(a) If a family $\left\{\left\langle G_{i_1}, G_{i_2}, G_{i_3}\right\rangle : i \in J\right\}$ of neutrosophic crisp α-compact sets in X satisfies the condition $A \subseteq \cup\left\{\left\langle G_{i_1}, G_{i_2}, G_{i_3}\right\rangle : i \in J\right\}$, then it is called a neutrosophic crisp open cover of A.

(b) Let's consider a finite subfamily of a neutrosophic crisp open subcover of $\left\{\left\langle G_{i_1}, G_{i_2}, G_{i_3}\right\rangle : i \in J\right\}$.

5.8 Corollary

Let $\left(X, \Gamma_1^{\alpha}\right)$, $\left(Y, \Gamma_2^{\alpha}\right)$ be NCαTSs and $f : X \to Y$ be a continuous surjection. If A is a neutrosophic crisp α-compact in $\left(X, \Gamma_1^{\alpha}\right)$, then so is $f(A)$ in $\left(Y, \Gamma_2^{\alpha}\right)$.

6. Conclusion

In this paper, we presented a generalization of the neutrosophic topological space. The basic definitions of the neutrosophic crisp α-topological space and the neutrosophic crisp α-compact space with some of their characterizations were deduced. Furthermore, we constructed a netrosophic crisp α-continuous function, with a study of a number its properties.

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Neutrosophic Goal Programming Applied to Bank: Three Investment Problem

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Abstract: This paper represents a new multi-objective Neutrosophic goal programming and Lexicographic goal programming to solve a multi-objective linear programming problem. Here we describe some basic properties of Neutrosophic sets. We have considered a multi-objective Bank Three Investment model to get optimal solution for different weights.

Keywords- Neutrosophic goal programming, Lexicographic goal programming, Bank Three model.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [1]. Since the fuzzy sets and fuzzy logic have been applied in many real applications to handle uncertainty. The traditional fuzzy sets uses one real value $\mu_A(x) \in [0, 1]$ to represents the truth membership function of fuzzy set A defined on universe X. Sometimes $\mu_A(x)$ itself is uncertain and hard to be defined by a crisp value. So the concept of interval valued fuzzy sets was proposed [2] to capture the uncertainty of truth membership. In some applications we should consider not only the truth membership supported by the evident but also the falsity membership against by the evident. That is beyond the scope of fuzzy sets and interval valued fuzzy sets. In 1986, Atanassov **[3]**, **[5]** devolved the idea of intuitionistic fuzzy set A characterized by the membership degree $\mu_A(x) \in [0, 1]$ as well as non-membership degree $\nu_A(x) \in [0, 1]$ with some restriction $0 \leq \mu_A(x)+\nu_A(x) \leq 1$. Therefore certain amount of indeterminacy 1 - $(\mu_A(x)+\nu_A(x))$ remains by default. However one may also consider the possibility $\mu_A(x)+\nu_A(x) > 1$, so that inconsistent beliefs are also allowed. In neutrosophic sets indeterminacy is quantified explicitly and truth membership, indeterminacy membership and falsity membership are independent. Neutrosophic set (NS) was introduced by Smarandache in 1995 **[4]** which is actually generalization of different types of FSs and IFSs. In 1978 a paper Fuzzy linear programming with several objective functions has been published by H.J Zimmermann **[11]**. In 2007 B.Jana and T.K.Roy **[9]** has studied multi-objective intuitionistic fuzzy linear programming problem and its application in Transportation model. In 1961 goal programming was introduced by Charnes and Cooper **[13]**, Aenaida and kwak **[14]** applied goal programming to find a solution for multi-objective transportation problem. Recently the authors used the fuzzy goal programming approach to solve multi-objective transportation problem **[15]**. Other authors used fuzzy goal programming technique to solve different types of multi-objective linear programming problems **[16, 17, 18, 19]**. Recently a paper named "neutrosophic goal programming" has published by Mohamed Abdel-Baset, Ibrahim M. Hezam and Florentin Smarandache in the journal Neutrosophic sets and sysatems **[20]**. The motivation of the present study is to give computational algorithm for solving multi-objective linear goal programming problem and Lexicographic goal programming problem by single valued neutrosophic optimization approach. We also aim to study the impact of truth membership, indeterminacy membership and falsity membership functions in such optimization process.

2. Some preliminaries

2.1 Definition -1 (Fuzzy set) [1]

Let X be a fixed set. A fuzzy set A of X is an object having the form $\tilde{A}$ = {(x,μ_A (x)), x ∈ X} where the function $\mu_A(x)$: X → [0, 1] define the truth membership of the element x ∈ X to the set A.

2.2 Definition-2 (Intuitionistic fuzzy set) [3]

Let a set X be fixed. An intuitionistic fuzzy set or IFS $\tilde{A}^i$ in X is an object of the form $\tilde{A}^i = \{< X, \mu_A(x), \nu_A(x) > / x \in X\}$ where $\mu_A(x)$: X→ [0, 1] and $\nu_A(x)$: X→ [0, 1] define the Truth-membership and Falsity-membership respectively, for every element of x∈ X , $0 \le \mu_A(x) + \nu_A(x) \le 1$.

2.3 Definition-3 (Neutrosophic set) [4]

Let X be a space of points (objects) and $x \in X$. A neutrosophic set $\tilde{A}^n$ in X is defined by a Truth-membership function$\mu_A(x)$, an indeterminacy-membership function $\sigma_A(x)$ and a falsity-membership function $\nu_A(x)$ and having the form $\tilde{A}^n = \{< X, \mu_A(x), \sigma_A(x), \nu_A(x) > / x \in X\}$.

$\mu_A(x), \sigma_A(x)$ *and* $\nu_A(x)$ are real standard or non-standard subsets of

] 0^-, 1^+ [. that is

$\mu_A(x)$: X→] 0^-, 1^+ [

$\sigma_A(x)$: X→] 0^-, 1^+ [

$\nu_A(x)$: X→] 0^-, 1^+ [

There is no restriction on the sum of $\mu_A(x), \sigma_A(x)$ *and* $\nu_A(x)$, so

$$0^- \le \sup \mu_A(x) + \sup \sigma_A(x) + \sup \nu_A(x) \le 3^+$$

2.4 Definition-4 (Single valued Neutrosophic sets) [6]

Let X be a universe of discourse. A single valued neutrosophic set $\tilde{A}^n$ over X is an object having the form $\tilde{A}^n = \{< X, \mu_A(x), \sigma_A(x), \nu_A(x) > / x \in X\}$where$\mu_A(x)$: X→ [0, 1],$\sigma_A(x)$: X→[0, 1] and$\nu_A(x)$: X→ [0, 1] with $0 \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 3$ for all x ∈ X.

Example 1 Assume that X = [x_1, x_2, x_3]. X_1 is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, $x_2$and x_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". $\tilde{A}^n$ is a single valued neutrosophic set of X defined by

$\tilde{A}^n$ =⟨0.3,0.4,0.5⟩/x_1 + ⟨0.5,0.2,0.3⟩/x_2 + ⟨0.7,0.2,0.2⟩/x_3. $\tilde{B}^n$ is a single valued neutrosophic set of X defined by $\tilde{B}^n$=⟨0.6,0.1,0.2⟩/x_1 + ⟨0.3,0.2,0.6⟩/x_2 + ⟨0.4,0.1,0.5⟩/x_3

2.5 Definition 5(Complement): [6].

The complement of a single valued neutrosophic set $\tilde{A}^n$ is denoted by c($\tilde{A}^n$) and is defined by

$$\mu_{c(\tilde{A}^n)}(x) = \nu_{\tilde{A}^n}(x)$$

$$\sigma_{c(\tilde{A}^n)}(x) = 1 - \sigma_{\tilde{A}^n}(x)$$

$$\nu_{c(\tilde{A}^n)}(x) = \mu_{\tilde{A}^n}(x) \quad \text{for all x in X}$$

Example 2 Let $\tilde{A}^n$ be a single valued neutrosophic set defined in example 1. Then

c($\tilde{A}^n$) = ⟨0.5,0.6,0.3⟩/x_1 + ⟨0.3,0.8,0.5⟩/x_2 + ⟨0.2,0.8,0.7⟩/X_3

2.6 Definition 6 (Union): [6] The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, Written as C = A ∪ B, whose truth-membership, indeterminacy-membership and falsity-membership functions are given by

$\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$

$\sigma_C(x) = \max\{\sigma_A(x), \sigma_B(x)\}$

$\nu_C(x) = \min\{\nu_A(x), \nu_B(x)\}$ for all x in X

Example 3 Let A and B be two single valued neutrosophic sets defined in example 1. Then A ∪ B = ⟨0.6,0.4,0.2⟩/x_1 + ⟨0.5,0.2,0.3⟩/x_2 + ⟨0.7,0.2,0.2⟩ /X_3

2.7 Definition 7(Intersection): [7] The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, Written as C = A ∩ B, whose truth-membership, indeterminacy-membership and falsity-membership functions are given by

$\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}$

$\sigma_C(x) = \min\{\sigma_A(x), \sigma_B(x)\}$

$\nu_C(x) = \max\{\nu_A(x), \nu_B(x)\}$ for all x in X

Example 4 Let A and B be two single valued neutrosophic sets defined in example 1. Then A ∩ B = ⟨0.3,0.1,0.5⟩/x_1 + ⟨0.3,0.2,0.6⟩/x_2 + ⟨0.4,0.1,0.5⟩ /X_3

3. Multi-objective linear programming problem (MOLPP) [19]

A general multi-objective linear programming problem with n objectives, m constraints and q decision variables may be taken in the following form

Minimize $f_1(x) = c_1$ X

Minimize $f_2(x) = c_2$ X

……………………………….

……………………………....

Minimize $f_n(x) = c_n$ X

Subject to A X $\le b$ and X ≥ 0

Where C= $(c_{i1}, c_{i2}, \dots\dots\dots, c_{iq})$ for i=1,2,……….n.

A= $[a_{ji}]_{m.n}$, X = $(x_1, x_2, \dots\dots\dots, x_q)^T$, b=$(b_1, b_2, \dots\dots\dots, b_m)^T$.

for j=1,2,………m; i=1,2,……….n.

4. A. Neutrosophic linear goal programming problem (NLGPP) [21]

Find X = $(x_1, x_2, \dots\dots\dots, x_q)^T$ (4.1)

So as to

Minimize $f_i(x)$

with target value c_i , Truth tolerance a_i , Falsity tolerance t_i, Indeterminacy tolerance p_i i= 1, 2, ………..n.

subject to $g_j(x) \le b_j$ j= 1, 2, ………..m.

$x_k \ge 0$ k= 1,2, ………..q.

With Truth-membership, Falsity-membership, Indeterminacy-membership functions are

$$\mu_i(f_i(x)) = \begin{cases} 1 & if\ f_i(x) \le c_i \\ \frac{a_i + c_i - f_i(x)}{a_i} & if\ c_i \le f_i(x) \le a_i + c_i \\ 0 & if\ f_i(x) \ge a_i + c_i \end{cases} \quad (4.2)$$

$$\nu_i(f_i(x)) = \begin{cases} 0 & if\ f_i(x) \le c_i \\ \frac{f_i(x) - c_i}{t_i} & if\ c_i \le f_k(x) \le c_i + t_i \\ 1 & if\ f_i(x) \ge c_i + t_i \end{cases}$$

(4.3)

$$\sigma_i(f_i(x)) = \begin{cases} 1 & if\ f_i(x) \le c_i' \\ \frac{p_i + c_i' - f_i(x)}{p_i} & if\ c_i' \le f_i(x) \le p_i + c_i' \\ 0 & if\ f_i(x) \ge p_i + c_i' \end{cases} \quad (4.4)$$

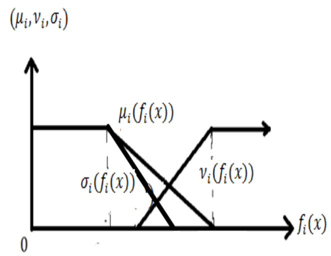

Fig1. Truth membership, Falsity membership and Indeterminacy membership functions of a minimization-type objective function.

Neutrosophic goal programming can be transformed into crisp linear programming problem using Truth-membership, Falsity-membership, Indeterminacy-membership functions as

Maximize $\sum_{i=1}^{n} w_i \, \mu_i(f_i(x))$

(4.5)

Minimize $\sum_{i=1}^{n} w_i \, \nu_i(f_i(x))$

Maximize$\sum_{i=1}^{n} w_i \, \sigma_i(f_i(x))$

i= 1, 2, ………..,n.

Subject to $g_j(x) \le b_j$

j= 1, 2,………..,m.

$x_k \ge 0$

k= 1,2, ………..q.

$\sum_{i=1}^{n} w_i = 1$

Which is equivalent to

Max $\sum_{i=1}^{n} w_i \, (\mu_i(f_i(x)) - \nu_i(f_i(x)) + \sigma_i(f_i(x)))$ (4.6)

Subject to $g_j(x) \le b_j$

j= 1, 2,………..,m.

$\sum_{i=1}^{n} w_i = 1$

B. Neutrosophic Lexicographic goal programming

The Lexicographic optimization takes objective in order: optimizing one, then a second subject to the first achieving its optimal value, and so on.

Step-1. Max $\mu_1 - \nu_1 + \sigma_1$

Subject to $g_j(x) \le b_j$ j= 1, 2,………..,m.

$x_k \ge 0$ k= 1,2, ………..q.

Solving we get optimal solution $f_1^* = F_1$

Step-2 Max $\mu_2 - \nu_2 + \sigma_2$

Subject to $g_j(x) \le b_j$ j= 1, 2,………..,m.

$f_1 \le F_1$

$x_k \ge 0$ k= 1,2, ………..q.

Solving we get optimal solution $f_2^* = F_2$

Step-3 Max $\mu_3 - \nu_3 + \sigma_3$

Subject to $g_j(x) \le b_j$ j= 1, 2,………..,m.

$f_1 \le F_1$

$f_2 \le F_2$

$x_k \ge 0$ k= 1,2, ………..q.

Solving we get optimal solution $f_3^* = F_3$

And so on.

Proceeding in this way finally we get optimal decision variables and all the optimal objective values.

5. Application of Neutrosophic goal programming to Bank Three Investment Problem

Every investor must trade off return versus risk in deciding how to allocate his or her available funds. The opportunities that promise the greatest profits are almost the ones that present the most serious risks.

Commercial banks must be especially careful in balancing return and risk because legal and ethical obligations demand that they avoid undue hazards, yet their goal as a business enterprise is to maximize profit. This dilemma leads naturally to multi-objective optimization of investment that includes both profit and risk criteria.

Our investment example **[12]** adopts this multi-objective approach to a fictitious Bank Three. Bank Three has a modest \$ 20 million capital, with \$ 150 million in demand deposits and \$80 million in time deposits (savings accounts and certificates of deposit). Table 1 displays the categories among which the bank must divide its capital and deposited funds. Rates of return are also provided for each category together with other information related to risk.

Table 1 Bank Three Investment Opportunities

Investment Category, j	Return Rate (%)	Liquid Part (%)	Required Capital Asset (%)	Risk ?
1: Cash	0.0	100.0	0.0	No
2: Short term	4.0	99.5	0.5	No
3: Government: 1 to 5 years	4.5	96.0	4.0	No
4: Government: 5 to 10 years	5.5	90.0	5.0	No
5: Government: over 10 years	7.0	85.0	7.5	No
6: Installment loans	*10.5*	0.0	10.0	Yes
7: Mortgage loans	8.5	0.0	10.0	Yes
8: Commercial loans	9.2	0.0	10.0	Yes

The first goal of any private business is to maximize profit. Using rates of return from table 1, this produces objective function

$$\text{Max} \quad 0.04x_2 + 0.045x_3 + 0.055x_4 + 0.070x_5 + 0.105x_6 + 0.085x_7 + 0.092x_8 \quad \text{(Profit)}$$

It is less clear how to quantify investment risk. We employ two common ratio measures.

One is the capital-adequacy ratio, expressed as the ratio of required capital for bank solvency to actual capital. A low value indicates minimum risk. The "required capital" rates of table 1 approximate U.S. government formulas used to compute this ratio, and Bank Three's present capital is $ 20 million. Thus we will express a second objective as

Min $\frac{1}{20}(0.005x_2 + 0.040x_3 + 0.050x_4 + 0.075x_5 + 0.100x_6 + 0.100x_7 + 0.100x_8)$

(Capital - adequacy)

Another measure of risk focuses on illiquid risk assets. A low risk asset/capital ratio indicates a financially secure institution. For our example, this third measure of success is expressed as

Min $\frac{1}{20}$ $(x_6 + x_7 + x_8)$ (Risk - asset)

To complete a model of Bank Three's investment plans, we must describe the relevant constraints.

1. Investments must sum to the available capital and deposit funds.
2. Cash reserves must be at least 14% of demand deposits plus 4 % of time Deposits.
3. The portion of investments considered liquid should be at least 47 % of Demand deposits plus 36 % of time deposits.
4. At least 5 % of funds should be invested in each of the eight categories.
5. At least 30 % of funds should be invested in commercial loans, to Maintain the bank's community status.

Combining the 3 objective functions above with these 5 constraints completes a multi-objective linear programming model of Bank Three's investment problem:

Max $0.04x_2 + 0.045x_3 + 0.055x_4 + 0.070x_5 + 0.105x_6 + 0.085x_7 + 0.092x_8$ (Profit)

Min $\frac{1}{20}(0.005x_2 + 0.040x_3 + 0.050x_4 + 0.075x_5 + 0.100x_6 + 0.100x_7 + 0.100x_8)$

(Capital - adequacy)

Min $\frac{1}{20}$ $(x_6 + x_7 + x_8)$ (Risk - asset)

Such that $x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8= (20+150+80)$ (Invest all)

$x_1 \geq 0.14 (150) + 0.04 (80)$ (Cash reserve)

$1.00x_1+0.995x_2+0.960x_3+0.900x_4+0.850x_5 \geq 0.47(150) + 0.36(80)$

(Liquidity)

$x_j \geq 0.05 (20+150+80)$ for all $j=1,\ldots,8$ (Diversification)

$x_8 \geq 0.30 (20+150+80)$ (Commercial)

$x_1, x_2, \ldots, x_8 \geq 0$

6. Numerical Example

$c_1 = 12$, $a_1 = 6.67$, $t_1 = 3$, $c_1' = 13$, $p_1 = 5.67$

$c_2 = 0.58$, $a_2 = 0.22$, $t_2 = 0.20$, $c_2' = 0.60$, $p_2 = 0.20$

$c_3 = 5$, $a_3 = 1.5$, $t_3 = 1.0$, $c_3' = 5.5$, $p_3 = 1.0$

Table 1. Goal Programming Solution of the Bank Three Problem

Weights	Optimal Primal Variables	Optimal Objectives
$W_1=0.8$, $w_2=0.1$, $w_3=0.1$	$x_1=24.2$, $x_2=12.5$, $x_3=12.5$, $x_4=12.5$, $x_5=46.367665$, $x_6=54.4323$, $x_7=12.5$, $x_8=75$	$f_1=18.67363$, $f_2=0.942915$, $f_3=7.096$
$W_1=0.05$, $w_2=0.9$, $w_3=0.05$	$x_1=100$, $x_2=12.5$, $x_3=12.5$, $x_4=12.5$, $x_5=12.5$, $x_6=12.5$, $x_7=12.5$, $x_8=75$	$f_1=11.9$, $f_2=0.60625$, $f_3=5.00$
$W_1=0.1$, $w_2=0.1$, $w_3=0.8$	$x_1=24.2$, $x_2=88.30$, $x_3=12.5$, $x_4=12.5$, $x_5=12.5$, $x_6=12.5$, $x_7=12.5$, $x_8=75$	$f_1=14.932$, $f_2=0.6252$, $f_3=5.00$
$W_1=1/3$, $w_2=1/3$, $w_3=1/3$	$x_1=24.2$, $x_2=88.30$, $x_3=12.5$, $x_4=12.5$, $x_5=12.5$, $x_6=12.5$, $x_7=12.5$, $x_8=75$	$f_1=14.932$, $f_2=0.6252$, $f_3=5.00$

Table 2. Lexicographic Goal Programming Solution of the Bank Three Problem

Optimal Primal variables	Optimal Objective functions
$x_1=24.2$, $x_2=22.51454$, $x_3=12.5$, $x_4=12.5$, $x_5=34.64474$, $x_6=56.14072$, $x_7=12.5$, $x_8=75$	$f_1=18.43299$, $f_2=0.9100$, $f_3=7.182036$

7. Conclusion

In this paper, we presents simple Neutrosophic optimization approach to solve Multi-objective linear goal programming problem and Lexicographic goal programming problem. It can be considered as an extension of fuzzy and intuitionistic fuzzy optimization. This proposed method Neutrosophic Goal Programming can also be applied for multi-objective non-linear programming problem.

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Neutrosophic Hyperideals of Semihyperrings

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Abstract. In this paper, we have introduced neutrosophic hyperideals of a semihyperring and considered some operations on them to study its basic notions and properties.

Keywords: Cartesian Product, Composition, Ideal, Intersection, Neutrosophic, Semihyperrimg.

1 Introduction

Hyperrings extend the classical notion of rings, substituting both or only one of the binary operations of addition and multiplication by hyperoperations. Hyperrings were introduced by several authors in different ways. If only the addition is a hyperoperation and the multiplication is a binary operation, then we say that R is a Krasner hyperring [4]. Davvaz [5] has defined some relations in hyperrings and proved isomorphism theorems. For a more comprehensive introduction about hyperrings, we refer to [9]. As a generalization of a ring, semiring was introduced by Vandiver [17] in 1934. A semiring is a structure $(R;+;\cdot;0)$ with two binary operations $+$ and $\cdot$ such that $(R;+;0)$ is a commutative semigroup, $(R;\cdot)$ a semigroup, multiplication is distributive from both sides over addition and $0\cdot x = 0 = x\cdot 0$ for all $x \in R$. In [18], Vougiouklis generalizes the notion of hyperring and named it as semihyperring, where both the addition and multiplication are hyperoperation. Semihyperrings are a generalization of Krasner hyperrings. Note that a semiring with zero is a semihyperring. Davvaz in [12] studied the notion of semihyperrings in a general form.

Hyperstructures, in particular hypergroups, were introduced in 1934 by Marty [11] at the eighth congress of Scandinavian Mathematicians. The notion of algebraic hyperstructure has been developed in the following decades and nowadays by many authors, especially Corsini [2, 3], Davvaz [5, 6, 7, 8, 9], Mittas [12], Spartalis [15], Stratigopoulos [16] and Vougiouklis [19]. Basic definitions and notions concerning hyperstructure theory can be found in [2].

The concept of a fuzzy set, introduced by Zadeh in his classical paper [20], provides a natural framework for generalizing some of the notions of classical algebraic structures.As a generalization of fuzzy sets, the intuitionistic fuzzy set was introduced by Atanassov [1] in 1986, where besides the degree of membership of each element there was considered a degree of non-membership with (membership value + non-membership value)≤ 1. There are also several well-known theories, for instances, rough sets, vague sets, interval-valued sets etc. which can be considered as mathematical tools for dealing with uncertainties.

In 2005, inspired from the sport games (winning/tie/ defeating), votes, from (yes /NA /no),from decision making(making a decision/ hesitating/not making), from (accepted /pending /rejected) etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics, F. Smarandache [14] combined the non-standard analysis [8,18] with a tri-component logic/set/probability theory and with philosophy and introduced Neutrosophic set which represents the main distinction between fuzzy and intuitionistic fuzzy logic/set. Here he included the middle component, i.e., the neutral/ indeterminate/ unknown part (besides the truth/membership and falsehood/non-membership components that both appear in fuzzy logic/set) to distinguish between 'absolute membership and relative membership' or 'absolute non-membership and relative non-membership'.

Using this concept, in this paper, we have defined neutrosophic ideals of semihyperrings and study some of its basic properties.

2 Preliminaries

Let H be a non-empty set and let $P(H)$ be the set of all non-empty subsets of H. A hyperoperation on H is a map $\circ : H \times H \to P(H)$ and the couple $(H,\circ)$ is called a hypergroupoid.
If A and B are non-empty subsets of H and $x \in H$, then we denote $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$,

$x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H,\circ)$ is called a semihypergroup if for all $x, y, z \in H$ we have $(x \circ y) \circ z = x \circ (y \circ z)$ which means that $\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$.

A semihyperring is an algebraic structure $(R;+;\cdot)$ which satisfies the following properties:

(i) $(R;+)$ is a commutative semihypergroup

(ii) $(R;\cdot)$ is a semihypergroup

(iii) Multiplication is distributive with respect to hyperoperation + that is $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot z + y \cdot z$

(iv) $0 \cdot x = 0 = x \cdot 0$ for all $x \in R.$

A semihyperring $(R;+;\cdot)$ is called commutative if and only if $a \cdot b = b \cdot a$ for all $a, b \in R$.

Vougiouklis in [18] and Davvaz in [6] studied the notion of semihyperrings in a general form, i.e., both the sum and product are hyperoperations.

A semihyperring $(R;+;\cdot)$ with identity $1_R \in R$ means that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$.

An element $x \in R$ is called unit if there exists $y \in R$ such that $1_R = x \cdot y = y \cdot x$.

