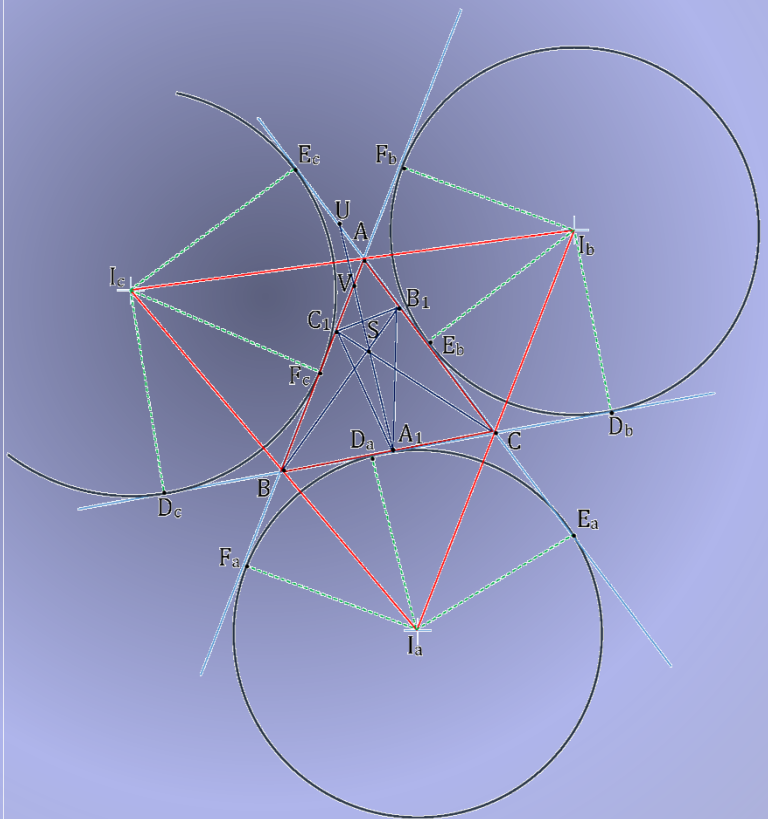


Ion Patrascu | Florentin Smarandache

Complements to Classic Topics of Circles Geometry



Pons Editions
Brussels | 2016

Ion Patrascu | Florentin Smarandache

**Complements to Classic Topics
of Circles Geometry**

*In the memory of the first author's father
Mihail Patrascu and the second author's
mother Maria (Marioara) Smarandache,
recently passed to eternity...*

Ion Patrascu | Florentin Smarandache

**Complements to Classic Topics
of Circles Geometry**

Pons Editions

Brussels | 2016

© 2016 Ion Patrascu & Florentin Smarandache

All rights reserved. This book is protected by copyright. No part of this book may be reproduced in any form or by any means, including photocopying or using any information storage and retrieval system without written permission from the copyright owners.

ISBN 978-1-59973-465-1

Contents

Introduction	15
Lemoine's Circles	17
1 st Theorem.....	17
Proof.	17
2 nd Theorem.....	19
Proof.	19
Remark.	21
References.....	22
Lemoine's Circles Radius Calculus	23
1 st Theorem.....	23
Proof.	23
Consequences.....	25
1 st Proposition.....	25
Proof.	25
2 nd Proposition.	26
Proof.	27
Remarks.....	28
3 rd Proposition.....	28
Proof.	28
Remark.	29
References.....	29
Radical Axis of Lemoine's Circles	31
1 st Theorem.....	31
2 nd Theorem.....	31
1 st Remark.	31
1 st Proposition.....	32
Proof.	32

Comment.....	34
2 nd Remark.	34
2 nd Proposition.	34
References.....	34
Generating Lemoine’s circles	35
1 st Definition.....	35
1 st Proposition.....	35
2 nd Definition.....	35
1 st Theorem.....	36
3 rd Definition.	36
1 st Lemma.	36
Proof.	36
Remark.	38
2 nd Theorem.....	38
Proof.	38
Further Remarks.....	40
References.....	40
The Radical Circle of Ex-Inscribed Circles of a Triangle	41
1 st Theorem.....	41
Proof.	41
2 nd Theorem.....	43
Proof.	43
Remark.	44
3 rd Theorem.....	44
Proof.	45
References.....	48
The Polars of a Radical Center	49
1 st Theorem.....	49
Proof.	50

2 nd Theorem.....	51
Proof.	52
Remarks.	52
References.	53
Regarding the First Droz-Farny's Circle	55
1 st Theorem.....	55
Proof.	55
Remarks.	57
2 nd Theorem.....	57
Remark.	58
Definition.	58
Remark.	58
3 rd Theorem.....	58
Proof.	59
Remark.	60
4 th Theorem.....	60
Proof.	60
Reciprocally.	61
Remark.	62
References.	62
Regarding the Second Droz-Farny's Circle.....	63
1 st Theorem.....	63
Proof.	63
1 st Proposition.....	65
Proof.	65
Remarks.	66
2 nd Theorem.....	66
Proof.	67
Reciprocally.	67
Remarks.	68

2 nd Proposition.	68
Proof.	69
References.	70
Neuberg’s Orthogonal Circles	71
1 st Definition.	71
2 nd Definition.	72
3 rd Definition.	72
1 st Proposition.	72
Proof.	73
Consequence.	74
2 nd Proposition.	74
Proof.	74
4 th Definition.	75
3 rd Proposition.	75
Proof.	75
Reciprocally.	76
4 th Proposition.	76
Proof.	77
References.	78
Lucas’s Inner Circles	79
1. Definition of the Lucas’s Inner Circles	79
Definition.	80
2. Calculation of the Radius of the A-Lucas Inner Circle	80
Note.	81
1 st Remark.	81
3. Properties of the Lucas’s Inner Circles	81
1 st Theorem.	81
Proof.	81
2 nd Definition.	83

Remark.	83
2 nd Theorem.....	84
3 rd Definition.	84
3 rd Theorem.....	84
Proof.	85
Remarks.....	86
4 th Definition.	87
1 st Property.....	87
Proof.	87
2 nd Property.....	88
Proof.	88
References.....	88
Theorems with Parallels Taken through a	
Triangle's Vertices and Constructions Performed	
only with the Ruler..... 89	
1 st Problem.....	89
2 nd Problem.....	89
3 rd Problem.....	90
1 st Lemma.	90
Proof.	91
Remark.	92
2 nd Lemma.	92
Proof.	92
3 rd Lemma.	93
Solution.....	93
Construction's Proof.	93
Remark.	94
1st Theorem.....	95
Proof.	96
2nd Theorem.	97

Proof.	98
Remark.	99
3 rd Theorem.	99
Proof.	99
Reciprocally.	100
Solution to the 1 st problem.	101
Solution to the 2 nd problem.	101
Solution to the 3 rd problem.	102
References.	102
Apollonius's Circles of k^{th} Rank.....	103
1 st Definition.	103
2 nd Definition.	103
1 st Theorem.....	103
Proof.	104
3 rd Definition.	105
2 nd Theorem.....	105
Proof.	106
3 rd Theorem.	107
Proof.	107
4 th Theorem.	108
Proof.	108
1 st Remark.	108
5 th Theorem.	108
Proof.	109
6 th Theorem.	109
Proof.	109
4 th Definition.	110
1 st Property.	110
2 nd Remark.	110
7 th Theorem.	111

Proof.	111
References.	112
Apollonius's Circle of Second Rank	113
1 st Definition.	113
1 st Theorem.	113
1 st Proposition.	114
Proof.	114
Remarks.	115
2 nd Definition.	115
3 rd Definition.	116
2 nd Proposition.	116
Proof.	116
Remarks.	118
Open Problem.	119
References.	120
A Sufficient Condition for the Circle of the 6 Points to Become Euler's Circle	121
1 st Definition.	121
1 st Remark.	122
2 nd Definition.	122
2 nd Remark.	122
1 st Proposition.	122
Proof.	122
3 rd Remark.	123
3 rd Definition.	123
4 th Remark.	124
1 st Theorem.	124
Proof.	124
4 th Definition.	126
2 nd Proposition.	126

1 st Proof.....	126
2 nd Proof.....	127
5 th Remark.	128
References.....	128
An Extension of a Problem of Fixed Point	129
Proof.	130
1 st Remark.	132
2 nd Remark.	132
3 rd Remark.	133
References.....	133
Some Properties of the Harmonic Quadrilateral	135
1 st Definition.....	135
2 nd Definition.....	135
1 st Proposition.....	135
2 nd Proposition.	136
Proof.	136
Remark 1.....	137
3 rd Proposition.....	137
Remark 2.....	138
Proposition 4.....	138
Proof.	139
3 rd Remark.	140
5 th Proposition.....	140
6 th Proposition.....	140
Proof.	140
3 rd Definition.	142
4 th Remark.	142
7 th Proposition.....	143
Proof.	143
5 th Remark.	144

8 th Proposition.....	145
Proof.	145
4 th Definition.	146
6 th Remark.	146
9 th Proposition.....	146
Proof.	146
10 th Proposition.	147
Proof.	147
7 th Remark.....	148
11 th Proposition.....	148
Proof.	149
References.....	149
Triangulation of a Triangle with Triangles having Equal Inscribed Circles	151
Solution.	151
Building the point <i>D</i>	153
Proof.	154
Discussion.....	155
Remark.	156
Open problem.....	156
An Application of a Theorem of Orthology	157
Proposition.....	157
Proof.	158
Remark.	160
Note.	160
References.....	162
The Dual of a Theorem Relative to the Orthocenter of a Triangle.....	163
1 st Theorem.....	163
Proof.	164

Note.	165
2 nd Theorem.....	165
Proof.	166
References.....	168
The Dual Theorem Concerning Aubert’s Line	169
1 st Theorem.....	169
Proof.	169
Note.	171
1 st Definition.....	171
Note.	171
2 nd Definition.....	172
2 nd Theorem.....	172
Note.	172
3 rd Theorem.....	172
Proof.	173
Notes.....	174
4 th Theorem.....	174
Proof.	174
References.....	176
Bibliography.....	177

Introductory Note

We approach several themes of classical geometry of the circle and complete them with some original results, showing that not everything in traditional math is revealed, and that it still has an open character.

The topics were chosen according to authors' aspiration and attraction, as a poet writes lyrics about spring according to his emotions.

Lemoine's Circles

In this article, we get to **Lemoine's circles** in a different manner than the known one.

1st Theorem.

Let ABC a triangle and K its simedian center. We take through K the parallel A_1A_2 to BC , $A_1 \in (AB)$, $A_2 \in (AC)$; through A_2 we take the antiparallels A_2B_1 to AB in relation to CA and CB , $B_1 \in (BC)$; through B_1 we take the parallel B_1B_2 to AC , $B_2 \in AB$; through B_2 we take the antiparallels B_1C_1 to BC , $C_1 \in (AC)$, and through C_1 we take the parallel C_1C_2 to AB , $C_2 \in (BC)$. Then:

- i. C_2A_1 is an antiparallel of AC ;
- ii. $B_1B_2 \cap C_1C_2 = \{K\}$;
- iii. The points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclical (the first Lemoine's circle).

Proof.

- i. The quadrilateral BC_2KA is a parallelogram, and its center, i.e. the middle of the segment (C_2A_1) , belongs to the simedian BK ; it follows that C_2A_2 is an antiparallel to AC (see *Figure 1*).

- ii. Let $\{K'\} = A_1A_2 \cap B_1B_2$, because the quadrilateral $K'B_1CA_2$ is a parallelogram; it follows that CK' is a simedian; also, CK is a simedian, and since $K, K' \in A_1A_2$, it follows that we have $K' = K$.
- iii. B_2C_1 being an antiparallel to BC and $A_1A_2 \parallel BC$, it means that B_2C_1 is an antiparallel to A_1A_2 , so the points B_2, C_1, A_2, A_1 are concyclical. From $B_1B_2 \parallel AC$, $\sphericalangle B_2C_1A \equiv \sphericalangle ABC$, $\sphericalangle B_1A_2C \equiv \sphericalangle ABC$ we get that the quadrilateral $B_2C_1A_2B_1$ is an isosceles trapezoid, so the points B_2, C_1, A_2, B_1 are concyclical. Analogously, it can be shown that the quadrilateral $C_2B_1A_2A_1$ is an isosceles trapezoid, therefore the points C_2, B_1, A_2, A_1 are concyclical.

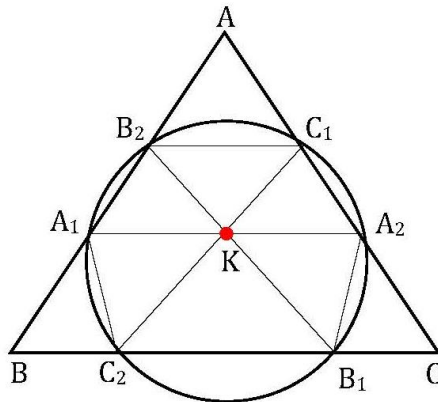


Figure 1

From the previous three quartets of concyclical points, it results the concyclicity of the points belonging to the first Lemoine's circle.

2nd Theorem.

In the scalene triangle ABC , let K be the simedian center. We take from K the antiparallel A_1A_2 to BC ; $A_1 \in AB, A_2 \in AC$; through A_2 we build $A_2B_1 \parallel AB$; $B_1 \in (BC)$, then through B_1 we build B_1B_2 the antiparallel to AC , $B_2 \in (AB)$, and through B_2 we build $B_2C_1 \parallel BC$, $C_1 \in AC$, and, finally, through C_1 we take the antiparallel C_1C_2 to AB , $C_2 \in (BC)$.

Then:

- i. $C_2A_1 \parallel AC$;
- ii. $B_1B_2 \cap C_1C_2 = \{K\}$;
- iii. The points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclical (the second Lemoine's circle).

Proof.

- i. Let $\{K'\} = A_1A_2 \cap B_1B_2$, having $\sphericalangle AA_1A_2 = \sphericalangle ACB$ and $\sphericalangle BB_1B_2 \equiv \sphericalangle BAC$ because A_1A_2 și B_1B_2 are antiparallels to BC , AC , respectively, it follows that $\sphericalangle K'A_1B_2 \equiv \sphericalangle K'B_2A_1$, so $K'A_1 = K'B_2$; having $A_1B_2 \parallel B_1A_2$ as well, it follows that also $K'A_2 = K'B_1$, so $A_1A_2 = B_1B_2$. Because C_1C_2 and B_1B_2 are antiparallels to AB and AC , we

have $K''C_2 = K''B_1$; we noted $\{K''\} = B_1B_2 \cap C_1C_2$; since $C_1B_2 \parallel B_1C_2$, we have that the triangle $K''C_1B_2$ is also isosceles, therefore $K''C_1 = C_1B_2$, and we get that $B_1B_2 = C_1C_2$. Let $\{K'''\} = A_1A_2 \cap C_1C_2$; since A_1A_2 and C_1C_2 are antiparallels to BC and AB , we get that the triangle $K'''A_2C_1$ is isosceles, so $K'''A_2 = K'''C_1$, but $A_1A_2 = C_1C_2$ implies that $K'''C_2 = K'''A_1$, then $\sphericalangle K'''A_1C_2 \equiv \sphericalangle K'''A_2C_1$ and, accordingly, $C_2A_1 \parallel AC$.

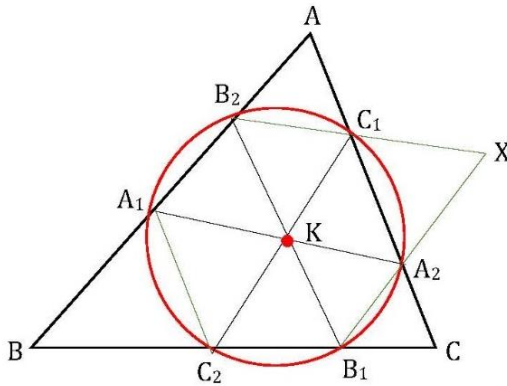


Figure 2

- ii. We noted $\{K'\} = A_1A_2 \cap B_1B_2$; let $\{X\} = B_2C_1 \cap B_1A_2$; obviously, BB_1XB_2 is a parallelogram; if K_0 is the middle of (B_1B_2) , then BK_0 is a simedian, since B_1B_2 is an antiparallel to AC , and the middle of the antiparallels of AC are situated on the

simedian BK . If $K_0 \neq K$, then $K_0K \parallel A_1B_2$ (because $A_1A_2 = B_1B_2$ and $B_1A_2 \parallel A_1B_2$), on the other hand, B, K_0, K are collinear (they belong to the simedian BK), therefore K_0K intersects AB in B , which is absurd, so $K_0 = K$, and, accordingly, $B_1B_2 \cap A_1A_2 = \{K\}$. Analogously, we prove that $C_1C_2 \cap A_1A_2 = \{K\}$, so $B_1B_2 \cap C_1C_2 = \{K\}$.

- iii. K is the middle of the congruent antiparallels A_1A_2, B_1B_2, C_1C_2 , so $KA_1 = KA_2 = KB_1 = KB_2 = KC_1 = KC_2$. The simedian center K is the center of the second Lemoine's circle.

Remark.

The center of the first Lemoine's circle is the middle of the segment $[OK]$, where O is the center of the circle circumscribed to the triangle ABC . Indeed, the perpendiculars taken from A, B, C on the antiparallels B_2C_1, A_1C_2, B_1A_2 respectively pass through O , the center of the circumscribed circle (the antiparallels have the directions of the tangents taken to the circumscribed circle in A, B, C). The mediatrix of the segment B_2C_1 pass through the middle of B_2C_1 , which coincides with the middle of AK , so is the middle line in the triangle AKO passing through the middle of (OK) . Analogously, it follows that the mediatrix of A_1C_2 pass through the middle L_1 of $[OK]$.

References.

- [1] D. Efremov: *Noua geometrie a triunghiului* [*The New Geometry of the Triangle*], translation from Russian into Romanian by Mihai Miculița, Cril Publishing House, Zalau, 2010.
- [2] Gh. Mihalescu: *Geometria elementelor remarcabile* [*The Geometry of the Outstanding Elements*], Tehnica Publishing House, Bucharest, 1957.
- [3] Ion Patrascu, Gh. Margineanu: *Cercul lui Tucker* [*Tucker's Circle*], in *Alpha* journal, year XV, no. 2/2009.

Lemoine's Circles Radius

Calculus

For the calculus of the **first Lemoine's circle**, we will first prove:

1st Theorem

(E. Lemoine - 1873)

The first Lemoine's circle divides the sides of a triangle in segments proportional to the squares of the triangle's sides.

Each extreme segment is proportional to the corresponding adjacent side, and the chord-segment in the Lemoine's circle is proportional to the square of the side that contains it.

Proof.

We will prove that $\frac{BC_2}{c^2} = \frac{C_2B_1}{a^2} = \frac{B_1C}{b^2}$.

In figure 1, K represents the symmedian center of the triangle ABC , and A_1A_2 ; B_1B_2 ; C_1C_2 represent Lemoine parallels.

The triangles BC_2A_1 ; CB_1A_2 and KC_2A_1 have heights relative to the sides BC_2 ; B_1C and C_2B_1 equal ($A_1A_2 \parallel BC$).

Hence:

$$\frac{Area_{\Delta} BA_1C_2}{BC_2} = \frac{Area_{\Delta} KC_2A_1}{C_2B_1} = \frac{Area_{\Delta} CB_1A_2}{B_1C} . \quad (1)$$

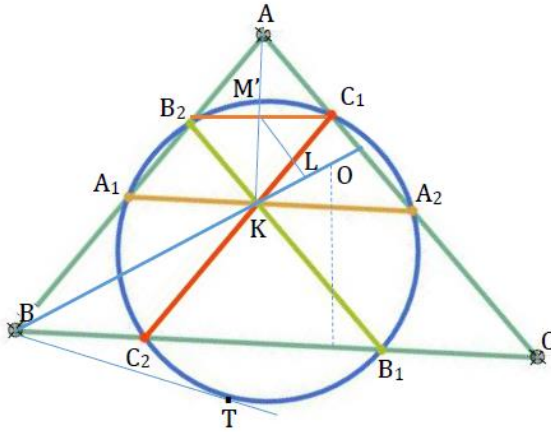


Figure 1

On the other hand: A_1C_2 and B_1A_2 being antiparallels with respect to AC and AB , it follows that $\Delta BC_2A_1 \sim \Delta BAC$ and $\Delta CB_1A_2 \sim \Delta CAB$, likewise $KC_2 \parallel AC$ implies: $\Delta KC_2B_1 \sim \Delta ABC$.

We obtain:

$$\begin{aligned} \frac{Area_{\Delta} BC_2A_1}{Area_{\Delta} ABC} &= \frac{BC_2^2}{c^2} ; \\ \frac{Area_{\Delta} KC_2B_1}{Area_{\Delta} ABC} &= \frac{C_2B_1^2}{a^2} ; \\ \frac{Area_{\Delta} CB_1A_2}{Area_{\Delta} ABC} &= \frac{CB_1^2}{b^2} . \end{aligned} \quad (2)$$

If we denote $Area_{\Delta} ABC = S$, we obtain from the relations (1) and (2) that:

$$\frac{BC_2}{c^2} = \frac{C_2B_1}{a^2} = \frac{B_1C}{b^2} .$$

Consequences.

1. According to the 1st Theorem, we find that:

$$BC_2 = \frac{ac^2}{a^2+b^2+c^2}; B_1C = \frac{ab^2}{a^2+b^2+c^2}; B_1C_2 = \frac{a^3}{a^2+b^2+c^2}.$$

2. We also find that:

$$\frac{B_1C_2}{a^3} = \frac{A_2C_1}{b^3} = \frac{A_1B_2}{c^3},$$

meaning that:

“The chords determined by the first Lemoine’s circle on the triangle’s sides are proportional to the cubes of the sides.”

Due to this property, the first Lemoine’s circle is known in England by the name of *triplicate ratio circle*.

1st Proposition.

The radius of the first Lemoine’s circle, R_{L_1} is given by the formula:

$$R_{L_1}^2 = \frac{1}{4} \cdot \frac{R^2(a^2+b^2+c^2) + a^2b^2c^2}{(a^2+b^2+c^2)^2}, \quad (3)$$

where R represents the radius of the circle inscribed in the triangle ABC .

Proof.

Let L be the center of the first Lemoine’s circle that is known to represent the middle of the segment (OK) – O being the center of the circle inscribed in the triangle ABC .

Considering C_1 , we obtain $BB_1 = \frac{a(c^2+a^2)}{a^2+b^2+c^2}$.

