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About Bernoulli's Numbers

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About Bernoulli's Numbers[1]

Many methods to compute the sum of the same powers of the first n natural numbers (see ([4]) are well-known.

In this paper we present a simple proof of the method from [3].

The Bernoulli's numbers are defined by

(1) $$B_n = \frac{-1}{n+1}\left(C^0_{n+1}B_0 + C^1_{n+1}B_1 + \dots + C^{n-1}_{n+1}B_{n-1}\right),$$

where $B_0 = 1$. It is known that $B_{n+1} = 0$ if $n \geq 1$. By calculation we find that (2) $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $B_{18} = 43867/798$, $B_{20} = -174611/330$, $B_{22} = 854513/138$, $B_{24} = -236364091/2730$ etc.

Let $S^k_n = 1^k + 2^k + \dots + n^k$ sum of the first n natural numbers which have the same power.

Theorem.

(3) $$S^k_n = \frac{1}{k+1}\left(n^{k+1} + \frac{1}{2}C^1_{k+1}n^k + C^2_{k+1}B_2n^{k-1} + \dots + C^k_{k+1}B_nn\right)$$

Proof. (1) can be written as:

(4) $$\sum_{i=0}^{n} C^i_{n+1}B_i = 0, \quad n \geq 1.$$

If $P(x) = \sum_{i=0}^{k} C^i_{k+1}B_ix^{k+1-i}$, then P(n+1) - P(n) =

$$= \sum_{i=0}^{k} C^i_{k+1}B_i\left((n+1)^{k+1-i} - n^{k+1-i}\right) =$$

$$= \sum_{i=0}^{k} C^i_{k+1}B_i\left(\sum_{j=1}^{k+1-i} C^j_{k+1-i}n^{k+1-i-j}\right).$$

Let A_t be the coeficients of n^{k-t}, where $t \in \{0, 1, \dots, k\}$.

$$A_t = \sum_{i=0}^{t} C^i_{k+1}C^{t-i+1}_{k+1-i}B_i = C^{t+1}_{k+1}\left(\sum_{i=0}^{t} C^i_{t+1}B_i\right).$$

If $t \geq 1$, then $A_t = 0$, only $A_0 = C^1_{k+1}$. On behalf of these

$P(n+1) - P(n) = C^1_{k+1}n^k$. Using this

$$\sum_{i=0}^{n-1} i^k = \frac{1}{k+1}\sum_{i=0}^{n-1}\left(P(i+1) - P(i)\right) = \frac{1}{k+1}P(n),$$

[1] Together with Mihály Bencze

because P(0) = 0. Then $S_n^k = \frac{1}{k+1}P(n) + n^k$. From here one gets (3).

Note. From the previous result we can also find the formula $S_n^k = \frac{1}{k+1}P(n+1)$. Using the previsious, we find the next equalities:

$S_n^0 = n$, $S_n^1 = \frac{1}{2}n(n+1)$, $S_n^2 = \frac{1}{6}n(n+1)(2n+1)$, $S_n^3 = \frac{1}{4}n^2(n+1)^2$,

$$S_n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$S_n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

$$S_n^6 = \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$$

$$S_n^7 = \frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$$

$$S_n^8 = \frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)$$

$$S_n^9 = \frac{1}{20}(2n^{10}+10n^9+15n^8-14n^6+10n^4-3n^2)$$

$$S_n^{10} = \frac{1}{66}(6n^{11}+33n^{10}+55n^9-66n^7+66n^5-33n^3+5n)$$

$$S_n^{11} = \frac{1}{24}(2n^{12}+12n^{11}+22n^{10}-33n^8+44n^6-33n^4+10n^2)$$

$$S_n^{12} = \frac{1}{2730}(210n^{13}+1365n^{12}+3630n^{11}-4935n^9+115n^8+9640n^7$$

$$+1960n^6-5899n^5+35n^4+4550n^3+1382n^2-691n) \text{ etc.}$$

Problems.

1). Using the mathematical induction on the base of (1), we prove that $B_{2n+1} = 0$, if $n \geq 1$.

2). Prove that S_n^k is divisible by n(n + 1).

3). Prove that S_n^{2k+1} is divisible by $n^2(n+1)^2$.

4). Determine those natural numbers n, k for which S_n^{2k} is divisible n(n + 1)(2n + 1).

5). Detach in parts the sums S_n^9, S_n^{10} S_n^{11}, S_n^{12}.

6). Using (2), (3), compute the sums $S_n^{13}, \ldots, S_n^{21}$.

References:

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[3] Z. I. Borevici, I. R. Safarevici, Teoria Numerelor, Ed. Ştiinţifică şi Enciclopedică, Bucureşti, România, 1985.

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