A nonempty subset S of a semihyperring $(R;+;\cdot)$ is called a sub-semihyperring if $a + b \subseteq S$ and $a \cdot b \subseteq S$ for all $a, b \in S$. A left hyperideal of a semihyperring R is a non-empty subset I of R satisfying

(i) If $a, b \in I$ then $a + b \subseteq I$

(ii) If $a \in I$ and $s \in R$ then $s \cdot a \subseteq I$

(iii) $I \neq R.$

A right hyperideal of R is defined in an analogous manner and an hyperideal of R is a nonempty subset which is both a left hyperideal and a right hyperideal of R.

For more results on semihyperrings and neutrosophic sets we refer to [6, 10] and [14] respectively.

3. Main Results

Definition 3.1. [14] A neutrosophic set A on the universe of discourse X is defined as $A = \{< x : A^T(x), A^I(x), A^F(x) >, x \in X\}$ where $A^T, A^I, A^F : X \to]^-0,1[^+$ and $^-0 \leq A^T(x) + A^I(x) + A^F(x) \leq 3^+$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $^-]0,1[^+$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $^-]0,1[^+$. Hence we consider the neutrosophic set which takes the value from the subset of [0, 1].

Throughout this section unless otherwise mentioned R denotes a semihyperring.

Definition 3.2. Let $\mu = (\mu^T, \mu^I, \mu^F)$ be a non empty neutrosophic subset of a semihyperring R (i.e. anyone of $\mu^T(x)$, $\mu^I(x)$ or $\mu^F(x)$ not equal to zero for some $x \in R$).Then μ is called a neutrosophic left hyperideal of R if

(i) $\inf_{z \in x+y} \mu^T(z) \geq \min\{\mu^T(x), \mu^T(y)\}$,

(ii) $\inf_{z \in x+y} \mu^I(z) \geq \dfrac{\mu^I(x) + \mu^I(y)}{2}$,

(iii) $\sup_{z \in x+y} \mu^F(z) \leq \max\{\mu^F(x), \mu^F(y)\}$,

(iv) $\inf_{z \in xy} \mu^T(z) \geq \mu^T(y)$,

(v) $\inf_{z \in xy} \mu^I(z) \geq \mu^I(y)$,

(vi) $\sup_{z \in xy} \mu^F(z) \leq \mu^F(y).$

for all $x, y \in R.$

Similarly we can define neutrosophic right hyperideal of R.

Example 3.3. Let $R = \{0, a, b, c\}$ be a set with the hyperoperation $\oplus$ and the multiplication $\bullet$ defined as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	{a,b}	b	c
b	b	b	{0,b}	c
c	c	c	c	{0,c}

and

$\bullet$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c

c	0	a	c	c

Then $(R,\oplus,\bullet)$ is a semihyperring.

Define neutrosophic subset μ of R by $\mu(0)=(1,0.6,0.1)$, $\mu(a)=(0.7,0.4,0.3)$, $\mu(b)=(0.8,0.5,0.2)$ $\mu(c)=(0.6,0.2,0.4)$. Then μ is a neutrosophic left hyperideal of R.

Theorem 3.4. A neutrosophic set μ of R is a neutrosophic left hyperideal of R if and only if any level subsets $\mu_t^T := \{x \in R : \mu^T(x) \geq t, t \in [0.1]\}$, $\mu_t^I := \{x \in R : \mu^I(x) \geq t, t \in [0.1]\}$ and $\mu_t^F := \{x \in R : \mu^T(x) \leq t, t \in [0.1]\}$ are left hyperideals of R.

Proof. Assume that the neutrosophic set μ of R is a neutrosophic left hyperideal of R.

Then anyone of μ^T, μ^I or μ^F is not equal to zero for some $x \in R$ i.e., in other words anyone of $\mu_t^{\ T}, \mu_t^{\ I}$ or $\mu_t^{\ F}$ is not empty for some $t \in [0,1]$. So, it is sufficient to consider that all of them are not empty.

Suppose $x, y \in \mu_t = (\mu_t^{\ T}, \mu_t^{\ I}, \mu_t^{\ F})$ and $s \in R$. Then

$$\inf_{z \in x+y} \mu^T(z) \geq \min\{\mu^T(x), \mu^T(y)\} = \min\{t,t\} = t$$

$$\inf_{z \in x+y} \mu^I(z) \geq \frac{\mu^I(x)+\mu^I(y)}{2} \geq \frac{t+t}{2} = t$$

$$\sup_{z \in x+y} \mu^F(z) \leq \max\{\mu^F(x), \mu^F(y)\} \leq \max\{t,t\} = t$$

which implies $x+y \subseteq \mu_t^{\ T}, \mu_t^{\ I}, \mu_t^{\ F}$ i.e., $x+y \subseteq \mu_t$.

Also

$$\inf_{z \in sx} \mu^T(z) \geq \mu^T(x) \geq t,$$

$$\inf_{z \in sx} \mu^I(z) \geq \mu^I(x) \geq t,$$

$$\sup_{z \in sx} \mu^F(z) \leq \mu^F(x) \leq t,$$

Hence $sx \subseteq \mu_t$.

Therefore μ_t is a left hyperideal of R.

Conversely, suppose $\mu_t (\neq \phi)$ is a left hyperideal of R. If possible μ is not a neutrosophic left hyperideal. Then for $x, y \in R$ anyone of the following inequality is true.

$$\inf_{z \in x+y} \mu^T(z) < \min\{\mu^T(x), \mu^T(y)\}$$

$$\inf_{z \in x+y} \mu^I(z) < \frac{\mu^I(x)+\mu^I(y)}{2}$$

$$\sup_{z \in x+y} \mu^F(z) > \max\{\mu^F(x), \mu^F(y)\}$$

For the first inequality, choose $t_1 = \frac{1}{2}[\inf_{z \in x+y} \mu^T(z) + \min\{\mu^T(x), \mu^T(y)\}]$. Then $\inf_{z \in x+y} \mu^T(z) < t_1 < \min\{\mu^T(x), \mu^T(y)\}$ which implies $x, y \in \mu_{t_1}^T$ but $x+y \not\subseteq \mu_{t_1}^T$ - a contradiction.

For the second inequality, choose $t_2 = \frac{1}{2}[\inf_{z \in x+y} \mu^I(z) + \min\{\mu^I(x), \mu^I(y)\}]$. Then

$\inf_{z \in x+y} \mu^I(z) < t_2 < \frac{\mu^I(x)+\mu^I(y)}{2}$ which implies $x, y \in \mu_{t_2}^I$ but $x+y \not\subseteq \mu_{t_2}^I$ - a contradiction.

For the third inequality, choose $t_3 = \frac{1}{2}[\sup_{z \in x+y} \mu^F(z) + \max\{\mu^F(x), \mu^F(y)\}]$. Then $\sup_{z \in x+y} \mu^F(z) > t_3 > \max\{\mu^F(x), \mu^F(y)\}$ which implies $x, y \in \mu_{t_3}^F$ but $x+y \not\subseteq \mu_{t_3}^F$ - a contradiction.

So, in any case we have a contradiction to the fact that μ_t is a left hyperideal of R.

Hence the result follows.

Definition 3.5. Let μ and ν be two neutrosophic subsets of R. The intersection of μ and is ν defined by

$$(\mu^T \cap \nu^T)(x) = \min\{\mu^T(x), \nu^T(x)\}$$

$$(\mu^I \cap \nu^I)(x) = \min\{\mu^I(x), \nu^I(x)\}$$

$$(\mu^F \cap \nu^F)(x) = \max\{\mu^F(x), \nu^F(x)\}$$

for all $x \in R$.

Proposition 3.6. Intersection of a nonempty collection of neutrosophic left hyperideals is a neutrosophic left hyperideal of R.

Proof. Let $\{\mu_i : i \in I\}$ be a non-empty family of neutrosophic left hyperideals of R and $x, y \in R$. Then

$$\inf_{z \in x+y} (\bigcap_{i \in I} \mu_i^T)(z)$$

$$= \inf_{z \in x+y} \inf_{i \in I} \mu_i^T(z)$$

$$\geq \inf_{i\in I}\{\min\{\mu_i^T(x),\mu_i^T(y)\}\}$$
$$=\min\{\inf_{i\in I}\mu_i^T(x),\inf_{i\in I}\mu_i^T(y)\}$$
$$=\min\{\bigcap_{i\in I}\mu_i^T(x),\bigcap_{i\in I}\mu_i^T(y)\}$$

$$\inf_{z\in x+y}(\bigcap_{i\in I}\mu_i^I)(z)$$
$$=\inf_{z\in x+y}\inf_{i\in I}\mu_i^I(z)$$
$$\geq \inf_{i\in I}\frac{\mu_i^I(x)+\mu_i^I(y)}{2}$$
$$=\frac{\inf_{i\in I}\mu_i^I(x)+\inf_{i\in I}\mu_i^I(y)}{2}$$
$$=\frac{\bigcap_{i\in I}\mu_i^I(x)+\bigcap_{i\in I}\mu_i^I(y)}{2}.$$

$$\sup_{z\in x+y}(\bigcap_{i\in I}\mu_i^F)(z)$$
$$=\sup_{z\in x+y}\sup_{i\in I}\mu_i^F(z)$$
$$\leq \sup_{i\in I}\{\max\{\mu_i^F(x),\mu_i^F(y)\}\}$$
$$=\max\{\sup_{i\in I}\mu_i^F(x),\sup_{i\in I}\mu_i^F(y)\}$$
$$=\max\{\bigcap_{i\in I}\mu_i^F(x),\bigcap_{i\in I}\mu_i^F(y)\}$$

$$\inf_{z\in sx}(\bigcap_{i\in I}\mu_i^T)(z)$$
$$=\inf_{z\in sx}\inf_{i\in I}\mu_i^T(z)$$
$$\geq \inf_{i\in I}\mu_i^T(x)$$
$$=\bigcap_{i\in I}\mu_i^T(x)$$
$$\inf_{z\in sx}(\bigcap_{i\in I}\mu_i^I)(z)$$
$$=\inf_{z\in sx}\inf_{i\in I}\mu_i^I(z)$$
$$\geq \inf_{i\in I}\mu_i^I(x)$$
$$=\bigcap_{i\in I}\mu_i^I(x)$$

$$\sup_{z\in sx}(\bigcap_{i\in I}\mu_i^F)(z)$$
$$=\sup_{z\in sx}\sup_{i\in I}\mu_i^F(z)$$
$$\leq \sup_{i\in I}\mu_i^F(x)=\bigcap_{i\in I}\mu_i^F(x)$$

Hence $\bigcap_{i\in I}\mu_i$ is a neutrosophic left hyperideal of R.

Definition 3.7. Let R, S be semihyperrings and $f:R\to S$ be a function. Then f is said to be a homomorphism if for all $a,b\in R$

$(i)\, f(a+b)\subseteq f(a)+f(b)$

$(ii)\, f(ab)\subseteq f(a)f(b)$

$(iii)\, f(0_R)=0_S$

where 0_R and 0_S are the zeros of R and S respectively.

Proposition 3.8. Let $f:R\to S$ be a morphism of semihyperrings. Then

(i) If ϕ is a neutrosophic left hyperideal of S, then $f^{-1}(\phi)$ [13] is a neutrosophic left hyperideal of R.

(ii) If f is surjective morphism and μ is a neutronsophic left hyperideal of R, then $f(\mu)$ [13] is a neutrosophic left hyperideal of S.

Proof. Let $f:R\to S$ be a morphism of semihyperrings. Let ϕ be a neutrosophic left hyperideal of S and $r,s\in R$.

$$\inf_{z\in r+s}f^{-1}(\phi^T)(z)$$
$$=\inf_{z\in r+s}\phi^T(f(z))$$
$$=\inf_{f(z)\subseteq f(r)+f(s)}\phi^T(f(z))$$
$$\geq \min\{\phi^T(f(r)),\phi^T(f(s))\}$$
$$=\min\{f^{-1}(\phi^T)(r),f^{-1}(\phi^T)(s)\}.$$
$$\inf_{z\in r+s}f^{-1}(\phi^I)(z)$$
$$=\inf_{z\in r+s}\phi^I(f(z))$$
$$=\inf_{f(z)\subseteq f(r)+f(s)}\phi^I(f(z))$$

$$\geq \frac{\phi^I(f(r)) + \phi^I(f(s))}{2}$$
$$= \frac{f^{-1}(\phi^I)(r) + f^{-1}(\phi^I)(s)}{2}.$$
$$\sup_{z \in r+s} f^{-1}(\phi^F)(z)$$
$$= \sup_{z \in r+s} \phi^F(f(z))$$
$$= \sup_{f(z) \subseteq f(r)+f(s)} \phi^F(f(z))$$
$$\leq \max\{\phi^F(f(r)), \phi^F(f(s))\}$$
$$\leq \max\{f^{-1}(\phi^F)(r), f^{-1}(\phi^F)(s)\}.$$

Again

$$\inf_{z \in rs} f^{-1}(\phi^T)(z)$$
$$= \inf_{z \in rs} \phi^T(f(z))$$
$$= \inf_{f(z) \subseteq f(r)f(s)} \phi^T(f(z))$$
$$\geq \phi^T(f(s)) = f^{-1}(\phi^T)(s).$$

$$\inf_{z \in rs} f^{-1}(\phi^I)(z)$$
$$= \inf_{z \in rs} \phi^I(f(z))$$
$$= \inf_{f(z) \subseteq f(r)f(s)} \phi^I(f(z))$$
$$\geq \phi^I(f(s)) = f^{-1}(\phi^I)(s).$$

$$\sup_{z \in rs} f^{-1}(\phi^F)(z)$$
$$= \sup_{z \in rs} \phi^F(f(z))$$
$$= \sup_{f(z) \subseteq f(r)f(s)} \phi^F(f(z))$$
$$\leq \phi^F(f(s)) = f^{-1}(\phi^F)(s).$$

Thus $f^{-1}(\phi)$ is a neutrosophic left hyperideal of R.

(ii) Suppose μ be a neutrosophic left hyperideal of R and $x', y' \in S$. Then

$$\inf_{z' \in x'+y'} (f(\mu^T))(z')$$
$$= \inf_{z' \in x'+y'} \sup_{z \in f^{-1}(z')} \mu^T(z)$$
$$= \inf_{z' \in x'+y'} \sup_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu^T(z)$$
$$\geq \sup_{x \in f^{-1}(x'), y \in f^{-1}(y')} \{\min\{\mu^T(x), \mu^T(y)\}\}$$
$$= \min\{\sup_{x \in f^{-1}(x')} \mu^T(x), \sup_{y \in f^{-1}(y')} \mu^T(y)\}$$
$$= \min\{(f(\mu^T))(x'), (f(\mu^T))(y')\}.$$
$$\inf_{z' \in x'+y'} (f(\mu^I))(z')$$
$$= \inf_{z' \in x'+y'} \sup_{z \in f^{-1}(z')} \mu^I(z)$$
$$= \inf_{z' \in x'+y'} \sup_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu^I(z)$$
$$\geq \sup_{x \in f^{-1}(x'), y \in f^{-1}(y')} \frac{\mu^I(x) + \mu^I(y)}{2}$$
$$= \frac{1}{2}[\sup_{x \in f^{-1}(x')} \mu^I(x) + \sup_{y \in f^{-1}(y')} \mu^I(y)]$$
$$= \frac{1}{2}[(f(\mu^I))(x') + (f(\mu^I))(y')].$$
$$\sup_{z' \in x'+y'} (f(\mu^F))(z')$$
$$= \sup_{z' \in x'+y'} \inf_{z \in f^{-1}(z')} \mu^F(z)$$
$$\leq \sup_{z' \in x'+y'} \inf_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu^F(z)$$
$$\leq \inf_{x \in f^{-1}(x'), y \in f^{-1}(y')} \{\max\{\mu^F(x), \mu^F(y)\}\}$$
$$= \max\{\inf_{x \in f^{-1}(x')} \mu^F(x), \inf_{y \in f^{-1}(y')} \mu^F(y)\}$$
$$= \max\{(f(\mu^F))(x'), (f(\mu^F))(y')\}.$$

Again

$$\inf_{z' \in x'y'} (f(\mu^T))(z')$$
$$= \inf_{z' \in x'y'} \sup_{z \in f^{-1}(z')} \mu^T(z)$$
$$= \sup_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu^T(z)$$
$$\geq \sup_{y \in f^{-1}(y')} \mu^T(y) = (f(\mu^T))(y').$$

$$\inf_{z' \in x'y'} (f(\mu^I))(z')$$
$$= \inf_{z' \in x'y'} \sup_{z \in f^{-1}(z')} \mu^I(z)$$

$$= \sup_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu^I(z)$$
$$\geq \sup_{y \in f^{-1}(y')} \mu^I(y) = (f(\mu^I))(y').$$
$$\sup_{z' \in x'y'} (f(\mu^F))(z')$$
$$= \sup_{z' \in x'y'} \inf_{z \in f^{-1}(z')} \mu^F(z)$$
$$\leq \inf_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu^F(z)$$
$$\leq \inf_{y \in f^{-1}(y')} \mu^F(y) = (f(\mu^F))(y')$$

Thus $f(\mu)$ is a neutrosophic left hyperideal of S.

Definition 3.9. Let μ and ν be two neutrosophic subsets of R. Then the Cartesian product of μ and ν is defined by
$$(\mu^T \times \nu^T)(x, y) = \min\{\mu^T(x), \nu^T(y)\}$$
$$(\mu^I \times \nu^I)(x, y) = \frac{\mu^I(x) + \nu^I(y)}{2}$$
$$(\mu^F \times \nu^F)(x, y) = \max\{\mu^F(x), \nu^F(y)\}$$
for all $x, y \in R$.

Theorem 3.10. Let μ and ν be two neutrosophic left hyperideals of R. Then $\mu \times \nu$ is a neutrosophic left hyperideal of $R \times R$.

Proof. Let (x_1, x_2), $(y_1, y_2) \in R \times R$. Then
$$\inf_{(z_1, z_2) \in (x_1, x_2) + (y_1, y_2)} (\mu^T \times \nu^T)(z_1, z_2)$$
$$= \inf_{z_1 \in (x_1 + y_1), z_2 \in (x_2 + y_2)} (\mu^T \times \nu^T)(z_1, z_2)$$
$$= \inf_{z_1 \in (x_1 + y_1), z_2 \in (x_2 + y_2)} \min \{\mu^T(z_1), \nu^T(z_2)\}$$
$$\geq \min\{\min\{\mu^T(x_1), \mu^T(y_1)\}, \min\{\nu^T(x_2), \nu^T(y_2)\}\}$$
$$= \min\{\min\{\mu^T(x_1), \nu^T(x_2)\}, \min\{\mu^T(y_1), \nu^T(y_2)\}\}$$
$$= \min\{(\mu^T \times \nu^T)(x_1, x_2), (\mu^T \times \nu^T)(y_1, y_2)\}.$$

$$\inf_{(z_1, z_2) \in (x_1, x_2) + (y_1, y_2)} (\mu^I \times \nu^I)(z_1, z_2)$$
$$= \inf_{z_1 \in (x_1 + y_1), z_2 \in (x_2 + y_2)} (\mu^I \times \nu^I)(z_1, z_2)$$
$$= \inf_{z_1 \in (x_1 + y_1), z_2 \in (x_2 + y_2)} \frac{\mu^I(z_1) + \nu^I(z_2)}{2}$$
$$\geq \frac{1}{2}\left[\frac{\mu^I(x_1) + \mu^I(y_1)}{2} + \frac{\nu^I(x_2) + \nu^I(y_2)}{2}\right]$$
$$= \frac{1}{2}\left[\frac{\mu^I(x_1) + \nu^I(x_2)}{2} + \frac{\mu^I(y_1) + \nu^I(y_2)}{2}\right]$$
$$= \frac{1}{2}[(\mu^I \times \nu^I)(x_1, x_2) + (\mu^I \times \nu^I)(y_1, y_2)].$$

$$\sup_{(z_1, z_2) \in (x_1, x_2) + (y_1, y_2)} (\mu^F \times \nu^F)(z_1, z_2)$$
$$= \sup_{z_1 \in (x_1 + y_1), z_2 \in (x_2 + y_2)} (\mu^F \times \nu^F)(z_1, z_2)$$
$$= \sup_{z_1 \in (x_1 + y_1), z_2 \in (x_2 + y_2)} \max \{\mu^F(z_1), \nu^F(z_2)\}$$
$$\leq \max\{\max\{\mu^F(x_1), \mu^F(y_1)\}, \max\{\nu^F(x_2), \nu^F(y_2)\}\}$$
$$= \max\{\max\{\mu^F(x_1), \nu^F(x_2)\}, \max\{\mu^F(y_1), \nu^F(y_2)\}\}$$
$$= \max\{(\mu^F \times \nu^F)(x_1, x_2), (\mu^F \times \nu^F)(y_1, y_2)\}.$$

$$\inf_{(z_1, z_2) \in (x_1, x_2)(y_1, y_2)} (\mu^T \times \nu^T)(z_1, z_2)$$
$$= \inf_{z_1 \in x_1 y_1, z_2 \in x_2 y_2} (\mu^T \times \nu^T)(z_1, z_2)$$
$$= \inf_{z_1 \in x_1 y_1, z_2 \in x_2 y_2} \min \{\mu^T(z_1), \nu^T(z_2)\}$$
$$\geq \min\{\mu^T(y_1), \nu^T(y_2)\} = (\mu^T \times \nu^T)(y_1, y_2).$$

$$\inf_{(z_1, z_2) \in (x_1, x_2)(y_1, y_2)} (\mu^I \times \nu^I)(z_1, z_2)$$
$$= \inf_{z_1 \in x_1 y_1, z_2 \in x_2 y_2} (\mu^I \times \nu^I)(z_1, z_2)$$
$$= \inf_{z_1 \in x_1 y_1, z_2 \in x_2 y_2} \frac{\mu^I(z_1) + \nu^I(z_2)}{2}$$
$$\geq \frac{\mu^I(y_1) + \nu^I(y_2)}{2} = (\mu^I \times \nu^I)(y_1, y_2).$$
$$\sup_{(z_1, z_2) \in (x_1, x_2)(y_1, y_2)} (\mu^F \times \nu^F)(z_1, z_2)$$
$$= \sup_{z_1 \in x_1 y_1, z_2 \in x_2 y_2} (\mu^F \times \nu^F)(z_1, z_2)$$
$$= \sup_{z_1 \in x_1 y_1, z_2 \in x_2 y_2} \max \{\mu^F(z_1), \nu^F(z_2)\}$$

$\leq \max\{\mu^F(y_1), \nu^F(y_2)\} = (\mu^F \times \nu^F)(y_1, y_2)$.

Hence $\mu \times \nu$ is a neutrosophic left hyperideal of $R \times R$.

Definition 3.11. Let μ and ν be two neutrosophic sets of a semiring R. Define composition of μ and ν by

$$(\mu^T o \nu^T)(x) = \sup_{x \in \sum_{i=1}^n a_i b_i} \{\min_i \{\mu^T(a_i), \nu^T(b_i)\}\}$$

$= 0$ if x cannot be expressed as above

$$(\mu^I o \nu^I)(x) = \sup_{x \in \sum_{i=1}^n a_i b_i} \sum_{i=1}^n \frac{\mu^I(a_i) + \nu^I(b_i)}{2n}$$

$= 0$ if x cannot be expressed as above

$$(\mu^F o \nu^F)(z) = \inf_{x \in \sum_{i=1}^n a_i b_i} \{\max_i \{\mu^F(a_i), \nu^F(b_i)\}\}$$

$= 0$ if x cannot be expressed as above

where $x, a_i, b_i \in R$ for $i = 1, \ldots, n.$

Theorem 3.12. If μ and ν be two neutrosophic left hyperideals of , R then $\mu o \nu$ is a neutrosophic left hyperideal of R.