Taking into account the power of point B in relation to the first Lemoine's circle, we have:

$$BC_2 \cdot BB_1 = BT^2 - LT^2,$$

(BT is the tangent traced from B to the first Lemoine's circle, see *Figure 1*).

$$\text{Hence: } R_{L_1}^2 = BL^2 - BC_2 \cdot BB_1. \quad (4)$$

The median theorem in triangle BOK implies that:

$$BL^2 = \frac{2 \cdot (BK^2 + BO^2) - OK^2}{4}.$$

It is known that $K = \frac{(a^2+c^2) \cdot S_b}{a^2+b^2+c^2}$; $S_b = \frac{2ac \cdot m_b}{a^2+c^2}$, where S_b and m_b are the lengths of the symmedian and the median from B , and $OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)}$, see (3).

Consequently: $BK^2 = \frac{2a^2c^2(a^2+c^2) - a^2b^2c^2}{(a^2+b^2+c^2)^2}$, and

$$4BL^2 = R^2 + \frac{4a^2c^2(a^2+c^2) + a^2b^2c^2}{(a^2+b^2+c^2)^2}.$$

As: $BC_2 \cdot BB_1 = \frac{a^2c^2(a^2+c^2)}{(a^2+b^2+c^2)^2}$, by replacing in (4),

we obtain formula (3).

2nd Proposition.

The radius of the second Lemoine's circle, R_{L_2} , is given by the formula:

$$R_{L_2} = \frac{abc}{a^2+b^2+c^2}. \quad (5)$$

Proof.

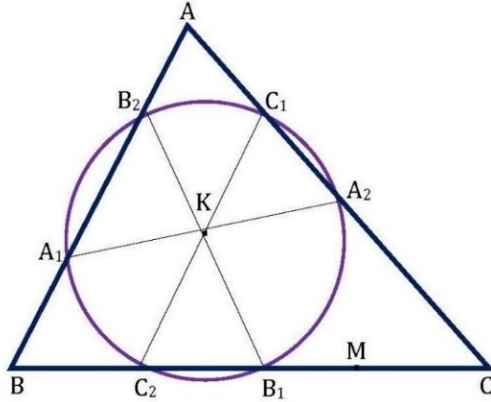


Figure 2

In Figure 2 above, A_1A_2 ; B_1B_2 ; C_1C_2 are Lemoine antiparallels traced through symmedian center K that is the center of the second Lemoine's circle, thence:

$$R_{L_2} = KA_1 = KA_2.$$

If we note with S and M the feet of the symmedian and the median from A , it is known that:

$$\frac{AK}{KS} = \frac{b^2+c^2}{a^2}.$$

From the similarity of triangles AA_2A_1 and ABC , we have: $\frac{A_1A_2}{BC} = \frac{AK}{AM}$.

$$\text{But: } \frac{AK}{AS} = \frac{b^2+c^2}{a^2+b^2+c^2} \text{ and } AS = \frac{2bc}{b^2+c^2} \cdot m_a.$$

$$A_1A_2 = 2R_{L_2}, \quad BC = a, \text{ therefore:}$$

$$R_{L_2} = \frac{AK \cdot a}{2m_a},$$

and as $AK = \frac{2bc \cdot m_a}{a^2+b^2+c^2}$, formula (5) is a consequence.

Remarks.

1. If we use $tg\omega = \frac{4S}{a^2+b^2+c^2}$, ω being the Brocard's angle (see [2]), we obtain: $R_{L_2} = R \cdot tg\omega$.

2. If, in *Figure 1*, we denote with M_1 the middle of the antiparallel B_2C_1 , which is equal to R_{L_2} (due to their similarity), we thus find from the rectangular triangle LM_1C_1 that:

$LC_1^2 = LM_1^2 + M_1C_1^2$, but $LM_1^2 = \frac{1}{4}a^2$ and $M_1C_1 = \frac{1}{2}R_{L_2}$; it follows that:

$$R_{L_1}^2 = \frac{1}{4}(R^2 + R_{L_2}^2) = \frac{R^2}{4}(1 + tg^2\omega).$$

We obtain:

$$R_{L_1} = \frac{R}{2} \cdot \sqrt{1 + tg^2\omega}.$$

3rd Proposition.

The chords determined by the sides of the triangle in the second Lemoine's circle are respectively proportional to the opposing angles cosines.

Proof.

KC_2B_1 is an isosceles triangle, $\sphericalangle KC_2B_1 = \sphericalangle KB_1C_2 = \sphericalangle A$; as $KC_2 = R_{L_2}$ we have that $\cos A = \frac{C_2B_1}{2R_{L_2}}$,
 deci $\frac{C_2B_1}{\cos A} = 2R_{L_2}$, similary: $\frac{A_2C_1}{\cos B} = \frac{B_2A_1}{\cos C} = 2R_{L_2}$.

Remark.

Due to this property of the Lemoine's second circle, in England this circle is known as the *cosine circle*.

References.

- [1] D. Efremov, *Noua geometrie a triunghiului* [The New Geometry of the Triangle], translation from Russian into Romanian by Mihai Miculița, Cril Publishing House, Zalau, 2010.
- [2] F. Smarandache and I. Patrascu, *The Geometry of Homological Triangles*, The Education Publisher, Ohio, USA, 2012.
- [3] I. Patrascu and F. Smarandache, *Variance on Topics of Plane Geometry*, Educational Publisher, Ohio, USA, 2013.

Radical Axis of Lemoine's Circles

In this article, we emphasize the **radical axis of the Lemoine's circles**.

For the start, let us remind:

1st Theorem.

The parallels taken through the simmedian center K of a triangle to the sides of the triangle determine on them six concyclic points (the first Lemoine's circle).

2nd Theorem.

The antiparallels taken through the triangle's simmedian center to the sides of a triangle determine six concyclic points (the second Lemoine's circle).

1st Remark.

If ABC is a scalene triangle and K is its simmedian center, then L , the center of the first Lemoine's circle, is the middle of the segment $[OK]$, where O is the center of the circumscribed circle, and

the center of the second Lemoine's circle is K . It follows that the radical axis of Lemoine's circles is perpendicular on the line of the centers LK , therefore on the line OK .

1st Proposition.

The radical axis of Lemoine's circles is perpendicular on the line OK raised in the simmedian center K .

Proof.

Let A_1A_2 be the antiparallel to BC taken through K , then KA_1 is the radius R_{L_2} of the second Lemoine's circle; we have:

$$R_{L_2} = \frac{abc}{a^2+b^2+c^2}.$$

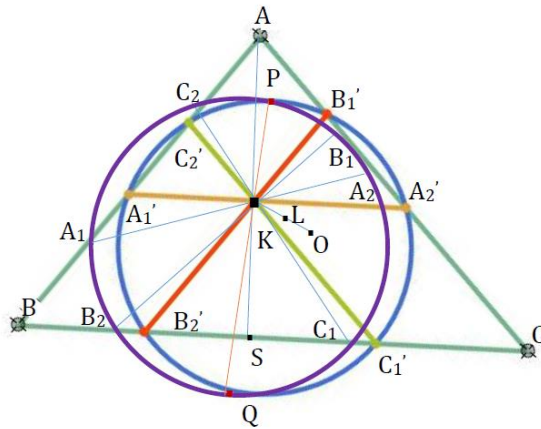


Figure 1

Let $A'_1A'_2$ be the Lemoine's parallel taken to BC ; we evaluate the power of K towards the first Lemoine's circle. We have:

$$\overrightarrow{KA'_1} \cdot \overrightarrow{KA'_2} = LK^2 - R_{L_1}^2. \quad (1)$$

Let S be the simmedian leg from A ; it follows that:

$$\frac{KA'_1}{BS} = \frac{AK}{AS} - \frac{KA'_2}{SC}.$$

We obtain:

$$KA'_1 = BS \cdot \frac{AK}{AS} \text{ and } KA'_2 = SC \cdot \frac{AK}{AS},$$

but $\frac{BS}{SC} = \frac{c^2}{b^2}$ and $\frac{AK}{AS} = \frac{b^2+c^2}{a^2+b^2+c^2}$.

Therefore:

$$\begin{aligned} \overrightarrow{KA'_1} \cdot \overrightarrow{KA'_2} &= -BS \cdot SC \cdot \left(\frac{AK}{AS}\right)^2 = \frac{-a^2b^2c^2}{(b^2+c^2)^2}; \\ \frac{(b^2+c^2)^2}{(a^2+b^2+c^2)^2} &= -R_{L_2}^2. \end{aligned} \quad (2)$$

We draw the perpendicular in K on the line LK and denote by P and Q its intersection to the first Lemoine's circle.

We have $\overrightarrow{KP} \cdot \overrightarrow{KQ} = -R_{L_2}^2$; by the other hand, $KP = KQ$ (PQ is a chord which is perpendicular to the diameter passing through K).

It follows that $KP = KQ = R_{L_2}$, so P and Q are situated on the second Lemoine's circle.

Because PQ is a chord which is common to the Lemoine's circles, it obviously follows that PQ is the radical axis.

Comment.

Equalizing (1) and (2), or by the Pythagorean theorem in the triangle PKL , we can calculate R_{L_1} .

It is known that: $OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}$, and since $LK = \frac{1}{2}OK$, we find that:

$$R_{L_1}^2 = \frac{1}{4} \cdot \left[R^2 + \frac{a^2b^2c^2}{(a^2+b^2+c^2)^2} \right].$$

2nd Remark.

The 1st Proposition, ref. the radical axis of the Lemoine's circles, is a particular case of the following *Proposition*, which we leave to the reader to prove.

2nd Proposition.

If $\mathcal{C}(O_1, R_1)$ și $\mathcal{C}(O_2, R_2)$ are two circles such as the power of center O_1 towards $\mathcal{C}(O_2, R_2)$ is $-R_1^2$, then the radical axis of the circles is the perpendicular in O_1 on the line of centers O_1O_2 .

References.

- [1] F. Smarandache, Ion Patrascu: *The Geometry of Homological Triangles*, Education Publisher, Ohio, USA, 2012.
- [2] Ion Patrascu, F. Smarandache: *Variance on Topics of Plane Geometry*, Education Publisher, Ohio, USA, 2013.

Generating Lemoine's circles

In this paper, we generalize the theorem relative to the first Lemoine's circle and thereby highlight a **method to build Lemoine's circles**.

Firstly, we review some notions and results.

1st Definition.

It is called a simedian of a triangle the symmetric of a median of the triangle with respect to the internal bisector of the triangle that has in common with the median the peak of the triangle.

1st Proposition.

In the triangle ABC , the cevian AS , $S \in (BC)$, is a simedian if and only if $\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2$. For *Proof*, see [2].

2nd Definition.

It is called a simedian center of a triangle (or Lemoine's point) the intersection of triangle's simedians.

1st Theorem.

The parallels to the sides of a triangle taken through the simedian center intersect the triangle's sides in six concyclic points (the first Lemoine's circle - 1873).

A *Proof* of this theorem can be found in [2].

3rd Definition.

We assert that in a scalene triangle ABC the line MN , where $M \in AB$ and $N \in AC$, is an antiparallel to BC if $\sphericalangle MNA \equiv \sphericalangle ABC$.

1st Lemma.

In the triangle ABC , let AS be a simedian, $S \in (BC)$. If P is the middle of the segment (MN) , having $M \in (AB)$ and $N \in (AC)$, belonging to the simedian AS , then MN and BC are antiparallels.

Proof.

We draw through M and N , $MT \parallel AC$ and $NR \parallel AB$, $R, T \in (BC)$, see *Figure 1*. Let $\{Q\} = MT \cap NR$; since $MP = PN$ and $AMQN$ is a parallelogram, it follows that $Q \in AS$.

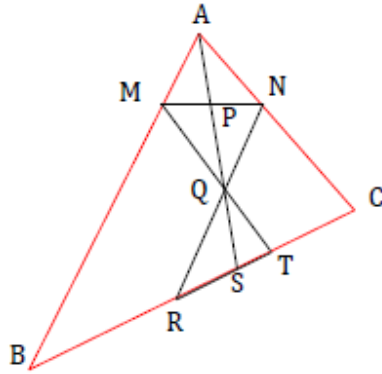


Figure 1.

Thales's Theorem provides the relations:

$$\frac{AN}{AC} = \frac{BR}{BC}; \quad (1)$$

$$\frac{AB}{AM} = \frac{BC}{CT}. \quad (2)$$

From (1) and (2), by multiplication, we obtain:

$$\frac{AN}{AM} \cdot \frac{AB}{AC} = \frac{BR}{TC}. \quad (3)$$

Using again Thales's Theorem, we obtain:

$$\frac{BR}{BS} = \frac{AQ}{AS}, \quad (4)$$

$$\frac{TC}{SC} = \frac{AQ}{AS}. \quad (5)$$

From these relations, we get

$$\frac{BR}{BS} = \frac{TC}{SC}, \quad (6)$$

or

$$\frac{BS}{SC} = \frac{BR}{TC}. \quad (7)$$

In view of *Proposition 1*, the relations (7) and (3) lead to $\frac{AN}{AB} = \frac{AB}{AC}$, which shows that $\Delta AMN \sim \Delta ACB$, so $\sphericalangle AMN \equiv \sphericalangle ABC$.

Therefore, MN and BC are antiparallels in relation to AB and AC .

Remark.

1. The reciprocal of *Lemma 1* is also valid, meaning that if P is the middle of the antiparallel MN to BC , then P belongs to the simedian from A .

2nd Theorem.

(Generalization of the 1st Theorem)

Let ABC be a scalene triangle and K its simedian center. We take $M \in AK$ and draw $MN \parallel AB, MP \parallel AC$, where $N \in BK, P \in CK$. Then:

- i. $NP \parallel BC$;
- ii. MN, NP and MP intersect the sides of triangle ABC in six concyclic points.

Proof.

In triangle ABC , let AA_1, BB_1, CC_1 the simedians concurrent in K (see *Figure 2*).

We have from Thales' Theorem that:

$$\frac{AM}{MK} = \frac{BN}{NK}, \quad (1)$$

$$\frac{AM}{MK} = \frac{CP}{PK}. \quad (2)$$

From relations (1) and (2), it follows that

$$\frac{BN}{NK} = \frac{CP}{PK}, \quad (3)$$

which shows that $NP \parallel BC$.

$VW \parallel AC$, it follows that the trapeze $WSUV$ is isosceles, therefore the points W, S, U, V are concyclic (6).

The relations (4), (5), (6) drive to the concyclicity of the points R, U, V, S, W, T , and the theorem is proved.

Further Remarks.

2. For any point M found on the simedian AA_1 , by performing the constructions from hypothesis, we get a circumscribed circle of the 6 points of intersection of the parallels taken to the sides of triangle.

3. The 2nd *Theorem* generalizes the 1st *Theorem* because we get the second in the case the parallels are taken to the sides through the simedian center k .

4. We get a circle built as in 2nd *Theorem* from the first Lemoine's circle by homothety of pole k and of ratio $\lambda \in \mathbb{R}$.

5. The centers of Lemoine's circles built as above belong to the line OK , where O is the center of the circle circumscribed to the triangle ABC .

References.

- [1] *Exercices de Géométrie*, par F.G.M., Huitième édition, Paris VI^e, Librairie Générale, 77, Rue Le Vaugirard.
- [2] Ion Patrascu, Florentin Smarandache: *Variance on topics of Plane Geometry*, Educational Publishing, Columbus, Ohio, 2013.

The Radical Circle of Ex-Inscribed Circles of a Triangle

In this article, we prove several theorems about the **radical center** and the **radical circle of ex-inscribed circles of a triangle** and calculate the **radius of the circle from vectorial considerations**.

1st Theorem.

The radical center of the ex-inscribed circles of the triangle ABC is the Spiecker's point of the triangle (the center of the circle inscribed in the median triangle of the triangle ABC).

Proof.

We refer in the following to the notation in *Figure 1*. Let I_a, I_b, I_c be the centers of the ex-inscribed circles of a triangle (the intersections of two external bisectors with the internal bisector of the other angle). Using tangents property taken from a point to a circle to be congruent, we calculate and find that:

$$\begin{aligned}AF_a &= AE_a = BD_b = BF_b = CD_c = CE_c = p, \\BD_c &= BF_c = CD_b = CE_b = p - a,\end{aligned}$$

$$CE_a = CD_a = AF_c = AE_c = p - b,$$

$$AF_b = AE_b = BF_c = BD_c = p - c.$$

If A_1 is the middle of segment D_cD_b , it follows that A_1 has equal powers to the ex-inscribed circles (I_b) and (I_c) . Of the previously set, we obtain that A_1 is the middle of the side BC .

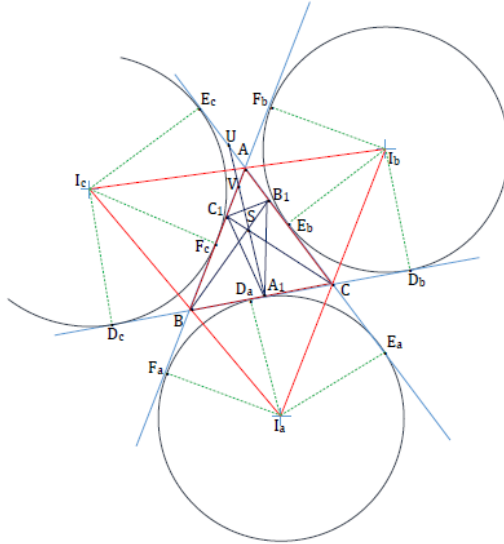


Figure 1.

Also, the middles of the segments E_bE_c and F_bF_c , which we denote U and V , have equal powers to the circles (I_b) and (I_c) .

The radical axis of the circles (I_b) , (I_c) will include the points A_1, U, V .

Because $AE_b = AF_b$ and $AE_c = AF_c$, it follows that $AU = AV$ and we find that $\sphericalangle AUV = \frac{1}{2}\sphericalangle A$, therefore the

radical axis of the ex-inscribed circles (F_b) and (I_c) is the parallel taken through the middle A_1 of the side BC to the bisector of the angle BAC .

Denoting B_1 and C_1 the middles of the sides AC , AB , respectively, we find that the radical center of the ex-inscribed circles is the center of the circle inscribed in the median triangle $A_1B_1C_1$ of the triangle ABC .

This point, denoted S , is the Spiecker's point of the triangle ABC .

2nd Theorem.

The radical center of the inscribed circle (I) and of the B –ex-inscribed and C –ex-inscribed circles of the triangle ABC is the center of the A_1 – ex-inscribed circle of the median triangle $A_1B_1C_1$, corresponding to the triangle ABC).

Proof.

If E is the contact of the inscribed circle with AC and E_b is the contact of the B –ex-inscribed circle with AC , it is known that these points are isotomic, therefore the middle of the segment EE_b is the middle of the side AC , which is B_1 .

This point has equal powers to the inscribed circle (I) and to the B –ex-inscribed circle (I_b), so it belongs to their radical axis.

Analogously, C_1 is on the radical axis of the circles (I) and (I_c) .

The radical axis of the circles (I) , (I_b) is the perpendicular taken from B_1 to the bisector II_b .

This bisector is parallel with the internal bisector of the angle $A_1B_1C_1$, therefore the perpendicular in B_1 on II_b is the external bisector of the angle $A_1B_1C_1$ from the median triangle.

Analogously, it follows that the radical axis of the circles (I) , (I_c) is the external bisector of the angle $A_1C_1B_1$ from the median triangle.

Because the bisectors intersect in the center of the circle A_1 -ex-inscribed to the median triangle $A_1B_1C_1$, this point S_a is the center of the radical center of the circles (I) , (I_b) , (I_c) .

Remark.

The theorem for the circles (I) , (I_a) , (I_b) and (I) , (I_a) , (I_c) can be proved analogously, obtaining the points S_c and S_b .

3rd Theorem.

The radical circle's radius of the circles ex-inscribed to the triangle ABC is given by the formula: $\frac{1}{2}\sqrt{r^2 + p^2}$, where r is the radius of the inscribed circle.

Proof.

The position vector of the circle I of the inscribed circle in the triangle ABC is:

$$\vec{PI} = \frac{1}{2p}(a\vec{PA} + b\vec{PB} + c\vec{PC}).$$

Spiecker's point S is the center of radical circle of ex-inscribed circle and is the center of the inscribed circle in the median triangle $A_1B_1C_1$, therefore:

$$\vec{PS} = \frac{1}{p}\left(\frac{1}{2}a\vec{PA}_1 + \frac{1}{2}b\vec{PB}_1 + \frac{1}{2}c\vec{PC}_1\right).$$

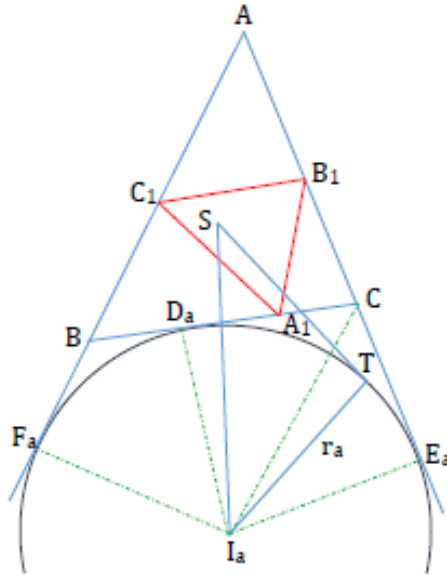


Figure 2.

We denote by T the contact point with the A -ex-inscribed circle of the tangent taken from S to this circle (see *Figure 2*).