Proof. Suppose μ, ν be two neutrosophic hyperideals of R and $x, y \in R$. If $x + y \notin \sum_{i=1}^n a_i b_i$ for $a_i, b_i \in R,$ then there is nothing to proof. So, assume that $x + y \in \sum_{i=1}^n a_i b_i$ for $a_i, b_i \in R,$. Then

$$\inf_{z \in x+y} (\mu^T o \nu^T)(z)$$

$$= \inf_{z \in x+y} \sup_{x+y \in \sum_{i=1}^n a_i b_i} \{\min_i \{\mu^T(a_i), \nu^T(b_i)\}\}$$

$$\geq \sup_{x \in \sum_{i=1}^n c_i d_i, y \in \sum_{i=1}^n e_i f_i} \{\min_i \{\mu^T(c_i), \nu^T(d_i), \mu^T(e_i), \nu^T(f_i)\}\}$$

$$= \min\{\sup_{x \in \sum_{i=1}^n c_i d_i} \{\min_i \{\mu^T(c_i), \nu^T(d_i)\}\},$$
$$\sup_{y \in \sum_{i=1}^n e_i f_i} \{\min_i \{\mu^T(e_i), \nu^T(f_i)\}\}\}$$

$$= \min\{(\mu^T o \nu^T)(x), (\mu^T o \nu^T)(y)\}$$

$$\inf_{z \in x+y} (\mu^I o \nu^I)(z)$$

$$= \inf_{z \in x+y} \sup_{x+y \in \sum_{i=1}^n a_i b_i} \sum_{i=1}^n \frac{\mu^I(a_i) + \nu^I(b_i)}{2n}$$

$$\geq \sup_{x \in \sum_{i=1}^n c_i d_i, y \in \sum_{i=1}^n e_i f_i} \sum_{i=1}^n \frac{\mu^I(c_i) + \nu^I(d_i) + \mu^I(e_i) + \nu^I(f_i)}{4n}$$

$$\geq \frac{1}{2}[\sup_{x \in \sum_{i=1}^n c_i d_i,} \sum_{i=1}^n \frac{\mu^I(c_i) + \nu^I(d_i)}{2n},$$
$$\sup_{y \in \sum_{i=1}^n e_i f_i,} \sum_{i=1}^n \frac{\mu^I(e_i) + \nu^I(f_i)}{2n}]$$

$$= \frac{(\mu^I o \nu^I)(x) + (\mu^I o \nu^I)(y)}{2}$$

$$\sup_{z \in x+y} (\mu^F o \nu^F)(z)$$

$$= \sup_{z \in x+y} \inf_{x+y \in \sum_{i=1}^n a_i b_i} \{\max_i \{\mu^F(a_i), \nu^F(b_i)\}\}$$

$$\leq \inf_{x \in \sum_{i=1}^n c_i d_i, y \in \sum_{i=1}^n e_i f_i} \{\max_i \{\mu^F(c_i), \nu^F(d_i), \mu^F(e_i), \nu^F(f_i)\}\}$$

$$= \max\{\inf_{x \in \sum_{i=1}^n c_i d_i} \{\max_i \{\mu^F(c_i), \nu^F(d_i)\}\},$$
$$\inf_{y \in \sum_{i=1}^n e_i f_i} \{\max_i \{\mu^F(e_i), \nu^F(f_i)\}\}\}$$

$$= \max\{(\mu^F o \nu^F)(x), (\mu^F o \nu^F)(y)\}$$

$$\inf_{z \in xy} (\mu^T o \nu^T)(z)$$

$$= \inf_{z \in xy} \sup_{xy \in \sum_{i=1}^{n} a_i b_i} \{\min_i \{\mu^T(a_i), \nu^T(b_i)\}\}$$

$$\geq \sup_{z \in xy \in \sum_{i=1}^{n} xe_i f_i} \{\min_i \{\mu^T(xe_i), \nu^T(f_i)\}\}$$

$$\geq \sup_{y \in \sum_{i=1}^{n} e_i f_i} \{\min_i \{\mu^T(e_i), \nu^T(f_i)\}\}$$

$$= (\mu^T o \nu^T)(y)$$

$$\inf_{z \in xy} (\mu^I o \nu^I)(z)$$

$$= \inf_{z \in xy} \sup_{xy \in \sum_{i=1}^{n} a_i b_i} \sum_{i=1}^{n} \frac{\mu^I(a_i) + \nu^I(b_i)}{2n}$$

$$\geq \sup_{z \in xy \in \sum_{i=1}^{n} xe_i f_i} \sum_{i=1}^{n} \frac{\mu^I(xe_i) + \nu^I(f_i)}{2n}$$

$$\geq \sup_{y \in \sum_{i=1}^{n} e_i f_i,_i} \sum_{i=1}^{n} \frac{\mu^I(e_i) + \nu^I(f_i)}{2n}]$$

$$= (\mu^I o \nu^I)(y)$$

$$\sup_{z \in xy} (\mu^F o \nu^F)(z)$$

$$= \sup_{z \in xy} \inf_{xy \in \sum_{i=1}^{n} a_i b_i} \{\max_i \{\mu^F(a_i), \nu^F(b_i)\}\}$$

$$\leq \inf_{xy \in \sum_{i=1}^{n} xe_i f_i} \{\max_i \{\mu^F(xe_i), \nu^F(f_i)\}\}$$

$$\leq \inf_{y \in \sum_{i=1}^{n} e_i f_i} \{\max_i \{\mu^F(e_i), \nu^F(f_i)\}\}\}$$

$$= (\mu^F o \nu^F)(y) .$$

Hence $\mu o \nu$ is a neutrosophic left hyperideal of R .

Conclusion

This is the introductory paper on neutrosophic hyperideals of semihyperrings in the sense of Smarandache[14]. Our next aim to use these results to study some other properties such prime neutrosophic hyperideal, semiprime neutrosophic hyperideal,neutrosophic bi-hyperideal, neutrosophic quasi-hyperideal, radicals etc.

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N_ω –Closed Sets in Neutrosophic Topological Spaces

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Abstract. Neutrosophic set and Neutrosophic Topological spaces has been introduced by Salama[5]. Neutrosophic Closed set and Neutrosophic Continuous Functions were introduced by Salama et. al.. In this paper, we introduce the concept of Nω- closed sets and their properties in Neutrosophic topological spaces.

Keywords: Intuitionistic Fuzzy set, Neutrosophic set, Neutrosophic Topology, N_s-open set, N_s-closed set, N_ω- closed set, N_ω-open set and Nω-closure.

1. Introduction

Many theories like, Theory of Fuzzy sets[10], Theory of Intuitionistic fuzzy sets[1], Theory of Neutrosophic sets[8] and The Theory of Interval Neutrosophic sets[4] can be considered as tools for dealing with uncertainities. However, all of these theories have their own difficulties which are pointed out in[8].

In 1965, Zadeh[10] introduced fuzzy set theory as a mathematical tool for dealing with uncertainities where each element had a degree of membership. The Intuitionistic fuzzy set was introduced by Atanassov[1] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of nonmembership of each element. The neutrosophic set was introduced by Smarandache[7] and explained, neutrosophic set is a generalization of intuitionistic fuzzy set.

In 2012, Salama, Alblowi[5] introduced the concept of Neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of intuitionistic fuzzy topological space and a neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of nonmembership of each element. In 2014 Salama, Smarandache and Valeri [6] were introduced the concept of neutrosophic closed sets and neutrosophic continuous functions. In this paper, we introduce the concept of N_ω- closed sets and their properties in neutrosophic topological spaces.

2. Preliminaries

In this paper, X denote a topological space (X, τ_N) on which no separation axioms are assumed unless otherwise explicitly mentioned. We recall the following definitions, which will be used throughout this paper. For a subset A of X, Ncl(A), Nint(A) and A^c denote the neutrosophic closure, neutrosophic interior, and the complement of neutrosophic set A respectively.

Definition 2.1.[3] Let X be a non-empty fixed set. A neutrosophic set(NS for short) A is an object having the form $A = \{<x, \mu_A(x), \sigma_A(x), \upsilon_A(x)>: \text{for all } x \in X\}$. Where $\mu_A(x)$, $\sigma_A(x)$, $\upsilon_A(x)$ which represent the degree of membership, the degree of indeterminacy and the degree of nonmembership of each element $x \in X$ to the set A.

Definition 2.2.[5] Let A and B be NSs of the form $A = \{<x, \mu_A(x), \sigma_A(x), \upsilon_A(x)>: \text{for all } x \in X\}$ and $B = \{<x, \mu_B(x), \sigma_B(x), \upsilon_B(x)>: \text{for all } x \in X\}$. Then

i. $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$, $\sigma_A(x) \ge \sigma_B(x)$ and $\upsilon_A(x) \ge \upsilon_B(x)$ for all $x \in X$,
ii. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
iii. $A^c = \{<x, \upsilon_A(x), 1- \sigma_A(x), \mu_A(x)>: \text{for all } x \in X \}$,
iv. $A \cup B = \{<x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \upsilon_A(x) \wedge \upsilon_B(x): \text{for all } x \in X >\}$,
v. $A \cap B = \{<x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \upsilon_A(x) \vee \upsilon_B(x): \text{for all } x \in X >\}$.

Definition 2.3.[5] A neutrosophic topology(NT for short) on a non empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms:

i) $0_N, 1_N \in \tau$,
ii) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$,
iii) $\cup G_i \in \tau$, for all $G_i: i \in J \subseteq \tau$

In this pair (X, τ) is called a neutrosophic topological space (NTS for short) for neutrosophic set (NOS for short) τ in X. The elements of τ are called open neutrosophic sets. A neutrosophic set F is called closed if and only if the complement of F(F^c for short) is neutrosophic open.

Definition 2.4.[5] Let (X, τ) be a neutrosophic topological space. A neutrosophic set A in (X, τ) is said to be neutrosophic closed(N-closed for short) if $Ncl(A) \subseteq G$ whenever $A \subseteq G$ and G is neutrosophic open.

Definition 2.5.[5] The complement of N-closed set is N-open set.

Proposition 2.6.[6] In a neutrosophic topological space (X, T), T = $\mathfrak{I}$ (the family of all neutrosophic closed sets) iff every neutrosophic subset of (X, T) is a neutrosophic closed set.

3. N_ω -closed sets

In this section, we introduce the concept of N_ω-closed set and some of their properties. Throughout this paper (X, τ_N) represent a neutrosophic topological spaces.

Definition 3.1. Let (X, τ_N) be a neutrosophic topological space. Then A is called neutrosophic semi open set(N_s-open set for short) if A ⊂ Ncl(Nint(A)).

Definition 3.2. Let (X, τ_N) be a neutrosophic topological space. Then A is called neutrosophic semi closed set(N_s-closed set for short) if Nint(Ncl(A)) ⊂ A.

Definition 3.3. Let A be a neutrosophic set of a neutrosophic topological space (X, τ_N). Then,

i. The neutrosophic semi closure of A is defined as $N_s cl(A) = \cap$ {K: K is a N_s-closed in X and A ⊆ K}
ii. The neutrosophic semi interior of A is defined as $N_s int(A) = \cup$ {G: G is a N_s-open in X and G ⊆ A}

Definition 3.4. Let (X, τ_N) be a neutrosophic topological space. Then A is called Neutrosophic ω closed set(N_ω-closed set for short) if Ncl(A) ⊆ G whenever A ⊆ G and G is N_s-open set.

Theorem 3.5. Every neutrosophic closed set is N_ω-closed set, but the converse may not be true.
Proof: If A is any neutrosophic set in X and G is any N_s-open set containing A, then Ncl (A) ⊆ G. Hence A is N_ω-closed set.

The converse of the above theorem need not be true as seen from the following example.

Example 3.6. Let X = {a,b,c} and $\tau_N = \{0_N, G_1, 1_N\}$ is a neutrosophic topology and (X, τ_N) is a neutrosophic topological spaces. Take G_1 = <x, (0.5, 0.6, 0.4), (0.4, 0.5, 0.2), (0.7, 0.6, 0.9)>, A = <x, (0.2, 0.2, 0.1), (0, 1, 0.2), (0.8, 0.6,0.9)>. Then the set A is N_ω-closed set but A is not a neutrosophic closed.

Theorem 3.7. Every N_ω-closed set is N-closed set but not conversely.
Proof: Let A be any N_ω-closed set in X and G be any neutrosophic open set such that A ⊆ G. Then G is N_s-open, A ⊆ G and Ncl (A) ⊆ G. Thus A is N-closed.

The converse of the above theorem proved by the following example.

Example 3.8. Let X = {a,b,c} and $\tau_N = \{0_N, G_1, 1_N\}$ is a neutrosophic topology and (X, τ_N) is a neutrosophic topological spaces. Let G_1 = <x, (0.5, 0.6, 0.4), (0.4, 0.5, 0.2), (0.7, 0.6, 0.9)> and A = <x, (0.55, 0.45, 0.6), (0.11, 0.3, 0.1), (0.11, 0.25, 0.2)>. Then the set A is N-closed but A is not a N_ω-closed set.

Remark 3.9. The concepts of N_ω-closed sets and N_s-closed sets are independent.

Example 3.10. Let X = {a,b,c} and $\tau_N = \{0_N, G_1, 1_N\}$ is a neutrosophic topology and (X, τ_N) is a neutrosophic topological spaces. Take G_1 = <x, (0.5, 0.6, 0.4), (0.4, 0.5, 0.2), (0.7, 0.6, 0.9)>, A = <x, (0.2, 0.2, 0.1), (0, 1, 0.2), (0.8, 0.6,0.9)>. Then the set A is N_ω-closed set but A is not a N_s-closed set.

Example 3.11. Let X = {a,b} and $\tau_N = \{0_N, G_1, G_2, 1_N\}$ is a neutrosophic topology and (X, τ_N) is a neutrosophic topological spaces. Take G_1 = <x, (0.6, 0.7), (0.3, 0.2), (0.2, 0.1)> and A = <x, (0.3, 0.4), (0.6, 0.7), (0.9,0.9)>. Then the set A is N_s-closed set but A is not a N_ω-closed.

Theorem 3.12. If A and B are N_ω-closed sets, then A ∪ B is N_ω-closed set.
Proof: If A ∪ B ⊆ G and G is N_s-open set, then A ⊆ G and B ⊆ G. Since A and B are N_ω-closed sets, Ncl(A) ⊆ G and Ncl(B) ⊆ G and hence Ncl(A) ∪ Ncl(B) ⊆ G. This implies Ncl(A ∪ B) ⊆ G. Thus A ∪ B is N_ω-closed set in X.

Theorem 3.13. A neutrosophic set A is N_ω-closed set then Ncl(A) – A does not contain any nonempty neutrosophic closed sets.
Proof: Suppose that A is N_ω-closed set. Let F be a neutrosophic closed subset of Ncl(A) – A. Then A ⊆ F^c. But A is N_ω-closed set. Therefore Ncl(A) ⊆ F^c. Consequently F ⊆ $(Ncl(A))^c$. We have F ⊆ Ncl(A). Thus F ⊆ Ncl(A)∩$(Ncl(A))^c$ = ϕ. Hence F is empty.

The converse of the above theorem need not be true as seen from the following example.

Example 3.14. Let X = {a,b,c} and $\tau_N = \{0_N, G_1, 1_N\}$ is a neutrosophic topology and (X, τ_N) is a neutrosophic topological spaces. Take G_1 = <x, (0.5, 0.6, 0.4), (0.4, 0.5, 0.2), (0.7, 0.6, 0.9)> and A = <x, (0.2, 0.2, 0.1), (0.6, 0.6, 0.6), (0.8, 0.9,0.9)>. Then the set A is not a N_ω-closed set and Ncl(A) – A = <x, (0.2, 0.2, 0.1), (0.6, 0.6, 0.6), (0.8, 0.9,0.9)> does not contain non-empty neutrosophic closed sets.

Theorem 3.15. A neutrosophic set A is N_ω-closed set if and only if Ncl(A) – A contains no non-empty N_s-closed set.

Proof: Suppose that A is N_ω-closed set. Let S be a N_s-closed subset of Ncl(A) – A. Then $A \subseteq S^c$. Since A is N_ω-closed set, we have $Ncl(A) \subseteq S^c$. Consequently $S \subseteq (Ncl(A))^c$. Hence $S \subseteq Ncl(A) \cap (Ncl(A))^c = \phi$. Therefore S is empty.
Conversely, suppose that Ncl(A) – A contains no non-empty N_s-closed set. Let $A \subseteq G$ and that G be N_s-open. If $Ncl(A) \nsubseteq G$, then $Ncl(A) \cap G^c$ is a non-empty N_s-closed subset of Ncl(A) – A. Hence A is N_ω-closed set.

Corollary 3.16. A N_ω-closed set A is N_s-closed if and only if $N_scl(A) – A$ is N_s-closed.
Proof: Let A be any N_ω-closed set. If A is N_s-closed set, then $N_scl(A) – A = \phi$. Therefore $N_scl(A) – A$ is N_s-closed set.
Conversely, suppose that Ncl(A) – A is N_s-closed set. But A is N_ω-closed set and Ncl(A) – A contains N_s-closed. By theorem 3.15, $N_scl(A) – A = \phi$. Therefore $N_scl(A) = A$. Hence A is N_s-closed set.

Theorem 3.17. Suppose that $B \subseteq A \subseteq X$, B is a N_ω-closed set relative to A and that A is N_ω-closed set in X. Then B is N_ω-closed set in X.
Proof: Let $B \subseteq G$, where G is N_s-open in X. We have $B \subseteq A \cap G$ and $A \cap G$ is N_s-open in A. But B is a N_ω-closed set relative to A. Hence $Ncl_A(B) \subseteq A \cap G$. Since $Ncl_A(B) = A \cap Ncl(B)$. We have $A \cap Ncl(B) \subseteq A \cap G$. It implies $A \subseteq G \cup (Ncl(B))^c$ and $G \cup (Ncl(B))^c$ is a N_s-open set in X. Since A is N_ω-closed in X, we have $Ncl(A) \subseteq G \cup (Ncl(B))^c$. Hence $Ncl(B) \subseteq G \cup (Ncl(B))^c$ and $Ncl(B) \subseteq G$. Therefore B is N_ω-closed set relative to X.

Theorem 3.18. If A is N_ω-closed and $A \subseteq B \subseteq Ncl(A)$, then B is N_ω-closed.
Proof: Since $B \subseteq Ncl(A)$, we have $Ncl(B) \subseteq Ncl(A)$ and $Ncl(B) – B \subseteq Ncl(A) – A$. But A is N_ω-closed. Hence Ncl(A) – A has no non-empty N_s-closed subsets, neither does Ncl(B) – B. By theorem 3.15, B is N_ω-closed.

Theorem 3.19. Let $A \subseteq Y \subseteq X$ and suppose that A is N_ω-closed in X. Then A is N_ω-closed relative to Y.
Proof: Let $A \subseteq Y \cap G$ where G is N_s-open in X. Then $A \subseteq G$ and hence $Ncl(A) \subseteq G$. This implies, $Y \cap Ncl(A) \subseteq Y \cap G$. Thus A is N_ω-closed relative to Y.

Theorem 3.20. If A is N_s-open and N_ω-closed, then A is neutrosophic closed set.
Proof: Since A is N_s-open and N_ω-closed, then $Ncl(A) \subseteq A$. Therefore Ncl(A) =A. Hence A is neutrosophic closed.

4. N_ω-open sets

In this section, we introduce and study about N_ω-open sets and some of their properties.

Definition 4.1. A Neutrosophic set A in X is called N_ω-open in X if A^c is N_ω-closed in X.

Theorem 4.2. Let (X, τ_N) be a neutrosophic topological space. Then

(i) Every neutrosophic open set is N_ω-open but not conversely.
(ii) Every N_ω-open set is N-open but not conversely.

The converse part of the above statements are proved by the following example.

Example 4.3. Let X = {a,b,c} and $\tau_N = \{0_N, G_1, 1_N\}$ is a neutrosophic topology and (X, τ_N) is a neutrosophic topological space. Take G_1 = <x, (0.7, 0.6, 0.9), (0.6, 0.5, 0.8), (0.5, 0.6, 0.4)> and A = <x, (0.8, 0.6, 0.9), (1, 0, 0.8), (0.2, 0.2, 0.1)>. Then the set A is N_ω-open set but not a neutrosophic open and B = <x, (0.11, 0.25, 0.2), (0.89, 0.7, 0.9), (0.55, 0.45, 0.6)> is N-open but not a N_ω-open set.

Theorem 4.4. A neutrosophic set A is N_ω-open if and only if $F \subseteq Nint(A)$ where F is N_s-closed and $F \subseteq A$.
Proof: Suppose that $F \subseteq Nint(A)$ where F is N_s-closed and $F \subseteq A$. Let $A^c \subseteq G$ where G is N_s-open. Then $G^c \subseteq A$ and G^c is N_s-closed. Therefore $G^c \subseteq Nint(A)$. Since $G^c \subseteq Nint(A)$, we have $(Nint(A))^c \subseteq G$. This implies $Ncl((A)^c) \subseteq G$. Thus A^c is N_ω-closed. Hence A is N_ω-open.
Conversely, suppose that A is N_ω-open, $F \subseteq A$ and F is N_s-closed. Then F^c is N_s-open and $A^c \subseteq F^c$. Therefore $Ncl((A)^c) \subseteq F^c$. But $Ncl((A)^c) = (Nint(A))^c$. Hence $F \subseteq Nint(A)$.

Theorem 4.5. A neutrosophic set A is N_ω-open in X if and only if G = X whenever G is N_s-open and $(Nint(A) \cup A^c) \subseteq G$.
Proof: Let A be a N_ω-open, G be N_s-open and $(Nint(A) \cup A^c) \subseteq G$. This implies $G^c \subseteq (Nint(A))^c \cap ((A)^c)^c = (Nint(A))^c – A^c = Ncl((A)^c) – A^c$. Since A^c is N_ω-closed and G^c is N_s-closed, by Theorem 3.15, it follows that $G^c = \phi$. Therefore X = G.
Conversely, suppose that F is N_s-closed and $F \subseteq A$. Then $Nint(A) \cup A^c \subseteq Nint(A) \cup F^c$. This implies $Nint(A) \cup F^c = X$ and hence $F \subseteq Nint(A)$. Therefore A is N_ω-open.

Theorem 4.6. If $Nint(A) \subseteq B \subseteq A$ and if A is N_ω-open, then B is N_ω-open.
Proof: Suppose that $Nint(A) \subseteq B \subseteq A$ and A is N_ω-open. Then $A^c \subseteq B^c \subseteq Ncl(A^c)$ and since A^c is N_ω-closed. We have by Theorem 3.15, B^c is N_ω-closed. Hence B is N_ω-open.

Theorem 4.7. A neutrosophic set A is N_ω-closed, if and only if Ncl(A) – A is N_ω-open.
Proof: Suppose that A is N_ω-closed. Let $F \subseteq Ncl(A) – A$ Where F is N_s-closed. By Theorem 3.15, $F = \phi$. Therefore $F \subseteq Nint((Ncl(A) – A)$ and by Theorem 4.4, we have Ncl(A) – A is N_ω-open.

Conversely, let A ⊆ G where G is a N_s-open set. Then $Ncl(A) \cap G^c \subseteq Ncl(A) \cap A^c = Ncl(A) - A$. Since $Ncl(A) \subseteq G^c$ is N_s-closed and $Ncl(A) - A$ is N_ω-open. By Theorem 4.4, we have $Ncl(A) \cap G^c \subseteq Nint(Ncl(A) - A) = \phi$. Hence A is N_ω-closed.

Theorem 4.8. For a subset A⊆X the following are equivalent:

(i) A is N_ω-closed.

(ii) $Ncl(A) - A$ contains no non-empty N_s-closed set.

(iii) $Ncl(A) - A$ is N_ω-open set.

Proof: Follows from Theorem 3.15 and Theorem 4.7.

5. N_ω-closure and Properties of N_ω-closure

In this section, we introduce the concept of N_ω-closure and some of their properties.

Definition 5.1. The N_ω-closure (briefly $N_\omega cl(A)$) of a subset A of a neutrosophic topological space (X, τ_N) is defined as follows:

$N_\omega cl(A) = \cap \{ F \subseteq X / A \subseteq F$ and F is N_ω-closed in $(X, \tau_N)\}$.

Theorem 5.2. Let A be any subset of (X, τ_N). If A is N_ω-closed in (X, τ_N) then $A = N_\omega cl(A)$.

Proof: By definition, $N_\omega cl(A) = \cap \{ F \subseteq X / A \subseteq F$ and F is a N_ω-closed in $(X, \tau_N)\}$ and we know that A ⊆ A. Hence $A = N_\omega cl(A)$.

Remark 5.3. For a subset A of (X, τ_N), $A \subseteq N_\omega cl(A) \subseteq Ncl(A)$.

Theorem 5.4. Let A and B be subsets of (X, τ_N). Then the following statements are true:

i. $N_\omega cl(A) = \phi$ and $N_\omega cl(A) = X$.

ii. If A ⊆ B, then $N_\omega cl(A) \subseteq N_\omega cl(B)$

iii. $N_\omega cl(A) \cup N_\omega cl(B) \subset N_\omega cl(A \cup B)$

iv. $N_\omega cl(A \cap B) \subset N_\omega cl(A) \cap N_\omega cl(B)$

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Expanding Comparative Literature into Comparative Sciences Clusters with Neutrosophy and Quad-stage Method

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Abstract. By using Neutrosophy and Quad-stage Method, the expansions of comparative literature include: comparative social sciences clusters, comparative natural sciences clusters, comparative interdisciplinary sciences clusters, and so on. Among them, comparative social sciences clusters include: comparative literature, comparative history, comparative philosophy, and so on; comparative natural sciences clusters include: comparative mathematics, comparative physics, comparative chemistry, comparative medicine, comparative biology, and so on. In addition, comparative literature itself can also be expanded. Under the two main categories of research and practice, comparative literature can be expanded into: comparative literature research, comparative literature practice (including comparative essay, comparative fiction, comparative poetry, comparative drama, and so on), comparative literature research and practice, and so on. This paper discusses the applications of comparative method in comparative sciences clusters and their various branches. Point out that in the existing fields of social sciences and natural sciences, many sprouts of comparative sciences clusters can be found, but a wide range of the achievements of comparative sciences clusters, still are the virgin lands to be developed.

Keywords: Comparative, comparative sciences clusters, comparative social sciences clusters, comparative natural sciences clusters, comparative interdisciplinary sciences clusters, comparative literature, comparative history, comparative philosophy, comparative mathematics, comparative physics, comparative chemistry, comparative medicine, comparative biology, comparative essay, comparative fiction, comparative poetry, comparative drama.

1 Introduction

Comparative literature is the literary branch running comparative study (research) about the relationship between two or more kinds of literatures. It consists of influence study, parallel study, interdisciplinary study, and so on.

At present, the research method of comparative literature has expanded into other areas, and establish many disciplines such as comparative sociology, comparative jurisprudence, and so on. But the expansion is not enough. In this paper, we try to expand comparative literature into comparative sciences clusters (including comparative social sciences clusters, comparative natural sciences clusters, comparative interdisciplinary sciences clusters, and so on).

2 Basic Contents of Neutrosophy and Basic Contents of Quad-stage

Neutrosophy is proposed by Prof. Florentin Smarandache in 1995.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <Anti-A> and the spectrum of "neutralities" <Neut-A> (i.e. notions or ideas located between the two extremes, supporting neither <A> nor <Anti-A>). The <Neut-A> and <Anti-A> ideas together are referred to as <Non-A>.

Neutrosophy is the base of neutrosophic logic, neutrosophic set, neutrosophic probability and statistics used in engineering applications (especially for software and information fusion), medicine, military, cybernetics, and physics.

Neutrosophic Logic is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionistic logic, etc. The main idea of NL is to characterize each logical statement in a 3D Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of]-0, 1+[without necessarily connection between them.

More information about Neutrosophy can be found in references [1，2].

Quad-stage (Four stages) is presented in reference [3], it is the expansion of Hegel’s triad-stage (triad thesis, antithesis, synthesis of development). The four stages are "general theses", "general antitheses", "the most important and the most complicated universal relations", and "general syntheses". They can be stated as follows.

The first stage, for the beginning of development (thesis), the thesis should be widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on; this is the stage of general theses. It should be noted that, here the thesis will be evolved into two or three, even more theses step by step. In addition, if in other stage we find that the first stage's work is not yet completed, then we may come back to do some additional work for the first stage.

The second stage, for the appearance of opposite (antithesis), the antithesis should be also widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on; this is the stage of general antitheses. It should be also noted that, here the antithesis will be evolved into two or three, even more antitheses step by step.

The third stage is the one that the most important and the most complicated universal relations, namely the seedtime inherited from the past and carried on for the future. Its purpose is to establish the universal relations in the widest scope. This widest scope contains all the regions related and non-related to the "general theses", "general antitheses", and the like. This stage's foundational works are to contact, grasp, discover, dig, and even create the opportunities, pieces of information, and so on as many as possible. The degree of the universal relations may be different, theoretically its upper limit is to connect all the existences, pieces of information and so on related to matters, spirits and so on in the universe; for the cases such as to create science fiction, even may connect all the existences, pieces of information and so on in the virtual world. Obviously, this stage provides all possibilities to fully use the complete achievements of nature and society, as well as all the humanity's wisdoms in the past, present and future. Therefore this stage is shortened as "universal relations" (for other stages, the universal relations are also existed, but their importance and complexity cannot be compared with the ones in this stage).

The fourth stage, to carry on the unification and synthesis regarding various opposites and the suitable pieces of information, factors, and so on; and reach one or more results which are the best or agreed with some conditions; this is the stage of "general syntheses". The results of this stage are called "synthesized second generation theses", all or partial of them may become the beginning of the next quad-stage.

For realizing the innovations in the areas such as science and technology, literature and art, and the like, it is a very useful tool to combine neutrosophy with quad-stage method. For example, in reference [4], expanding Newton mechanics with neutrosophy and quad-stage method, and establishing New Newton Mechanics taking law of conservation of energy as unique source law; in reference [5], negating four color theorem with neutrosophy and quad-stage method, and "the two color theorem" and "the five color theorem" are derived to replace "the four color theorem"; in reference [6], expanding Hegelian triad thesis, antithesis, synthesis with Neutrosophy and Quad-stage Method; in reference [7], interpretating and expanding Laozi's governing a large country is like cooking a small fish with Neutrosophy and Quad-stage Method; in reference [8], interpretating and expanding the meaning of "Yi" with Neutrosophy and Quad-stage Method; in reference [9], Creating Generalized and Hybrid Set and Library with Neutrosophy and Quad-stage Method.

Applying Neutrosophy and Quad-stage Method, will significantly help us to consider all possible situations. Therefore, Neutrosophy and Quad-stage Method can play a very important role to expand comparative literature.

3 Expanding Comparative Literature with Neutrosophy and Quad-stage Method

The process of expanding comparative literature can be divided into four stages.

The first stage (stage of "general theses"), for the beginning of development, the thesis (namely "comparative literature") should be widely, deeply, carefully and repeatedly contacted, explored, analyzed, perfected and so on.

Currently, "comparative literature" has become a complex subject. Its research achievements absorb the research results of traditional world literature, as well as a variety of other areas even including natural science research; in fact, the inherent discipline bounds have been broken, and beyond the limitations of region and time, put the Asian-African literature, European-American literature, and so on, as well as classical literature, modern literature, and so on, into one or more overall structures or frames.

For example, in the research (study) of comparative literature, the literature can be compared with social sciences (philosophy, psychology, linguistics, history, sociology, anthropology, and so on), and the natural sciences (mathematical statistics, computer technology, system theory, information theory, and so on), as well as other artistic disciplines (painting, sculpture, architecture, music, film, and so on).

Of course, we should also see that different scholars may have different viewpoints and interpretations for "comparative literature" and the related problems, and the different opinions and arguments will be endless from generation to generation.

In the second stage (the stage of "general antitheses"), the opposites (antitheses) should be discussed carefully. Obviously, there are more than one opposites (antitheses) of comparative literature here.

For example, according to the viewpoint of Neutrosophy, if "comparative literature" is considered as the concept <A>, the opposite <Anti-A> may be: "non-comparative literature" (such as comparative sociology, comparative jurisprudence, and so on); while the neutral (middle state) fields <Neut-A> including: "undetermined comparative literature" (neither "comparative literature", nor "non-comparative literature"; or, sometimes it is "comparative literature", and sometimes it is "non-comparative literature; and so on".

In the third stage, considering the most important and the most complicated universal relations to link with "comparative literature". The purpose of this provision stage is to establish the universal relations in the widest scope.

For "comparative literature", different people will have different research methods and findings; even if for the same person, at different times and in different situations, he or she may also apply different research methods and reach different research results. Therefore, pursuing the unique right research method and research result do not seem to make sense. So the advisable method of work is to collect all people's research methods and research results from ancient times to modern times, and plus own research methods and research results, to form the so-called "full research methods and research results", and to store up them as Think Tank; while once we need to apply them, then immediately the one or several best research methods and research results can be elected, or according to the information in Think Tank and the reality to obtain one or several best programmes temporarily, thus we can be invincible.

Now we list some specific research methods and results.

The first school of comparative literature in the world is France school. Characterized by respecting the facts, and emphasizing the textual studies; and the research achievements occupy a glorious page in the history of world literature.

Later, United States school is appeared and takig "parallel study" as the symbol, the scholars of this school consider that literature as a discipline should compare with other disciplines.

At present, in United Kingdom, Russia, China and other countries, comparative literature studies have achieved fruitful results.

In the fourth stage, the comprehensive results of the front three stages can be used to expand "comparative literature" with a variety of ways and means. Here we mainly according to Neutrosophy and Quad-stage method to seek expanded results.

According to Neutrosophy and Quad-stage method, if the social sciences can be considered as <A>, then the natural sciences can be considered as the opposite <Anti-A>, and the interdisciplinary sciences can be considered as <Neut-A> (neutral A).

Firstly, link to social sciences, "comparative literature" should be expanded into "comparative social sciences", or "comparative social sciences clusters" including comparative literature, comparative history, comparative philosophy, and so on.

Secondly, link to natural sciences, "comparative literature" should be expanded into "comparative natural sciences", or "comparative natural sciences clusters" including comparative mathematics, comparative physics, chemistry, comparative medicine, comparative biology, and so on.

Thirdly, link to interdisciplinary sciences, "comparative literature" should be expanded into "comparative interdisciplinary sciences", or "comparative interdisciplinary sciences clusters" including comparative mathematical medicine, comparative mathematical biology, and so on.

In addition, the "comparative literature" itself can also be expanded. In addition, comparative literature itself can also be expanded. In references [10], under the two main categories of research and practice, comparative literature can be expanded into: comparative literature research, comparative literature practice (Including comparative essay, comparative fiction, comparative poetry, comparative drama, and so on), comparative literature research and practice, and so on. For the sake of convenience of classification, and to distinguish with other forms of work, naming the essay created by comparative method as comparative essay, the fiction created by comparative method as comparative fiction, the poetry created by comparative method as comparative poetry, the drama created by comparative method as comparative drama, and so on.

4 Applications of comparative method in comparative sciences clusters and their branches

"Comparison" means: according to the certain standards and methodologies, to identify advantages and disadvantages, same and different, beauty and ugliness, and so on between two or more things.

The principle of comparison: there shall be the object to be compared with, as well as the common comparative foundation, and the certain standards and methods, and so on; as comparing, we should try to consider all possible situations.

Based on the above concepts and principles, comparative method can be widely used in comparative sciences clusters and their various branches, and provide a variety of ways and broad space for development.

Firstly we discuss the comparative objects. In comparative sciences clusters and their various branches, the comparative objects can be selected within the large range, the medium range, and the small range.

Secondly we discuss the comparative standards. The comparative standards can be selected as: advantages and disadvantages, same and different, beauty and ugliness, and so on. As taking advantages and disadvantages as the comparative standard, the comparative result can be decided by experts, by ordinary scholars and readers, and by all the people (including experts, ordinary scholars and readers).

As for the methods and ways for comparison, they are also numerous. For example, to compare according to the time sequence, according to the different spatial locations; or according to the longitudinal direction and transverse direction; as well as qualitative comparison, quantitative comparison, macro-comparison, micro-comparison; and the combination of different methods and ways.

It needs to be emphasized that, when comparing, we should try to consider all possible situations. This is also the great feature of comparative sciences clusters and their various branches.

As for how to consider all possible situations, we will discuss this problem in another paper.

It should be noted that, in the existing fields of social sciences and natural sciences, many sprouts of comparative sciences clusters can be found, but a wide range of the achievements of comparative sciences clusters, still are the virgin lands to be developed.

Conclusions

Applying comparative methods and ways in comparative sciences clusters and their various branches, will play an extremely important role to promote the development of social sciences, natural sciences, interdisciplinary sciences, and so on; and continue to make new achievements.

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On Neutrosophic Quadruple Algebraic Structures

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Abstract. In this paper we present the concept of neutrosophic quadruple algebraic structures. Specially, we study neutrosophic quadruple rings and we present their elementary properties.

Keywords: Neutrosophy, neutrosophic quadruple number, neutrosophic quadruple semigroup, neutrosophic quadruple group, neutrosophic quadruple ring, neutrosophic quadruple ideal, neutrosophic quadruple homomorphism.

1 Introduction

The concept of neutrosophic quadruple numbers was introduced by Florentin Smarandache [3]. It was shown in [3] how arithmetic operations of addition, subtraction, multiplication and scalar multiplication could be performed on the set of neutrosophic quadruple numbers. In this paper, we studied neutrosophic sets of quadruple numbers together with binary operations of addition and multiplication and the resulting algebraic structures with their elementary properties are presented. Specially, we studied neutrosophic quadruple rings and we presented their basic properties.

Definition 1.1 [3]

A neutrosophic quadruple number is a number of the form (a, bT, cI, dF), where T, I, F have their usual neutrosophic logic meanings and $a, b, c, d \in \mathbb{R}$ or $\mathbb{C}$. The set NQ defined by

$$NQ = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\} \quad (1)$$

is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number (a, bT, cI, dF), representing any entity which may be a number, an idea, an object, etc., a is called the known part and (bT, cI, dF) is called the unknown part.

Definition 1.2

Let

$a = (a_1, a_2T, a_3I, a_4F)$,
$b = (b_1, b_2T, b_3I, b_4F) \in NQ$.

We define the following:

$$a + b = (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F) \quad (2)$$

$$a - b = (a_1 - b_1, (a_2 - b_2)T, (a_3 - b_3)I, (a_4 - b_4)F). \quad (3)$$

Definition 1.3

Let

$a = (a_1, a_2T, a_3I, a_4F) \in NQ$

and let α be any scalar which may be real or complex, the scalar product $\alpha.a$ is defined by

$$\alpha.a = \alpha.(a_1, a_2T, a_3I, a_4F) = (\alpha a_1, \alpha a_2T, \alpha a_3I, \alpha a_4F) \quad (4)$$

If $\alpha = 0$, then we have $0.a = (0, 0, 0, 0)$ and for any non-zero scalars m and n and $b = (b_1, b_2T, b_3I, b_4F)$, we have:

$(m + n)a = ma + na$,
$m(a + b) = ma + mb$,
$mn(a) = m(na)$,
$-a = (-a_1, -a_2T, -a_3I, -a_4F)$.

Definition 1.4 [3] [Absorbance Law]

Let X be a set endowed with a total order $x < y$, named "x prevailed by y" or "x less strong than y" or "x less preferred than y". $x \leq y$ is considered as "x prevailed by or equal to y" or "x less strong than or equal to y" or "x less preferred than or equal to y".

For any elements $x, y \in X$, with $x \leq y$, absorbance law is defined as

$$x \cdot y = y \cdot x = \text{absorb}(x, y) = \max\{x, y\} = y \quad (5)$$

which means that the bigger element absorbs the smaller element (*the big fish eats the small fish*). It is clear from (5) that

$$x \cdot x = x^2 = absorb(x, x) = max\{x, x\} = x \quad (6)$$

and

$$x_1 \cdot x_2 \cdots x_n = max\{x_1, x_2, \cdots, x_n\}. \quad (7)$$

Analogously, if $x > y$, we say that "x prevails to y" or "x is stronger than y" or "x is preferred to y". Also, if $x \geq y$, we say that "x prevails or is equal to y" or "x is stronger than or equal to y" or "x is preferred or equal to y".

Definition 1.5

Consider the set $\{T, I, F\}$. Suppose in an optimistic way we consider the prevalence order $T > I > F$. Then we have:

$$TI = IT = \max\{T, I\} = T, \tag{8}$$
$$TF = FT = \max\{T, F\} = T, \tag{9}$$
$$IF = FI = \max\{I, F\} = I, \tag{10}$$
$$TT = T^2 = T, \tag{11}$$
$$II = I^2 = I, \tag{12}$$
$$FF = F^2 = F. \tag{13}$$

Analogously, suppose in a pessimistic way we consider the prevalence order $T < I < F$. Then we have:

$$TI = IT = max\{T, I\} = I, \tag{14}$$
$$TF = FT = max\{T, F\} = F, \tag{15}$$
$$IF = FI = max\{I, F\} = F, \tag{16}$$
$$TT = T^2 = T, \tag{17}$$
$$II = I^2 = I, \tag{18}$$
$$FF = F^2 = F. \tag{19}$$

Definition 1.6

Let

$$a = (a_1, a_2T, a_3I, a_4F),$$
$$b = (b_1, b_2T, b_3I, b_4F) \in NQ.$$

Then (20)

$$a.b = (a_1, a_2T, a_3I, a_4F).(b_1, b_2T, b_3I, b_4F) = (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, (a_1b_4 + a_2b_4, a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F).$$

2 Main Results

All neutrosophic quadruple numbers to be considered in this section will be real neutrosophic quadruple numbers i.e $a, b, c, d \in \mathbb{R}$ for any neutrosophic quadruple number $(a, bT, cI, dF) \in NQ$.

Theorem 2.1

$(NQ, +)$ is an abelian group.

Proof.

Suppose that

$$a = (a_1, a_2T, a_3I, a_4F),$$
$$b = (b_1, b_2T, b_3I,$$
$$c = (c_1, c_2T, c_3I, c_4F \in NQ$$

are arbitrary.

It can easily be shown that

$$a + b = b + a \cdot a + (b + c) = (a + b) + c \cdot a + (0,0,0,0) = (0,0,0,0) = a$$

and

$$a + (-a) = -a + a = (0,0,0,0).$$

Thus, $0 = (0,0,0,0)$ is the additive identity element in $(NQ, +)$ and for any $a \in NQ, -a$ is the additive inverse. Hence, $(NQ, +)$ is an abelian group.

Theorem 2.2

$(NQ, .)$ is a commutative monoid.

Proof.

Let

$$a = (a_1, a_2T, a_3I, a_4F),$$
$$b = (b_1, b_2T, b_3I,$$
$$c = (c_1, c_2T, c_3I, c_4F$$

be arbitrary elements in NQ. It can easily be shown that

$$ab = ba \cdot a(bc) = (ab)c \cdot a \cdot (1, 0, 0, 0) = a.$$

Thus, $e = (1, 0, 0, 0)$ is the multiplicative identity element in $(NQ, .)$. Hence, $(NQ, .)$ is a commutative monoid.

Theorem 2.3

$(NQ, .)$ is not a group.

Proof.

Let

$$x = (a, bT, cI, dF)$$

be any arbitrary element in NQ.

Since we cannot find any element $y = (p, qT, rI, sF) \in NQ$ such that $xy = yx = e = (1, 0, 0, 0)$, it follows that $x - 1$ does not exist in NQ for any given $a, b, c, d \in \mathbb{R}$ and consequently, $(NQ, .)$ cannot be a group.

Example 1.

Let $X = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{Z}_n\}$. Then $(X, +)$ is an abelian group.

Example 2.

Let

$$(M_{2\times2}, .) = \left\{ \begin{matrix} \begin{bmatrix} (a, bT, cI, dF) & (e, fT, gI, hF) \\ (i, jT, kI, lF) & (m, nT, pI, qF) \end{bmatrix} : \\ \text{a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, q} \in \mathbb{R} \end{matrix} \right\}$$

Then $(M_{2\times2}, .)$ is a non-commutative monoid.

Theorem 2.4

$(NQ, +, .)$ is a commutative ring.

Proof.

It is clear that $(NQ, +)$ is an abelian group and $(NQ, .)$ is a semigroup. To complete the proof, suppose that

$a = (a_1, a_2T, a_3I, a_4F)$,
$b = (b_1, b_2T, b_3I,$
$c = (c_1, c_2T, c_3I, c_4F \in NQ$

are arbitrary. It can easily be shown that $a(b + c) = ab + ac, (b + c)a = ba + ca$ and $ab = ba$. Hence, $(NQ, +, .)$ is a commutative ring.

From now on, the ring $(NQ, +, .)$ will be called neutrosophic quadruple ring and it will be denoted by NQR. The zero element of NQR will be denoted by (0, 0, 0, 0) and the unity of NQR will be denoted by (1, 0, 0, 0).

Example 3.

(i) Let X be as defined in EXAMPLE 1. Then $(X, +, .)$ is a commutative neutrosophic quadruple ring called a neutrosophic quadruple ring of integers modulo n.

It should be noted that $NQR(\mathbb{Z}_n)$ has 4^n elements and for $NQR(\mathbb{Z}_2)$we have

$NQR(\mathbb{Z}_2) =$
$= \{(0,0,0,0), (1,0,0,0), (0, T, 0,0), (0,0, I, 0), (0,0,0, F),$
$(0, T, I, F), (0,0, I, F), (0, T, I, 0), (0, T, 0, F), (1, T, 0,0),$
$(1,0, I, 0), (1,0,0, F), (1, T, 0, F), (1,0, I, F), (1, T, I, 0),$
$(1, T, I, F)\}$.

(ii) Let $M_{2\times2}$ be as defined in EXAMPLE 2. Then $(M_{2\times2},.)$ is a non-commutative neutrosophic quadruple ring.

Definition 2.5

Let NQR be a neutrosophic quadruple ring.

(i) An element $a \in NQR$ is called idempotent if $a^2 = a$.

(ii) An element $a \in NQR$ is called nilpotent if there $exists\ n \in Z^+$ such that $a^n = 0$.

Example 4.

(i) In $NQR(\mathbb{Z}_2)$, $(1, T, I, F)$ and $(1, T, I, 0)$ are idempotent elements.

(ii) In $NQR(\mathbb{Z}_4)$, $(2,2T, 2I, 2F)$ is a nilpotent element.

Definition 2.6

Let NQR be a neutrosophic quadruple ring.

NQR is called a neutrosophic quadruple integral domain if for $x, y \in NQR$, $xy = 0$ implies that $x = 0$ or $y = 0$.

Example 5.

$NQR(\mathbb{Z})$ the neutrosophic quadruple ring of integers is a neutrosophic quadruple integral domain.

Definition 2.7

Let NQR be a neutrosophic quadruple ring.

An element $x \in NQR$ is called a zero divisor if there exists a nonzero element $y \in NQR$ such that $xy = 0$. For example in $NQR(\mathbb{Z}_2)$, $(0, 0, I, F)$ and $(0, T, I, 0)$ are zero divisors even though $\mathbb{Z}_2$ has no zero divisors.

This is one of the distinct features that characterize neutrosophic quadruple rings.

Definition 2.8

Let NQR be a neutrosophic quadruple ring and let NQS be a nonempty subset of NQR. Then NQS is called a neutrosophic quadruple subring of NQR if $(NQS, +, .)$ is itself a neutrosophic quadruple ring. For example, $NQR(n\mathbb{Z})$ is a neutrosophic quadruple subring of $NQR(\mathbb{Z})$ for $n = 1, 2, 3,\cdots$.

Theorem 2.9

Let NQS be a nonempty subset of a neutrosophic quadruple ring NQR. Then NQS is a neutrosophic quadruple subring if and only if for all $x, y \in NQS$, the following conditions hold:

(i) $x - y \in NQS$

and

(ii) $xy \in NQS$.

Proof.

Same as the classical case and so omitted.

Definition 2.10

Let NQR be a neutrosophic quadruple ring.
Then the set

$Z(NQR) = \{x \in NQR: xy = yx\ \forall\ y \in NQR\}$

is called the centre of NQR.

Theorem 2.11

Let NQR be a neutrosophic quadruple ring.
Then $Z(NQR)$ is a neutrosophic quadruple subring of NQR.

Proof.

Same as the classical case and so omitted.

Theorem 2.12

Let NQR be a neutrosophic quadruple ring and let NQS_j be families of neutrosophic quadruple subrings of NQR. Then

$$\bigcap_{j=1}^{n} NQS_j$$

is a neutrosophic quadruple subring of NQR.

Definition 2.13

Let NQR be a neutrosophic quadruple ring.
If there exists a positive integer n such that $nx = 0$ for

each $x \in NQR$, then the smallest such positive integer is called the characteristic of NQR. If no such positive integer exists, then NQR is said to have characteristic zero. For example, $NQR(\mathbb{Z})$ has characteristic zero and $NQR(\mathbb{Z}_n)$ has characteristic n.

Definition 2.14

Let NQJ be a nonempty subset of a neutrosophic quadruple ring NQR. NQJ is called a neutrosophic quadruple ideal of NQR if for all $x, y \in NQJ$, $r \in NQR$, the following conditions hold:

(i) $x - y \in NQJ$.

(ii) $xr \in NQJ$ and $rx \in NQJ$.

Example 6.

(i) $NQR(3\mathbb{Z})$ is a neutrosophic quadruple ideal of $NQR(\mathbb{Z})$.

(ii) Let

$$NQJ = \{(0,0,0,0), (2,0,0,0), (0,2T,2I,2F), (2,2T,2I,2F)\}$$

be a subset of $NQR(\mathbb{Z}_4)$. Then NQJ is a neutrosophic quadruple ideal.

Theorem 2.15

Let NQJ and NQS be neutrosophic quadruple ideals of NQR and let

$$\{NQJ_j\}_{j=1}^{n}$$

be a family of neutrosophic quadruple ideals of NQR. Then:

(i) $NQJ + NQJ = NQJ$.

(ii) $x + NQJ = NQJ$ for all $x \in NQJ$.

(iii)

$$\bigcap_{j=1} n\, NQS_j$$

is a neutrosophic quadruple ideal of NQR.

(iv) $NQJ + NQS$ is a neutrosophic quadruple ideal of NQR.

Definition 2.16

Let NQJ be a neutrosophic quadruple ideal of NQR. The set

$$NQR/NQJ = \{x + NQJ : x \in NQR\}$$

is called a neutrosophic quadruple quotient ring.

If $x + NQJ$ and $y + NQJ$ are two arbitrary elements of NQR/NQJ and if $\oplus$ and $\odot$ are two binary operations on NQR/NQJ defined by:

$$(x + NQJ) \oplus (y + 4\,NQJ) = (x + y) + NQJ,$$
$$(x + NQJ) \odot (y + NQJ) = (xy) + NQJ,$$

it can be shown that $\oplus$ and $\odot$ are well defined and that (NQR/NQJ, $\oplus$, $\odot$) is a neutrosophic quadruple ring.

Example 7.

Consider the neutrosophic quadruple ring $NQR(\mathbb{Z})$ and its neutrosophic quadruple ideal $NQR(2\mathbb{Z})$. Then

$$\frac{NQR(\mathbb{Z})}{NQR(2\mathbb{Z})} = \{NQR(2\mathbb{Z}), (1,0,0,0) + NQR(2\mathbb{Z}), (0,T,0,0) + NQR(2\mathbb{Z}), (0,0,I,0) + NQR(2\mathbb{Z}), (0,0,0,F) + NQR(2\mathbb{Z}), (0,T,I,F) + NQR(2\mathbb{Z}), (0,0,I,F) + NQR(2\mathbb{Z}), (0,T,I,0) + NQR(2\mathbb{Z}), (0,T,0,F) + NQR(2\mathbb{Z}), (1,T,0,0) + NQR(2\mathbb{Z}), (1,0,I,0) + NQR(2\mathbb{Z}), (1,0,0,F) + NQR(2\mathbb{Z}), (1,T,0,F) + NQR(2\mathbb{Z}), (1,0,I,F) + NQR(2\mathbb{Z}), (1,T,I,0) + NQR(2\mathbb{Z}), (1,T,I,F) + NQR(2\mathbb{Z})\}.$$

which is clearly a neutrosophic quadruple ring.