The radical circle's radius is given by:

$$ST = \sqrt{SI_a^2 - I_a^2}$$

$$\overrightarrow{I_a S} = \frac{1}{2p} (a\overrightarrow{I_a A_1} + b\overrightarrow{I_a B_1} + c\overrightarrow{I_a C_1}).$$

We evaluate the product of the scales $\overrightarrow{I_a S} \cdot \overrightarrow{I_a S}$;
we have:

$$I_a S^2 = \frac{1}{4p^2} (a^2 I_a A_1^2 + b^2 I_a B_1^2 + c^2 I_a C_1^2 + 2ab\overrightarrow{I_a A_1} \cdot \overrightarrow{I_a B_1} + 2bc\overrightarrow{I_a B_1} \cdot \overrightarrow{I_a C_1} + 2ac\overrightarrow{I_a A_1} \cdot \overrightarrow{I_a C_1}).$$

From the law of cosines applied in the triangle $I_a A_1 B_1$, we find that:

$$2\overrightarrow{I_a A_1} \cdot \overrightarrow{I_a B_1} = I_a A_1^2 + I_a B_1^2 - \frac{1}{4}c^2, \text{ therefore:}$$

$$2ab\overrightarrow{I_a A_1} \cdot \overrightarrow{I_a B_1} = ab(I_a A_1^2 + I_a B_1^2 - \frac{1}{4}abc^2).$$

Analogously, we obtain:

$$2bc\overrightarrow{I_a B_1} \cdot \overrightarrow{I_a C_1} = bc(I_a B_1^2 + I_a C_1^2 - \frac{1}{4}a^2bc),$$

$$2ac\overrightarrow{I_a A_1} \cdot \overrightarrow{I_a C_1} = ac(I_a A_1^2 + I_a C_1^2 - \frac{1}{4}ab^2c).$$

$$I_a S^2 = \frac{1}{4p^2} \left[(a^2 + ab + ac)I_a A_1^2 + (b^2 + ab + bc)I_a B_1^2 + (c^2 + bc + ac)I_a C_1^2 - \frac{abc}{4}(a + b + c) \right],$$

$$I_a S^2 = \frac{1}{4p^2} [2p(aI_a A_1^2 + bI_a B_1^2 + cI_a C_1^2) - 2RS_p],$$

$$I_a S^2 = \frac{1}{2p} (aI_a A_1^2 + bI_a B_1^2 + cI_a C_1^2) - \frac{1}{2}Rr.$$

From the right triangle $I_a D_a A_1$, we have that:

$$I_a A_1^2 = r_a^2 + A_1 D_a^2 = r_a^2 + \left[\frac{a}{2} - (p - c) \right]^2 =$$

$$= r_a^2 + \frac{(c-b)^2}{4}.$$

From the right triangles $I_a E_a B_1$ și $I_a F_a C_1$, we find:

$$\begin{aligned} I_a B_1^2 &= r_a^2 + B_1 E_a^2 = r_a^2 + \left[\frac{b}{2} - (p - b) \right]^2 = \\ &= r_a^2 + \frac{1}{4}(a + c)^2, \end{aligned}$$

$$I_a C_1^2 = r_a^2 + \frac{1}{4}(a + b)^2.$$

Evaluating $aI_a A_1^2 + bI_a B_1^2 + cI_a C_1^2$, we obtain:

$$\begin{aligned} aI_a A_1^2 + bI_a B_1^2 + cI_a C_1^2 &= \\ &= 2pr_a^2 + \frac{1}{2}p(ab + ac + bc) - \frac{1}{4}abc. \end{aligned}$$

But:

$$ab + ac + bc = r^2 + p^2 + 4Rr.$$

It follows that:

$$\frac{1}{2p} [aI_a A_1^2 + bI_a B_1^2 + cI_a C_1^2] = r_a^2 + \frac{1}{4}(r^2 + p^2) + \frac{1}{2}Rr$$

and

$$I_a S^2 = r_a^2 + \frac{1}{4}(r^2 + p^2).$$

Then, we obtain:

$$ST = \frac{1}{2}\sqrt{r^2 + p^2}.$$

References.

- [1] C. Barbu: *Teoreme fundamentale din geometria triunghiului* [Fundamental Theorems of Triangle Geometry]. Bacau, Romania: Editura Unique, 2008.
- [2] I. Patrascu, F. Smarandache: *Variance on topics of plane geometry*. Columbus: The Educational Publisher, Ohio, USA, 2013.

The Polars of a Radical Center

In [1], the late mathematician **Cezar Cosnita**, using the barycenter coordinates, proves two theorems which are the subject of this article.

In a remark made after proving the first theorem, C. Cosnita suggests an elementary proof by employing the **concept of polar**.

In the following, we prove the theorems based on the indicated path, and state that the second theorem is a particular case of the former. Also, we highlight other particular cases of these theorems.

1st Theorem.

Let ABC be a given triangle; through the pairs of points (B,C) , (C,A) and (A,B) we take three circles such that their radical center is on the outside.

The polar lines of the radical center of these circles in relation to each of them cut the sides BC , CA and AB respectively in three collinear points.

Proof.

We denote by D, P, F the second point of intersection of the pairs of circles passing through (A, B) and (B, C) ; (B, C) and (A, C) , (B, C) and (A, B) respectively (see *Figure 1*).

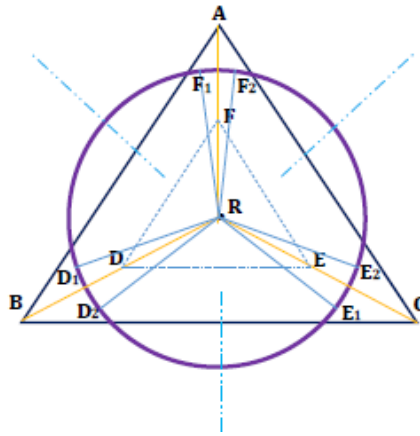


Figure 1

Let R be the radical center of those circles. In fact, $\{R\} = AF \cap BD \cap CE$.

We take from R the tangents $RD_1 = RD_2$ to the circle (B, C) , $RE_1 = RE_2$ to the circle (A, C) and $RF_1 = RF_2$ to the circle passing through (A, B) . Actually, we build the radical circle $\mathcal{C}(R, RD_1)$ of the given circles.

The polar lines of R to these circles are the lines D_1D_2, E_1E_2, F_1F_2 . These three lines cut BC, AC and AB in the points X, Y and Z , and these lines are respectively the polar lines of R in respect to the

circles passing through (B,C) , (C,A) and (A,B) . The polar lines are the radical axis of the radical circle with each of the circles passing through (B,C) , (C,A) , (A,B) , respectively. The points belong to the radical axis having equal powers to those circles, thereby $XD_1 \cdot XD_2 = XC \cdot XB$.

This relationship shows that the point X has equal powers relative to the radical circle and to the circle circumscribed to the triangle ABC ; analogically, the point Y has equal powers relative to the radical circle and to the circle circumscribed to the triangle ABC ; and, likewise, the point Z has equal powers relative to the radical circle and to the circle circumscribed to the triangle ABC .

Because the locus of the points having equal powers to two circles is generally a line, i.e. their radical axis, we get that the points X, Y and Z are collinear, belonging to the radical axis of the radical circle and to the circle circumscribed to the given triangle.

2nd Theorem.

If M is a point in the plane of the triangle ABC and the tangents in this point to the circles circumscribed to triangles C, MAC, MAB , respectively, cut BC, CA and AB , respectively, in the points X, Y, Z , then these points are collinear.

Proof.

The point M is the radical center for the circles (MBC) , (MAC) , and (MAB) , and the tangents in M to these circles are the polar lines to M in relation to these circles.

If X, Y, Z are the intersections of these tangents (polar lines) with BC, CA, AB , then they belong to the radical axis of the circumscribed circle to the triangle ABC and to the circle “reduced” to the point M ($XM^2 = XB \cdot XC$, etc.).

Being located on the radical axis of the two circles, the points X, Y, Z are collinear.

Remarks.

1. Another elementary proof of this theorem is to be found in [3].
2. If the circles from the 1st theorem are adjoint circles of the triangle ABC , then they intersect in Ω (the Brocard’s point). Therefore, we get that the tangents taken in Ω to the adjoint circles cut the sides BC, CA and AB in collinear points.

References.

- [1] C. Coşniţă: *Coordonnées baricentrique* [Barycentric Coordinates], Bucarest - Paris: Librairie Vuibert, 1941.
- [3] I. Pătraşcu: *Axe și centre radicale ale cercului adjuncte unui triunghi* [Axis and radical centers of the adjoin circle of a triangle], in “Recreații matematice”, year XII, no. 1, 2010.
- [3] C. Mihalescu: *Geometria elementelor remarcabile* [The Geometry of Outstanding Elements], Bucharest: Editura Tehnica, 1957.
- [4] F. Smarandache, I. Patrascu: *Geometry of Homological Triangle*, Columbus: The Educational Publisher Inc., 2012.

Regarding the First Droz-Farny's Circle

In this article, we define the **first Droz-Farny's circle**, we establish a connection between it and a **concylicity theorem**, then we generalize this theorem, leading to the **generalization of Droz-Farny's circle**. The first theorem of this article was enunciated by J. Steiner and it was proven by Droz-Farny (*Mathésis*, 1901).

1st Theorem.

Let ABC be a triangle, H its orthocenter and A_1, B_1, C_1 the means of sides $(BC), (CA), (AB)$.

If a circle, having its center H , intersects B_1C_1 in P_1, Q_1 ; C_1A_1 in P_2, Q_2 and A_1B_1 in P_3, Q_3 , then $AP_1 = AQ_1 = BP_2 = BQ_2 = CP_3 = CQ_3$.

Proof.

Naturally, $HP_1 = HQ_1, B_1C_1 \parallel BC, AH \perp BC$.
It follows that $AH \perp B_1C_1$.

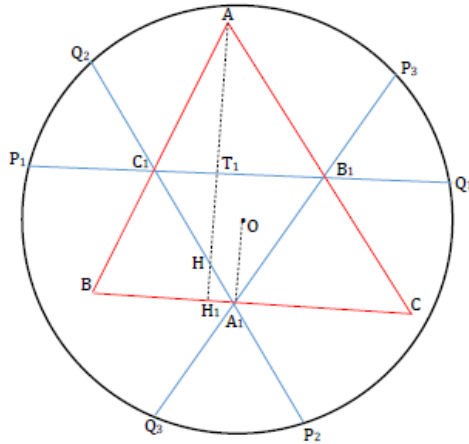


Figure 1.

Therefore, AH is the mediator of segment P_1Q_1 ; similarly, BH and CH are the mediators of segments P_2Q_2 and P_3Q_3 .

Let T_1 be the intersection of lines AH and B_1C_1 (see Figure 1); we have $Q_1A^2 - Q_1H^2 = T_1A^2 - T_1H^2$. We denote $R_H = HP_1$. It follows that $Q_1A^2 = R_H^2 + (T_1A + T_1H)(T_1A - T_1H) = R_H^2 + AH \cdot (T_1A - T_1H)$.

However, $T_1A = T_1H_1$, where H_1 is the projection of A on BC ; we find that $Q_1A^2 = R_H^2 + AH \cdot HH_1$.

It is known that the symmetric of orthocenter H towards BC belongs to the circle of circumscribed triangle ABC .

Denoting this point by H'_1 , we have $AH \cdot HH'_1 = R^2 - OH^2$ (the power of point H towards the circumscribed circle).

We obtain that $AH \cdot HH_1 = \frac{1}{2} \cdot (R^2 - OH^2)$, and therefore $AQ_1^2 = R_H^2 + \frac{1}{2} \cdot (R^2 - OH^2)$, where O is the center of the circumscribed triangle ABC .

Similarly, we find $BQ_2^2 = CQ_3^2 = R_H^2 + \frac{1}{2}(R^2 - OH^2)$, therefore $AQ_1 = BQ_2 = CQ_3$.

Remarks.

- a. The proof adjusts analogously for the obtuse triangle.
- b. *1st Theorem* can be equivalently formulated in this way:

2nd Theorem.

If we draw three congruent circles centered in a given triangle's vertices, the intersection points of these circles with the sides of the median triangle of given triangle (middle lines) are six points situated on a circle having its center in triangle's orthocenter.

If we denote by ρ the radius of three congruent circles having A, B, C as their centers, we get:

$$R_H^2 = \rho^2 + \frac{1}{2}(OH^2 - R^2).$$

However, in a triangle, $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$, R being the radius of the circumscribed circle; it follows that:

$$R_H^2 = \rho^2 + 4R^2 - \frac{1}{2}(a^2 + b^2 + c^2).$$

Remark.

A special case is the one in which $\rho = R$, where we find that $R_H^2 = R_1^2 = 5R^2 - \frac{1}{2}(a^2 + b^2 + c^2) = \frac{1}{2}(R^2 + OH^2)$.

Definition.

The circle $\mathcal{C}(H, R_1)$, where:

$$R_1 = \sqrt{5R^2 - \frac{1}{2}(a^2 + b^2 + c^2)},$$

is called the first Droz-Farny's circle of the triangle ABC .

Remark.

Another way to build the first Droz-Farny's circle is offered by the following theorem, which, according to [1], was the subject of a problem proposed in 1910 by V. Thébault in the *Journal de Mathématiques Elementaire*.

3rd Theorem.

The circles centered on the feet of a triangle's altitudes passing through the center of the circle circumscribed to the triangle cut the triangle's sides in six concyclical points.

Proof.

We consider ABC an acute triangle and H_1, H_2, H_3 the altitudes' feet. We denote by $A_1, A_2; B_1, B_2; C_1, C_2$ the intersection points of circles having their centers H_1, H_2, H_3 to BC, CA, AB , respectively.

We calculate HA_2 of the right angled triangle HH_1A_2 (see *Figure 2*). We have $HA_2^2 = HH_1^2 + H_1A_2^2$.

Because $H_1A_2 = H_1O$, it follows that $HA_2^2 = HH_1^2 + H_1O^2$. We denote by O_9 the mean of segment OH ; the median theorem in triangle H_1HO leads to $H_1H^2 + H_1O^2 = 2H_1O_9^2 + \frac{1}{2}OH^2$.

It is known that O_9H_1 is the nine-points circle's radius, so $H_1O_9 = \frac{1}{2}R$; we get: $HA_1^2 = \frac{1}{2}(R^2 + OH^2)$; similarly, we find that $HB_1^2 = HC_1^2 = \frac{1}{2}(R^2 + OH^2)$, which shows that the points $A_1, A_2; B_1, B_2; C_1, C_2$ belong to the first Droz-Farny's circle.

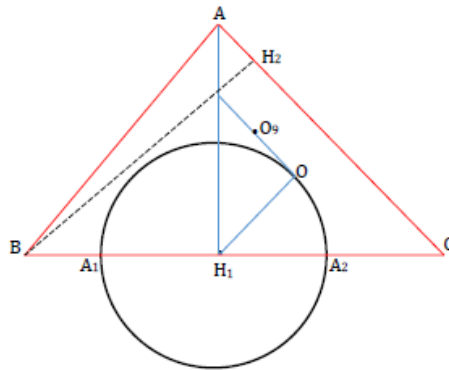


Figure 2.

Remark.

The 2nd and the 3rd theorems show that the first Droz-Farny's circle pass through 12 points, situated two on each side of the triangle and two on each side of the median triangle of the given triangle.

The following theorem generates the 3rd Theorem.

4th Theorem.

The circles centered on the feet of altitudes of a given triangle intersect the sides in six concyclical points if and only if their radical center is the center of the circle circumscribed to the given triangle.

Proof.

Let $A_1, A_2; B_1, B_2; C_1, C_2$ be the points of intersection with the sides of triangle ABC of circles having their centers in altitudes' feet H_1, H_2, H_3 .

Suppose the points are concyclical; it follows that their circle's radical center and of circles centered in H_2 and H_3 is the point A (sides AB and AC are radical axes), therefore the perpendicular from A on H_2H_3 is radical axis of centers having their centers H_2 and H_3 .

Since H_2H_3 is antiparallel to BC , it is parallel to tangent taken in A to the circle circumscribed to triangle ABC .

Consequently, the radical axis is the perpendicular taken in A on the tangent to the circumscribed circle, therefore it is AO .

Similarly, the other radical axis of circles centered in H_1, H_2 and of circles centered in H_1, H_3 pass through O , therefore O is the radical center of these circles.

Reciprocally.

Let O be the radical center of the circles having their centers in the feet of altitudes. Since AO is perpendicular on H_2H_3 , it follows that AO is the radical axis of circles having their centers in H_2, H_3 , therefore $AB_1 \cdot AB_2 = AC_1 \cdot AC_2$.

From this relationship, it follows that the points $B_1, B_2; C_1, C_2$ are concyclic; the circle on which these points are located has its center in the orthocenter H of triangle ABC .

Indeed, the mediators' chords B_1B_2 and C_1C_2 in the two circles are the altitudes from C and B of triangle ABC , therefore $HB_1 = HB_2 = HC_1 = HC_2$.

This reasoning leads to the conclusion that BO is the radical axis of circles having their centers H_1 and H_3 , and from here the concyclicity of the points $A_1, A_2; C_1, C_2$ on a circle having its center in H , therefore $HA_1 = HA_2 = HC_1 = HC_2$. We obtained that $HA_1 = HA_2 = HB_1 = HB_2 = HC_1 = HC_2$, which shows the concyclicity of points $A_1, A_2, B_1, B_2, C_1, C_2$.

Remark.

The circles from the 3rd Theorem, passing through O and having noncollinear centers, admit O as radical center, and therefore the 3rd Theorem is a particular case of the 4th Theorem.

References.

- [1] N. Blaha: *Asupra unor cercuri care au ca centre două puncte inverse* [On some circles that have as centers two inverse points], in “Gazeta Matematica”, vol. XXXIII, 1927.
- [2] R. Johnson: *Advanced Euclidean Geometry*. New York: Dover Publication, Inc. Mineola, 2004
- [3] Eric W. Weisstein: *First Droz-Farny's circle*. From Math World - A Wolfram WEB Resurse, <http://mathworld.wolfram.com/>.
- [4] F. Smarandache, I. Patrascu: *Geometry of Homological Triangle*. Columbus: The Education Publisher Inc., Ohio, SUA, 2012.

Regarding the Second Droz-Farny's Circle

In this article, we prove the theorem relative to the **second Droz-Farny's circle**, and a sentence that generalizes it.

The paper [1] informs that the following *Theorem* is attributed to J. Neuberg (*Mathesis*, 1911).

1st Theorem.

The circles with its centers in the middles of triangle ABC passing through its orthocenter H intersect the sides BC , CA and AB respectively in the points A_1, A_2, B_1, B_2 and C_1, C_2 , situated on a concentric circle with the circle circumscribed to the triangle ABC (the second Droz-Farny's circle).

Proof.

We denote by M_1, M_2, M_3 the middles of ABC triangle's sides, see *Figure 1*. Because $AH \perp M_2M_3$ and H belongs to the circles with centers in M_2 and M_3 , it follows that AH is the radical axis of these circles,

therefore we have $AC_1 \cdot AC_2 = AB_2 \cdot AB_1$. This relation shows that B_1, B_2, C_1, C_2 are concyclic points, because the center of the circle on which they are situated is O , the center of the circle circumscribed to the triangle ABC , hence we have that:

$$OB_1 = OC_1 = OC_2 = OB_2. \quad (1)$$

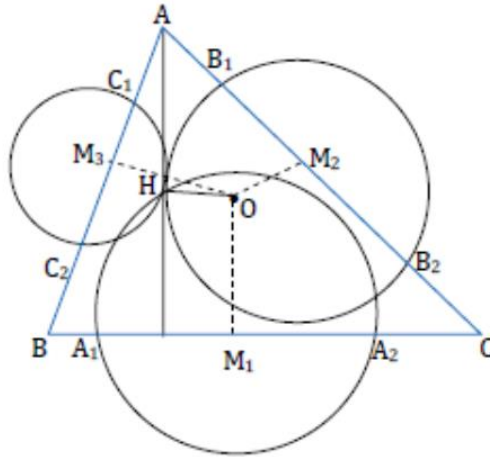


Figure 1.

Analogously, O is the center of the circle on which the points A_1, A_2, C_1, C_2 are situated, hence:

$$OA_1 = OC_1 = OC_2 = OA_2. \quad (2)$$

Also, O is the center of the circle on which the points A_1, A_2, B_1, B_2 are situated, and therefore:

$$OA_1 = OB_1 = OB_2 = OA_2. \quad (3)$$

The relations (1), (2), (3) show that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are situated on a circle having the center in O , called the second Droz-Farny's circle.

1st Proposition.

The radius of the second Droz-Farny's circle is given by:

$$R_2^2 = 5e^2 - \frac{1}{2}(a^2 + b^2 + c^2).$$

Proof.

From the right triangle OM_1A_1 , using Pitagora's theorem, it follows that:

$$OA_1^2 = OM_1^2 + A_1M_1^2 = OM_1^2 + M_1M_2.$$

From the triangle BHC , using the median theorem, we have:

$$HM_1^2 = \frac{1}{4}[2(BH^2 + CH^2) - BC^2].$$

But in a triangle,

$$AH = 2OM_1, BH = 2OM_2, CH = 2OM_3,$$

hence:

$$HM_1^2 = 2OM_2^2 + 2OM_3^2 = \frac{a^2}{4}.$$

But:

$$OM_1^2 = R^2 - \frac{a^2}{4};$$

$$OM_2^2 = R^2 - \frac{b^2}{4};$$

$$OM_3^2 = R^2 - \frac{c^2}{4},$$

where R is the radius of the circle circumscribed to the triangle ABC .

$$\text{We find that } OA_1^2 = R_2^2 = 5R^2 - \frac{1}{2}(a^2 + b^2 + c^2).$$

Remarks.

- a. We can compute $OM_1^2 + M_1M_2$ using the median theorem in the triangle OM_1H for the median M_1O_9 (O_9 is the center of the nine points circle, i.e. the middle of (OH)). Because $O_9M_1 = \frac{1}{2}R$, we obtain: $R_2^2 = \frac{1}{2}(OM^2 + R^2)$. In this way, we can prove the *Theorem* computing OB_1^2 and OC_1^2 .
- b. The statement of the *1st Theorem* was the subject no. 1 of the 49th International Olympiad in Mathematics, held at Madrid in 2008.
- c. The *1st Theorem* can be proved in the same way for an obtuse triangle; it is obvious that for a right triangle, the second Droz-Farny's circle coincides with the circle circumscribed to the triangle ABC .
- d. The *1st Theorem* appears as proposed problem in [2].

2nd Theorem.

The three pairs of points determined by the intersections of each circle with the center in the middle of triangle's side with the respective side are on a circle if and only these circles have as radical center the triangle's orthocenter.

Proof.

Let M_1, M_2, M_3 the middles of the sides of triangle ABC and let $A_1, A_2, B_1, B_2, C_1, C_2$ the intersections with BC, CA, AB respectively of the circles with centers in M_1, M_2, M_3 .

Let us suppose that $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic points. The circle on which they are situated has evidently the center in O , the center of the circle circumscribed to the triangle ABC .

The radical axis of the circles with centers M_2, M_3 will be perpendicular on the line of centers M_2M_3 , and because A has equal powers in relation to these circles, since $AB_1 \cdot AB_2 = AC_1 \cdot AC_2$, it follows that the radical axis will be the perpendicular taken from A on M_2M_3 , i.e. the height from A of triangle ABC .

Furthermore, it ensues that the radical axis of the circles with centers in M_1 and M_2 is the height from B of triangle ABC and consequently the intersection of the heights, hence the orthocenter H of the triangle ABC is the radical center of the three circles.

Reciprocally.

If the circles having the centers in M_1, M_2, M_3 have the orthocenter with the radical center, it follows that the point A , being situated on the height from A which is the radical axis of the circles of centers M_2, M_3

will have equal powers in relation to these circles and, consequently, $AB_1 \cdot AB_2 = AC_1 \cdot AC_2$, a relation that implies that B_1, B_2, C_1, C_2 are concyclic points, and the circle on which these points are situated has O as its center.

Similarly, $BA_1 \cdot BA_2 = BC_1 \cdot BC_2$, therefore A_1, A_2, C_1, C_2 are concyclic points on a circle of center O . Having $OB_1 = OB_2 = OC_1 = OC_2$ and $OA_1 \cdot OA_2 = OC_1 \cdot OC_2$, we get that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are situated on a circle of center O .

Remarks.