Definition 2.17

Let NQR and NQS be two neutrosophic quadruple rings and let $\varphi : NQR \to NQS$ be a mapping defined for all $x, y \in NQR$ as follows:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$.
(ii) $\varphi(xy) = \varphi(x)\varphi(y)$.
(iii) $\varphi(T) = T, \varphi(I) = I$ and $\varphi(F) = F$.
(iv) $\varphi(1,0,0,0) = (1,0,0,0)$.

Then φ is called a neutrosophic quadruple homomorphism. Neutrosophic quadruple monomorphism, endomorphism, isomorphism, and other morphisms can be defined in the usual way.

Definition 2.18

Let $\varphi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism.

(i) The image of φ denoted by $Im\varphi$ is defined by the set $Im\varphi = \{y \in NQS : y = \varphi(x)$, for some $x \in NQR\}$.

(ii) The kernel of φ denoted by $Ker\varphi$ is defined by the set $Ker\varphi = \{x \in NQR : \varphi(x) = (0,0,0,0)\}$.

Theorem 2.19

Let $\varphi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism. Then:

(i) $Im\varphi$ is a neutrosophic quadruple subring of NQS.
(ii) $Ker\varphi$ is not a neutrosophic quadruple ideal of NQR.

Proof.

(i) Clear.

(ii) Since T, I, F cannot have image (0,0,0,0) under φ, it follows that the elements $(0,T,0,0), (0,0,I,0), (0,0,0,F)$ cannot be in the $Ker\varphi$. Hence, $Ker\varphi$ cannot be a neutrosophic quadruple ideal of NQR.

Example 8.

Consider the projection map

$\varphi : NQR(\mathbb{Z}_2) \times NQR(\mathbb{Z}_2) \rightarrow NQR(\mathbb{Z}_2)$

defined by $\varphi(x, y) = x$ for all $x, y \in NQR(\mathbb{Z}_2)$.

It is clear that φ is a neutrosophic quadruple homomorphism and its kernel is given as

$$Ker\varphi = \{((0,0,0,0),(0,0,0,0)),((0,0,0,0),(1,0,0,0)), \big((0,0,0,0),(0,T,0,0)\big),\big((0,0,0,0),(0,0,I,0)\big), \big((0,0,0,0),(0,0,0,F)\big),\big((0,0,0,0),(0,T,I,F)\big), \big((0,0,0,0),(0,0,I,F)\big),\big((0,0,0,0),(0,T,I,0)\big), \big((0,0,0,0),(0,T,0,F)\big),\big((0,0,0,0),(1,T,0,0)\big), \big((0,0,0,0),(1,0,I,0)\big),\big((0,0,0,0),(1,0,0,F)\big), \big((0,0,0,0),(1,T,0,F)\big),\big((0,0,0,0),(1,0,I,F)\big), ((0,0,0,0),(1,T,I,0)),((0,0,0,0),(1,T,I,F))\}.$$

Theorem 2.20

Let φ: NQR(Z) → NQR(Z)/NQR(nZ) be a mapping defined by $\varphi(x) = x +$ NQR(nZ) for all $x \in$ NQR(Z) and $n = 1, 2, 3, \ldots$. Then φ is not a neutrosophic quadruple ring homomorphism.

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Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making

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Abstract: The objective of the paper ARE to introduce single-valued trapezoidal neutrosophic numbers(SVTrNNs), which is a special case of single-valued neutrosophic numbers and to develop a ranking method for ranking SVTrNNs. Some operational rules as well as cut sets of SVTrNNs have been introduced. The value and ambiguity indices of truth, indeterminacy, and falsity membership functions of SVTrNNs have been defined. A new ranking method has been proposed by using these two indices and applied the ranking method to multi attribute decision making problem in which the ratings of the alternatives over the attributes are expressed in terms of TrNFNs. Finally, an illustrative example has been provided to demonstrate the validity and applicability of the proposed approach.

Keywords: Single-valued neutrosophic number(SVNN), Single-valued trapezoidal neutrosophic number, Value index, Ambiguity index, Ranking of SVTrNNs, Multi attribute decision making.

1 Introduction

Fuzzy set [1] is capable of dealing with imprecise or vague information in decision making process, whose basic component is a membership function lying in the unit interval [0, 1]. Fuzzy number [2, 3] is a fuzzy subset of real numbers representing the expansion of assurance. Fuzzy numbers can be used to represent vagueness in multi-attribute decision making (MADM) [4, 5, 6, 7], data mining, pattern recognition, medical diagnosis, etc. However, in fuzzy numbers independence of non-membership function is not considered although it is equally important to represent imprecise numerical values in a flexible way. Intuitionistic fuzzy number [8], a generalization of fuzzy numbers, can present ill-known information with membership and non-membership function in the case where the available information is not sufficient to be expressed with fuzzy numbers. Shu et al.[9] defined a triangular intuitionistic fuzzy number(TIFN) and applied to fault tree analysis on printed board circuit assembly. Wang [10] extended TIFN to the trapezoidal intuitionistic fuzzy number(TrIFN) in a similar way as that of the fuzzy number. The concept of ranking of intuitionistic fuzzy numbers [11, 12, 13, 14, 15] has been employed in MADM under intuitionistic fuzzy environment. Li [16] proposed a ranking method for TIFNs by defining a ratio of value index to ambiguity index of TIFNSs and applied it to MADM problem. Zeng et al.[17] extended this ranking method by incorporating TrIFN and utilized it in MADM problems. For intuitionistic fuzzy number, indeterminate information is partially lost although hesitant information is taken into account by default. Therefore, indeterminate information should be considered in decision making process.

Smarandache [18, 19] defined neutrosophic set that can handle indeterminate and inconsistent information. Wang et al.[20] defined single valued neutrosophic set (SVNS), an instance of neutrosophic set, which simply represents uncertainty, imprecise, incomplete, indeterminate and inconsistent information. However, the domain of SVNSs is a discrete set where the truth membership degrees, indeterminacy membership degrees, and the falsity membership membership degrees are only expressed with fuzzy concept like "very good","good", "bad", etc. Taking the universe as a real line, we can develop the concept of single valued neutrosophic number (SVNN) whose domain is to be considered as a consecutive set. Hence, we can consider SVNNs as a special case of single-valued neutrosophic sets. These numbers can express ill-known quantity with uncertain numerical values in decision making problems. The nature of truth membership, indeterminacy membership, and falsity membership functions of SVNN may have different shape such as triangular shaped, trapezoidal shaped, bell shaped, etc. In the present study, we present only the case of trapezoidal shaped and leave others for future work. We define single-valued trapezoidal neutrosophic numbers (SVTrNN) in which its truth membership, indeterminacy membership, and falsity membership functions can be expressed as trapezoidal fuzzy numbers. Recently, the research on SVNNs has received a little attention and several definitions of SVNNs and its operational rules have been proposed. Ye [21] studied multiple attribute decision making problem by introducing trapezoidal fuzzy neutrosophic set. In his study Ye [21] also defined score function, accuracy function, and some operational rules of trapezoidal fuzzy neutrosophic sets. Biswas et al. [22] defined trapezoidal fuzzy neutrosophic number and their membership functions. Biswas et al. [22] also proposed relative expected value and cosine similarity measure for solving multiple attribute decision making problem.

Ranking method of SVTrNNs can play an important role in decision making problems involving indeterminate information which is beyond the scope of fuzzy numbers, intuitionistic fuzzy numbers. Literature review reflects that little attention has been received to the researchers regarding ranking method of SVTrNNs. Recently, Deli and Şubaş [23] proposed a ranking method for generalized SVTrNNs and presented a numerical example to solve multi-attribute decision making problem in neutrosophic environment. In the present study, We define normalized SVTrNNs and develop a ranking method of SVTrNNs to solve multi attribute decision making problem in neutrosophic environment.

Rest of the paper has been organized as follows: Section 2 provides some basic definitions of fuzzy sets, fuzzy numbers, single-valued neutrosophic sets. In Section 3, we propose SVNNs, SVTrNNs and study some of their properties. In Section 4, we present some arithmetic operations of SVTrNNs. Section 5 is devoted to present the concept of value index and ambiguity index of SVTrNNs and a ranking method of SVTrNNs. In Section 6, we formulate MADM model with the proposed ranking method of TrNNs. Section 7 presents an illustrative example. In Section 8, we present concluding remarks and future scope of research.

2 Preliminaries

In this Section, we recall some basic concepts of fuzzy sets, fuzzy number, single valued neutrosophic set.

Definition 1. *[1, 3] A fuzzy set $\tilde{A}$ in a universe of discourse X is defined by $\tilde{A}$=$\{\langle x, \mu_{\tilde{A}}(x)\rangle \,|x \in X\}$, where, $\mu_{\tilde{A}}(x)$: $X \to [0,1]$ is called the membership function of $\tilde{A}$ and the value of $\mu_{\tilde{A}}(x)$ is called the degree of membership for $x \in X$.*

The $\alpha-$cut of the fuzzy set A is the crisp set A_α given by $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$, $\alpha \in [0,1]$.

Definition 2. *[3] A fuzzy set $\tilde{A}$ of the real line $\mathbb{R}$ with membership function $\mu_{\tilde{A}}(x) : \mathbb{R} \to [0,1]$ is called a fuzzy number if*

1. *$\tilde{A}$ is normal, i.e. there exists an element x_0 such that $\mu_{\tilde{A}}(x_0) = 1$,*
2. *$\tilde{A}$ is convex, i.e. $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min\big(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\big)$ for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$,*
3. *$\mu_{\tilde{A}}$ is upper semi continuous, and*
4. *the support of $\tilde{A}$, i.e. $S(\tilde{A})$=$\{x \in X : \mu_{\tilde{A}}(x) > 0\}$ is bounded.*

Definition 3. *[2] A fuzzy number $\tilde{A}$ is called a trapezoidal fuzzy number(TrFN), if its membership function is defined by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \dfrac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & otherwise. \end{cases}$$

Figure 1: Trapezoidal fuzzy number $\tilde{A}$

The TrFN $\tilde{A}$ is denoted by the quadruplet $\tilde{A}$=(a_1, a_2, a_3, a_4) where a_1,a_2,a_3,a_4 are the real numbers and $a_1 \leq a_2 \leq a_3 \leq a_4$. The value of x at $[a_2, a_3]$ gives the maximum of $\mu_{\tilde{A}}(x)$, i.e., $\mu_{\tilde{A}}(x)$ =1; it is the most probable value of the evaluation data. The value of x outside the interval $[a_1, a_4]$ gives the minimum of $\mu_{\tilde{A}}(x)$, i.e., $\mu_{\tilde{A}}(x) = 0$; it is the least probable value of the evaluation data. Constants a_1 and a_4 are the lower and upper bounds of the available area for the evaluation data. The $\alpha-$cut of TrFN $\tilde{A}$=(a_1, a_2, a_3, a_4) is the closed interval

$$\begin{aligned} A_\alpha &= [L^\alpha(\tilde{A}), R^\alpha(\tilde{A})] \\ &= [(a_2-a_1)\alpha + a_1, -(a_4-a_3)\alpha + a_4], \alpha \in [0,1]. \end{aligned}$$

Definition 4. *[20] A single valued neutrosophic set $\tilde{A}$ in a universe of discourse X is given by*

$$\tilde{A} = \Big\{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle | x \in X\Big\},$$

where, $T_{\tilde{A}} : X \to [0,1]$, $I_{\tilde{A}} : X \to [0,1]$ and $F_{\tilde{A}} : X \to [0,1]$, with the condition

$$0 \leq T_{\tilde{A}}(x) + I_{\tilde{A}}(x) + F_{\tilde{A}}(x) \leq 3, \text{ for all } x \in X.$$

The numbers $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$ respectively represent the truth membership, indeterminacy membership and falsity membership degree of the element x to the set $\tilde{A}$.

Definition 5. *An (α, β, γ)-cut set of SVNS $\tilde{A}$, a crisp subset of $\mathbb{R}$ is defined by*

$$\tilde{A}_{\alpha,\beta,\gamma} = \big\{x | T_{\tilde{A}}(x) \geq \alpha, I_{\tilde{A}}(x) \leq \beta, F_{\tilde{A}}(x) \leq \gamma\big\} \qquad (1)$$

where, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, and $0 \leq \alpha + \beta + \gamma \leq 3$.

Definition 6. *A single-valued neutrosophic set $\tilde{A}$=$\Big\{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle | x \in X\Big\}$ is called neut-normal, if there exist at least three points $x_0, x_1, x_2 \in X$ such that $T_{\tilde{A}}(x_0) = 1$, $I_{\tilde{A}}(x_1) = 1$, $F_{\tilde{A}}(x_2) = 1$.*

Definition 7. *A single-valued neutrosophic set $\tilde{A}$=$\Big\{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle | x \in X\Big\}$ is a subset of the* *real line, called neut-convex if for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$ the following conditions are satisfied*

1. $T_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min\big(T_{\tilde{A}}(x_1), T_{\tilde{A}}(x_2)\big)$;
2. $I_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \leq max\big(I_{\tilde{A}}(x_1), I_{\tilde{A}}(x_2)\big)$;
3. $F_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \leq max\big(F_{\tilde{A}}(x_1), F_{\tilde{A}}(x_2)\big)$.

That is $\tilde{A}$ is neut-convex if its truth membership function is fuzzy convex, indeterminacy membership function is fuzzy concave and falsity membership function is fuzzy concave.

3 Single-valued neutrosophic number and some arithmetic operations

Single valued neutrosophic set is a flexible and practical tool to handle incomplete, indeterminate or uncertain type information. However, it is often hard to express this information with the truth membership degree, the indeterminacy degree, and the falsity degree represented by the exact real values. Thus extension of SVNSs is required to deal the issues.

Definition 8. *A single-valued neutrosophic set $\tilde{A}$=$\Big\{\big\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\big\rangle | x \in X\Big\}$, subset of the real line, is called single-valued neutrosophic number if*

1. *$\tilde{A}$ is neut-normal,*
2. *$\tilde{A}$ is neut-convex,*
3. *$T_A(x)$ is upper semi continuous, $I_A(x)$ is lower semi continuous, and $F_A(x)$ is lower semi continuous, and*
4. *the support of $\tilde{A}$, i.e. $S(\tilde{A})$=$\{x \in X : T_{\tilde{A}}(x) > 0, I_{\tilde{A}}(x) < 1, F_{\tilde{A}}(x) < 1\}$ is bounded.*

Thus for any SVNNs $\tilde{A}$, there exist twelve numbers a_{11}, a_{21}, a_{31}, a_{41}, b_{11}, b_{21}, b_{31}, b_{41}, c_{11}, c_{21}, c_{31}, c_{41} $\in \mathbb{R}$ such that $c_{11} \leq b_{11} \leq a_{11} \leq c_{21} \leq b_{21} \leq a_{21} \leq a_{31} \leq b_{31} \leq c_{31} \leq a_{41} \leq b_{41} \leq c_{41}$ and the six functions $T^L_{\tilde{A}}(x)$, $T^R_{\tilde{A}}(x)$,$I^L_{\tilde{A}}(x)$, $I^R_{\tilde{A}}(x)$,$F^L_{\tilde{A}}(x)$, $F^R_{\tilde{A}}(x) : \mathbb{R} \to [0,1]$ to represent the truth membership, indeterminacy membership,and falsity membership degree of $\tilde{A}$. The three non decreasing functions $T^L_{\tilde{A}}(x)$, $I^L_{\tilde{A}}(x)$, and $F^L_{\tilde{A}}(x)$ represent the left side of truth, indeterminacy, and falsity membership functions of a SVNN $\tilde{A}$ respectively. Similarly, the three non increasing functions $T^R_{\tilde{A}}(x)$, $I^R_{\tilde{A}}(x)$, $F^R_{\tilde{A}}(x)$ represent the right side of truth membership, indeterminacy, and falsity membership functions of a SVNN $\tilde{A}$, respectively.

Then the truth membership, indeterminacy membership and falsity membership functions of $\tilde{A}$ can be defined in the following form:

$$T_{\tilde{A}}(x) = \begin{cases} T^L_{\tilde{A}}(x), & a_{11} \leq x \leq a_{21}, \\ 1, & a_{21} \leq x \leq a_{31}, \\ T^U_{\tilde{A}}(x), & a_{31} \leq x \leq a_{41}, \\ 0, & otherwise. \end{cases} \tag{2}$$

$$I_{\tilde{A}}(x) = \begin{cases} I^L_{\tilde{A}}(x), & b_{11} \leq x \leq b_{21}, \\ 1, & b_{21} \leq x \leq b_{31}, \\ I^U_{\tilde{A}}(x), & b_{31} \leq x \leq b_{41}, \\ 0, & otherwise, \end{cases} \tag{3}$$

$$F_{\tilde{A}}(x) = \begin{cases} F^L_{\tilde{A}}(x), & c_{11} \leq x \leq c_{21}, \\ 1, & c_{21} \leq x \leq c_{31}, \\ F^U_{\tilde{A}}(x), & c_{31} \leq x \leq c_{41}, \\ 0, & otherwise. \end{cases} \tag{4}$$

The sum of three independent membership degrees of a SVNN $\tilde{A}$ lie between the interval $[0,3]$ i.e,

$$0 \leq T^U_{\tilde{A}}(x) + I^U_{\tilde{A}}(x) + F^U_{\tilde{A}}(x) \leq 3, x \in \tilde{A}.$$

Definition 9. *A single-valued trapezoidal neutrosophic number (SVTrNN) $\tilde{A}$ with the set of parameters $c_{11} \leq b_{11} \leq a_{11} \leq c_{21} \leq b_{21} \leq a_{21} \leq a_{31} \leq b_{31} \leq c_{31} \leq a_{41} \leq b_{41} \leq c_{41}$ is denoted as $\tilde{A}$= $\langle(a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41})\rangle$ in the set of real numbers $\mathbb{R}$. The truth membership, indeterminacy membership and falsity membership degree of $\tilde{A}$ can be defined as follows:*

$$T_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_{11}}{a_{21} - a_{11}}, & a_{11} \leq x \leq a_{21}, \\ 1, & a_{21} \leq x \leq a_{31}, \\ \dfrac{a_{41} - x}{a_{41} - a_{31}}, & a_{31} \leq x \leq a_{41}, \\ 0, & otherwise. \end{cases} \tag{5}$$

$$I_{\tilde{A}}(x) = \begin{cases} \dfrac{x - b_{11}}{b_{21} - b_{11}}, & b_{11} \leq x \leq b_{21}, \\ 1, & b_{21} \leq x \leq b_{31}, \\ \dfrac{x - b_{31}}{b_{41} - b_{31}}, & b_{31} \leq x \leq b_{41}, \\ 0, & otherwise. \end{cases} \tag{6}$$

$$F_{\tilde{A}}(x) = \begin{cases} \dfrac{x - c_{11}}{c_{21} - c_{11}}, & c_{11} \leq x \leq c_{21}, \\ 1, & c_{21} \leq x \leq c_{31}, \\ \dfrac{x - c_{31}}{c_{41} - c_{31}}, & c_{31} \leq x \leq c_{41}, \\ 0, & otherwise. \end{cases} \tag{7}$$

For a SVTrNN $\tilde{A}$, a_{21}=a_{31} for truth membership, b_{21}=b_{31} for indeterminacy membership, and c_{21}=c_{31} for falsity membership degree yield a single-valued triangular neutrosophic numbers which is a special case of SVTrNNs.

3.1 Cuts of single-valued trapezoidal neutrosophic numbers

Let $\tilde{A} = \langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$ be the SVTrNN in the set of real numbers $\mathbb{R}$, where $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$, and $F_{\tilde{A}}(x)$ be the truth, indeterminacy and falsity membership functions.

Definition 10. *A α-cut set of SVTrNN $\tilde{A}=\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$ is a crisp subset of $\mathbb{R}$ defined by $\tilde{A}_\alpha = \{x | T_{\tilde{A}}(x) \geq \alpha\}$, where $0 \leq \alpha \leq 1$. According to the definition of SVTrNN of $\tilde{A}$ and Definition 1, it can be shown that $\tilde{A}_\alpha$ is a closed interval. This interval is denoted by $\tilde{A}_\alpha = [L^\alpha(\tilde{A}), R^\alpha(\tilde{A})]$ and defined by*

$$[L^\alpha(\tilde{A}), R^\alpha(\tilde{A})] = [a_{11} + \alpha(a_{21} - a_{11}), a_{41} - \alpha(a_{41} - a_{31})]. \quad (8)$$

Definition 11. *A β-cut set of SVTrNN $\tilde{A}=\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$ is a crisp subset of $\mathbb{R}$ defined by $\tilde{A}_\beta = \{x | T_{\tilde{A}}(x) \leq \beta\}$, where $0 \leq \beta \leq 1$.*

Similarly, the close interval is denoted by $\tilde{A}_\beta = [L^\beta(\tilde{A}), R^\beta(\tilde{A})]$ and defined by

$$[L^\beta(\tilde{A}), R^\beta(\tilde{A})] = [b_{21} + \beta(b_{21} - b_{11}), b_{31} + \beta(b_{41} - b_{31})]. \quad (9)$$

Definition 12. *A γ-cut set of SVTrNN $\tilde{A}=\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$ is a crisp subset of $\mathbb{R}$ defined by $\tilde{A}_\alpha = \{x | T_{\tilde{A}}(x) \leq \gamma\}$, where $0 \leq \gamma \leq 1$. The close interval obtained from $\tilde{A}$ is denoted by $\tilde{A}_\gamma = [L^\gamma(\tilde{A}), R^\gamma(\tilde{A})]$ and defined by*

$$[L^\gamma(\tilde{A}), R^\gamma(\tilde{A})] = [c_{21} + \gamma(c_{21} - c_{11}), c_{31} + \gamma(c_{41} - c_{31})]. \quad (10)$$

The (α, β, γ)-cut set of SVTrNN $\tilde{A}=\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$ can be defined by using Eqs.(8),(9), and (10) simultaneously.

Definition 13. *An (α, β, γ)-cut set of SVTrNN $\tilde{A}=\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$ is a crisp subset of $\mathbb{R}$, which is defined by*

$$\begin{aligned}
\tilde{A}_{\alpha,\beta,\gamma} &= \{x | T_{\tilde{A}}(x) \geq \alpha, I_{\tilde{A}}(x) \leq \beta, F_{\tilde{A}}(x) \leq \gamma\} \\
&= \left\{ \begin{array}{r} [L^\alpha(\tilde{A}), R^\alpha(\tilde{A})], [L^\gamma(\tilde{A}), R^\gamma(\tilde{A})], \\ [L^\gamma(\tilde{A}), R^\gamma(\tilde{A})] \end{array} \right\} \quad (11) \\
&= \left\{ \begin{array}{c} [a_{11} + \alpha(a_{21} - a_{11}), a_{41} - \alpha(a_{41} - a_{31})], \\ [b_{21} + \beta(b_{21} - b_{11}), b_{31} + \beta(b_{41} - b_{31})], \\ [c_{21} + \gamma(c_{21} - c_{11}), c_{31} + \gamma(c_{41} - c_{31})] \end{array} \right\} \quad (12)
\end{aligned}$$

where, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, and $0 \leq \alpha + \beta + \gamma \leq 3$.

We observe for the (α, β, γ)-cut set of SVTrNN $\tilde{A}$ that

1. *$\frac{dL^\alpha(\tilde{A})}{d\alpha} > 0$, $\frac{dR^\alpha(\tilde{A})}{d\alpha} < 0$ for all $\alpha \in [0, 1]$, thus $L^1(\tilde{A}) \geq R^1(\tilde{A})$,*
2. *$\frac{dL^\beta(\tilde{A})}{d\beta} < 0$, $\frac{dR^\beta(\tilde{A})}{d\beta} > 0$ for all $\beta \in [0, 1]$, thus $L^0(\tilde{A}) \leq R^0(\tilde{A})$,*
3. *$\frac{dL^\gamma(\tilde{A})}{d\gamma} < 0$, $\frac{dR^\gamma(\tilde{A})}{d\gamma} > 0$ for all $\gamma \in [0, 1]$, thus $L^0(\tilde{A}) \leq R^0(\tilde{A})$.*

4 Some arithmetic operations of single-valued trapezoidal neutrosophic numbers

In this section, some arithmetic operations of SVTrNNs have been presented by using neutrosophic extension principle and (α, β, γ)-cuts method.

4.1 Arithmetic Operations of single-valued neutrosophic numbers based on extension principle

The arithmetic operation $(*)$ of two SVTrNNs is a mapping of an input vector $X = [x_1, x_2]^T$ defined in the Cartesian product space $\mathbb{R} \times \mathbb{R}$ on to an output y defined in the real space $\mathbb{R}$. Let $\tilde{A}$ and $\tilde{B}$ be two SVTrNNs, then their outcome of arithmetic operation is also an SVTrNN defined by the form

$$(\tilde{A} * \tilde{B})(y) = \left\{ \begin{array}{l} \Big(y, \sup\limits_{y = x_1 * x_2} \Big[\min\big(T_{\tilde{A}}(x_1), T_{\tilde{B}}(x_1)\big)\Big], \\ \quad \inf\limits_{y = x_1 * x_2} \Big[\max\big(I_{\tilde{A}}(x_1), I_{\tilde{B}}(x_1)\big)\Big], \\ \quad \inf\limits_{y = x_1 * x_2} \Big[\max\big(F_{\tilde{A}}(x_1), F_{\tilde{B}}(x_1)\big)\Big]\Big), \end{array} \right\}. \quad (13)$$

for all x_1, x_2 in $\mathbb{R}$.