1. The 1st Theorem is a particular case of the 2nd Theorem, because the three circles of centers M_1, M_2, M_3 pass through H , which means that H is their radical center.
2. The Problem 525 from [3] leads us to the following Proposition providing a way to construct the circles of centers M_1, M_2, M_3 intersecting the sides in points that belong to a Droz-Farny's circle of type 2.

2nd Proposition.

The circles $C\left(M_1, \frac{1}{2}\sqrt{k+a^2}\right)$, $C\left(M_2, \frac{1}{2}\sqrt{k+b^2}\right)$, $C\left(M_3, \frac{1}{2}\sqrt{k+c^2}\right)$ intersect the sides BC , CA , AB respectively in six concyclic points; k is a conveniently

chosen constant, and a, b, c are the lengths of the sides of triangle ABC .

Proof.

According to the 2nd Theorem, it is necessary to prove that the orthocenter H of triangle ABC is the radical center for the circles from hypothesis.

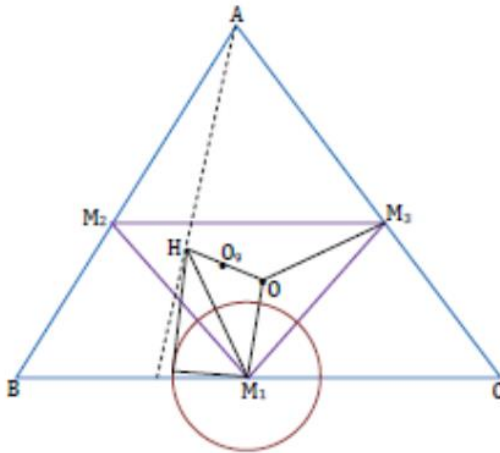


Figure 2.

The power of H in relation with $C\left(M_1, \frac{1}{2}\sqrt{k+a^2}\right)$ is equal to $HM_1^2 - \frac{1}{4}(k+a^2)$. We observed that $M_1^2 = 4R^2 - \frac{b^2}{2} - \frac{c^2}{2} - \frac{a^2}{4}$, therefore $HM_1^2 - \frac{1}{4}(k+a^2) = 4R^2 - \frac{a^2+b^2+c^2}{4} - \frac{1}{4}k$. We use the same expression for the power of H in relation to the circles of centers M_2, M_3 , hence H is the radical center of these three circles.

References.

- [1] C. Mihalescu: *Geometria elementelor remarcabile* [The Geometry of Outstanding Elements]. Bucharest: Editura Tehnică, 1957.
- [2] I. Pătrașcu: *Probleme de geometrie plană* [Some Problems of Plane Geometry], Craiova: Editura Cardinal, 1996.
- [3] C. Coșniță: *Teoreme și probleme alese de matematică* [Theorems and Problems], București: Editura de Stat Didactică și Pedagogică, 1958.
- [4] I. Pătrașcu, F. Smarandache: *Variance on Topics of Plane Geometry*, Educational Publishing, Columbus, Ohio, SUA, 2013.

Neuberg's Orthogonal Circles

In this article, we highlight some metric properties in connection with **Neuberg's circles** and **triangle**.

We recall some results that are necessary.

1st Definition.

It's called Brocard's point of the triangle ABC the point Ω with the property: $\sphericalangle\Omega AB = \sphericalangle\Omega BC = \sphericalangle\Omega CA$. The measure of the angle ΩAB is denoted by ω and it is called Brocard's angle. It occurs the relationship:

$$\text{ctg}\omega = \text{ctg}A + \text{ctg}B + \text{ctg}C \text{ (see [1]).}$$

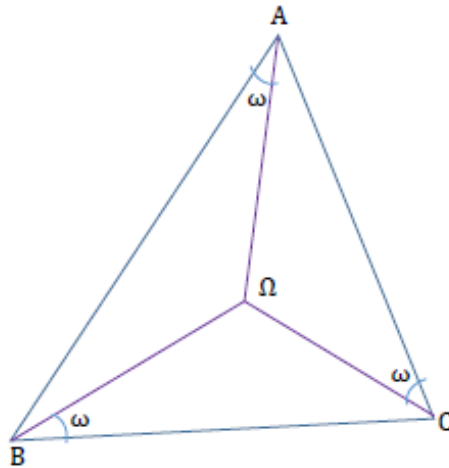


Figure 1.

2nd Definition.

Two triangles are called equibrocardian if they have the same Brocard's angle.

3rd Definition.

The locus of points M from the plane of the triangle located on the same side of the line BC as A and forming with BC an equibrocardian triangle with ABC , containing the vertex A of the triangle, it's called A-Neuberg' circle of the triangle ABC .

We denote by N_a the center of A-Neuberg' circle by radius n_a (analogously, we define B-Neuberg' and C-Neuberg' circles).

We get that $m(BN_aC) = 2\omega$ and $n_a = \frac{a}{2} \sqrt{\text{ctg}^2 \omega - 3}$ (see [1]).

The triangle $N_aN_bN_c$ formed by the centers of Neuberg's circles is called Neuberg's triangle.

1st Proposition.

The distances from the center circumscribed to the triangle ABC to the vertex of Neuberg's triangle are proportional to the cubes of ABC triangle's sides lengths.

Proof.

Let O be the center of the circle circumscribed to the triangle ABC (see *Figure 2*).

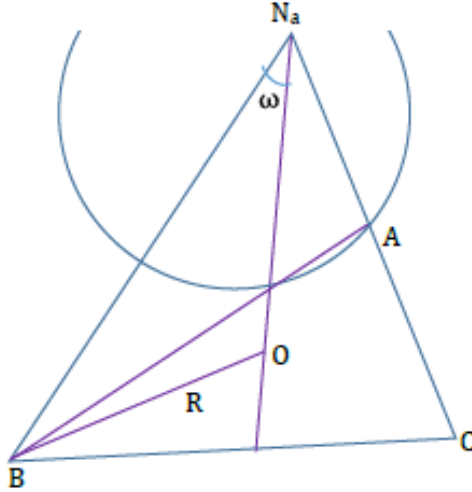


Figure 2.

The law of cosines applied in the triangle ON_aB provides:

$$\frac{ON_a}{\sin(N_aBO)} = \frac{R}{\sin\omega}.$$

$$\text{But } m(\sphericalangle N_aBO) = m(\sphericalangle N_aBC) - m(\sphericalangle OBC) = A - \omega.$$

$$\text{We have that } \frac{ON_a}{\sin(A-\omega)} = \frac{R}{\sin\omega}.$$

But

$$\frac{\sin(A-\omega)}{\sin\omega} = \frac{C\Omega}{A\Omega} = \frac{\frac{a}{c} \cdot 2R\sin\omega}{\frac{b}{a} \cdot 2R\sin\omega} = \frac{a^3}{abc} = \frac{a^3}{4RS},$$

S being the area of the triangle ABC .

It follows that $ON_a = \frac{a^3}{4S}$, and we get that $\frac{ON_a}{a^3} = \frac{ON_b}{b^3} = \frac{ON_c}{c^3}$.

Consequence.

In a triangle ABC, we have:

- 1) $ON_a \cdot ON_b \cdot ON_c = R^3$;
- 2) $\text{ctg}\omega = \frac{ON_a}{a} + \frac{ON_b}{b} + \frac{ON_c}{c}$.

2nd Proposition.

If $N_aN_bN_c$ is the Neuberg's triangle of the triangle ABC, then:

$$N_aN_b^2 = \frac{(a^2 + b^2)(a^4 + b^4) - a^2b^2c^2}{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}.$$

(The formulas for N_bN_c and N_cN_a are obtained from the previous one, by circular permutations.)

Proof.

We apply the law of cosines in the triangle N_aON_c :

$$ON_a = \frac{a^3}{4S}, ON_b = \frac{b^3}{4S}, m(\sphericalangle N_aON_b) = 180^\circ - \hat{c}.$$

$$\begin{aligned} N_aN_b^2 &= \frac{a^6 + b^6 - 2a^3b^3\cos(180^\circ - c)}{16S^2} \\ &= \frac{a^6 + b^6 + 2a^3b^3\cos C}{16S^2}. \end{aligned}$$

But the law of cosines in the triangle ABC provides

$$2\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

and, from Heron's formula, we find that

$$16S^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

Substituting the results above, we obtain, after a few calculations, the stated formula.

4th Definition.

Two circles are called orthogonal if they are secant and their tangents in the common points are perpendicular.

3rd Proposition.

(Gaultier - 18B)

Two circles $\mathcal{C}(O_1, r_1)$, $\mathcal{C}(O_2, r_2)$ are orthogonal if and only if

$$r_1^2 + r_2^2 = O_1O_2^2.$$

Proof.

Let $\mathcal{C}(O_1, r_1)$, $\mathcal{C}(O_2, r_2)$ be orthogonal (see *Figure 3*); then, if A is one of the common points, the triangle O_1AO_2 is a right triangle and the Pythagorean Theorem applied to it, leads to $r_1^2 + r_2^2 = O_1O_2^2$.

Reciprocally.

If the metric relationship from the statement occurs, it means that the triangle O_1AO_2 is a right triangle, therefore A is their common point (the relationship $r_1^2 + r_2^2 = O_1O_2^2$ implies $r_1^2 + r_2^2 > O_1O_2^2$), then $O_1A \perp O_2A$, so O_1A is tangent to the circle $C(O_2, r_2)$ because it is perpendicular in A on radius O_2A , and as well O_2A is tangent to the circle $C(O_1, r_1)$, therefore the circles are orthogonal.

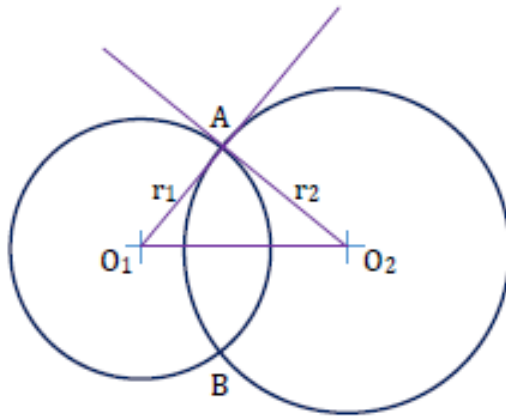


Figure 3.

4th Proposition.

B-Neuberg's and C-Neuberg's circles associated to the right triangle ABC (in A) are orthogonal.

Proof.

$$\text{If } m(\hat{A}) = 90^0, \text{ then } N_b N_c^2 = \frac{b^6 + c^6}{16S}.$$

$$n_b = \frac{b}{2} \sqrt{\text{ctg}^2 \omega - 3}; n_c = \frac{c}{2} \sqrt{\text{ctg}^2 \omega - 3}.$$

$$\text{But } \text{ctg} \omega = \frac{a^2 + b^2 + c^2}{4S} = \frac{b^2 + c^2}{2S} = \frac{a^2}{bc}.$$

It was taken into account that $a^2 = b^2 + c^2$ and $2S = bc$.

$$\text{ctg}^2 \omega - 3 = \frac{a^4}{b^2 c^2} - 3 = \frac{(b^2 + c^2)^2 - 3b^2 c^2}{b^2 c^2}$$

$$\text{ctg}^2 \omega - 3 = \frac{b^4 + c^4 - b^2 c^2}{b^2 c^2}$$

$$\begin{aligned} n_b^2 + n_c^2 &= \frac{b^4 + c^4 - b^2 c^2}{b^2 c^2} \left(\frac{b^2 + c^2}{4} \right) \\ &= \frac{(b^2 + c^2)(b^4 + c^4 - b^2 c^2)}{4b^2 c^2} = \frac{b^6 + c^6}{16S^2}. \end{aligned}$$

By $N_b^2 + N_c^2 = N_b N_c^2$, it follows that B-Neuberg's and C-Neuberg's circles are orthogonal.

References.

- [1] F. Smarandache, I. Patrascu: *The Geometry of Homological Triangles*. Columbus: The Educational Publisher, Ohio, USA, 2012.
- [2] T. Lalescu: *Geometria triunghiului* [The Geometry of the Triangle]. Craiova: Editura Apollo, 1993.
- [3] R. A. Johnson: *Advanced Euclidian Geometry*. New York: Dover Publications, 2007.

Lucas's Inner Circles

In this article, we define the **Lucas's inner circles** and we highlight some of their properties.

1. Definition of the Lucas's Inner Circles

Let ABC be a random triangle; we aim to construct the square inscribed in the triangle ABC , having one side on BC .

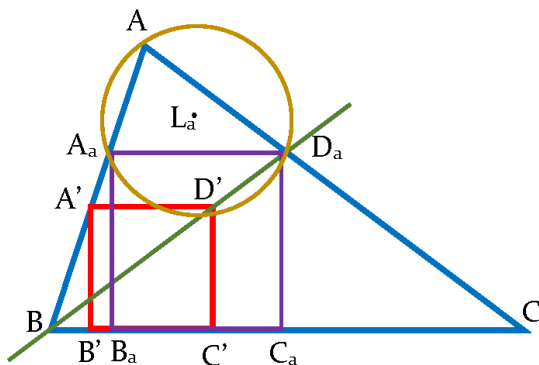


Figure 1.

In order to do this, we construct a square $A'B'C'D'$ with $A' \in (AB)$, $B', C' \in (BC)$ (see Figure 1).

We trace the line BD' and we note with D_a its intersection with (AC) ; through D_a we trace the

parallel $D_a A_a$ to BC with $A_a \in (AB)$ and we project onto BC the points A_a, D_a in B_a respectively C_a .

We affirm that the quadrilateral $A_a B_a C_a D_a$ is the required square.

Indeed, $A_a B_a C_a D_a$ is a square, because $\frac{D_a C_a}{D' C'} = \frac{B D_a}{B D'}$ and, as $D' C' = A' D'$, it follows that $A_a D_a = D_a C_a$.

Definition.

It is called A-Lucas's inner circle of the triangle ABC the circle circumscribed to the triangle $AA_a D_a$.

We will note with L_a the center of the A-Lucas's inner circle and with l_a its radius.

Analogously, we define the B-Lucas's inner circle and the C-Lucas's inner circle of the triangle ABC .

2. Calculation of the Radius of the A-Lucas Inner Circle

We note $A_a D_a = x$, $BC = a$; let h_a be the height from A of the triangle ABC .

The similarity of the triangles $AA_a D_a$ and ABC leads to: $\frac{x}{a} = \frac{h_a - x}{h_a}$, therefore $x = \frac{a h_a}{a + h_a}$.

$$\text{From } \frac{l_a}{R} = \frac{x}{a} \text{ we obtain } l_a = \frac{R \cdot h_a}{a + h_a}. \quad (1)$$

Note.

Relation (1) and the analogues have been deduced by Eduard Lucas (1842-1891) in 1879 and they constitute the “birth certificate of the Lucas’s circles”.

1st Remark.

If in (1) we replace $h_a = \frac{2S}{a}$ and we also keep into consideration the formula $abc = 4RS$, where R is the radius of the circumscribed circle of the triangle ABC and S represents its area, we obtain:

$$l_a = \frac{R}{1 + \frac{2aR}{bc}} \text{ [see Ref. 2].}$$

3. Properties of the Lucas’s Inner Circles

1st Theorem.

The Lucas’s inner circles of a triangle are inner tangents of the circle circumscribed to the triangle and they are exteriorly tangent pairwise.

Proof.

The triangles AA_aD_a and ABC are homothetic through the homothetic center A and the rapport: $\frac{h_a}{a+h_a}$.

Because $\frac{l_a}{R} = \frac{h_a}{a+h_a}$, it means that the A-Lucas's inner circle and the circle circumscribed to the triangle ABC are inner tangents in A .

Analogously, it follows that the B-Lucas's and C-Lucas's inner circles are inner tangents of the circle circumscribed to ABC .

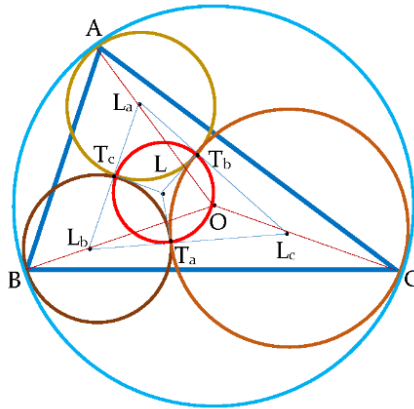


Figure 2.

We will prove that the A-Lucas's and C-Lucas's circles are exterior tangents by verifying

$$L_a L_c = l_a + l_c. \quad (2)$$

We have:

$$OL_a = R - l_a;$$

$$OL_c = R - l_c$$

and

$$m(\widehat{AOC}) = 2B$$

(if $m(\widehat{B}) > 90^\circ$ then $m(\widehat{AOC}) = 360^\circ - 2B$).

The theorem of the cosine applied to the triangle OL_aL_c implies, keeping into consideration (2), that:

$$(R - l_a)^2 + (R - l_c)^2 - 2(R - l_a)(R - l_c)\cos 2B = (l_a + l_c)^2.$$

Because $\cos 2B = 1 - 2\sin^2 B$, it is found that (2) is equivalent to:

$$\sin^2 B = \frac{l_a l_c}{(R - l_a)(R - l_c)}. \quad (3)$$

$$\text{But we have: } l_a l_c = \frac{R^2 ab^2 c}{(2aR + bc)(2cR + ab)},$$

$$l_a + l_c = Rb \left(\frac{c}{2aR + bc} + \frac{a}{2cR + ab} \right).$$

By replacing in (3), we find that $\sin^2 B = \frac{ab^2 c}{4acR^2} = \frac{b^2}{4a^2} \Leftrightarrow \sin B = \frac{b}{2R}$ is true according to the sines theorem. So, the exterior tangent of the A-Lucas's and C-Lucas's circles is proven.

Analogously, we prove the other tangents.

2nd Definition.

It is called an A-Apollonius's circle of the random triangle ABC the circle constructed on the segment determined by the feet of the bisectors of angle A as diameter.

Remark.

Analogously, the B-Apollonius's and C-Apollonius's circles are defined. If ABC is an isosceles triangle with $AB = AC$ then the A-Apollonius's circle

isn't defined for ABC , and if ABC is an equilateral triangle, its Apollonius's circle isn't defined.

2nd Theorem.

The A-Apollonius's circle of the random triangle is the geometrical point of the points M from the plane of the triangle with the property: $\frac{MB}{MC} = \frac{c}{b}$.

3rd Definition.

We call a fascicle of circles the bunch of circles that do not have the same radical axis.

- a. If the radical axis of the circles' fascicle is exterior to them, we say that the fascicle is of the first type.
- b. If the radical axis of the circles' fascicle is secant to the circles, we say that the fascicle is of the second type.
- c. If the radical axis of the circles' fascicle is tangent to the circles, we say that the fascicle is of the third type.

3rd Theorem.

The A-Apollonius's circle and the B-Lucas's and C-Lucas's inner circles of the random triangle ABC form a fascicle of the third type.

Proof.

Let $\{O_A\} = L_b L_c \cap BC$ (see *Figure 3*).

Menelaus's theorem applied to the triangle ABC implies that:

$$\frac{O_A B}{O_A C} \cdot \frac{L_b B}{L_b O} \cdot \frac{L_c O}{L_c C} = 1,$$

so:

$$\frac{O_A B}{O_A C} \cdot \frac{l_b}{R-l_b} \cdot \frac{R-l_c}{l_c} = 1$$

and by replacing l_b and l_c , we find that:

$$\frac{O_A B}{O_A C} = \frac{b^2}{c^2}.$$

This relation shows that the point O_A is the foot of the exterior symmedian from A of the triangle ABC (so the tangent in A to the circumscribed circle), namely the center of the A-Apollonius's circle.

Let N_1 be the contact point of the B-Lucas's and C-Lucas's circles. The radical center of the B-Lucas's, C-Lucas's circles and the circle circumscribed to the triangle ABC is the intersection T_A of the tangents traced in B and in C to the circle circumscribed to the triangle ABC .

It follows that $BT_A = CT_A = N_1 T_A$, so N_1 belongs to the circle \mathcal{C}_A that has the center in T_A and orthogonally cuts the circle circumscribed in B and C . The radical axis of the B-Lucas's and C-Lucas's circles is $T_A N_1$, and $O_A N_1$ is tangent in N_1 to the circle \mathcal{C}_A . Considering the power of the point O_A in relation to \mathcal{C}_A , we have:

$$O_A N_1^2 = O_A B \cdot O_A C.$$

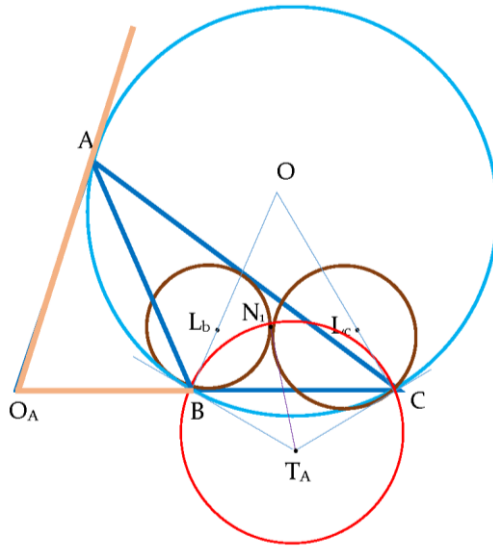


Figure 3.

Also, $O_A O^2 = O_A B \cdot O_A C$; it thus follows that $O_A A = O_A N_1$, which proves that N_1 belongs to the A-Apollonius's circle and is the radical center of the A-Apollonius's, B-Lucas's and C-Lucas's circles.

Remarks.

1. If the triangle ABC is right in A then $L_b L_c \parallel BC$, the radius of the A-Apollonius's circle is equal to: $\frac{abc}{|b^2 - c^2|}$. The point N_1 is the foot of the bisector from A . We find that $O_A N_1 = \frac{abc}{|b^2 - c^2|}$, so the theorem stands true.

2. The A-Apollonius's and A-Lucas's circles are orthogonal. Indeed, the radius of the A-Apollonius's circle is perpendicular to the radius of the circumscribed circle, OA , so, to the radius of the A-Lucas's circle also.

4th Definition.

The triangle $T_A T_B T_C$ determined by the tangents traced in A, B, C to the circle circumscribed to the triangle ABC is called the tangential triangle of the triangle ABC .

1st Property.

The triangle ABC and the Lucas's triangle $L_a L_b L_c$ are homological.

Proof.

Obviously, AL_a, BL_b, CL_c are concurrent in O , therefore O , the center of the circle circumscribed to the triangle ABC , is the homology center.

We have seen that $\{O_A\} = L_b L_c \cap BC$ and O_A is the center of the A-Apollonius's circle, therefore the homology axis is the Apollonius's line $O_A O_B O_C$ (the line determined by the centers of the Apollonius's circle).

2nd Property.

The tangential triangle and the Lucas's triangle of the triangle ABC are orthogonal triangles.

Proof.