To calculate the arithmetic operation of NTrFNs, it is sufficient to determine truth, indeterminacy and falsity membership function of resultant NTrFN as

$$T_{\tilde{A}*\tilde{B}}(y) = \sup_{y = x_1 * x_2} \Big[\min\big(T_{\tilde{A}}(x_1), T_{\tilde{B}}(x_1)\big)\Big];$$

$$I_{\tilde{A}*\tilde{B}}(y) = \inf_{y = x_1 * x_2} \Big[\max\big(I_{\tilde{A}}(x_1), I_{\tilde{B}}(x_1)\big)\Big]$$

$$\text{and } F_{\tilde{A}*\tilde{B}}(y) = \inf_{y = x_1 * x_2} \Big[\max\big(F_{\tilde{A}}(x_1), F_{\tilde{B}}(x_1)\big)\Big].$$

4.2 Arithmetic operations of single-valued trapezoidal neutrosophic numbers based on (α, β, γ)-cuts method

Some properties of SVTrNNs in the set of real numbers are presented here.

Property 1. *If $\tilde{A}=\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41}) \rangle$ and $\tilde{B}=\langle (a_{12}, a_{22}, a_{32}, a_{42}), (b_{12}, b_{22}, b_{32}, b_{42}), (c_{12}, c_{22}, c_{32}, c_{42}) \rangle$ be two SVTrNNs in the set of real numbers $\mathbb{R}$ then, $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is also a SVTrNN and*

$$\tilde{A} \oplus \tilde{B} = \left\langle \begin{array}{c} (a_{11} + a_{12}, a_{21} + a_{22}, a_{31} + a_{32}, a_{41} + a_{42}), \\ (b_{11} + b_{12}, b_{21} + b_{22}, b_{31} + b_{32}, b_{41} + b_{42}), \\ (c_{11} + c_{12}, c_{21} + c_{22}, c_{31} + c_{32}, c_{41} + c_{42}) \end{array} \right\rangle. \quad (14)$$

Proof. Based on the extensible principle of single valued neutrosophic set and (α, β, γ)-cut sets of $\tilde{A}$ and $\tilde{B}$ for $\alpha, \beta, \gamma \in [0, 1]$, it sufficient to prove that $A_{\alpha,\beta,\gamma} + B_{\alpha,\beta,\gamma} = (A + B)_{\alpha,\beta,\gamma}$. Using Eq.(12), the summation of (α,β,γ)-cut sets of $\tilde{A}$ and $\tilde{B}$ is

$$\begin{aligned}
&A_{\alpha,\beta,\gamma}+B_{\alpha,\beta,\gamma}\\
&=\begin{bmatrix}\big[a_{11}+\alpha(a_{21}-a_{11}),a_{41}-\alpha(a_{41}-a_{31})\big],\\ \big[b_{21}+\beta(b_{21}-b_{11}),b_{31}+\beta(b_{41}-b_{31})\big],\\ \big[c_{21}+\gamma(c_{21}-c_{11}),c_{31}+\gamma(c_{41}-c_{31})\big]\end{bmatrix} \qquad (15)\\
&\qquad+\begin{bmatrix}\big[a_{12}+\alpha(a_{22}-a_{12}),a_{42}-\alpha(a_{42}-a_{32})\big],\\ \big[b_{22}+\beta(b_{22}-b_{12}),b_{32}+\beta(b_{42}-b_{32})\big],\\ \big[c_{22}+\gamma(c_{22}-c_{12}),c_{32}+\gamma(c_{42}-c_{32})\big]\end{bmatrix}\\
&=\begin{bmatrix}\big[a_{11}+a_{12}+\alpha(a_{21}+a_{22}-a_{11}-a_{12}),\\ \quad a_{41}+a_{42}-\alpha(a_{41}+a_{42}-a_{31}-a_{32})\big],\\ \big[b_{21}+b_{22}+\beta(b_{21}+b_{22}-b_{11}-b_{12}),\\ \quad b_{31}+b_{32}+\beta(b_{41}+b_{42}-b_{31}-b_{32})\big],\\ \big[c_{21}+c_{22}+\gamma(c_{21}+c_{22}-c_{11}-c_{12}),\\ \quad c_{31}+c_{32}+\gamma(c_{41}+c_{42}-c_{31}-c_{32})\big]\end{bmatrix}\\
&=(A+B)_{\alpha,\beta,\gamma}. \qquad (16)
\end{aligned}$$

This establishes the property. □

Property 2. *If $\tilde{A}$=$\big\langle(a_{11},a_{21},a_{31},a_{41}),(b_{11},b_{21},b_{31},b_{41}),(c_{11},c_{21},c_{31},c_{41})\big\rangle$ be a SVTrNN in the set of real numbers $\mathbb{R}$ and k be a real number then, $k\tilde{A}$ is also a SVTrNN and*

$$k\tilde{A}=\begin{cases}\left\langle\begin{matrix}(ka_{11},ka_{21},ka_{31},ka_{41}),(kb_{11},kb_{21},kb_{31},b_{41}),\\ (kc_{11},kc_{21},kc_{31},kc_{41})\end{matrix}\right\rangle\\ \left\langle\begin{matrix}(ka_{41},ka_{31},ka_{21},ka_{11}),(kb_{41},kb_{31},kb_{21},b_{11}),\\ (kc_{41},kc_{31},kc_{21},kc_{11})\end{matrix}\right\rangle\end{cases}, \qquad (17)$$

for $k>0$ and $k<0$ respectively.

Proof. To establish this property, it has to be proved that $k\tilde{A}_{\alpha,\beta,\gamma}$=$(k\tilde{A})_{\alpha,\beta,\gamma}$.
From Eq.(12), the (α,β,γ)-cut sets of $\tilde{A}$ multiplied with the real number $k>0$ can be taken as

$$\begin{aligned}
k\tilde{A}_{\alpha,\beta,\gamma}&=k\begin{bmatrix}\big[a_{11}+\alpha(a_{21}-a_{11}),a_{41}-\alpha(a_{41}-a_{31})\big],\\ \big[b_{21}+\beta(b_{21}-b_{11}),b_{31}+\beta(b_{41}-b_{31})\big],\\ \big[c_{21}+\gamma(c_{21}-c_{11}),c_{31}+\gamma(c_{41}-c_{31})\big]\end{bmatrix}\\
&=\begin{bmatrix}[ka_{11}+\alpha(ka_{21}-ka_{11}),ka_{41}-\alpha(ka_{41}-ka_{31})],\\ [kb_{21}+\beta(kb_{21}-kb_{11}),kb_{31}+\beta(kb_{41}-kb_{31})],\\ [kc_{21}+\gamma(kc_{21}-kc_{11}),kc_{31}+\gamma(kc_{41}-kc_{31})]\end{bmatrix}\\
&=(k\tilde{A})_{\alpha,\beta,\gamma}.
\end{aligned}$$

Similarly, it can be shown that $k\tilde{A}_{\alpha,\beta,\gamma}$=$(k\tilde{A})_{\alpha,\beta,\gamma}$ for real number $k<0$. The two results for $k>0$ and $k<0$ prove this property. □

Now we define some arithmetical operation of SVTrNN.

Definition 14. *If $\tilde{A}$=$\big\langle(a_{11},a_{21},a_{31},a_{41}),(b_{11},b_{21},b_{31},b_{41}),(c_{11},c_{21},c_{31},c_{41})\big\rangle$ be a SVTrNN in the set of real numbers $\mathbb{R}$ and k be a real number, then the following operations are valid:*

1. $$\tilde{A}\oplus\tilde{B}=\left\langle\begin{matrix}(a_{11}+a_{12},a_{21}+a_{22},a_{31}+a_{32},a_{41}+a_{42}),\\ (b_{11}+b_{12},b_{21}+b_{22},b_{31}+b_{32},b_{41}+b_{42}),\\ (c_{11}+c_{12},c_{21}+c_{22},c_{31}+c_{32},c_{41}+c_{42})\end{matrix}\right\rangle$$

2. $$\tilde{A}\otimes\tilde{B}=\left\langle\begin{matrix}(a_{11}a_{12},a_{21}a_{22},a_{31}a_{32},a_{41}a_{42}),\\ (b_{11}b_{12},b_{21}b_{22},b_{31}b_{32},b_{41}b_{42}),\\ (c_{11}c_{12},c_{21}c_{22},c_{31}c_{32},c_{41}c_{42})\end{matrix}\right\rangle$$

3. $$\lambda\tilde{A}=\left\langle\begin{matrix}(\lambda a_{11},\lambda a_{21},\lambda a_{31},\lambda a_{41}),\\ (\lambda b_{11},\lambda b_{21},\lambda b_{31},\lambda b_{41}),\\ (\lambda c_{11},\lambda c_{21},\lambda c_{31},\lambda c_{41})\end{matrix}\right\rangle$$

4. $$\tilde{A}^{\lambda}=\left\langle\begin{matrix}(a_{11}^{\lambda},a_{21}^{\lambda},a_{31}^{\lambda},a_{41}^{\lambda}),(b_{11}^{\lambda},b_{21}^{\lambda},b_{31}^{\lambda},b_{41}^{\lambda}),\\ (c_{11}^{\lambda},c_{21}^{\lambda},c_{31}^{\lambda},c_{41}^{\lambda})\end{matrix}\right\rangle$$

5 Value and ambiguity index based ranking method for SVTrNNs

Definition 15. *Let $\tilde{A}_\alpha$, $\tilde{A}_\beta$, and $\tilde{A}_\gamma$ be the α-cut, β-cut, and γ-cut sets of a SVTrNN $\tilde{A}$. Then the value of truth($T_{\tilde{A}}(x)$), indeterminacy($I_{\tilde{A}}(x)$), and falsity($T_{\tilde{A}}(x)$) membership degree of $\tilde{A}$ are respectively defined by*

$$V_T(\tilde{A})=\int_0^1\big(L^{\alpha}(\tilde{A})+R^{\alpha}(\tilde{A})\big)f(\alpha)d\alpha; \qquad (18)$$

$$V_I(\tilde{A})=\int_0^1\big(L^{\beta}(\tilde{A})+R^{\beta}(\tilde{A})\big)g(\beta)d\beta; \qquad (19)$$

$$V_F(\tilde{A})=\int_0^1\big(L^{\gamma}(\tilde{A})+R^{\gamma}(\tilde{A})\big)h(\gamma)d\gamma. \qquad (20)$$

Weighting functions $f(\alpha)$, $g(\beta)$ and $h(\gamma)$ can be set according to nature of decision making in real situations. The function $f(\alpha)=\alpha(\alpha\in[0,1])$ gives different weights to elements in different α-cut sets which make less the contribution of the lower α-cut sets as these cut sets arising from values of $T_{\tilde{A}}(x)$ have a considerable amount of uncertainty. Thus, $V_T(\tilde{A})$ synthetically reflects the information on every membership degree and may be regarded as a central value that represents from the membership function point of view. Similarly, the function $g(\beta)=1-\beta$ has the effect of weighting on the different β-cut sets. $g(\beta)$ diminishes the contribution of the higher β-cut sets, which is reasonable since these cut sets arising from values of $I_{\tilde{A}}(x)$ have a considerable amount of uncertainty. $V_T(\tilde{A})$ synthetically reflects the information on every indeterminacy degree and may be regarded as a central value that represents from the indeterminacy function point of view. Similarly, the function $h(\gamma)=1-\gamma$ has the effect of weighting on the different γ-cut sets. $g(\gamma)$ diminishes the contribution of the higher γ-cut sets, which is reasonable since these cut sets arising from values of $F_{\tilde{A}}(x)$ have a considerable amount of uncertainty. $V_F(\tilde{A})$ synthetically reflects the information on every falsity degree and may be regarded as a central value that represents from the falsity membership function point of view.

Taking $f(\alpha)=\alpha$ in Eq.(18), the value of truth membership function can be obtained as:

$$\begin{aligned}
V_T&=\int_0^1\Big(L_{\tilde{A}}(\alpha)+R_{\tilde{A}}(\alpha)\Big)f(\alpha)\,\mathrm{d}\alpha\\
&=\int_0^1\Big[a_{11}+\alpha(a_{21}-a_{11})+a_{41}-\alpha(a_{21}-a_{11})\Big]\alpha\mathrm{d}\alpha\\
&=\frac{1}{6}\big(a_{11}+2a_{21}+2a_{31}+a_{41}\big). \qquad (21)
\end{aligned}$$

Similarly, considering $g(\beta)=1-\beta$ in Eq.(19), the value of indeterminacy membership function can be defined as:

$$\begin{aligned}V_I &= \int_0^1 \Big(L_{\tilde{A}}(\alpha) + R_{\tilde{A}}(\alpha)\Big) g(\beta)\,\mathrm{d}\beta \\ &= \int_0^1 \Big[b_{21} - \gamma(b_{21} - b_{11}) + b_{31} + \gamma(b_{41} - b_{31})\Big](1-\beta)\mathrm{d}\beta \\ &= \frac{1}{6}\big(b_{11} + 2b_{21} + 2b_{31} + b_{41}\big). \end{aligned} \tag{22}$$

and by considering $h(\gamma) = 1 - \gamma$ in Eq.(20), the value of falsity membership function is defined by

$$\begin{aligned}V_F &= \int_0^1 \Big(L_{\tilde{A}}(\gamma) + R_{\tilde{A}}(\gamma)\Big) g(\gamma)\,\mathrm{d}\gamma \\ &= \int_0^1 \Big[c_{21} - \gamma(c_{21} - c_{11}) + c_{31} + \gamma(c_{41} - c_{31})\Big](1-\gamma)\mathrm{d}\gamma \\ &= \frac{1}{6}\big(c_{11} + 2c_{21} + 2c_{31} + c_{41}\big). \end{aligned} \tag{23}$$

Definition 16. *Let $\tilde{A}_\alpha$, $\tilde{A}_\beta$, and $\tilde{A}_\gamma$ be the α-cut, β-cut, and γ-cut sets of a SVTrNN $\tilde{A}$. Then the ambiguity of truth($T_{\tilde{A}}(x)$), indeterminacy($I_{\tilde{A}}(x)$), and falsity($T_{\tilde{A}}(x)$) membership function of a SVTrNN $\tilde{A}$ are respectively defined by*

$$A_T(\tilde{A}) = \int_0^1 \big(R^\alpha(\tilde{A}) - L^\alpha(\tilde{A})\big) f(\alpha) d\alpha; \tag{24}$$

$$A_I(\tilde{A}) = \int_0^1 \big(R^\beta(\tilde{A}) - L^\beta(\tilde{A})\big) g(\beta) d\beta; \tag{25}$$

$$A_F(\tilde{A}) = \int_0^1 \big(R^\gamma(\tilde{A}) - L^\gamma(\tilde{A})\big) h(\gamma) d\gamma. \tag{26}$$

It is observed that $R^\alpha(\tilde{A}) - L^\alpha(\tilde{A})$, $R^\beta(\tilde{A}) - L^\beta(\tilde{A})$, and $R^\gamma(\tilde{A}) - L^\gamma(\tilde{A})$ represent the length of the intervals of $\tilde{A}_\alpha$, $\tilde{A}_\beta$, and $\tilde{A}_\gamma$ respectively. Thus, $A_T(\tilde{A})$, $A_I(\tilde{A})$, and $A_F(\tilde{A})$ can be regarded as the global spreads of the truth, indeterminacy, and falsity membership function respectively. The ambiguity of three membership functions determine the measure of vagueness of $\tilde{A}$.

Now, putting the values of α-cut of $\tilde{A}$ and $f(\alpha) = \alpha$ in Eq.(24), the ambiguity of membership function $T_{\tilde{A}}(x)$ can be determined as:

$$\begin{aligned}A_T(\tilde{A}) &= \int_0^1 \big(R^\alpha(\tilde{A}) - L^\alpha(\tilde{A})\big) f(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 \Big[a_{41} - \alpha(a_{41} - a_{31}) - a_{11} - \alpha(a_{21} - a_{11})\Big]\alpha\mathrm{d}\alpha \\ &= \frac{1}{6}\big(-a_{11} - 2a_{21} + 2a_{31} + a_{41}\big). \end{aligned} \tag{27}$$

Similarly, putting the values of β-cut of $\tilde{A}$ and $f(\beta) = 1 - \beta$ in Eq.(25), the ambiguity of membership function $I_{\tilde{A}}(x)$ can be determined as:

$$\begin{aligned}A_T(\tilde{A}) &= \int_0^1 \big(R^\beta(\tilde{A}) - L^\alpha(\tilde{A})\big) f(\beta)\mathrm{d}\beta \\ &= \int_0^1 \begin{bmatrix} b_{31} + \beta(b_{41} - b_{31}) - b_{21} \\ + \beta(b_{21} - b_{11}) \end{bmatrix} (1-\beta)\mathrm{d}\beta \\ &= \frac{1}{6}\big(-b_{11} - 2b_{21} + 2b_{31} + b_{41}\big); \end{aligned} \tag{28}$$

and setting the values of γ-cut of $\tilde{A}$ and $f(\gamma) = 1 - \gamma$ in Eq.(26), the ambiguity of membership function $I_{\tilde{A}}(x)$ can be determined as:

$$\begin{aligned}A_T(\tilde{A}) &= \int_0^1 \big(R^\beta(\tilde{A}) - L^\alpha(\tilde{A})\big) f(\gamma)\mathrm{d}\gamma \\ &= \int_0^1 \begin{bmatrix} b_{31} + \gamma(b_{41} - b_{31}) - b_{21} \\ + \gamma(b_{21} - b_{11}) \end{bmatrix} (1-\gamma)\mathrm{d}\gamma \\ &= \frac{1}{6}\big(-c_{11} - 2c_{21} + 2c_{31} + c_{41}\big). \end{aligned} \tag{29}$$

Definition 17. *Let $\tilde{A} = \big\langle(a_{11}, a_{21}, a_{31}, a_{41}),\ (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41})\big\rangle$ be a SVTrNN. A value index and ambiguity index for $\tilde{A}$ can be defined by*

$$V_{\lambda,\mu,\nu} = \lambda V_T + \mu V_I + \nu V_F \tag{30}$$

$$\begin{aligned}&= \frac{\lambda}{6}\Big(a_{11} + 2a_{21} + 2a_{31} + a_{41}\Big) \\ &\quad + \frac{\mu}{6}\Big(b_{11} + 2b_{21} + 2b_{31} + b_{41}\Big) \\ &\quad + \frac{\nu}{6}\Big(c_{11} + 2c_{21} + 2c_{31} + c_{41}\Big)\end{aligned} \tag{31}$$

$$A_{\lambda,\mu,\nu} = \lambda A_T + \mu A_I + \nu A_F \tag{32}$$

$$\begin{aligned}&= \frac{\lambda}{6}\Big(-a_{11} - 2a_{21} + 2a_{31} + a_{41}\Big) \\ &\quad + \frac{\mu}{6}\Big(-b_{11} - 2b_{21} + 2b_{31} + b_{41}\Big) \\ &\quad + \frac{\nu}{6}\Big(-c_{11} - 2c_{21} + 2c_{31} + c_{41}\Big)\end{aligned} \tag{33}$$

where, the co-efficients λ, μ, ν of $V_{\lambda,\mu,\nu}$ and $A_{\lambda,\mu,\nu}$ represent the decision makers' preference value with the condition $\lambda + \mu + \nu = 1$. The decision maker may intend to take decision pessimistically in uncertain environment for $\lambda \in [0, \frac{1}{3}]$ and $\mu + \nu \in [\frac{1}{3}, 1]$. On the contrary, the decision maker may intend to take decision optimistically in uncertain environment for $\lambda \in [\frac{2}{3}, 1]$ and $\mu + \nu \in [0, \frac{1}{3}]$. The impact of truth, indeterminacy, and falsity degree are same to the decision maker for $\lambda = \mu = \nu = \frac{1}{3}$. Therefore, the value index and the ambiguity index may reflect the decision makers attitude for SVTrNN.

In the following, some properties regarding value and ambiguity index have been presented.

Theorem 3. *Let $\tilde{A}_1 = \big\langle(a_{11}, a_{21}, a_{31}, a_{41}),\ (b_{11}, b_{21}, b_{31}, b_{41}), (c_{11}, c_{21}, c_{31}, c_{41})\big\rangle$ and $\tilde{A}_2 = \big\langle(a_{12}, a_{22}, a_{32}, a_{42}), (b_{12}, b_{22}, b_{32}, b_{42}), (c_{12}, c_{22}, c_{32}, c_{42})\big\rangle$ be two SVTrNN in the set of real numbers $\mathbb{R}$. Then for $\lambda, \mu, \nu \in [0, 1]$ and $\psi \in \mathbb{R}$, the following results hold good.*

$$V_{\lambda,\mu,\nu}\big(\tilde{A}_1 + \tilde{A}_2\big) = V_{\lambda,\mu,\nu}\big(\tilde{A}_1\big) + V_{\lambda,\mu,\nu}\big(\tilde{A}_2\big) \tag{34}$$

$$V_{\lambda,\mu,\nu}\big(\phi\tilde{A}_1\big) = \phi V_{\lambda,\mu,\nu}\big(\tilde{A}_1\big) \tag{35}$$

Proof. From definition-14, the sum of two NTrFNs $\tilde{A}_1$ and $\tilde{A}_2$ can be written as follows:

$$\begin{aligned}\tilde{A} \oplus \tilde{B} = \Big\langle&(a_{11} + a_{12} - a_{11}a_{12}, a_{21} + a_{22} - a_{21}a_{22}, \\ &a_{31} + a_{32} - a_{31}a_{32}, a_{41} + a_{42} - a_{41}a_{42}), \\ &(b_{11}b_{12}, b_{21}b_{22}, b_{31}b_{32}, b_{41}b_{42}), \\ &(c_{11}c_{12}, c_{21}c_{22}, c_{31}c_{32}, c_{41}c_{42})\Big\rangle\end{aligned}$$

Now, by Eq.(31) the value index of the sum of two SVTrNNs $\tilde{A}_1$ and

$\tilde{A}_2$ can be written as follows:

$$V_{\lambda,\mu,\nu}(\tilde{A}_1+\tilde{A}_2) \tag{36}$$
$$=\lambda V_T(\tilde{A}_1+\tilde{A}_2)+\mu V_I(\tilde{A}_1+\tilde{A}_2)+\nu V_F(\tilde{A}_1+\tilde{A}_2) \tag{37}$$
$$=\begin{bmatrix}\frac{\lambda}{6}\Big[(a_{11}+a_{21})+2(a_{12}+a_{22})+2(a_{13}+a_{23})+(a_{14}+a_{24})\Big]\\ +\frac{\mu}{6}\Big[(b_{11}+b_{21})+2(b_{12}+b_{22})+2(b_{13}+b_{23})+(b_{14}+b_{24})\Big]\\ +\frac{\nu}{6}\Big[(c_{11}+c_{21})+2(c_{12}+c_{22})+2(c_{13}+c_{23})+(c_{14}+c_{24})\Big]\end{bmatrix} \tag{38}$$
$$\begin{aligned}=&\frac{\lambda}{6}\Big(a_{11}+2a_{12}+2a_{13}+a_{14}\Big)+\frac{\lambda}{6}\Big(a_{21}+2a_{22}+2a_{23}+a_{24}\Big)\\ &+\frac{\mu}{6}\Big(b_{11}+2b_{12}+2b_{13}+b_{14}\Big)+\frac{\mu}{6}\Big(b_{21}+2b_{22}+2b_{23}+b_{24}\Big)\\ &+\frac{\nu}{6}\Big(c_{11}+2c_{12}+2c_{13}+c_{14}\Big)+\frac{\nu}{6}\Big(c_{21}+2c_{22}+2c_{23}+c_{24}\Big)\\ =&V_{\lambda,\mu,\nu}(\tilde{A}_1)+V_{\lambda,\mu,\nu}(\tilde{A}_2)\end{aligned} \tag{39}$$

For the second part of the theorem,

$$V_{\lambda,\mu,\nu}(\phi\tilde{A}_1) \tag{40}$$
$$\begin{aligned}&=\lambda V_T(\phi\tilde{A}_1)+\mu V_I(\phi\tilde{A}_1)+\nu V_I(\phi\tilde{A}_1)\\ &=\begin{bmatrix}\frac{\lambda}{6}(\phi a_{11}+2\phi a_{12}+2\phi a_{13}+\phi a_{14})\\ +\frac{\mu}{6}(\phi b_{11}+2\phi b_{12}+2\phi b_{13}+\phi b_{14})\\ +\frac{\nu}{6}(\phi c_{11}+2\phi c_{12}+2\phi c_{13}+\phi c_{14})\end{bmatrix}\\ &=\phi\begin{bmatrix}\frac{\lambda}{6}(a_{11}+2a_{12}+2a_{13}+a_{14})\\ +\frac{\mu}{6}(b_{11}+2b_{12}+2b_{13}+b_{14})\\ +\frac{\nu}{6}(c_{11}+2c_{12}+2c_{13}+c_{14})\end{bmatrix}\\ &=\phi V_{\lambda,\mu,\nu}(\tilde{A}_1)\end{aligned} \tag{41}$$