The line $T_A N_1$ is the radical axis of the B-Lucas's inner circle and the C-Lucas's inner circle, therefore it is perpendicular on the line of the centers $L_b L_c$. Analogously, $T_B N_2$ is perpendicular on $L_c L_a$, because the radical axes of the Lucas's circles are concurrent in L , which is the radical center of the Lucas's circles; it follows that $T_A T_B T_C$ and $L_a L_b L_c$ are orthological and L is the center of orthology. The other center of orthology is O the center of the circle circumscribed to ABC .

References.

- [1] D. Brânzei, M. Miculița. *Lucas circles and spheres*. In "The Didactics of Mathematics", volume 9/1994, Cluj-Napoca (Romania), p. 73-80.
- [2] P. Yiu, A. P. Hatzipolakis. *The Lucas Circles of a Triangle*. In "Amer. Math. Monthly", no. 108/2001, pp. 444-446. <http://www.math.fau.edu/yiu/monthly437-448.pdf>.
- [3] F. Smarandache, I. Patrascu. *The Geometry of Homological Triangles*. Educational Publisher, Columbus, Ohio, U.S.A., 2013.

Theorems with Parallels Taken through a Triangle's Vertices and Constructions Performed only with the Ruler

In this article, we solve problems of **geometric constructions only with the ruler**, using known theorems.

1st Problem.

Being given a triangle ABC , its circumscribed circle (its center known) and a point M fixed on the circle, construct, using only the ruler, a transversal line A_1, B_1, C_1 , with $A_1 \in BC, B_1 \in CA, C_1 \in AB$, such that $\sphericalangle MA_1C \equiv \sphericalangle MB_1C \equiv \sphericalangle MC_1A$ (the lines taken through M to generate congruent angles with the sides BC, CA and AB , respectively).

2nd Problem.

Being given a triangle ABC , its circumscribed circle (its center known) and A_1, B_1, C_1 , such that $A_1 \in$

$BC, B_1 \in CA, C_1 \in AB$ and A_1, B_1, C_1 collinear, construct, using only the ruler, a point M on the circle circumscribing the triangle, such that the lines MA_1, MB_1, MC_1 to generate congruent angles with BC, CA and AB , respectively.

3rd Problem.

Being given a triangle ABC inscribed in a circle of given center and AA' a given cevian, A' a point on the circle, construct, using only the ruler, the isogonal cevian AA_1 to the cevian AA' .

To solve these problems and to prove the theorems for problems solving, we need the following *Lemma*:

1st Lemma.

(Generalized Simpson's Line)

If M is a point on the circle circumscribed to the triangle ABC and we take the lines MA_1, MB_1, MC_1 which generate congruent angles ($A_1 \in BC, B_1 \in CA, C_1 \in AB$) with BC, CA and AB respectively, then the points A_1, B_1, C_1 are collinear.

Proof.

Let M on the circle circumscribed to the triangle ABC (see *Figure 1*), such that:

$$\sphericalangle MA_1C \equiv \sphericalangle MB_1C \equiv \sphericalangle MC_1A = \varphi. \quad (1)$$

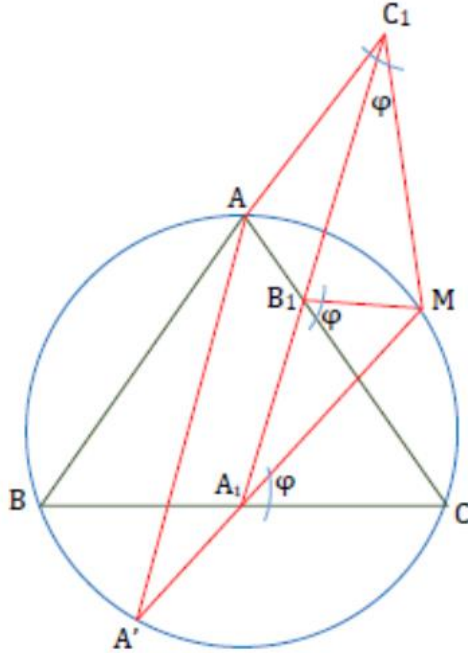


Figure 1.

From the relation (1), we obtain that the quadrilateral MB_1A_1C is inscriptible and, therefore:

$$\sphericalangle A_1BC \equiv \sphericalangle A_1MC. \quad (2).$$

Also from (1), we have that MB_1AC_1 is inscriptible, and so

$$\sphericalangle AB_1C_1 \equiv \sphericalangle AMC_1. \quad (3)$$

The quadrilateral MABC is inscribed, hence:

$$\sphericalangle MAC_1 \equiv \sphericalangle BCM. \quad (4)$$

On the other hand,

$$\sphericalangle A_1MC = 180^\circ - (\widehat{BCM} + \varphi),$$

$$\sphericalangle AMC_1 = 180^\circ - (\widehat{MAC_1} + \varphi).$$

The relation (4) drives us, together with the above relations, to:

$$\sphericalangle A_1MC \equiv \sphericalangle AMC_1. \quad (5)$$

Finally, using the relations (5), (2) and (3), we conclude that: $\sphericalangle A_1B_1C \equiv \sphericalangle AB_1C_1$, which justifies the collinearity of the points A_1, B_1, C_1 .

Remark.

The Simson's Line is obtained in the case when $\varphi = 90^\circ$.

2nd Lemma.

If M is a point on the circle circumscribed to the triangle ABC and A_1, B_1, C_1 are points on BC, CA and AB , respectively, such that $\sphericalangle MA_1C = \sphericalangle MB_1C = \sphericalangle MC_1A = \varphi$, and MA_1 intersects the circle a second time in A' , then $AA' \parallel A_1B_1$.

Proof.

The quadrilateral MB_1A_1C is inscriptible (see *Figure 1*); it follows that:

$$\sphericalangle CMA' \equiv \sphericalangle A_1B_1C. \quad (6)$$

On the other hand, the quadrilateral $MAA'C$ is also inscriptible, hence:

$$\sphericalangle CMA' \equiv \sphericalangle A'AC. \quad (7)$$

The relations (6) and (7) imply: $\sphericalangle A'MC \equiv \sphericalangle A'AC$, which gives $AA' \parallel A_1B_1$.

3rd Lemma.

(The construction of a parallel with a given diameter using a ruler)

In a circle of given center, construct, using only the ruler, a parallel taken through a point of the circle at a given diameter.

Solution.

In the given circle $\mathcal{C}(O, R)$, let be a diameter (AB) and let $M \in \mathcal{C}(O, R)$. We construct the line BM (see *Figure 2*). We consider on this line the point D (M between D and B). We join D with O , A with M and denote $DO \cap AM = \{P\}$.

We take BP and let $\{N\} = DA \cap BP$. The line MN is parallel to AB .

Construction's Proof.

In the triangle DAB , the cevians DO , AM and BN are concurrent.

Ceva's Theorem provides:

$$\frac{OA}{OB} \cdot \frac{MB}{MD} \cdot \frac{ND}{NA} = 1. \quad (8)$$

But DO is a median, $DO = BO = R$.

From (8), we get $\frac{MB}{MD} = \frac{NA}{ND}$, which, by Thales reciprocal, gives $MN \parallel AB$.

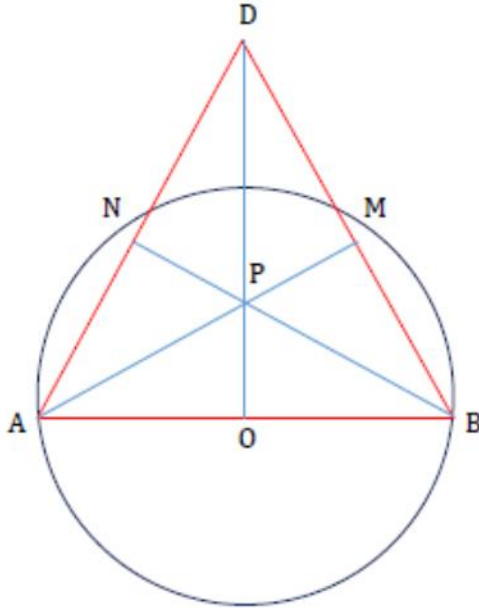


Figure 2.

Remark.

If we have a circle with given center and a certain line d , we can construct through a given point M a parallel to that line in such way: we take two diameters $[RS]$ and $[UV]$ through the center of the given circle (see Figure 3).

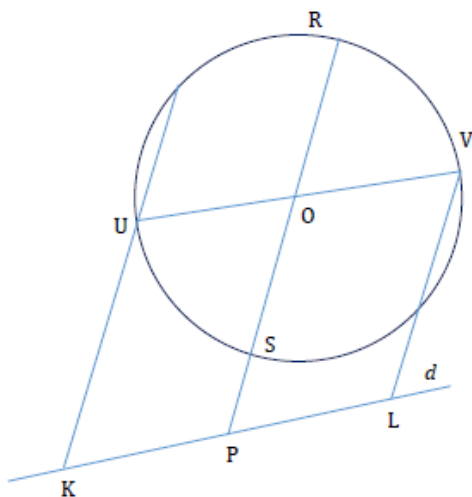


Figure 3.

We denote $RS \cap d = \{P\}$; because $[RO] \equiv [SO]$, we can construct, applying the 3^{rd} Lemma, the parallels through U and V to RS which intersect d in K and L , respectively. Since we have on the line d the points K, P, L , such that $[KP] \equiv [PL]$, we can construct the parallel through M to d based on the construction from 3^{rd} Lemma.

1st Theorem.

(P. Aubert - 1899)

If, through the vertices of the triangle ABC , we take three lines parallel to each other, which intersect the circumscribed circle in A', B' and C' , and M is a

point on the circumscribed circle, as well $MA' \cap BC = \{A_1\}$, $MB' \cap CA = \{B_1\}$, $MC' \cap AB = \{C_1\}$, then A_1, B_1, C_1 are collinear and their line is parallel to AA' .

Proof.

The point of the proof is to show that MA_1, MB_1, MC_1 generate congruent angles with BC, CA and AB , respectively.

$$m(\widehat{MA_1C}) = \frac{1}{2} [m(\widehat{MC}) + m(\widehat{BA'})] \quad (9)$$

$$m(\widehat{MB_1C}) = \frac{1}{2} [m(\widehat{MC}) + m(\widehat{AB'})] \quad (10)$$

But $AA' \parallel BB'$ implies $m(\widehat{BA'}) = m(\widehat{AB'})$, hence, from (9) and (10), it follows that:

$$\sphericalangle MA_1C \equiv \sphericalangle MB_1C, \quad (11)$$

$$m(\widehat{MC_1A}) = \frac{1}{2} [m(\widehat{BM}) - m(\widehat{AC'})]. \quad (12)$$

But $AA' \parallel CC'$ implies that $m(\widehat{AC'}) = m(\widehat{A'C})$; by returning to (12), we have that:

$$\begin{aligned} m(\widehat{MC_1A}) &= \frac{1}{2} [m(\widehat{BM}) - m(\widehat{AC'})] = \\ &= \frac{1}{2} [m(\widehat{BA'}) + m(\widehat{MC})]. \end{aligned} \quad (13)$$

The relations (9) and (13) show that:

$$\sphericalangle MA_1C \equiv \sphericalangle MC_1A. \quad (14)$$

From (11) and (14), we obtain: $\sphericalangle MA_1C \equiv \sphericalangle MB_1C \equiv \sphericalangle MC_1A$, which, by 1st Lemma, verifies the collinearity of points A_1, B_1, C_1 . Now, applying the 2nd Lemma, we obtain the parallelism of lines AA' and A_1B_1 .

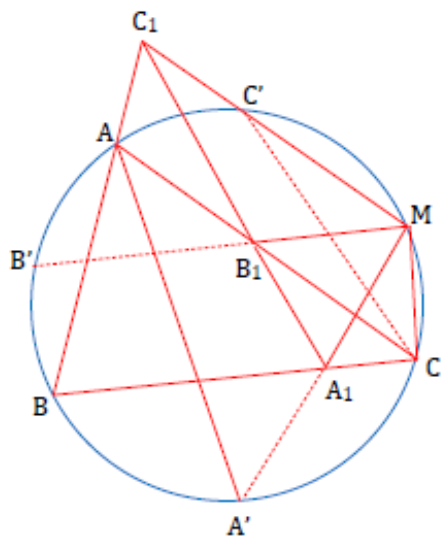


Figure 4.

2nd Theorem.

(M'Kensie - 1887)

If $A_1B_1C_1$ is a transversal line in the triangle ABC ($A_1 \in BC, B_1 \in CA, C_1 \in AB$), and through the triangle's vertices we take the chords AA', BB', CC' of a circle circumscribed to the triangle, parallels with the transversal line, then the lines AA', BB', CC' are concurrent on the circumscribed circle.

Proof.

We denote by M the intersection of the line A_1A' with the circumscribed circle (see *Figure 5*) and with B'_1 , respectively C'_1 the intersection of the line MB' with AC and of the line MC' with AB .

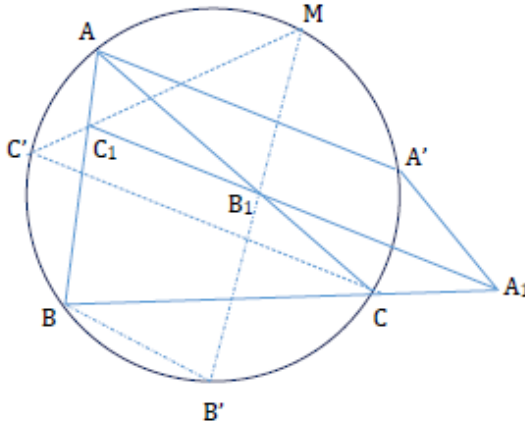


Figure 5.

According to the P. Aubert's theorem, we have that the points A_1, B'_1, C'_1 are collinear and that the line $A_1B'_1$ is parallel to AA' .

From hypothesis, we have that $A_1B_1 \parallel AA'$; from the uniqueness of the parallel taken through A_1 to AA' , it follows that $A_1B_1 \equiv A_1B'_1$, therefore $B'_1 = B_1$, and analogously $C'_1 = C_1$.

Remark.

We have that: MA_1, MB_1, MC_1 generate congruent angles with BC, CA and AB , respectively.

3rd Theorem.

(Beltrami - 1862)

If three parallels are taken through the three vertices of a given triangle, then their isogonals intersect each other on the circle circumscribed to the triangle, and vice versa.

Proof.

Let AA', BB', CC' the three parallel lines with a certain direction (see *Figure 6*).

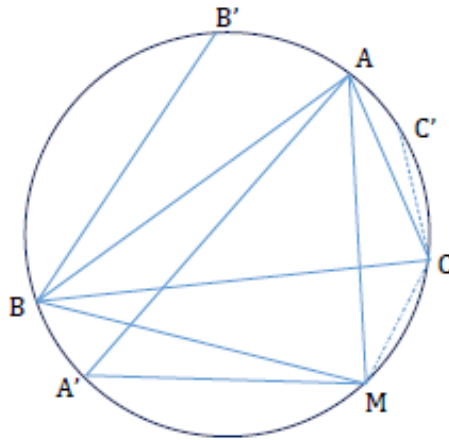


Figure 6.

To construct the isogonal of the cevian AA' , we take $A'M \parallel BC$, M belonging to the circle circumscribed to the triangle, having $\widetilde{BA'} \equiv \widetilde{CM}$, it follows that AM will be the isogonal of the cevian AA' . (Indeed, from $\widetilde{BA'} \equiv \widetilde{CM}$ it follows that $\sphericalangle BAA' \equiv \sphericalangle CAM$.)

On the other hand, $BB' \parallel AA'$ implies $\widetilde{BA'} \equiv \widetilde{AB'}$, and since $\widetilde{BA'} \equiv \widetilde{CM}$ we have that $\widetilde{AB'} \equiv \widetilde{CM}$, which shows that the isogonal of the parallel BB' is BM . From $CC' \parallel AA'$, it follows that $A'C \equiv AC'$, having $\sphericalangle B'CM \equiv \sphericalangle ACC'$, therefore the isogonal of the parallel CC' is CM' .

Reciprocally.

If AM, BM, CM are concurrent cevians in M , the point on the circle circumscribed to the triangle ABC , let us prove that their isogonals are parallel lines. To construct an isogonal of AM , we take $MA' \parallel BC$, A' belonging to the circumscribed circle. We have $\widetilde{MC} \equiv \widetilde{BA'}$. Constructing the isogonal BB' of BM , with B' on the circumscribed circle, we will have $\widetilde{CM} \equiv \widetilde{AB'}$, it follows that $\widetilde{BA'} \equiv \widetilde{AB'}$ and, consequently, $\sphericalangle ABB' \equiv \sphericalangle BAA'$, which shows that $AA' \parallel BB'$. Analogously, we show that $CC' \parallel AA'$.

We are now able to solve the proposed problems.

Solution to the 1st problem.

Using the 3rd *Lemma*, we construct the parallels AA', BB', CC' with a certain directions of a diameter of the circle circumscribed to the given triangle.

We join M with A', B', C' and denote the intersection between MA' and BC , A_1 ; $MB' \cap CA = \{B_1\}$ and $MA' \cap AV = \{C_1\}$.

According to the Aubert's Theorem, the points A_1, B_1, C_1 will be collinear, and MA', MB', MC' generate congruent angles with BC, CA and AB , respectively.

Solution to the 2nd problem.

Using the 3rd *Lemma* and the remark that follows it, we construct through A, B, C the parallels to A_1B_1 ; we denote by A', B', C' their intersections with the circle circumscribed to the triangle ABC . (It is enough to build a single parallel to the transversal line $A_1B_1C_1$, for example AA').

We join A' with A_1 and denote by M the intersection with the circle. The point M will be the point we searched for. The construction's proof follows from the M'Kensie Theorem.

Solution to the 3rd problem.

We suppose that A' belongs to the little arc determined by the chord \overline{BC} in the circle circumscribed to the triangle ABC .

In this case, in order to find the isogonal AA_1 , we construct (by help of the 3rd Lemma and of the remark that follows it) the parallel $A'A_1$ to BC , A_1 being on the circumscribed circle, it is obvious that AA' and AA_1 will be isogonal cevians.

We suppose that A' belongs to the high arc determined by the chord \overline{BC} ; we consider $A' \in \overline{AB}$ (the arc \overline{AB} does not contain the point C). In this situation, we firstly construct the parallel BP to AA' , P belongs to the circumscribed circle, and then through P we construct the parallel PA_1 to AC , A_1 belongs to the circumscribed circle. The isogonal of the line AA' will be AA_1 . The construction's proof follows from 3rd Lemma and from the proof of Beltrami's Theorem.

References.

- [1] F. G. M.: *Exercices de Géométrie*. VIII-e ed., Paris, VI-e Librairie Vuibert, Rue de Vaugirard, 77.
- [2] T. Lalesco: *La Géométrie du Triangle*. 13-e ed., Bucarest, 1937; Paris, Librairie Vuibert, Bd. Saint Germain, 63.
- [3] C. Mihalescu: *Geometria elementelor remarcabile* [The Geometry of remarkable elements]. Bucharest: Editura Tehnică, 1957.

Apollonius's Circles of k^{th} Rank

The purpose of this article is to introduce the notion of **Apollonius's circle of k^{th} rank**.

1st Definition.

It is called an internal cevian of k^{th} rank the line AA_k where $A_k \in (BC)$, such that $\frac{BA}{A_kC} = \left(\frac{AB}{AC}\right)^k$ ($k \in \mathbb{R}$).

If A'_k is the harmonic conjugate of the point A_k in relation to B and C , we call the line AA'_k an external cevian of k^{th} rank.

2nd Definition.

We call Apollonius's circle of k^{th} rank with respect to the side BC of ABC triangle the circle which has as diameter the segment line $A_kA'_k$.

1st Theorem.

Apollonius's circle of k^{th} rank is the locus of points M from ABC triangle's plan, satisfying the relation: $\frac{MB}{MC} = \left(\frac{AB}{AC}\right)^k$.

Proof.

Let O_{A_k} the center of the Apollonius's circle of rank k^{th} relative to the side BC of ABC triangle (see *Figure 1*) and U, V the points of intersection of this circle with the circle circumscribed to the triangle ABC . We denote by D the middle of arc BC , and we extend DA_k to intersect the circle circumscribed in U' .

In $BU'C$ triangle, $U'D$ is bisector; it follows that $\frac{BA_k}{A_kC} = \frac{U'B}{U'C} = \left(\frac{AB}{AC}\right)^k$, so U' belongs to the locus.

The perpendicular in U' on $U'A_k$ intersects BC on A''_k , which is the foot of the BUC triangle's outer bisector, so the harmonic conjugate of A_k in relation to B and C , thus $A''_k = A'_k$.

Therefore, U' is on the Apollonius's circle of rank k relative to the side BC , hence $U' = U$.

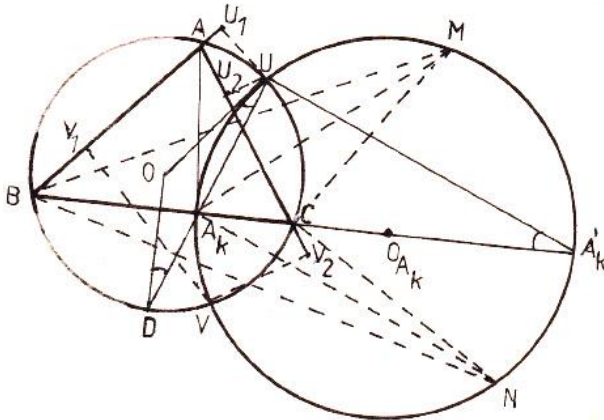


Figure 3

Let M a point that satisfies the relation from the statement; thus $\frac{MB}{MC} = \frac{BA_k}{A_kC}$; it follows - by using the reciprocal of bisector's theorem - that MA_k is the internal bisector of angle BMC . Now let us proceed as before, taking the external bisector; it follows that M belongs to the Apollonius's circle of center O_{A_k} . We consider now a point M on this circle, and we construct C' such that $\sphericalangle BNA_k \equiv \sphericalangle A_kNC'$ (thus NA_k is the internal bisector of the angle $\widehat{BNC'}$). Because $A'_kN \perp NA_k$, it follows that A_k and A'_k are harmonically conjugated with respect to B and C' . On the other hand, the same points are harmonically conjugated with respect to B and C ; from here, it follows that $C' = C$, and we have $\frac{NB}{NC} = \frac{BA_k}{A_kC} = \left(\frac{AB}{AC}\right)^k$.

3rd Definition.

It is called a complete quadrilateral the geometric figure obtained from a convex quadrilateral by extending the opposite sides until they intersect. A complete quadrilateral has 6 vertices, 4 sides and 3 diagonals.

2nd Theorem.

In a complete quadrilateral, the three diagonals' middles are collinear (Gauss - 1810).

Proof.

Let $ABCDEF$ a given complete quadrilateral (see *Figure 2*). We denote by H_1, H_2, H_3, H_4 respectively the orthocenters of ABF, ADE, CBE, CDF triangles, and let A_1, B_1, F_1 the feet of the heights of ABF triangle.