Therefore, $V_{\lambda,\mu,\nu}(\tilde{A}_1+\tilde{A}_2)$=$V_{\lambda,\mu,\nu}(\tilde{A}_1)+V_{\lambda,\mu,\nu}(\tilde{A}_2)$ and $V_{\lambda,\mu,\nu}(\phi\tilde{A}_1)$= $\phi V_{\lambda,\mu,\nu}(\tilde{A}_1)$. □

Theorem 4. *Let* $\tilde{A}_1$=$\langle(a_{11},a_{21},a_{31},a_{41}),\ (b_{11},b_{21},b_{31},b_{41}),\ (c_{11},c_{21},c_{31},c_{41})\rangle$ *and* $\tilde{A}_2$ =$\langle(a_{12},a_{22},a_{32},a_{42}),\ (b_{12},b_{22},b_{32},b_{42}),(c_{12},c_{22},c_{32},c_{42})\rangle$ *be two SVTrNNs in the set of real numbers* $\mathbb{R}$*. Then for* $\lambda,\mu,\nu\in[0,1]$ *and* $\psi\in\mathbb{R}$*, the following equations hold good.*

$$A_{\lambda,\mu,\nu}(\tilde{A}_1+\tilde{A}_2)=A_{\lambda,\mu,\nu}(\tilde{A}_1)+A_{\lambda,\mu,\nu}(\tilde{A}_2) \tag{42}$$
$$A_{\lambda,\mu,\nu}(\phi\tilde{A}_1)=\phi A_{\lambda,\mu,\nu}(\tilde{A}_1) \tag{43}$$

Proof. From definition-14, the sum of two SVTrNNs $\tilde{A}_1$ and $\tilde{A}_2$, the ambiguity index of the sum of two SVTrNNs $\tilde{A}_1$ and $\tilde{A}_2$ can be written as:

$$A_{\lambda,\mu,\nu}(\tilde{A}_1+\tilde{A}_2) \tag{44}$$
$$=\lambda A_T(\tilde{A}_1+\tilde{A}_2)+\mu A_I(\tilde{A}_1+\tilde{A}_2)+\nu A_F(\tilde{A}_1+\tilde{A}_2) \tag{45}$$
$$=\begin{bmatrix}\frac{\lambda}{6}\Big[-(a_{11}+a_{21})-2(a_{12}+a_{22})+2(a_{13}+a_{23})+(a_{14}+a_{24})\Big]\\ +\frac{\mu}{6}\Big[-(b_{11}+b_{21})-2(b_{12}+b_{22})+2(b_{13}+b_{23})+(b_{14}+b_{24})\Big]\\ +\frac{\nu}{6}\Big[-(c_{11}+c_{21})-2(c_{12}+c_{22})+2(c_{13}+c_{23})+(c_{14}+c_{24})\Big]\end{bmatrix} \tag{46}$$
$$\begin{aligned}=&\frac{\lambda}{6}\Big(-a_{11}-2a_{12}+2a_{13}+a_{14}\Big)+\frac{\lambda}{6}\Big(-a_{21}-2a_{22}+2a_{23}+a_{24}\Big)\\ &+\frac{\mu}{6}\Big(-b_{11}-2b_{12}+2b_{13}+b_{14}\Big)+\frac{\mu}{6}\Big(-b_{21}-2b_{22}+2b_{23}+b_{24}\Big)\\ &+\frac{\nu}{6}\Big(-c_{11}-2c_{12}+2c_{13}+c_{14}\Big)+\frac{\nu}{6}\Big(-c_{21}-2c_{22}+2c_{23}+c_{24}\Big)\\ =&A_{\lambda,\mu,\nu}(\tilde{A}_1)+A_{\lambda,\mu,\nu}(\tilde{A}_2)\end{aligned}$$

For the second part of the theorem,

$$A_{\lambda,\mu,\nu}(\phi\tilde{A}_1) \tag{47}$$
$$\begin{aligned}&=\lambda A_T(\phi\tilde{A}_1)+\mu A_I(\phi\tilde{A}_1)+\nu A_I(\phi\tilde{A}_1)\\ &=\begin{bmatrix}\frac{\lambda}{6}(-\phi a_{11}-2\phi a_{12}+2\phi a_{13}+\phi a_{14})\\ +\frac{\mu}{6}(-\phi b_{11}-2\phi b_{12}+2\phi b_{13}+\phi b_{14})\\ +\frac{\nu}{6}(-\phi c_{11}-2\phi c_{12}+2\phi c_{13}+\phi c_{14})\end{bmatrix}\\ &=\phi\begin{bmatrix}\frac{\lambda}{6}(-a_{11}-2a_{12}+2a_{13}+a_{14})\\ +\frac{\mu}{6}(-b_{11}-2b_{12}+2b_{13}+b_{14})\\ +\frac{\nu}{6}(-c_{11}-2c_{12}+2c_{13}+c_{14})\end{bmatrix}\\ &=\phi A_{\lambda,\mu,\nu}(\tilde{A}_1)\end{aligned} \tag{48}$$

Therefore, $A_{\lambda,\mu,\nu}(\tilde{A}_1+\tilde{A}_2)$=$A_{\lambda,\mu,\nu}(\tilde{A}_1)+V_{\lambda,\mu,\nu}(\tilde{A}_2)$ and $A_{\lambda,\mu,\nu}(\phi\tilde{A}_1)$= $\phi A_{\lambda,\mu,\nu}(\tilde{A}_1)$. □

Proposition 1. *Let* $\tilde{A}_1$=$\langle(a_{11},a_{21},a_{31},a_{41}),\ (b_{11},b_{21},b_{31},b_{41}),\ (c_{11},c_{21},c_{31},c_{41})\rangle$ *and* $\tilde{A}_2$ =$\langle(a_{12},a_{22},a_{32},a_{42}),\ (b_{12},b_{22},b_{32},b_{42}),(c_{12},c_{22},c_{32},c_{42})\rangle$ *be two SVTrNNs in the set of real numbers* $\mathbb{R}$*. Then ranking of two SVTrNNs* $\tilde{A}_1$ *and* $\tilde{A}_2$ *can be done by using the value and ambiguity of SVTrNN. The procedures have been defined as follows:*

P1. *If* $V_{\lambda,\mu,\nu}(A_1)\le V_{\lambda,\mu,\nu}(A_1)$*, then* $\tilde{A}_1$ *is smaller than* $\tilde{A}_2$*, i.e,* $\tilde{A}_1\prec\tilde{A}_2$.

P2. *If* $V_{\lambda,\mu,\nu}(A_1)\ge V_{\lambda,\mu,\nu}(A_1)$*, then* $\tilde{A}_1$ *is greater than* $\tilde{A}_2$*, i.e,* $\tilde{A}_1\succ\tilde{A}_2$.

P3. *If* $V_{\lambda,\mu,\nu}(A_1)=V_{\lambda,\mu,\nu}(A_1)$ *and* $A_{\lambda,\mu,\nu}(A_1)\ge V_{\lambda,\mu,\nu}(A_1)$*, then* $\tilde{A}_1$ *is smaller than* $\tilde{A}_2$*, i.e,* $\tilde{A}_1\prec\tilde{A}_2$.

P4. *If* $V_{\lambda,\mu,\nu}(A_1)=V_{\lambda,\mu,\nu}(A_1)$ *and* $A_{\lambda,\mu,\nu}(A_1)\le V_{\lambda,\mu,\nu}(A_1)$*, then* $\tilde{A}_1$ *is grater than* $\tilde{A}_2$*, i.e,* $\tilde{A}_1\succ\tilde{A}_2$.

P5. *If* $V_{\lambda,\mu,\nu}(A_1)=V_{\lambda,\mu,\nu}(A_1)$ *and* $A_{\lambda,\mu,\nu}(A_1)=V_{\lambda,\mu,\nu}(A_1)$*, then* $\tilde{A}_1$ *is equal* $\tilde{A}_2$*, i.e,* $\tilde{A}_1\approx\tilde{A}_2$.

Theorem 5. *Let* $\tilde{A}_1$=$\langle(a_{11},a_{21},a_{31},a_{41}),\ (b_{11},b_{21},b_{31},b_{41}),\ (c_{11},c_{21},c_{31},c_{41})\rangle$ *and* $\tilde{A}_2$ =$\langle(a_{12},a_{22},a_{32},a_{42}),$

$(b_{12}, b_{22}, b_{32}, b_{42}), (c_{12}, c_{22}, c_{32}, c_{42})\rangle$ *be two NTrFNs in the set of real numbers* $\mathbb{R}$.*If* $a_{11} > a_{42}$, $b_{11} > b_{42}$ *and* $c_{11} > c_{42}$ *then* $\tilde{A}_1 > \tilde{A}_2$.

Proof. We can obtain the following results from Eq.(21), (22) and (23):

$$V_T(\tilde{A}_1) = \frac{\lambda}{6}\big(a_{11} + 2a_{21} + 2a_{31} + a_{41}\big) > a_{11},$$
$$V_T(\tilde{A}_2) = \frac{\lambda}{6}\big(a_{12} + 2a_{22} + 2a_{32} + a_{42}\big) < a_{42}$$
$$V_I(\tilde{A}_1) = \frac{\lambda}{6}\big(b_{11} + 2b_{21} + 2b_{31} + b_{41}\big) > b_{11},$$
$$V_I(\tilde{A}_2) = \frac{\lambda}{6}\big(b_{12} + 2b_{22} + 2b_{32} + b_{42}\big) < b_{42}$$
$$\text{and} \quad V_F(\tilde{A}_1) = \frac{\lambda}{6}\big(c_{11} + 2c_{21} + 2c_{31} + c_{41}\big) > c_{11},$$
$$V_F(\tilde{A}_2) = \frac{\lambda}{6}\big(c_{12} + 2c_{22} + 2c_{32} + c_{42}\big) < c_{42}$$

With the relations $a_{11} > a_{42}$, $b_{11} > b_{42}$ and $c_{11} > c_{42}$, it follows that $V_T(\tilde{A}_1) > V_T(\tilde{A}_2)$, $V_I(\tilde{A}_1) > V_I(\tilde{A}_2)$, and $V_F(\tilde{A}_1) > V_F(\tilde{A}_2)$. Therefore from Eq.(30), we can obtain

$$\begin{aligned} V_{\lambda,\mu,\nu}(\tilde{A}_1) &= \lambda V_T(\tilde{A}_1) + \mu V_I(\tilde{A}_1) + \nu V_F(\tilde{A}_1) \\ &> \lambda V_T(\tilde{A}_2) + \mu V_I(\tilde{A}_2) + \nu V_F(\tilde{A}_2) = V_{\lambda,\mu,\nu}(\tilde{A}_2) \end{aligned}$$

This completes the proof. □

Theorem 6. *Let* A_1, A_2 *and* A_3 *be three SVTrNNs, where* $\tilde{A}_i = \langle(a_{1i}, a_{2i}, a_{3i}, a_{4i}), (b_{1i}, b_{2i}, b_{3i}, b_{4i}), (c_{1i}, c_{2i}, c_{3i}, c_{4i})\rangle$ *for* $i = 1, 2, 3$. *If* $\tilde{A}_1 > \tilde{A}_2$, *then* $\tilde{A}_1 + \tilde{A}_3 > \tilde{A}_2 + \tilde{A}_3$.

Proof. For A_1, A_2 and A_3, we can write the following results from Eq.(30):

$$\begin{aligned} V_{\lambda,\mu,\nu}(\tilde{A}_1 + \tilde{A}_2) &= V_{\lambda,\mu,\nu}(\tilde{A}_1) + V_{\lambda,\mu,\nu}(\tilde{A}_2) \\ \text{and } V_{\lambda,\mu,\nu}(\tilde{A}_2 + \tilde{A}_3) &= V_{\lambda,\mu,\nu}(\tilde{A}_2) + V_{\lambda,\mu,\nu}(\tilde{A}_3). \end{aligned}$$

Since $\tilde{A}_1 > \tilde{A}_2$, then we have

$$\begin{aligned} V_{\lambda,\mu,\nu}(\tilde{A}_1 + \tilde{A}_2) &= V_{\lambda,\mu,\nu}(\tilde{A}_1) + V_{\lambda,\mu,\nu}(\tilde{A}_2) \\ &> V_{\lambda,\mu,\nu}(\tilde{A}_2) + V_{\lambda,\mu,\nu}(\tilde{A}_3) \\ &= V_{\lambda,\mu,\nu}(\tilde{A}_2 + \tilde{A}_3). \end{aligned}$$

This completes the proof. □

6 Formulation of MADM model under SVTrNNs information

In this section, we present value and ambiguity based ranking method to MADM in which the ratings of alternatives over the attributes have been expressed with NTrFNs. Assume that for a MADM problem, $A = \{A_1, A_2, \ldots, A_m\}$ be a set of m alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ be a set of n attributes. The weight vector of the attributes provided the decision makers is $W = (w_1, w_2, \ldots, w_n)^T$, where $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$ and w_j is the degree of importance for the attribute C_j. The rating of alternative A_i with respect to attribute C_j has been expressed with NTrFN $d_{ij} = \langle(a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}), (b^1_{ij}, b^2_{ij}, b^3_{ij}, b^4_{ij}), (c^1_{ij}, c^2_{ij}, c^3_{ij}, c^4_{ij})\rangle$, where a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}, b^1_{ij}, b^2_{ij}, b^3_{ij}, b^4_{ij}, c^1_{ij}, c^2_{ij}, c^3_{ij}, $c^4_{ij} \in \mathbb{R}$ and $c^1_{ij} \le b^1_{ij} \le a^1_{ij} \le c^2_{ij} \le b^2_{ij} \le a^2_{ij} \le a^3_{ij} \le b^3_{ij} \le c^3_{ij} \le a^4_{ij} \le b^4_{ij} \le c^4_{ij}$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. The component $(a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij})$, $(b^1_{ij}, b^2_{ij}, b^3_{ij}, b^4_{ij})$, and $(c^1_{ij}, c^2_{ij}, c^3_{ij}, c^4_{ij})$ represent the truth membership degree, the indeterminacy membership degree and the falsity membership degree, respectively, of the alternative A_i with respect to the attribute C_j.

In a MADM problem, the rating values $\tilde{d}_{ij} = \langle(a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}), (b^1_{ij}, b^2_{ij}, b^3_{ij}, b^4_{ij}), (c^1_{ij}, c^2_{ij}, c^3_{ij}, c^4_{ij})\rangle$ can be arranged in a matrix format, we call it neutrosophic decision matrix $D = \big(\tilde{d}_{ij}\big)_{m\times n}$ where,

$$\big(\tilde{d}_{ij}\big)_{m\times n} = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_n \\ \hline A_1 & \tilde{d}_{11} & \tilde{d}_{12} & \cdots & \tilde{d}_{13} \\ A_2 & \tilde{d}_{21} & \tilde{d}_{22} & \cdots & \tilde{d}_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_m & \tilde{d}_{m1} & \tilde{d}_{m2} & \cdots & \tilde{d}_{mn} \end{array} \tag{49}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Here, value index and ambiguity index of SVTrNN have been applied to solve a MADM problem with SVTrNN by the following steps:

Step 1. Normalization of SVTrNNs based decision matrix

The decision matrix $\big(\tilde{d}_{ij}\big)_{m\times n}$ needs to be normalized into $\big(\tilde{r}_{ij}\big)_{m\times n}$ to eliminate the effect of different physical dimensions during final decision making process, where $\tilde{r}_{ij} = \langle(x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}), (y^1_{ij}, y^2_{ij}, y^3_{ij}, y^4_{ij}), (z^1_{ij}, z^2_{ij}, z^3_{ij}, z^4_{ij})\rangle$. Linear normalization technique has been used to normalize the decision matrix for the benefit type attribute (B) and cost type attribute (C) by the following formulas:

$$\begin{aligned} \tilde{r}_{ij} = \Bigg\langle &\left(\frac{x^1_{ij}}{x^{4+}_j}, \frac{x^2_{ij}}{x^{4+}_j}, \frac{x^3_{ij}}{x^{4+}_j}, \frac{x^4_{ij}}{x^{4+}_j}\right), \left(\frac{y^1_{ij}}{y^{4+}_j}, \frac{y^2_{ij}}{y^{4+}_j}, \frac{y^3_{ij}}{y^{4+}_j}, \frac{y^4_{ij}}{y^{4+}_j}\right), \\ &\left(\frac{z^1_{ij}}{z^{4+}_j}, \frac{z^2_{ij}}{z^{4+}_j}, \frac{z^3_{ij}}{z^{4+}_j}, \frac{z^4_{ij}}{z^{4+}_j}\right)\Bigg\rangle \text{ for } j \in B; \end{aligned} \tag{50}$$

$$\begin{aligned} \tilde{r}_{ij} = \Bigg\langle &\left(\frac{x^{1-}_j}{x^1_{ij}}, \frac{x^{1-}_j}{x^2_{ij}}, \frac{x^{1-}_j}{x^3_{ij}}, \frac{x^{1-}_j}{x^4_{ij}}\right), \left(\frac{y^{1-}_j}{y^1_{ij}}, \frac{y^{1-}_j}{y^2_{ij}}, \frac{y^{1-}_j}{y^3_{ij}}, \frac{y^{1-}_j}{y^4_{ij}}\right), \\ &\left(\frac{z^{1-}_j}{z^1_{ij}}, \frac{z^{1-}_j}{z^2_{ij}}, \frac{z^{1-}_j}{z^3_{ij}}, \frac{z^{1-}_j}{z^4_{ij}}\right)\Bigg\rangle \text{ for } j \in C \end{aligned} \tag{51}$$

where, $x^{4+}_j = \max_i(x^4_{ij})$, $y^{4+}_j = \max_i(y^4_{ij})$, $z^{4+}_j = \max_i(z^4_{ij})$, $x^{1-}_j = \max_i(x^1_{ij})$, $y^{1-}_j = \max_i(y^1_{ij})$, and $z^{1-}_j = \max_i(z^1_{ij})$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 2. Aggregation of the weighted rating values of alternatives

According to definition-14, the aggregated weighted rating values of the alternatives $A_i (i = 1, 2, \ldots, m)$ can be determined as

$$\tilde{S}_i = \sum_{j=1}^{n} w_j \tilde{r}_{ij}, \tag{52}$$

respectively. Here, the aggregated weighted rating values $\tilde{S}_i (i = 1, 2, \ldots, m)$ are considered as SVTrNNs.

Step 3. Ranking of all alternatives

According to Eq.(52) and Proposition-1, ranking of all alternatives can be determined to the non-increasing order of SVTrNNs $\tilde{A}_i (i = 1, 2, \ldots, m)$ by using the value and ambiguity index of SVTrNN.

7 An illustrative Example

Consider a decision making problem in which a customer intends to buy a tablet from the set of primarily chosen five tablets $A_i (i = 1, 2, 3, 4, 5)$.The customer takes into account of the four attributes namely:

1. features (C_1);
2. hardware specification (C_2);
3. affordable price (C_3);
4. customer care (C_4).

Assume that the weight vector of the four attribute is W=$\{0.25, 0.25, 0.30, 0.20\}$ and the evaluations of the five alternatives with respect to the four attributes have been considered as SVTrNNs. Then we have a SVTrNNs based decision matrix $\left(\tilde{d}_{ij}\right)_{5\times 4}$ presented in Table-1.

Table 1: NTrFNs based decision matrix

	C_1
A_1	$\langle(0.5, 0.6, 0.7, 0.8), (0.1, 0.1, 0.2, 0.3), (0.1, 0.2, 0.2, 0.3)\rangle$
A_2	$\langle(0.3, 0.4, 0.5, 0.5), (0.1, 0.2, 0.2, 0.4), (0.1, 0.1, 0.2, 0.3)\rangle$
A_3	$\langle(0.3, 0.3, 0.3, 0.3), (0.2, 0.3, 0.4, 0.4), (0.6, 0.7, 0.8, 0.9)\rangle$
A_4	$\langle(0.7, 0.8, 0.8, 0.9), (0.1, 0.2, 0.3, 0.3), (0.2, 0.2, 0.2, 0.2)\rangle$
A_5	$\langle(0.1, 0.2, 0.2, 0.3), (0.2, 0.2, 0.3, 0.4), (0.6, 0.6, 0.7, 0.8)\rangle$
	C_2
A_1	$\langle(0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.4), (0.4, 0.5, 0.6, 0.7)\rangle$
A_2	$\langle(0.2, 0.3, 0.4, 0.5), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.3)\rangle$
A_3	$\langle(0.1, 0.2, 0.2, 0.3), (0.2, 0.3, 0.3, 0.4), (0.4, 0.5, 0.6, 0.6)\rangle$
A_4	$\langle(0.5, 0.6, 0.7, 0.7), (0.2, 0.2, 0.2, 0.2), (0.1, 0.1, 0.2, 0.2)\rangle$
A_5	$\langle(0.5, 0.6, 0.6, 0.7), (0.1, 0.2, 0.3, 0.4), (0.2, 0.2, 0.3, 0.4)\rangle$
	C_3
A_1	$\langle(0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.2, 0.3), (0.2, 0.2, 0.3, 0.4)\rangle$
A_2	$\langle(0.2, 0.2, 0.2, 0.2), (0.1, 0.1, 0.1, 0.1), (0.6, 0.7, 0.8, 0.8)\rangle$
A_3	$\langle(0.2, 0.3, 0.4, 0.5), (0.2, 0.3, 0.3, 0.4), (0.3, 0.4, 0.4, 0.5)\rangle$
A_4	$\langle(0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.2, 0.3), (0.1, 0.2, 0.3, 0.4)\rangle$
A_5	$\langle(0.6, 0.7, 0.8, 0.8), (0.2, 0.2, 0.3, 0.3), (0.1, 0.1, 0.2, 0.3)\rangle$
	C_4
A_1	$\langle(0.4, 0.5, 0.6, 0.7), (0.2, 0.2, 0.3, 0.4), (0.1, 0.2, 0.3, 0.4)\rangle$
A_2	$\langle(0.4, 0.5, 0.6, 0.6), (0.2, 0.2, 0.3, 0.3), (0.2, 0.3, 0.4, 0.4)\rangle$
A_3	$\langle(0.2, 0.2, 0.3, 0.4), (0.3, 0.3, 0.3, 0.3), (0.3, 0.4, 0.5, 0.6)\rangle$
A_4	$\langle(0.1, 0.2, 0.3, 0.4), (0.2, 0.2, 0.3, 0.3), (0.5, 0.6, 0.7, 0.8)\rangle$
A_5	$\langle(0.2, 0.3, 0.4, 0.4), (0.1, 0.2, 0.3, 0.4), (0.3, 0.4, 0.4, 0.5)\rangle$

Step 1. Normalization of SVTrNNs based decision matrix

Using Eq.(50), the decision matrix $\left(\tilde{d}_{ij}\right)_{5\times 4}$ has been normalized to the decision matrix $\left(\tilde{d}^N_{ij}\right)_{5\times 4}$ by considering the selected four attributes as benefit type attributes. Then the normalized decision matrix $\left(\tilde{d}^N_{ij}\right)_{5\times 4}$ can be obtained in Table-2.

Step 2. Aggregation of the weighted normalized rating values of alternatives

The weighted normalized rating values of the alternative $A_i (i = 1, 2, 3, 4, 5)$ can be determined by using Eq.(52). Table-3 shows the aggregated weighted normalized rating values of alternatives.

Step 3. Ranking of all alternatives

The value index and ambiguity index of NTrFNs $\tilde{A}_i (i = 1, 2, \ldots, m)$ are determined by using Definition-17 and Proposition-1 as

$$\begin{aligned}
V_{\lambda,\mu,\nu}(A_1) &= 0.5428\lambda + 0.5542\mu + 0.4536\nu;\\
V_{\lambda,\mu,\nu}(A_2) &= 0.6041\lambda + 0.4396\mu + 0.5365\nu;\\
V_{\lambda,\mu,\nu}(A_3) &= 0.5667\lambda + 0.7708\mu + 0.5898\nu;\\
V_{\lambda,\mu,\nu}(A_4) &= 0.5871\lambda + 0.7278\mu + 0.3656\nu;\\
V_{\lambda,\mu,\nu}(A_5) &= 0.6083\lambda + 0.6354\mu + 0.4542\nu;
\end{aligned}$$

and

$$\begin{aligned}
A_{\lambda,\mu,\nu}(A_1) &= 0.0802\lambda + 0.1417\mu + 0.0941\nu\\
A_{\lambda,\mu,\nu}(A_2) &= 0.0847\lambda + 0.0979\mu + 0.1260\nu\\
A_{\lambda,\mu,\nu}(A_3) &= 0.0933\lambda + 0.0875\mu + 0.0713\nu\\
A_{\lambda,\mu,\nu}(A_4) &= 0.0574\lambda + 0.1222\mu + 0.0677\nu\\
A_{\lambda,\mu,\nu}(A_5) &= 0.0625\lambda + 0.1729\mu + 0.0750\nu.
\end{aligned}$$

To rank the alternatives $A_i (i = 1, 2, 3, 4, 5)$, the value index and ambiguity index of each alternative have been examined for different values for $\lambda, \mu, \nu \in [0, 1]$. The results have been shown in the Table-4. For different values of $\lambda, \mu, \nu \in [0, 1]$, the ranking order of alternatives has been obtained as follows:

$$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1.$$

Thus A_5 is the best alternative.