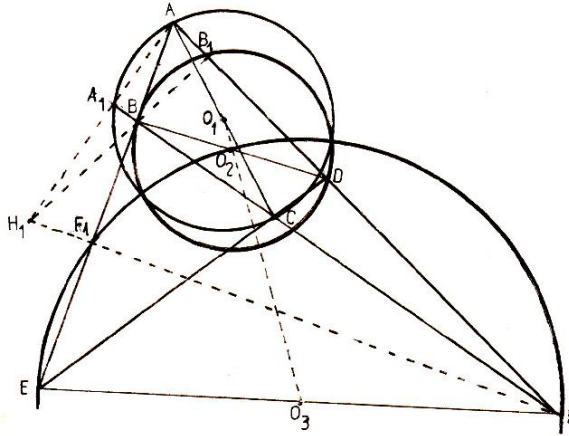


Figure 4

As previously shown, the following relations occur: $H_1A \cdot H_1A_1 - H_1B \cdot H_1B_1 = H_1F \cdot H_1F_1$; they express that the point H_1 has equal powers to the circles of diameters AC, BD, EF , because those circles contain respectively the points A_1, B_1, F_1 , and H_1 is an internal point.

It is shown analogously that the points H_2, H_3, H_4 have equal powers to the same circles, so those points are situated on the radical axis (common to the circles), therefore the circles are part of a fascicle, as

such their centers - which are the middles of the complete quadrilateral's diagonals - are collinear.

The line formed by the middles of a complete quadrilateral's diagonals is called Gauss's line or Gauss-Newton's line.

3rd Theorem.

The Apollonius's circle of k^{th} rank of a triangle are part of a fascicle.

Proof.

Let AA_k, BB_k, CC_k be concurrent cevians of k^{th} rank and AA'_k, BB'_k, CC'_k be the external cevians of k^{th} rank (see *Figure 3*). The figure $B'_kC_kB_kC'_kA_kA'_k$ is a complete quadrilateral and 2nd theorem is applied.

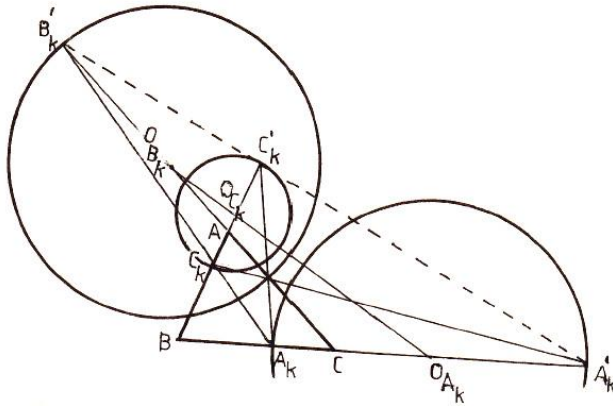


Figure 5

4th Theorem.

The Apollonius's circle of k^{th} rank of a triangle are the orthogonals of the circle circumscribed to the triangle.

Proof.

We unite O to D and U (see *Figure 1*), $OD \perp BC$ and $m(\widehat{A_k U A'_k}) = 90^\circ$, it follows that $\widehat{U A'_k A_k} = \widehat{O D A_k} = \widehat{O U A_k}$.

The congruence $\widehat{U A'_k A_k} \equiv \widehat{O U A_k}$ shows that OU is tangent to the Apollonius's circle of center O_{A_k} .

Analogously, it can be demonstrated for the other Apollonius's Circle.

1st Remark.

The previous theorem indicates that the radical axis of Apollonius's circle of k^{th} rank is the perpendicular taken from O to the line $O_{A_k} O_{B_k}$.

5th Theorem.

The centers of Apollonius's Circle of k^{th} rank of a triangle are situated on the trilinear polar associated to the intersection point of the cevians of $2k^{th}$ rank.

Proof.

From the previous theorem, it results that $OU \perp UO_{A_k}$, so UO_{A_k} is an external cevian of rank 2 for BCU triangle, thus an external symmedian. Henceforth, $\frac{O_{A_k}B}{O_{A_k}C} = \left(\frac{BU}{CU}\right)^2 = \left(\frac{AB}{AC}\right)^{2k}$ (the last equality occurs because U belong to the Apollonius's circle of rank k associated to the vertex A).

6th Theorem.

The Apollonius's circle of k^{th} rank of a triangle intersects the circle circumscribed to the triangle in two points that belong to the internal and external cevians of $k+1^{th}$ rank.

Proof.

Let U and V points of intersection of the Apollonius's circle of center O_{A_k} with the circle circumscribed to the ABC (see *Figure 1*). We take from U and V the perpendiculars UU_1, UU_2 and VV_1, VV_2 on AB and AC respectively. The quadrilaterals $ABVC$, $ABCU$ are inscribed, it follows the similarity of triangles BVV_1, CVV_2 and BUU_1, CUU_2 , from where we get the relations:

$$\frac{BV}{CV} = \frac{VV_1}{VV_2}, \quad \frac{UB}{UC} = \frac{UU_1}{UU_2}.$$

But $\frac{BV}{CV} = \left(\frac{AB}{AC}\right)^k$, $\frac{UB}{UC} = \left(\frac{AB}{AC}\right)^k$, $\frac{VV_1}{VV_2} = \left(\frac{AB}{AC}\right)^k$ and $\frac{UU_1}{UU_2} = \left(\frac{AB}{AC}\right)^k$, relations that show that V and U belong respectively to the internal cevian and the external cevian of rank $k + 1$.

4th Definition.

If the Apollonius's circle of k^{th} rank associated with a triangle has two common points, then we call these points isodynamic points of k^{th} rank (and we denote them W_k, W'_k).

1st Property.

If W_k, W'_k are isodynamic centers of k^{th} rank, then:

$$W_k A \cdot BC^k = W_k B \cdot AC^k = W_k C \cdot AB^k;$$

$$W'_k A \cdot BC^k = W'_k B \cdot AC^k = W'_k C \cdot AB^k.$$

The proof of this property follows immediately from 1st Theorem.

2nd Remark.

The Apollonius's circle of 1st rank is the investigated Apollonius's circle (the bisectors are cevians of 1st rank). If $k = 2$, the internal cevians of 2nd rank are the symmedians, and the external cevians of 2nd rank are the external symmedians, i.e. the tangents

in triangle's vertices to the circumscribed circle. In this case, for the Apollonius's circle of 2nd rank, the 3rd *Theorem* becomes:

7th Theorem.

The Apollonius's circle of 2nd rank intersects the circumscribed circle to the triangle in two points belonging respectively to the antibisector's isogonal and to the cevian outside of it.

Proof.

It follows from the proof of the 6th theorem. We mention that the antibisector is isotomic to the bisector, and a cevian of 3rd rank is isogonic to the antibisector.

References.

- [1] N. N. Mihăileanu: *Lecții complementare de geometrie* [Complementary Lessons of Geometry], Editura Didactică și Pedagogică, București, 1976.
- [2] C. Mihalescu: *Geometria elementelor remarcabile* [The Geometry of Outstanding Elements], Editura Tehnică, București, 1957.
- [3] V. Gh. Vodă: *Triunghiul – ringul cu trei colțuri* [The Triangle-The Ring with Three Corners], Editura Albatros, București, 1979.
- [4] F. Smarandache, I. Pătrașcu: *Geometry of Homological Triangle*, The Education Publisher Inc., Columbus, Ohio, SUA, 2012.

Apollonius's Circle of Second Rank

This article highlights some properties of **Apollonius's circle of second rank** in connection with the **adjoint circles** and the **second Brocard's triangle**.

1st Definition.

It is called Apollonius's circle of second rank relative to the vertex A of the triangle ABC the circle constructed on the segment determined on the simedians' feet from A on BC as diameter.

1st Theorem.

The Apollonius's circle of second rank relative to the vertex A of the triangle ABC intersect the circumscribed circle of the triangle ABC in two points belonging respectively to the cevian of third rank (antibisector's isogonal) and to its external cevian.

The theorem's proof follows from the theorem relative to the Apollonius's circle of k^{th} rank (see [1]).

1st Proposition.

The Apollonius's circle of second rank relative to the vertex A of the triangle ABC intersects the circumscribed circle in two points Q and P (Q on the same side of BC as A). Then, $(QS$ is a bisector in the triangle QBC , S is the simedian's foot from A of the triangle ABC .

Proof.

Q belongs to the Apollonius's circle of second rank, therefore:

$$\frac{QB}{QC} = \left(\frac{AB}{AC}\right)^2. \quad (1)$$

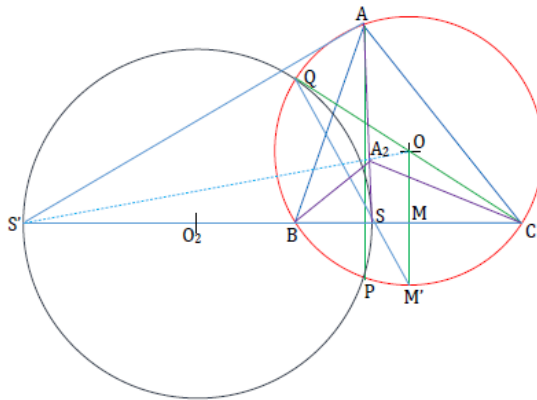


Figure 1.

On the other hand, S being the simedian's foot, we have:

$$\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2. \quad (2)$$

From relations (1) and (2), we note that

$$\frac{QB}{QC} = \frac{SB}{SC},$$

a relation showing that QS is bisector in the triangle QBC .

Remarks.

1. The Apollonius's circle of second relative to the vertex A of the triangle ABC (see *Figure 1*) is an Apollonius's circle for the triangle QBC . Indeed, we proved that QS is an internal bisector in the triangle QBC , and since S' , the external simedian's foot of the triangle ABC , belongs to the Apollonius's Circle of second rank, we have $m(\sphericalangle S'QS) = 90^\circ$, therefore QS' is an external bisector in the triangle QBC .
2. QP is a simedian in QBC . Indeed, the Apollonius's circle of second rank, being an Apollonius's circle for QBC , intersects the circle circum-scribed to QBC after QP , which is simedian in this triangle.

2nd Definition.

It is called adjoint circle of a triangle the circle that passes through two vertices of the triangle and in

one of them is tangent to the triangle's side. We denote $(B\bar{A})$ the adjoint circle that passes through B and A , and is tangent to the side AC in A .

About the circles $(B\bar{A})$ and $(C\bar{A})$, we say that they are adjoint to the vertex A of the triangle ABC .

3rd Definition.

It is called the second Brocard's triangle the triangle $A_2B_2C_2$ whose vertices are the projections of the center of the circle circumscribed to the triangle ABC on triangle's simedians.

2nd Proposition.

The Apollonius's circle of second rank relative to the vertex A of triangle ABC and the adjoint circles relative to the same vertex A intersect in vertex A_2 of the second Brocard's triangle.

Proof.

It is known that the adjoint circles $(B\bar{A})$ and $(C\bar{A})$ intersect in a point belonging to the simedian AS ; we denote this point A_2 (see [3]).

We have:

$$\sphericalangle BA_2S = \sphericalangle A_2BA + \sphericalangle A_2AB,$$

but:

$$\sphericalangle A_2BA \equiv \sphericalangle BA_2S = \sphericalangle A_2AB + \sphericalangle A_2AC = \sphericalangle A.$$

Analogously, $\sphericalangle CA_2S = \sphericalangle A$, therefore $(A_2S$ is the bisector of the angle BA_2C . The bisector's theorem in this triangle leads to:

$$\frac{SB}{SC} = \frac{BA_2}{CA_2},$$

but:

$$\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2,$$

consequently:

$$\frac{BA_2}{CA_2} = \left(\frac{AB}{AC}\right)^2,$$

so A_2 is a point that belongs to the Apollonius's circle of second rank.

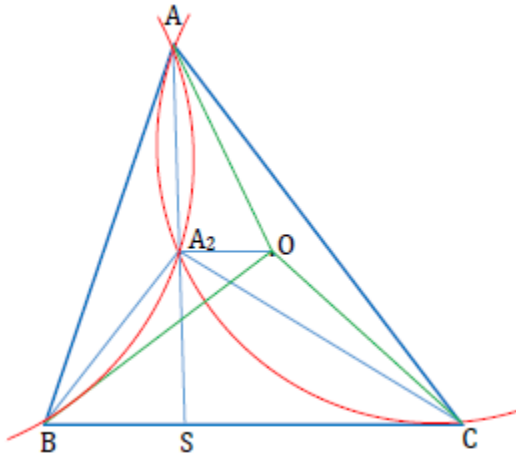


Figure 2.

We prove that A_2 is a vertex in the second Brocard's triangle, i.e. $OA_2 \perp AS$, O the center of the circle circumscribed to the triangle ABC .

We pointed (see *Figure 2*) that $m(\widehat{BA_2C}) = 2A$, if $\sphericalangle A$ is an acute angle, then also $m(\widehat{BOC}) = 2A$, therefore the quadrilateral OCA_2B is inscriptible.

Because $m(\widehat{OCB}) = 90^\circ - m(\hat{A})$, it follows that $m(\widehat{BA_2O}) = 90^\circ + m(\hat{A})$.

On the other hand, $m(\widehat{AA_2B}) = 180^\circ - m(\hat{A})$, so $m(\widehat{BA_2O}) + m(\widehat{AA_2B}) = 270^\circ$ and, consequently, $OA_2 \perp AS$.

Remarks.

1. If $m(\hat{A}) < 90^\circ$, then four remarkable circles pass through A_2 : the two circles adjoint to the vertex A of the triangle ABC , the circle circumscribed to the triangle BOC (where O is the center of the circumscribed circle) and the Apollonius's circle of second rank corresponding to the vertex A .
2. The vertex A_2 of the second Brocard's triangle is the middle of the chord of the circle circumscribed to the triangle ABC containing the simedian AS .
3. The points O , A_2 and S' (the foot of the external simedian to ABC) are collinear. Indeed, we proved that $OA_2 \perp AS$; on the other hand, we proved that A_2S is an internal bisector in the triangle BA_2C , and

since $S'A_2 \perp AS$, the outlined collinearity follows from the uniqueness of the perpendicular in A_2 on AS .

Open Problem.

The Apollonius's circle of second rank relative to the vertex A of the triangle ABC intersects the circle circumscribed to the triangle ABC in two points P and Q (P and A apart of BC).

We denote by X the second point of intersection between the line AP and the Apollonius's circle of second rank.

What can we say about X ?

Is X a remarkable point in triangle's geometry?

References.

- [1] I. Patrascu, F. Smarandache: *Cercurile Apollonius de rangul k* [The Apollonius's Circle of k^{th} rank]. In: "Recreații matematice", Anul XVIII, nr. 1/2016, Iași, România, p. 20-23.
- [2] I. Patrascu: *Axe și centre radicale ale cercurilor adjuncte ale unui triunghi* [Axes and Radical Centers of Adjoint Circles of a Triangle]. In: "Recreații matematice", Anul XVII, nr. 1/2010, Iași, România.
- [3] R. Johnson: *Advanced Euclidean Geometry*. New York: Dover Publications, Inc. Mineola, 2007.
- [4] I. Patrascu, F. Smarandache: *Variance on topics of plane geometry*. Columbus: The Educational Publisher, Ohio, USA, 2013.

A Sufficient Condition for the Circle of the 6 Points to Become Euler's Circle

In this article, we prove the theorem relative to the **circle of the 6 points** and, requiring on this circle to have three other remarkable triangle's points, we obtain the **circle of 9 points** (the Euler's Circle).

1st Definition.

It is called *cevian* of a triangle the line that passes through the vertex of a triangle and an opposite side's point, other than the two left vertices.

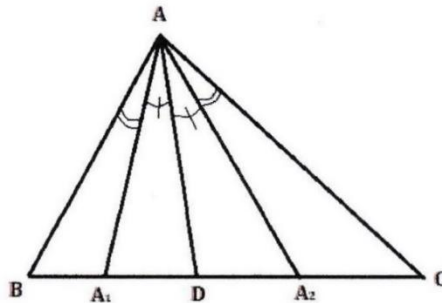


Figure 1.

1st Remark.

The name *cevian* was given in honor of the italian geometrician Giovanni Ceva (1647 - 1734).

2nd Definition.

Two cevians of the same triangle's vertex are called *isogonal cevians* if they create congruent angles with triangle's bisector taken through the same vertex.

2nd Remark.

In the *Figure 1* we represented the bisector AD and the isogonal cevians AA_1 and AA_2 . The following relations take place:

$$\begin{aligned}\widehat{A_1AD} &\equiv \widehat{A_2AD}; \\ \widehat{BAA_1} &\equiv \widehat{CAA_2}.\end{aligned}$$

1st Proposition.

In a triangle, the height and the radius of the circumscribed circle corresponding to a vertex are isogonal cevians.

Proof.

Let ABC an acute-angled triangle with the height AA' and the radius AO (see *Figure 2*).

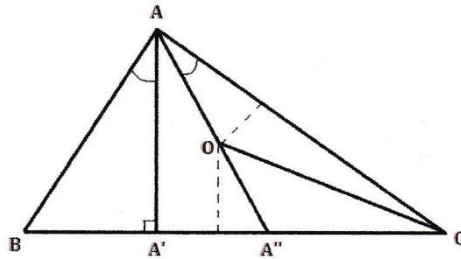


Figure 2.

The angle AOC is a central angle, and the angle ABC is an inscribed angle, so $\widehat{AOC} = 2\widehat{ABC}$. It follows that $\widehat{AOC} = 90^\circ - \hat{B}$.

On the other hand, $\widehat{BAA'} = 90^\circ - \hat{B}$, so AA' and AO are isogonal cevians.

The theorem can be analogously proved for the obtuse triangle.

3rd Remark.

One can easily prove that in a triangle, if AO is circumscribed circle's radius, its isogonal cevian is the triangle's height from vertex A .

3rd Definition.

Two points P_1, P_2 in the plane of triangle ABC are called *isogonals* (isogonal conjugated) if the cevians' pairs (AP_1, AP_2) , (BP_1, BP_2) , (CP_1, CP_2) , are composed of isogonal cevians.

4th Remark.

In a triangle, the orthocenter and circumscribed circle's center are isogonal points.

1st Theorem.

(The 6 points circle)

If P_1 and P_2 are two isogonal points in the triangle ABC , and A_1, B_1, C_1 respectively A_2, B_2, C_2 are their projections on the sides BC, CA and AB of the triangle, then the points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclical.

Proof.

The mediator of segment $[A_1A_2]$ passes through the middle P of segment $[P_1, P_2]$ because the trapezoid $P_1A_1A_2P_2$ is rectangular and the mediator of $[A_1A_2]$ contains its middle line, therefore (see *Figure 3*), we have: $PA_1 = PA_2$ (1). Analogously, it follows that the mediators of segments $[B_1B_2]$ and $[C_1C_2]$ pass through P , so $PB_1 = PB_2$ (2) and $PC_1 = PC_2$ (3). We denote by A_3 and A_4 respectively the middles of segments $[AP_1]$ and $[AP_2]$. We prove that the triangles PA_3C_1 and B_2A_4P are congruent. Indeed, $PA_3 = \frac{1}{2}AP_2$ (middle line), and $B_2A_4 = \frac{1}{2}AP_2$, because it is a median in the rectangular triangle P_2B_2A , so $PA_3 = B_2A_4$; analogously, we obtain that $A_4P = A_3C_1$.

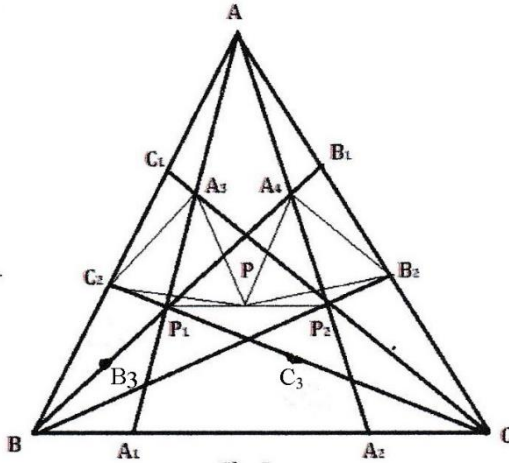


Figure 3.

We have that:

$$\begin{aligned} \widehat{PA_3C_1} &= \widehat{PA_3P_1} + \widehat{P_1A_3C_1} = \widehat{P_1AP_2} + 2\widehat{P_1AC_1} = \\ &= \hat{A} + \widehat{P_1AB}; \\ \widehat{B_2A_4P} &= \widehat{B_2A_4P_2} + \widehat{PA_4P_2} = \widehat{P_1AP_2} + 2\widehat{P_2AB_2} = \\ &= \hat{A} + \widehat{P_2AC}. \end{aligned}$$

But $\widehat{P_1AB} = \widehat{P_2AC}$, because the cevians AP_1 and AP_2 are isogonal and therefore $\widehat{PA_3C_1} = \widehat{B_2A_4P}$. Since $\Delta PA_3C_1 = \Delta B_2A_4P$, it follows that $PB_2 = PC_1$ (4).

Repeating the previous reasoning for triangles PB_3C_1 and A_2B_4P , where B_3 and B_4 are respectively the middles of segments (BP_1) and (BP_2) , we find that they are congruent and it follows that $PC_1 = PA_2$ (5).

The relations (1) - (5) lead to $PA_1 = PA_2 = PB_1 = PB_2 = PC_1 = PC_2$, which shows that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are located on a circle centered in P ,

the middle of the segment determined by isogonal points given by P_1 and P_2 .

4th Definition.

It is called the 9 points circle or Euler's circle of the triangle ABC the circle that contains the middles of triangle's sides, the triangle heights' feet and the middles of the segments determined by the orthocenter with triangle's vertex.

2nd Proposition.

If P_1, P_2 are isogonal points in the triangle ABC and if on the circle of their corresponding 6 points there also are the middles of the segments (AP_1) , (BP_1) , (CP_1) , then the 6 points circle coincides with the Euler's circle of the triangle ABC .

1st Proof.

We keep notations from *Figure 3*; we proved that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are on the 6 points circle of the triangle ABC , having its center in P , the middle of segment $[P_1P_2]$.

If on this circle are also situated the middles A_3, B_3, C_3 of segments (AP_1) , (BP_1) , (CP_1) , then we have $PA_3 = PB_3 = PC_3$.

We determined that PA_3 is middle line in the triangle P_1P_2A , therefore $PA_3 = \frac{1}{2}AP_2$, analogously $PB_3 = \frac{1}{2}BP_2$ and $PC_3 = \frac{1}{2}CP_2$, and we obtain that $P_2A = P_2B = P_2C$, consequently P is the center of the circle circumscribed to the triangle ABC , so $P_2 = O$.

Because P_1 is the isogonal of O , it follows that $P_1 = H$, therefore the circle of 6 points of the isogonal points O and H is the circle of 9 points.