8 Conclusions

In the present study, we have introduced the concept of SVTrNN and defined some operational rules. We have also defined value index and ambiguity index of SVTrNN and established some of their properties. Then we have proposed a ranking method SVTrNN by using these two indices of SVTrNN. The proposed method has been applied to MADM problem with SVTrNN information. The method is simple, attractive and effective to determine the ranking order of alternatives used in neutrosophic MADM problems. The proposed concept can be easily extended to rank single-valued triangular neutrosophic numbers. The proposed MADM approach can be extended to solve the problem of medical diagnosis, pattern recognition, personal selection, etc.

Table 2: SVTrNNs based normalized decision matrix

	C_1
A_1	$\langle(0.6250, 0.7500, 0.8750, 1.000), (0.2500, 0.2500, 0.5000, 0.7500), (0.1429, 0.2857, 0.2857, 0.4286)\rangle$
A_2	$\langle(0.5000, 0.6667, 0.8333, 0.8333), (0.2500, 0.5000, 0.5000, 1.0000), (0.1250, 0.1250, 0.2500, 0.3750)\rangle$
A_3	$\langle(0.6000, 0.6000, 0.6000, 0.6000), (0.5000, 0.7500, 1.0000, 1.0000), (0.6667, 0.7778, 0.8889, 1.0000)\rangle$
A_4	$\langle(0.7778, 0.8889, 0.8889, 1.0000), (0.3333, 0.6667, 1.0000, 1.0000), (0.2500, 0.2500, 0.2500, 0.2500)\rangle$
A_5	$\langle(0.1250, 0.2500, 0.2500, 0.3750), (0.5000, 0.5000, 0.7500, 1.0000), (0.7500, 0.7500, 0.8750, 1.0000)\rangle$
	C_2
A_1	$\langle(0.1250, 0.1250, 0.2500, 0.3750), (0.5000, 0.5000, 0.7500, 1.0000), (0.5714, 0.7143, 0.8571, 1.0000)\rangle$
A_2	$\langle(0.3333, 0.5000, 0.6667, 0.8333), (0.2500, 0.2500, 0.5000, 0.7500), (0.2500, 0.2500, 0.3750, 0.3750)\rangle$
A_3	$\langle(0.2000, 0.4000, 0.4000, 0.6000), (0.5000, 0.7500, 0.7500, 1.0000), (0.4444, 0.5556, 0.6667, 0.6667)\rangle$
A_4	$\langle(0.5556, 0.6667, 0.7778, 0.7778), (0.6667, 0.6667, 0.6667, 0.6667), (0.1250, 0.1250, 0.2500, 0.2500)\rangle$
A_5	$\langle(0.6250, 0.7500, 0.7500, 0.8750), (0.2500, 0.5000, 0.7500, 1.0000), (0.2500, 0.2500, 0.3750, 0.5000)\rangle$
	C_3
A_1	$\langle(0.3750, 0.5000, 0.5000, 0.6250), (0.2500, 0.5000, 0.5000, 0.7500), (0.2857, 0.2857, 0.4286, 0.5714)\rangle$
A_2	$\langle(0.3333, 0.3333, 0.3333, 0.3333), (0.2500, 0.2500, 0.2500, 0.2500), (0.7500, 0.8750, 1.0000, 1.0000)\rangle$
A_3	$\langle(0.4000, 0.6000, 0.8000, 1.0000), (0.5000, 0.7500, 0.7500, 1.0000), (0.3333, 0.4444, 0.4444, 0.5556)\rangle$
A_4	$\langle(0.3333, 0.4444, 0.4444, 0.5556), (0.3333, 0.6667, 0.6667, 1.0000), (0.1250, 0.2500, 0.3750, 0.5000)\rangle$
A_5	$\langle(0.7500, 0.8750, 1.0000, 1.0000), (0.5000, 0.5000, 0.7500, 0.7500), (0.1250, 0.1250, 0.2500, 0.3750)\rangle$
	C_4
A_1	$\langle(0.5000, 0.6250, 0.7500, 0.8750), (0.5000, 0.5000, 0.7500, 1.0000), (0.1429, 0.2857, 0.4286, 0.5714)\rangle$
A_2	$\langle(0.6667, 0.8333, 1.0000, 1.0000), (0.5000, 0.5000, 0.7500, 0.7500), (0.2500, 0.3750, 0.5000, 0.5000)\rangle$
A_3	$\langle(0.4000, 0.4000, 0.6000, 0.8000), (0.7500, 0.7500, 0.7500, 0.7500), (0.3333, 0.4444, 0.5556, 0.6667)\rangle$
A_4	$\langle(0.1111, 0.2222, 0.3333, 0.4444), (0.6667, 0.6667, 1.0000, 1.0000), (0.6250, 0.7500, 0.8750, 1.0000)\rangle$
A_5	$\langle(0.2500, 0.3750, 0.5000, 0.5000), (0.2500, 0.5000, 0.7500, 1.0000), (0.3750, 0.5000, 0.5000, 0.6250)\rangle$

Table 3: Aggregated rating values of attributes

Alternative	Aggregated rating values of Attributes
A_1	$\langle(0.4000, 0.4938, 0.5813, 0.7063), (0.3625, 0.4375, 0.6125, 0.8625), (0.2929, 0.3928, 0.5000, 0.6429)\rangle$
A_2	$\langle(0.4417, 0.5583, 0.6750, 0.7166), (0.3000, 0.3625, 0.4750, 0.6625), (0.3688, 0.4313, 0.5563, 0.8750)\rangle$
A_3	$\langle(0.4000, 0.5100, 0.6100, 0.7600), (0.5500, 0.7500, 0.8125, 0.9500), (0.4444, 0.5556, 0.6333, 0.7167)\rangle$
A_4	$\langle(0.4556, 0.5667, 0.6167, 0.7000), (0.4833, 0.6667, 0.8167, 0.9167), (0.2563, 0.3188, 0.4125, 0.4750)\rangle$
A_5	$\langle(0.4625, 0.5875, 0.6500, 0.7125), (0.3875, 0.5000, 0.7500, 0.9250), (0.3625, 0.3875, 0.4875, 0.6125)\rangle$

Table 4: Ranking results for alternatives

Alternative	Value of λ, μ, ν	Value index	Ambiguity index	Ranking order
A_1		0.5027	0.1117	
A_2		0.5045	0.1107	
A_3	$\lambda = .10; \mu = .40;$	0.6599	0.0800	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
A_4	$\nu = .50$	0.5327	0.0885	
A_5		0.5421	0.1129	
A_1		0.5125	0.1051	
A_2		0.5258	0.1046	
A_3	$\lambda = .30; \mu = .32;$	0.6408	0.0831	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
A_4	$\nu = .38$	0.5480	0.0821	
A_5		0.5584	0.1026	
A_1		0.5168	0.1053	
A_2		0.5267	0.1029	
A_3	$\lambda = \frac{1}{3}; \mu = \frac{1}{3};$	0.6424	0.0840	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
A_4	$\nu = \frac{1}{3}$	0.5602	0.0824	
A_5		0.5660	0.1035	
A_1		0.5283	0.1014	
A_2		0.5412	0.0969	
A_3	$\lambda = .50; \mu = .30;$	0.6325	0.0872	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
A_4	$\nu = .20$	0.5850	0.0789	
A_5		0.5856	0.0981	
A_1		0.5361	0.0939	
A_2		0.5645	0.0915	
A_3	$\lambda = .70; \mu = .20;$	0.6098	0.0900	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
A_4	$\nu = .10$	0.5931	0.0714	
A_5		0.5983	0.0858	

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A Projection Model of Neutrosophic Numbers for Multiple Attribute Decision Making of Clay-Brick Selection

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Abstract. Brick plays a significant role in building construction. So we should use the effective mathematical decision making tool to select quality clay-bricks for building construction. The purpose of this paper is to present a projection model of neutrosophic numbers and its decision-making method for the selecting problems of clay-bricks with neutrosophic number information. The projection method of neutrosophic numbers is one useful tool that can deal with decision-making problems with indeterminacy data. By the projection measure between each alternative and the ideal alternative, all the alternatives can be ranked to select the best one. Finally, an actual example on clay-brick selection in construction field demonstrates the application and effectiveness of the projection method.

Keywords: Neutrosophic number, projection method, clay-brick selection, decision making.

1 Introduction

As we know, in realistic decision making situations, some information cannot be described only by unique crisp numbers, and then may imply indeterminacy. In order to deal with this situation, Smarandache [1-3] introduced neutrosophic numbers. To apply them in real situations, Ye [4, 5] proposed the method of de-neutrosophication and possibility degree ranking order of neutrosophic numbers and the bidirectional projection method respectively, and then applied them to multiple attribute group decision-making problems under neutrosophic number environments. Then, Ye [6] developed a fault diagnosis method of steam turbine using the exponential similarity measure of neutrosophic numbers. Further Kong et al. [7] presented the misfire fault diagnosis method of gasoline engine by using the cosine similarity measure of neutrosophic numbers.

Clay-brick selection problem in construction field is a multiple attribute decision-making problem. Hence, Mondal and Pramanik [8] presented a quality clay-brick selection approach based on multiple attribute decision making with single valued neutrosophic grey relational analysis. However, so far neutrosophic numbers are not applied to decision making problems in construction field. To do it, this paper introduces a projection-based model of neutrosophic numbers and applies it to the multiple attribute decision-making problem of clay-brick selection in construction field under neutrosophic number environment.

The rest of the paper is organized as the following. Section 2 reviews basic concepts of neutrosophic numbers. Section 3 introduces a projection measure of neutrosophic numbers. Section 4 presents a multiple attribute decision-making method based on the projection model under neutrosophic number environment. In section 5, an actual example is provided for the decision-making problem of clay-brick selection to illustrate the application of the proposed method. Section 6 presents conclusions and future research direction.

2 Basic concept of neutrosophic numbers

A neutrosophic number, proposed by Smarandache [1-3], consists of the determinate part and the indeterminate part, which is denoted by $N = d + uI$, where d and u are real numbers and I is indeterminacy, such that $I^n = I$ for $n > 0$, $0 \times I = 0$, and uI/kI = undefined for any real number k.

For example, assume that there is a neutrosophic number $N = 2 + 2I$. If $I \in [0, 0.2]$, it is equivalent to $N \in [2, 2.4]$ for sure $N \geq 2$, this means that its determinate part is 2 and its indeterminate part is $2I$ with the indeterminacy $I \in [0, 0.2]$ and the possibility for the number ''N'' is within the interval [2, 2.4]. In general, a neutrosophic number may be considered as a changeable interval.

Let $N = d + uI$ be a neutrosophic number. If d, $u \geq 0$, then N is called positive neutrosophic numbers. In the following, all neutrosophic numbers are considered as positive neutrosophic numbers, which are called neutrosophic numbers for short, unless they are stated. Based on the cosine measure and projection model [5, 7], we introduce the following definitions.

Let $N_1 = d_1 + u_1I$ and $N_2 = d_2 + u_2I$ be two neutrosophic numbers, then there are the following operational relations of neutrosophic numbers [1-3]:

(1) $N_1 + N_2 = d_1 + d_2 + (u_1 + u_2)I$;

(2) $N_1 - N_2 = d_1 - d_2 + (u_1 - u_2)I$;;

(3) $N_1 \times N_2 = d_1 d_2 + (d_1 u_2 + u_1 d_2 + u_1 u_2)I$;

(4) $N_1^2 = (d_1 + u_1 I)^2 = d_1^2 + (2d_1 u_1 + u_1^2)I$;

(5) $\frac{N_1}{N_2} = \frac{d_1 + u_1 I}{d_2 + u_2 I} = \frac{d_1}{d_2} + \frac{d_2 u_1 - d_1 u_2}{d_2(d_2 + u_2)} I$ for $d_2 \neq 0$ and $d_2 \neq -u_2$;

(6) $$\sqrt{N_1} = \sqrt{d_1 + u_1 I} = \begin{cases} \sqrt{d_1} - (\sqrt{d_1} + \sqrt{d_1 + u_1})I \\ \sqrt{d_1} - (\sqrt{d_1} - \sqrt{d_1 + u_1})I \\ -\sqrt{d_1} + (\sqrt{d_1} + \sqrt{d_1 + u_1})I \\ -\sqrt{d_1} + (\sqrt{d_1} - \sqrt{d_1 + u_1})I \end{cases}.$$

Definition 1 [7]. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two neutrosophic number vectors, where $a_j = [d_{aj} + u_{aj}I^L, d_{aj} + u_{aj}I^U]$ and $b_j = [d_{bj} + u_{bj}I^L, d_{bj} + u_{bj}I^U]$ for $I \in [I^L, I^U]$ and $j = 1, 2, \ldots, n$. Then, the modules of a and b are defined as $\|a\| = \sqrt{\sum_{j=1}^{n} \left(d_{aj} + u_{aj}I^L\right)^2 + \left(d_{aj} + u_{aj}I^U\right)^2}$ and $\|b\| = \sqrt{\sum_{j=1}^{n} \left(d_{bj} + u_{bj}I^L\right)^2 + \left(d_{bj} + u_{bj}I^U\right)^2}$, the inner product between a and b is defined as $a \cdot b = \sum_{j-1}^{n} \left((d_{aj} + u_{aj}I^L)(d_{bj} + u_{bj}I^L) + (d_{aj} + u_{aj}I^U)(d_{bj} + u_{bj}I^U)\right)$. Thus, a cosine measure is defined as

$$\cos(a,b) = \frac{a \cdot b}{\|a\|\|b\|} \quad (1)$$

which is called the cosine of the included angle between a and b.

3 Projection measure of neutrosophic numbers

Definition 2 [5]. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two neutrosophic number vectors, where $a_j = [d_{aj} + u_{aj}I^L, d_{aj} + u_{aj}I^U]$ and $b_j = [d_{bj} + u_{bj}I^L, d_{bj} + u_{bj}I^U]$ for $I \in [I^L, I^U]$ and $j = 1, 2, \ldots, n$. Then the projection of the vector a on the vector b is defined as

$$\mathrm{Pr}oj_b(a) = \|a\|\cos(a,b) = \frac{a \cdot b}{\|b\|} = \frac{\sum_{j=1}^{n} [(d_{aj} + u_{aj}I^L)(d_{bj} + u_{bj}I^L) + (d_{aj} + u_{aj}I^U)(d_{bj} + u_{bj}I^U)]}{\sqrt{\sum_{j=1}^{n} [(d_{bj} + u_{bj}I^L)^2 + (d_{bj} + u_{bj}I^U)^2]}}. \quad (2)$$

If one considers the importance of each element in neutrosophic number vectors a and b, the weight of each element can be introduced by w_j ($j = 1, 2, \ldots, n$) with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Thus, we introduce the following definition.

Definition 3. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two neutrosophic number vectors, where $a_j = [d_{aj} + u_{aj}I^L, d_{aj} + u_{aj}I^U]$ and $b_j = [d_{bj} + u_{bj}I^L, d_{bj} + u_{bj}I^U]$ for $I \in [I^L, I^U]$ and $j = 1, 2, \ldots, n$. The weight of the elements is w_j ($j = 1, 2, \ldots, n$) with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then the projection of the vector a on the vector b is defined as

$$W\,\mathrm{Pr}oj_b(a) = \|a\|_w \cos_w(a,b) = \frac{(a \cdot b)_w}{\|b\|_w} = \frac{\sum_{j=1}^{n} w_j^2 [(d_{aj} + u_{aj}I^L)(d_{bj} + u_{bj}I^L) + (d_{aj} + u_{aj}I^U)(d_{bj} + u_{bj}I^U)]}{\sqrt{\sum_{j=1}^{n} w_j^2 [(d_{bj} + u_{bj}I^L)^2 + (d_{bj} + u_{bj}I^U)^2]}} \quad (3)$$

Based on the projection model of interval numbers improved by Xu and Liu [9], the projection model of Eq. (3) is improved as the following form:

$$WP_b(a) = \frac{(a \cdot b)_w}{\|b\|_w^2} = \frac{\sum_{j=1}^{n} w_j^2 [(d_{aj} + u_{aj}I^L)(d_{bj} + u_{bj}I^L) + (d_{aj} + u_{aj}I^U)(d_{bj} + u_{bj}I^U)]}{\sum_{j=1}^{n} w_j^2 [(d_{bj} + u_{bj}I^L)^2 + (d_{bj} + u_{bj}I^U)^2]} \quad (4)$$

Obviously, the closer the value of $WP_b(a)$ is to 1, the closer the vector a is to the vector b.

4 Decision-making method based on the projection measure

In this section, we present a handling method for multiple attribute decision-making problems by using the proposed projection measure under neutrosophic number environment.

In a multiple attribute decision-making problem, let $S = \{S_1, S_2, ..., S_m\}$ be a set of alternatives and $A = \{A_1, A_2, ..., A_n\}$ be a set of attributes. If the decision maker provides an evaluation value of the attribute A_j (j=1,2,...,n) for the alternative S_i ($i = 1, 2, \ldots, m$) by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy I, which is represented by the form of a neutrosophic number $a_{ij} = d_{ij} + u_{ij}I$ for $I \in [I^L, I^U]$ and constructed as a set of neutrosophic numbers $S_i = \{a_{i1}, a_{i2}, ..., a_{in}\}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Thus, we can establish the neutrosophic number decision matrix $M = (a_{ij})_{m \times n}$.

If the weights of attributes are considered as the different importance of each attribute A_j (j = 1, 2, …, n), the weight vector of attributes is $W = (w_1, w_2, …, w_n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Then, the procedure of the decision-making problem is described as follows:

Step 1: Specify the indeterminacy $I \in [I^L, I^U]$ according to decision makers' preference and real requirements, each neutrosophic number $a_{ij} = d_{ij} + u_{ij}I$ in the neutrosophic number decision matrix M can be transformed into an interval numbers $a_{ij} = [d_{ij} + u_{ij}I^L, d_{ij} + u_{ij}I^U]$ for $I \in [I^L, I^U]$ for i = 1, 2, …, m and j = 1, 2, …, n. By $a_j^* = [a_j^{L*}, a_j^{U*}] = [\max_i(d_{ij} + u_{ij}I^L), \max_i(d_{ij} + u_{ij}I^U)]$ (j = 1, 2, …, n), the ideal solution (ideal neutrosophic numbers) can be determined as the ideal alternative $S^* = \{a_1^*, a_2^*, ..., a_n^*\}$.

Step 2: According to Eq. (4), the projection measure between each alternative S_i (i = 1, 2, …, m) and the ideal alternative S^* can be calculated by

$$WP_{S^*}(S_i) = \frac{(S_i \cdot S^*)_w}{\left\|S^*\right\|_w^2} = \frac{\sum_{k=1}^{n} w_j^2[(d_{ij} + u_{ij}I^L)a_j^{L*} + (d_{ij} + u_{ij}I^U)a_j^{U*}]}{\sum_{j=1}^{n} w_j^2[(a_j^{L*})^2 + (a_j^{U*})^2]} \quad (5)$$

Step 3: The alternatives are ranked in a descending order according to the values of $WP_{S^*}(S_i)$ for i = 1, 2, …, m. The greater value of $WP_{S^*}(S_i)$ means the better alternative S_i.

Step 4: End.

5 Actual example of clay-brick selection

In this section, an actual example on clay-brick selection in construction field adapted from [8] illustrates the application of the projection method.

Let us consider a set of four possible alternatives (providers of clay-bricks) $S = \{S_1, S_2, ..., S_m\}$ in construction field, which need to satisfy six attributes (criteria) of clay-bricks: solidity (A_1), color (A_2), size and shape (A_3), and strength of brick (A_4), brick cost (A_5), carrying cost (A_6) [8]. Then, the weighting vector of the six attributes is W = (0.275, 0.175, 0.2, 0.1, 0.05, 0.2).

When the four alternatives with respect to the six attributes are evaluated by the expert corresponding to a scale from 1 (less fit) to 10 (more fit) with indeterminacy I, we can obtain the evaluation values of neutrosophic numbers. For example, the expert give the neutrosophic number of an attribute A_1 for an alternative S_1 as $a_{11} = 7 + 2I$ by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy I, which indicates that the evaluation value of the attribute A_1 for the alternative S_1 is the determinate degree 7 with the indeterminate degree $2I$ with some indeterminacy $I \in [I^L, I^U]$. By the similar evaluation process, we can obtain the following decision matrix:

$$M = (a_{ij})_{4\times 6} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\ \begin{bmatrix} 7+2I & 8+I & 7+I & 6+2I & 7 & 5+3I \\ 7+I & 7+2I & 8+I & 7+2I & 8+I & 7+2I \\ 8+I & 8 & 7+2I & 6+2I & 7+I & 6+2I \\ 7 & 9+I & 7+3I & 8+2I & 6+2I & 7+3I \end{bmatrix} \end{array}.$$

Assume $I \in [0,1]$, then the above neutrosophic number decision matrix can be transformed into the following de-neutrosophication matrix:

$$M = (a_{ij})_{4\times 6} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\ \begin{bmatrix} [7,9] & [8,9] & [7,8] & [6,8] & [7,7] & [5,8] \\ [7,8] & [7,9] & [8,9] & [7,9] & [8,9] & [7,9] \\ [8,9] & [8,8] & [7,9] & [6,8] & [7,8] & [6,8] \\ [7,7] & [9,10] & [7,10] & [8,10] & [6,8] & [7,10] \end{bmatrix} \end{array}.$$

By $a_j^* = [a_j^{L*}, a_j^{U*}] = [\max_i(d_{ij} + u_{ij}I^L), \max_i(d_{ij} + u_{ij}I^U)]$ (j = 1, 2, …, 6), the ideal solution (ideal neutrosophic numbers) can be determined as the following ideal alternative:

S^* ={[8, 9], [9, 10], [8, 10], [8, 10], [8, 9], [7, 10]}.

According to Eq. (5), the weighted projection measure values between each alternative S_i (i = 1, 2, 3, 4) and the ideal alternative S^* can be obtained as follows:

$WP_{S^*}(S_1)$ = 0.8554, $WP_{S^*}(S_2)$ =0.9026, $WP_{S^*}(S_3)$ = 0.8826, and $WP_{S^*}(S_4)$ = 0.9366.

Since the values of the projection measure are $WP_{S^*}(S_4) > WP_{S^*}(S_2) > WP_{S^*}(S_3) > WP_{S^*}(S_1)$, the ranking order of the four alternatives is $S_4 > S_2 > S_3 > S_1$. Hence, the alternative S_4 is the best choice among all the alternatives.

Compared with the neutrosophic grey relational analysis for clay-brick selection [8], the proposed approach is more convenient and less calculation steps.

6 Conclusion

This paper presented a projection measure of neutrosophic numbers and a projection model-based multiple attribute decision-making method under a neutrosophic number environment. In the decision-making process, through the projection measure between each alternative and the ideal alternative, the ranking order of all alternatives can be determined in order to select the best alternative. Finally, an actual example on the selecting problem of clay-bricks demonstrated the application of the proposed method. However, the main advantage of the proposed approach is easy evaluation and calculation in actual applications. In the future work, we shall extend the proposed decision-making method with neutrosophic numbers to the decision-making method with refine neutrosophic numbers.

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Abstract

This volume is a collection of seventeen papers, written by different authors and co-authors (listed in the order of the papers): F. Smarandache, K. Bhutani, M. Kumar, G. Garg, S. Aggarwal, P. Biswas, S. Pramanik, B. C. Giri, J. Ye, A. Mukherjee, M. Datta, S. Sarkar, N. Shah, M. K. EL Gayyar, S. K. Patro, B. C. Cuong, P. H. Phong, A. A. Salama, I. M. Hanafy, H. Elghawalby and M. S. Dabash, R. Roy, P. Das, D. Mandal, Santhi R., Udhayarani N., F. Yuhua, S. A. Akinleye, A.A.A. Agboola, and J. Chen.

In first paper, the authors studied Degrees of Membership > 1 and < 0 of the Elements With Respect to a Neutrosophic Off-Set. Assessing IT Projects Success with Extended Fuzz Cognitive Maps & Neutrosophic Cognitive Maps in comparison to Fuzzy Cognitive Maps are discussed in the second paper. Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making is studied in third paper. In fourth paper, Similarity Measure of Refined Single-Valued Neutrosophic Sets and Its Multicriteria Decision Making Method . Similarly in fifth paper, Restricted Interval Valued Neutrosophic Sets and Restricted Interval Valued Neutrosophic Topological Spaces is discussed. In paper six, Some Studies in Neutrosophic Graphs is studied by the author. Smooth Neutrosophic Topological Spaces is proposed in the next paper. The Neutrosophic Statistical Distribution, More Problems, More Solutions in the next paper. Further, Standard Neutrosophic Soft Theory: Some First Results are discussed by the authors in the tenth paper. In eleventh paper, Neutrosophic Crisp α-Topological Spaces have been studied by the author. In the next paper, Neutrosophic Goal Programming applied to Bank: Three Investment Problem. In thirteenth paper, Neutrosophic Hyperideals of Semihyperrings are introduced by the authors. In fourteenth paper, the author studied Nω-Closed Sets in Neutrosophic Topological Spaces. In the last paper, Expanding Comparative Literature into Comparative Sciences Clusters with Neutrosophy and Quad-stage Method. On Neutrosophic Quadruple Algebraic Structures in fifteenth paper. Sixteenth paper is about Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multiattribute decision making. Las paper is about A Projection Model of Neutrosophic Numbers for Multiple Attribute Decision Making of Clay-Brick Selection

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