2nd Proof.

Because A_3B_3 is middle line in the triangle P_1AB , it follows that

$$\sphericalangle P_1AB \equiv \sphericalangle P_1A_2B_3. \quad (1)$$

Also, A_3C_3 is middle line in the triangle P_1AC , and A_3C_3 is middle line in the triangle P_1AP_2 , therefore we get

$$\sphericalangle PA_3C_3 \equiv \sphericalangle P_2AC. \quad (2)$$

The relations (1), (2) and the fact that AP_1 and AP_2 are isogonal cevians lead to:

$$\sphericalangle P_1A_2B_3 \equiv \sphericalangle PA_3C_3. \quad (3)$$

The point P is the center of the circle circumscribed to $A_3B_3C_3$; then, from (3) and from isogonal cevians' properties, one of which is circumscribed circle radius, it follows that in the triangle $A_3B_3C_3$ the line P_1A_3 is a height, as $B_3C_3 \parallel BC$, we get that P_1A is a height in the triangle ABC and, therefore, P_1 will be the orthocenter of the triangle

ABC , and P_2 will be the center of the circle circumscribed to the triangle ABC .

5th Remark.

Of those shown, we get that the center of the circle of 9 points is the middle of the line determined by triangle's orthocenter and by the circumscribed circle's center, and the fact that Euler's circle radius is half the radius of the circumscribed circle.

References.

- [1] Roger A. Johnson: *Advanced Euclidean Geometry*. New York: Dover Publications, 2007.
- [2] Cătălin Barbu: *Teoreme fundamentale din geometria triunghiului* [Fundamental theorems of triangle's geometry]. Bacău: Editura Unique, 2008.

An Extension of a Problem of Fixed Point

In this article, we extend the requirement of the *Problem 9.2* proposed at *Varna 2015 Spring Competition*, both in terms of membership of the measure γ , and the case for the problem for the ex-inscribed circle C . We also try to guide the student in the search and identification of the fixed point, for succeeding in solving any problem of this type.

The statement of the problem is as follows:

“We fix an angle $\gamma \in (0, 90^\circ)$ and the line AB which divides the plane in two half-planes γ and $\bar{\gamma}$. The point C in the half-plane γ is situated such that $m(\widehat{ACB}) = \gamma$. The circle inscribed in the triangle ABC with the center I is tangent to the sides AC and BC in the points F and E , respectively. The point P is located on the segment line (IE) , the point E between I and P such that $PE \perp BC$ and $PE = AF$. The point Q is situated on the segment line (IF) , such that F is between I and Q ; $QF \perp AC$ and $QF = BE$. Prove

that the mediator of segment PQ passes through a fixed point.” (Stanislav Chobanov)

Proof.

Firstly, it is useful to note that the point C varies in the half-plane ψ on the arc capable of angle γ ; we know as well that $m(\widehat{ATB}) = 90^\circ + \frac{\gamma}{2}$, so I varies on the arc capable of angle of measure $90^\circ + \frac{\gamma}{2}$ situated in the half-plane ψ .

Another useful remark is about the segments AF and BE , which in a triangle have the lengths $p - a$, respectively $p - b$, where p is the half-perimeter of the triangle ABC with $AB = c$ - constant; therefore, we have $\triangle PEB \cong \triangle AFQ$ with the consequence $PB = QA$. Considering the vertex C of the triangle ABC the middle of the arc capable of angle γ built on AB , we observe that PQ is parallel to AB ; more than that, $ABPQ$ is an isosceles trapezoid, and segment PQ mediator will be a symmetry axis of the trapezoid, so it will coincide with the mediator of AB , which is a fixed line, so we're looking for the fixed point on mediator of AB .

Let D be the intersection of the mediators of segments PQ and AB , see *Figure 1*, where we considered $m(\hat{A}) < m(\hat{B})$. The point D is on the mediator of AB , so we have $DA = DB$; the point D is also on the mediator of PQ , so we have $DP = DQ$; it

follows that: $\Delta PBD \equiv \Delta QAD$, a relation from where we get that $\sphericalangle QAD = \sphericalangle PBD$.

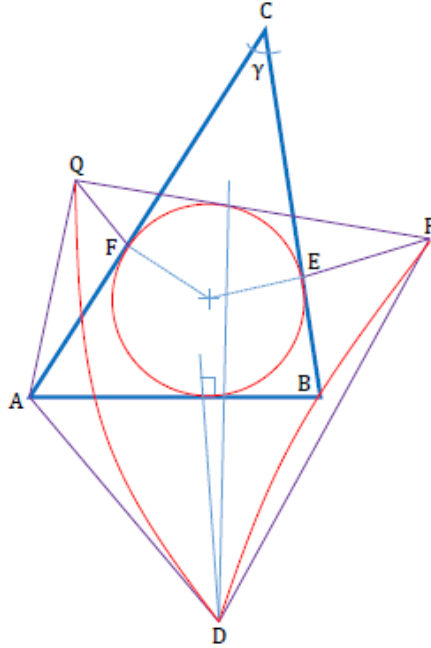


Figure 1

If we denote $m(\widehat{QAF}) = x$ and $m(\widehat{DAB}) = y$, we have $\widehat{QAD} = x + A + y$, $\widehat{PBD} = 360^\circ - B - y - (90^\circ - x)$.

From $x + A + y = 360^\circ - B - y - 90^\circ + x$, we find that $A + B + 2y = 270^\circ$, and since $A + B = 180^\circ - \gamma$, we find that $2y = 90^\circ - \gamma$, therefore the requested fixed point D is the vertex of triangle DAB , situated in $\bar{\psi}$ such that $m(\widehat{ADB}) = 90^\circ - \gamma$.

1st Remark.

If $\gamma = 90^0$, we propose to the reader to prove that the quadrilateral $ABPQ$ is a parallelogram; in this case, the requested fixed point does not exist (or is the point at infinity of the perpendicular lines to AB).

2nd Remark.

If $\gamma \in (90^0, 180^0)$, the problem applies, and we find that the fixed point D is located in the half-plane ψ , such that the triangle DAB is isosceles, having $m(\widehat{AOB}) = \gamma - 90^0$.

We suggest to the reader to solve the following problem:

We fix an angle $\gamma \in (0^0, 180^0)$ and the line AB which divides the plane in two half-planes, ψ and $\bar{\psi}$. The point C in the half-plane ψ is located such that $m(\widehat{ACB}) = \gamma$. The circle C - ex-inscribed to the triangle ABC with center I_c is tangent to the sides AC and BC in the points F and E , respectively. The point P is located on the line segment $(I_cE$, E is between I_c and P such that $PE \perp BC$ and $PE = AF$. The point Q is located on the line segment $(I_cF$ such that F is between I and Q , $QF \perp AC$ and $QF = BE$. Prove that the mediator of the segment PQ passes through a fixed point.

3rd Remark.

As seen, this problem is also true in the case $\gamma = 90^\circ$, more than that, in this case, the fixed point is the middle of AB . Prove!

References.

- [1] Ion Patrascu: *Probleme de geometrie plană* [Planar Geometry Problems]. Craiova: Editura Cardinal, 1996.

Some Properties of the Harmonic Quadrilateral

In this article, we review some properties of the **harmonic quadrilateral** related to triangle simedians and to Apollonius's Circle.

1st Definition.

A convex circumscribable quadrilateral $ABCD$ having the property $AB \cdot CD = BC \cdot AD$ is called harmonic quadrilateral.

2nd Definition.

A triangle simedian is the isogonal cevian of a triangle median.

1st Proposition.

In the triangle ABC , the cevian AA_1 , $A_1 \in (BC)$ is a simedian if and only if $\frac{BA_1}{A_1C} = \left(\frac{AB}{AC}\right)^2$. For *Proof* of this property, see *infra*.

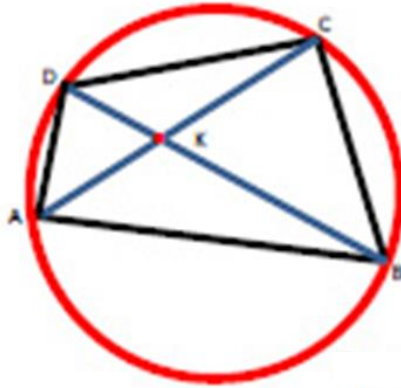


Figura 1.

2nd Proposition.

In an harmonic quadrilateral, the diagonals are simedians of the triangles determined by two consecutive sides of a quadrilateral with its diagonal.

Proof.

Let $ABCD$ be an harmonic quadrilateral and $\{K\} = AC \cap BD$ (see *Figure 1*). We prove that BK is simedian in the triangle ABC .

From the similarity of the triangles ABK and DCK , we find that:

$$\frac{AB}{DC} = \frac{AK}{DK} = \frac{BK}{CK}. \quad (1)$$

From the similarity of the triangles BCK și ADK , we conclude that:

$$\frac{BC}{AD} = \frac{CK}{DK} = \frac{BK}{AK}. \quad (2)$$

From the relations (1) and (2), by division, it follows that:

$$\frac{AB}{BC} \cdot \frac{AD}{DC} = \frac{AK}{CK}. \quad (3)$$

But $ABCD$ is an harmonic quadrilateral; consequently,

$$\frac{AB}{BC} = \frac{AD}{DC};$$

substituting this relation in (3), it follows that:

$$\left(\frac{AB}{BC}\right)^2 = \frac{AK}{CK};$$

As shown by Proposition 1, BK is a simedian in the triangle ABC . Similarly, it can be shown that AK is a simedian in the triangle ABD , that CK is a simedian in the triangle BCD , and that DK is a simedian in the triangle ADC .

Remark 1.

The converse of the 2nd Proposition is proved similarly, i.e.:

3rd Proposition.

If in a convex circumscribable quadrilateral, a diagonal is a simedian in the triangle formed by the other diagonal with two consecutive sides of the quadrilateral, then the quadrilateral is an harmonic quadrilateral.

Remark 2.

From 2nd and 3rd Propositions above, it results a simple way to build an harmonic quadrilateral.

In a circle, let a triangle ABC be considered; we construct the simedian of A , be it AK , and we denote by D the intersection of the simedian AK with the circle. The quadrilateral $ABCD$ is an harmonic quadrilateral.

Proposition 4.

In a triangle ABC , the points of the simedian of A are situated at proportional lengths to the sides AB and AC .

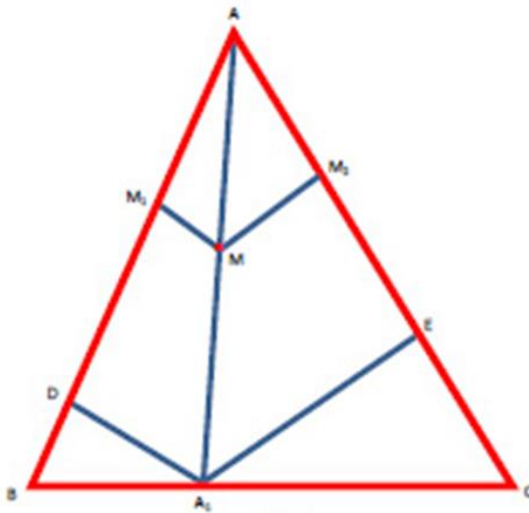


Figura 2.

Proof.

We have the simedian AA_1 in the triangle ABC (see *Figure 2*). We denote by D and E the projections of A_1 on AB , and AC respectively.

We get:

$$\frac{BA_1}{CA_1} = \frac{Area_{\Delta}(ABA_1)}{Area_{\Delta}(ACA_1)} = \frac{AB \cdot A_1D}{AC \cdot A_1E}.$$

Moreover, from 1st Proposition, we know that

$$\frac{BA_1}{A_1C} = \left(\frac{AB}{AC}\right)^2.$$

Substituting in the previous relation, we obtain that:

$$\frac{A_1D}{A_1E} = \frac{AB}{AC}.$$

On the other hand, $DA_1 = AA_1$. From BAA_1 and $A_1E = AA_1 \cdot \sin\widehat{CAA_1}$, hence:

$$\frac{A_1D}{A_1E} = \frac{\sin\widehat{BAA_1}}{\sin\widehat{CAA_1}} = \frac{AB}{AC}. \quad (4)$$

If M is a point on the simedian and MM_1 and MM_2 are its projections on AB , and AC respectively, we have:

$$MM_1 = AM \cdot \sin\widehat{BAA_1}, \quad MM_2 = AM \cdot \sin\widehat{CAA_1},$$

hence:

$$\frac{MM_1}{MM_2} = \frac{\sin\widehat{BAA_1}}{\sin\widehat{CAA_1}}.$$

Taking (4) into account, we obtain that:

$$\frac{MM_1}{MM_2} = \frac{AB}{AC}.$$

3rd Remark.

The converse of the property in the statement above is valid, meaning that, if M is a point inside a triangle, its distances to two sides are proportional to the lengths of these sides. The point belongs to the simedian of the triangle having the vertex joint to the two sides.

5th Proposition.

In an harmonic quadrilateral, the point of intersection of the diagonals is located towards the sides of the quadrilateral to proportional distances to the length of these sides. The *Proof* of this Proposition relies on 2nd and 4th Propositions.

6th Proposition.

(R. Tucker)

The point of intersection of the diagonals of an harmonic quadrilateral minimizes the sum of squares of distances from a point inside the quadrilateral to the quadrilateral sides.

Proof.

Let $ABCD$ be an harmonic quadrilateral and M any point within.

We denote by x, y, z, u the distances of M to the AB, BC, CD, DA sides of lengths $a, b, c,$ and d (see Figure 3).

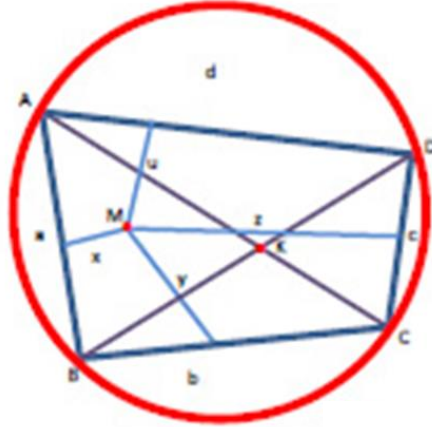


Figure 3.

Let S be the $ABCD$ quadrilateral area.

We have:

$$ax + by + cz + du = 2S.$$

This is true for x, y, z, u and a, b, c, d real numbers.

Following Cauchy-Buniakowski-Schwarz Inequality, we get:

$$(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + u^2) \geq (ax + by + cz + du)^2,$$

and it is obvious that:

$$x^2 + y^2 + z^2 + u^2 \geq \frac{4S^2}{a^2 + b^2 + c^2 + d^2}.$$

We note that the minimum sum of squared distances is:

$$\frac{4S^2}{a^2 + b^2 + c^2 + d^2} = \text{const.}$$

In Cauchy-Buniakowski-Schwarz Inequality, the equality occurs if:

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{u}{d}.$$

Since $\{K\} = AC \cap BD$ is the only point with this property, it ensues that $M = K$, so K has the property of the minimum in the statement.

3rd Definition.

We call external simedian of ABC triangle a cevian AA_1' corresponding to the vertex A , where A_1' is the harmonic conjugate of the point A_1 – simedian's foot from A relative to points B and C .

4th Remark.

In *Figure 4*, the cevian AA_1 is an internal simedian, and AA_1' is an external simedian.

We have:

$$\frac{A_1B}{A_1C} = \frac{A_1'B}{A_1'C}.$$

In view of 1st Proposition, we get that:

$$\frac{A_1'B}{A_1'C} = \left(\frac{AB}{AC}\right)^2.$$

7th Proposition.

The tangents taken to the extremes of a diagonal of a circle circumscribed to the harmonic quadrilateral intersect on the other diagonal.

Proof.

Let P be the intersection of a tangent taken in D to the circle circumscribed to the harmonic quadrilateral $ABCD$ with AC (see *Figure 4*).

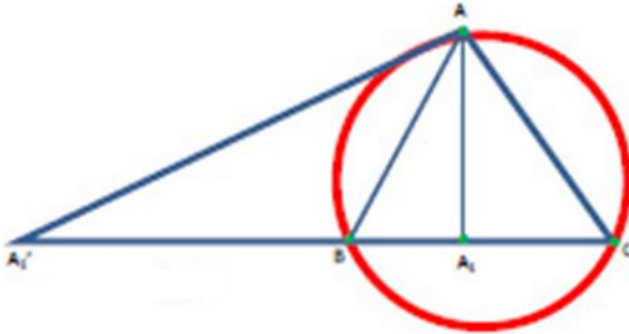


Figure 4.

Since triangles PDC and PAD are alike, we conclude that:

$$\frac{PD}{PA} = \frac{PC}{PD} = \frac{DC}{AD}. \quad (5)$$

From relations (5), we find that:

$$\frac{PA}{PC} = \left(\frac{AD}{DC}\right)^2. \quad (6)$$

This relationship indicates that P is the harmonic conjugate of K with respect to A and C , so DP is an external simedian from D of the triangle ADC .

Similarly, if we denote by P' the intersection of the tangent taken in B to the circle circumscribed with AC , we get:

$$\frac{P'A}{P'C} = \left(\frac{BA}{BC}\right)^2. \quad (7)$$

From (6) and (7), as well as from the properties of the harmonic quadrilateral, we know that:

$$\frac{AB}{BC} = \frac{AD}{DC},$$

which means that:

$$\frac{PA}{PC} = \frac{P'A}{P'C},$$

hence $P = P'$.

Similarly, it is shown that the tangents taken to A and C intersect at point Q located on the diagonal BD .

5th Remark.

- a. The points P and Q are the diagonal poles of BD and AC in relation to the circle circumscribed to the quadrilateral.
- b. From the previous *Proposition*, it follows that in a triangle the internal simedian of an angle is consecutive to the external simedians of the other two angles.

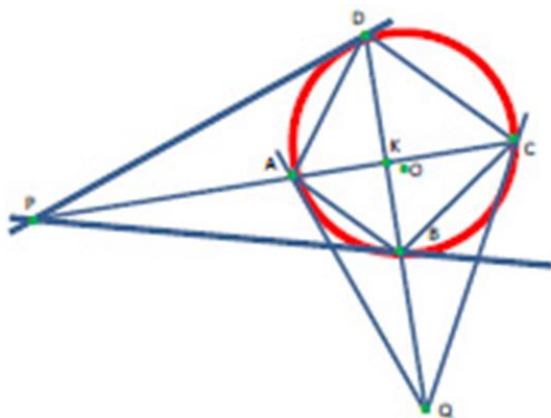


Figure 5.

8th Proposition.

Let $ABCD$ be an harmonic quadrilateral inscribed in the circle of center O and let P and Q be the intersections of the tangents taken in B and D , respectively in A and C to the circle circumscribed to the quadrilateral.

If $\{K\} = AC \cap BD$, then the orthocenter of triangle PKQ is O .

Proof.

From the properties of tangents taken from a point to a circle, we conclude that $PO \perp BD$ and $QO \perp AC$. These relations show that in the triangle PKQ , PO and QO are heights, so O is the orthocenter of this triangle.

4th Definition.

The Apollonius's circle related to the vertex A of the triangle ABC is the circle built on the segment $[DE]$ in diameter, where D and E are the feet of the internal, respectively external, bisectors taken from A to the triangle ABC .

6th Remark.

If the triangle ABC is isosceles with $AB = AC$, the Apollonius's circle corresponding to vertex A is not defined.

9th Proposition.

The Apollonius's circle relative to the vertex A of the triangle ABC has as center the feet of the external simedian taken from A .

Proof.

Let O_a be the intersection of the external simedian of the triangle ABC with BC (see *Figure 6*).

Assuming that $m(\hat{B}) > m(\hat{C})$, we find that:

$$m(\widehat{EAB}) = \frac{1}{2}[m(\hat{B}) + m(\hat{C})].$$

O_a being a tangent, we find that $m(\widehat{O_aAB}) = m(\hat{C})$.

Withal,

$$m(\widehat{EAO_a}) = \frac{1}{2}[m(\hat{B}) - m(\hat{C})]$$

and

$$m(\widehat{AEO_a}) = \frac{1}{2} [m(\widehat{B}) - m(\widehat{C})].$$

It results that $O_aE = O_aA$; onward, EAD being a right angled triangle, we obtain: $O_aA = O_aD$; hence O_a is the center of Apollonius's circle corresponding to the vertex A .

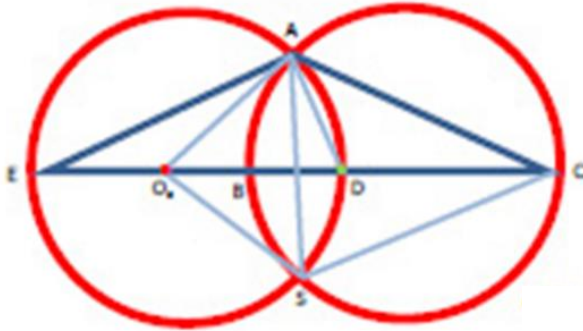


Figura 6.

10th Proposition.

Apollonius's circle relative to the vertex A of triangle ABC cuts the circle circumscribed to the triangle following the internal simedian taken from A .

Proof.

Let S be the second point of intersection of Apollonius's Circle relative to vertex A and the circle circumscribing the triangle ABC .

Because O_aA is tangent to the circle circumscribed in A , it results, for reasons of symmetry, that O_aS will be tangent in S to the circumscribed circle.

For triangle ACS , O_aA and O_aS are external simedians; it results that CO_a is internal simedian in the triangle ACS , furthermore, it results that the quadrilateral $ABSC$ is an harmonic quadrilateral.

Consequently, AS is the internal simedian of the triangle ABC and the property is proven.

7th Remark.

From this, in view of *Figure 5*, it results that the circle of center Q passing through A and C is an Apollonius's circle relative to the vertex A for the triangle ABD . This circle (of center Q and radius QC) is also an Apollonius's circle relative to the vertex C of the triangle BCD .

Similarly, the Apollonius's circle corresponding to vertexes B and D and to the triangles ABC , and ADC respectively, coincide.

We can now formulate the following:

11th Proposition.

In an harmonic quadrilateral, the Apollonius's circle - associated with the vertexes of a diagonal and to the triangles determined by those vertexes to the other diagonal - coincide.

Radical axis of the Apollonius's circle is the right determined by the center of the circle circumscribed to the harmonic quadrilateral and by the intersection of its diagonals.

Proof.

Referring to *Fig. 5*, we observe that the power of O towards the Apollonius's Circle relative to vertexes B and C of triangles ABC and BCU is:

$$OB^2 = OC^2.$$

So O belongs to the radical axis of the circles.

We also have $KA \cdot KC = KB \cdot KD$, relatives indicating that the point K has equal powers towards the highlighted Apollonius's circle.

References.

- [1] Roger A. Johnson: *Advanced Euclidean Geometry*, Dover Publications, Inc. Mineola, New York, USA, 2007.
- [2] F. Smarandache, I. Patrascu: *The Geometry of Homological Triangles*, The Education Publisher, Inc. Columbus, Ohio, USA, 2012.
- [2] F. Smarandache, I. Patrascu: *Variance on Topics of plane Geometry*, The Education Publisher, Inc. Columbus, Ohio, USA, 2013.

Triangulation of a Triangle with Triangles having Equal Inscribed Circles

In this article, we solve the following problem:

Any triangle can be divided by a cevian in two triangles that have congruent inscribed circles.

Solution.

We consider a given triangle ABC and we show that there is a point D on the side (BC) so that the inscribed circles in the triangles ABD , ACD are congruent. If ABC is an isosceles triangle ($AB = AC$), where D is the middle of the base (BC) , we assume that ABC is a non-isosceles triangle. We note I_1, I_2 the centers of the inscribed congruent circles; obviously, I_1I_2 is parallel to the BC . (1)

We observe that:

$$m(\sphericalangle I_1 A I_2) = \frac{1}{2} m(\hat{A}). \quad (2)$$

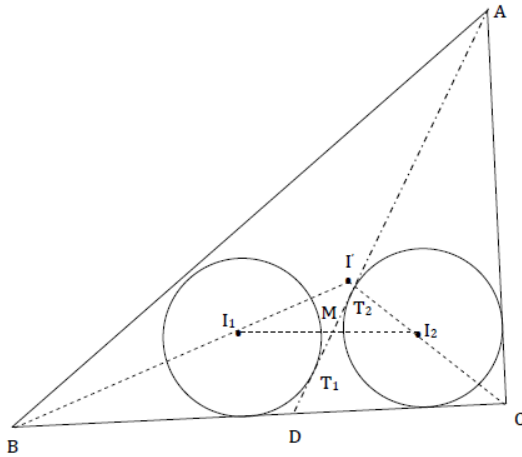


Figure 1.

If T_1, T_2 are contacts with the circles of the cevian AD , we have $\Delta I_1 T_1 M \equiv \Delta I_2 T_2 M$; let M be the intersection of $I_1 I_2$ with AD , see Figure 1.

From this congruence, it is obvious that:

$$(I_1 M) \equiv (I_2 M). \quad (3)$$

Let I be the center of the circle inscribed in the triangle ABC ; we prove that: AI is a simedian in the triangle $I_1 A I_2$. (4)

Indeed, noting $\alpha = m(\widehat{BAI_1})$, it follows that $m(\sphericalangle I_1 A M) = \alpha$. From $\sphericalangle I_1 A I_2 = \sphericalangle B A I$, it follows that $\sphericalangle B A I_1 \equiv \sphericalangle I A I_2$, therefore $\sphericalangle I_1 A M \equiv \sphericalangle I A I_2$, indicating that AM and AI are isogonal cevians in the triangle $I_1 A I_2$.

Since in this triangle AM is a median, it follows that AI is a simedian.

Now, we show how we build the point D , using the conditions (1) – (4), and then we prove that this construction satisfies the enunciation requirements.

Building the point D .

- 1°: We build the circumscribed circle of the given triangle ABC ; we build the bisector of the angle BAC and denote by P its intersection with the circumscribed circle (see *Figure 2*).
- 2°: We build the perpendicular on C to CP and (BC) side mediator; we denote O_1 the intersection of these lines.
- 3°: We build the circle $C(O_1; O_1C)$ and denote A' the intersection of this circle with the bisector AI (A' is on the same side of the line BC as A).
- 4°: We build through A the parallel to $A'O_1$ and we denote it IO_1 .
- 5°: We build the circle $C(O'_1; O'_1A)$ and we denote I_1, I_2 its intersections with BI , and CI respectively.
- 6°: We build the middle M of the segment (I_1I_2) and denote by D the intersection of the lines AM and BC .

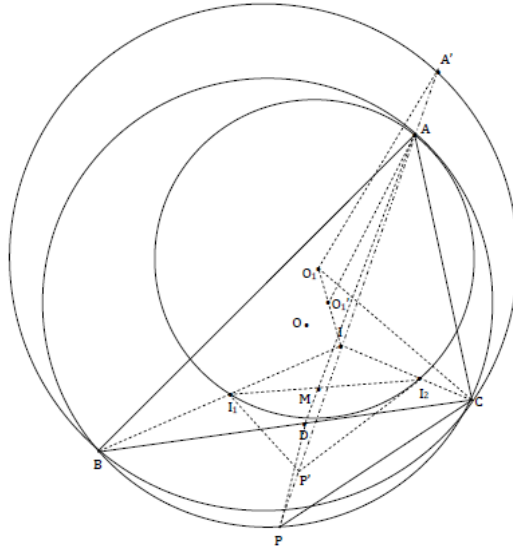


Figure 2.

Proof.

The point P is the middle of the arc \widehat{BC} , then $m(\widehat{PCB}) = \frac{1}{2}m(\widehat{A})$.

The circle $C(O_1; O_1C)$ contains the arc from which points the segment (BC) „is shown” under angle measurement $\frac{1}{2}m(\widehat{A})$.

The circle $C(O'_1; O'_1A)$ is homothetical to the circle $C(O_1; O_1C)$ by the homothety of center I and by the report $\frac{IA'}{IA}$; therefore, it follows that I_1I_2 will be parallel to the BC , and from the points of circle $C(O'_1; O'_1A)$ of the same side of BC as A , the segment

(I_1I_2) „is shown” at an angle of measure $\frac{1}{2}m(\hat{A})$. Since the tangents taken in B and C to the circle $C(O_1; O_1C)$ intersect in P , on the bisector AI , as from a property of simedians, we get that $A'I$ is a simedian in the triangle $A'BC$.

Due to the homothetical properties, it follows also that the tangents in the points I_1, I_2 to the circle $C(O'_1; O'_1A)$ intersect in a point P' located on AI , i.e. AP' contains the simedian (AS) of the triangle I_1AI_2 , noted $\{S\} = AP' \cap I_1I_2$.

In the triangle I_1AI_2 , AM is a median, and AS is simedian, therefore $\sphericalangle I_1AM \equiv I_2AS$; on the other hand, $\sphericalangle BAS \equiv \sphericalangle I_1AI_2$; it follows that $\sphericalangle BAI_1 \equiv I_2AS$, and more: $\sphericalangle I_1AM \equiv BAI_1$, which shows that AI_1 is a bisector in the triangle BAD ; plus, I_1 , being located on the bisector of the angle B , it follows that this point is the center of the circle inscribed in the triangle BAD .

Analogous considerations lead to the conclusion that I_2 is the center of the circle inscribed in the triangle ACD . Because I_1I_2 is parallel to BC , it follows that the rays of the circles inscribed in the triangles ABD and ACD are equal.

Discussion.

The circles $C(O_1; O_1A')$, $C(O'_1; O'_1A)$ are unique; also, the triangle I_1AI_2 is unique; therefore, determined as before, the point D is unique.

Remark.

At the beginning of the *Proof*, we assumed that ABC is a non-isosceles triangle with the stated property. There exist such triangles; we can construct such a triangle starting "backwards". We consider two given congruent external circles and, by tangent constructions, we highlight the ABC triangle.

Open problem.

Given a scalene triangle ABC , could it be triangulated by the cevians AD , AE , with D , E belonging to (BC) , so that the inscribed circles in the triangles ABD , DAE and the EAC to be congruent?

An Application of a Theorem of Orthology

In this short article, we make connections between Problem 21 of [1] and the theory of orthological triangles.

The enunciation of the competition problem is:

Let (T_A) , (T_B) , (T_C) be the tangents in the peaks A, B, C of the triangle ABC to the circle circumscribed to the triangle. Prove that the perpendiculars drawn from the middles of the opposite sides on (T_A) , (T_B) , (T_C) are concurrent and determine their concurrent point.

We formulate and we demonstrate below a sentence containing in its proof the solution of the competition problem in this case.

Proposition.

The tangential triangle and the median triangle of a given triangle ABC are orthological. The orthological centers are O – the center of the circle circumscribed to the triangle ABC , and O_9 – the center of the circle of ABC triangle's 9 points.

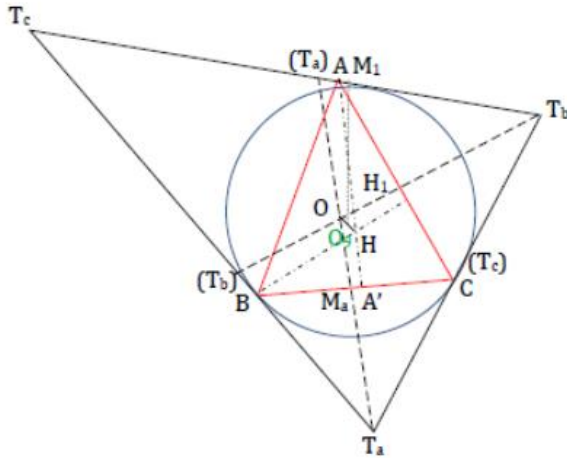


Figure 1.

Proof.

Let $M_a M_b M_c$ be the median triangle of triangle ABC and $T_a T_b T_c$ the tangential triangle of the triangle ABC . It is obvious that the triangle $T_a T_b T_c$ and the triangle ABC are orthological and that O is the orthological center.

Verily, the perpendiculars taken from T_a, T_b, T_c on $BC; CA; AB$ respectively are internal bisectors in the triangle $T_a T_b T_c$ and consequently passing through O , which is center of the circle inscribed in triangle $T_a T_b T_c$. Moreover, $T_a O$ is the mediator of (BC) and accordingly passing through M_a , and $T_a M_c$ is perpendicular on BC , being a mediator, but also on $M_b M_c$ which is a median line.

From orthological triangles theorem, it follows that the perpendiculars taken from M_a, M_b, M_c on T_bT_c, T_cT_a, T_aT_b respectively, are concurrent. The point of concurrency is the second orthological center of triangles $T_aT_bT_c$ and $M_aM_bM_c$. We prove that this point is O_9 - the center of Euler circle of triangle ABC . We take $M_aM_1 \perp T_bT_c$ and denote by $\{H_1\} = M_aM_1 \cap AH$, H being the orthocenter of the triangle ABC . We know that $AH = 2OM_a$; we prove this relation vectorially.

From Sylvester's relation, we have that: $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, but $\overrightarrow{OB} + \overrightarrow{OC} = 2\overrightarrow{OM_a}$; it follows that $\overrightarrow{OH} - \overrightarrow{OA} = 2\overrightarrow{OM_a}$, so $\overrightarrow{AH} = 2\overrightarrow{OM_a}$; changing to module, we have $AH = 2OM_a$. Uniting O to A , we have $OA \perp T_bT_c$, and because $M_aM_1 \perp T_bT_c$ and $AH \parallel OM_a$, it follows that the quadrilateral OM_aH_1A is a parallelogram.

From $AH_1 = OM_a$ and $AH = 2OM_a$ we get that H_1 is the middle of (AH) , so H_1 is situated on the circle of the 9 points of triangle ABC . On this circle, we find as well the points A' - the height foot from A and M_a ; since $\sphericalangle AA'M_a = 90^\circ$, it follows that M_aH_1 is the diameter of Euler circle, therefore the middle of (M_aH_1) , which we denote by O_9 , is the center of Euler's circle; we observe that the quadrilateral H_1HM_aO is as well a parallelogram; it follows that O_9 is the middle of segment $[OH]$.

In conclusion, the perpendicular from M_a on T_bT_c pass through O_9 .

Analogously, we show that the perpendicular taken from M_b on T_aT_c pass through O_9 and consequently O_9 is the enunciated orthological center.

Remark.

The triangles $M_aM_bM_c$ and $T_aT_bT_c$ are homological as well, since T_aM_a , T_bM_b , T_cM_c are concurrent in O, as we already observed, therefore the triangles $T_aT_bT_c$ and $M_aM_bM_c$ are orthohomological of rank I (see [2]).

From P. Sondat theorem (see [4]), it follows that the Euler line OO_9 is perpendicular on the homological axis of the median triangle and of the tangential triangle corresponding to the given triangle ABC .

Note.

(Regarding the triangles that are simultaneously orthological and homological)

In the article *A Theorem about Simultaneous Orthological and Homological Triangles*, by Ion Patrascu and Florentin Smarandache, we stated and proved a theorem which was called by Mihai Dicu in [2] and [3] *The Smarandache-Patrascu Theorem of Orthohomological Triangle*; then, in [4], we used the term *orthohomological triangles* for the triangles that are simultaneously orthological and homological.

The term orthohomological triangles was used by J. Neuberg in *Nouvelles Annales de Mathematiques*

(1885) to name the triangles that are simultaneously orthogonal (one has the sides perpendicular to the sides of the other) and homological.

We suggest that the triangles that are simultaneously orthogonal and homological to be called orthohomological triangles of first rank, and triangles that are simultaneously orthological and homological to be called orthohomological triangles of second rank.

References.

- [1] Titu Andreescu and Dorin Andrica: *360 de probleme de matematică pentru concursuri* [360 Competitional Math Problem], Editura GIL, Zalau, Romania, 2002.
- [2] Mihai Dicu, *The Smarandache-Patrascu Theorem of Orthohomological Triangle*, <http://www.scrib.com/doc/28311880/>, 2010
- [3] Mihai Dicu, *The Smarandache-Patrascu Theorem of Orthohomological Triangle*, in “Revista de matematică ALPHA”, Year XVI, no. 1/2010, Craiova, Romania.
- [4] Ion Patrascu, Florentin Smarandache: *Variance on topics of Plane Geometry*, Educational Publishing, Columbus, Ohio, 2013.
- [5] *Multispace and multistructure neutrosophic transdisciplinarity (100 collected papers of sciences)*, Vol IV, Edited by prof. Florentin Smarandache, Dept. of Mathematics and Sciences, University of New Mexico, USA - North European Scientific Publishers, Hanco, Finland, 2010.

The Dual of a Theorem Relative to the Orthocenter of a Triangle

In [1] we introduced the notion of **Bobillier's transversal relative to a point O in the plane of a triangle ABC** ; we use this notion in what follows.

We transform by duality with respect to a circle $\mathbb{C}(o,r)$ the following theorem relative to the orthocenter of a triangle.

1st Theorem.

If ABC is a nonisosceles triangle, H its orthocenter, and AA_1, BB_1, CC_1 are cevians of a triangle concurrent at point Q different from H , and M, N, P are the intersections of the perpendiculars taken from H on given cevians respectively, with BC, CA, AB , then the points M, N, P are collinear.

Proof.

We note with $\alpha = m(\sphericalangle BAA_1)$; $\beta = m(\sphericalangle CBB_1)$, $\gamma = m(\sphericalangle ACC_1)$, see *Figure 1*.

According to Ceva's theorem, trigonometric form, we have the relation:

$$\frac{\sin \alpha}{\sin(A-\alpha)} \cdot \frac{\sin \beta}{\sin(B-\beta)} \cdot \frac{\sin \gamma}{\sin(C-\gamma)} = 1. \quad (1)$$

We notice that:

$$\frac{MB}{MC} = \frac{\text{Area}(MHB)}{\text{Area}(MHC)} = \frac{MH \cdot HB \cdot \sin(\widehat{MHB})}{MH \cdot HC \cdot \sin(\widehat{MHC})}.$$

Because:

$$\sphericalangle MHB \equiv \sphericalangle A_1AC,$$

as angles of perpendicular sides, it follows that

$$m(\sphericalangle MHB) = m(\widehat{A}) - \alpha.$$

Therewith:

$$m(\sphericalangle MHC) = m(\widehat{MHB}) + m(\widehat{BHC}) = 180^\circ \alpha.$$

We thus get that:

$$\frac{MB}{MC} = \frac{\sin(A - \alpha)}{\sin \alpha} \cdot \frac{HB}{HC}.$$

Analogously, we find that:

$$\frac{NC}{NA} = \frac{\sin(B-\beta)}{\sin \beta} \cdot \frac{HC}{HA};$$

$$\frac{PA}{PB} = \frac{\sin(C-\gamma)}{\sin \gamma} \cdot \frac{HA}{HB}.$$

Applying the reciprocal of Menelaus' theorem, we find, in view of (1), that:

$$\frac{MB}{MC} \cdot \frac{HC}{HA} \cdot \frac{PA}{PB} = 1.$$

This shows that M, N, P are collinear.

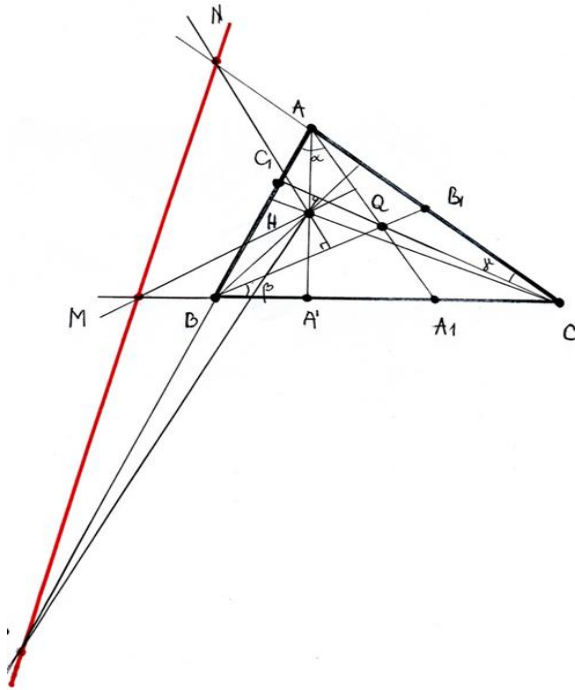


Figure 1.

Note.

1^{st} Theorem is true even if ABC is an obtuse, nonisosceles triangle.

The proof is adapted analogously.

2nd Theorem.

(The Dual of the Theorem 1)

If ABC is a triangle, O a certain point in his plan, and A_1, B_1, C_1 Bobillier's transversals relative to O of

ABC triangle, as well as $A_2 - B_2 - C_2$ a certain transversal in ABC , and the perpendiculars in O , and on OA_2, OB_2, OC_2 respectively, intersect the Bobillier's transversals in the points A_3, B_3, C_3 , then the cevians AA_3, BB_3, CC_3 are concurrent.

Proof.

We convert by duality with respect to a circle $\mathbb{C}(o, r)$ the figure indicated by the statement of this theorem, i.e. *Figure 2*. Let a, b, c be the polars of the points A, B, C with respect to the circle $\mathbb{C}(o, r)$. To the lines BC, CA, AB will correspond their poles $\{A'4 = bnc; \{B'4 = cna; \{C'4 = anb$.

To the points A_1, B_1, C_1 will respectively correspond their polars a_1, b_1, c_1 concurrent in transversal's pole $A_1 - B_1 - C_1$.

Since $OA_1 \perp OA$, it means that the polars a and a_1 are perpendicular, so $a_1 \perp B'C'$, but a_1 pass through A' , which means that Q' contains the height from A' of $A'B'C'$ triangle and similarly b_1 contains the height from B' and c_1 contains the height from C' of $A'B'C'$ triangle.

Consequently, the pole of $A_1 - B_1 - C_1$ transversal is the orthocenter H' of $A'B'C'$ triangle. In the same way, to the points A_2, B_2, C_2 will correspond the polars to a_2, b_2, c_2 which pass respectively through A', B', C' and are concurrent in a point Q' , the pole of the line $A_2 - B_2 - C_2$ with respect to the circle $\mathbb{C}(o, r)$.

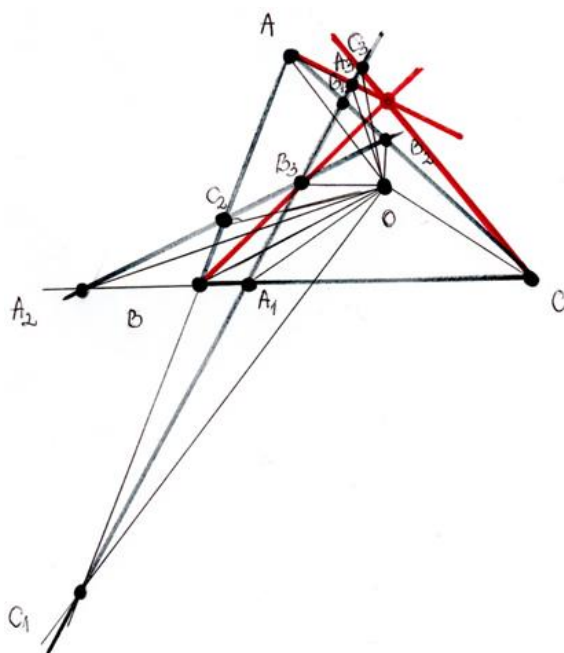


Figure 2.

Given $OA_2 \perp OA_3$, it means that the polars a_2 and a_3 are perpendicular, a_2 correspond to the cevian $A'Q'$, also a_3 passes through the the pole of the transversal $A_1 - B_1 - C_1$, so through H' , in other words Q_3 is perpendicular taken from H' on $A'Q'$; similarly, $b_2 \perp b_3, c_2 \perp c_3$, so b_3 is perpendicular taken from H' on $C'Q'$. To the cevian AA_3 will correspond by duality considered to its pole, which is the intersection of the polars of A and A_3 , i.e. the intersection of lines a and a_3 , namely the intersection of $B'C'$ with the perpendicular taken from H' on $A'Q'$; we denote this

point by M' . Analogously, we get the points N' and P' . Thereby, we got the configuration from *1st Theorem* and *Figure 1*, written for triangle $A'B'C'$ of orthocenter H' . Since from *1st Theorem* we know that M', N', P' are collinear, we get the the cevians AA_3, BB_3, CC_3 are concurrent in the pole of transversal $M' - N' - P'$ with respect to the circle $\mathbb{C}(o, r)$, and *2nd Theorem* is proved.

References.

- [1] Ion Patrascu, Florentin Smarandache: *The Dual Theorem concerning Aubert's Line*, below.
- [2] Florentin Smarandache, Ion Patrascu: *The Geometry of Homological Triangles*. The Education Publisher Inc., Columbus, Ohio, USA - 2012.

The Dual Theorem Concerning Aubert's Line

In this article we introduce the concept of **Bobillier's transversal of a triangle with respect to a point in its plan**; we prove the Aubert's Theorem about the collinearity of the orthocenters in the triangles determined by the sides and the diagonals of a complete quadrilateral, and we obtain the Dual Theorem of this Theorem.

1st Theorem.

(E. Bobillier)

Let ABC be a triangle and M a point in the plane of the triangle so that the perpendiculars taken in M , and MA, MB, MC respectively, intersect the sides BC, CA and AB at Am, Bm și Cm . Then the points Am, Bm and Cm are collinear.

Proof.

We note that $\frac{AmB}{AmC} = \frac{\text{aria}(BMAM)}{\text{aria}(CMAM)}$ (see *Figure 1*).

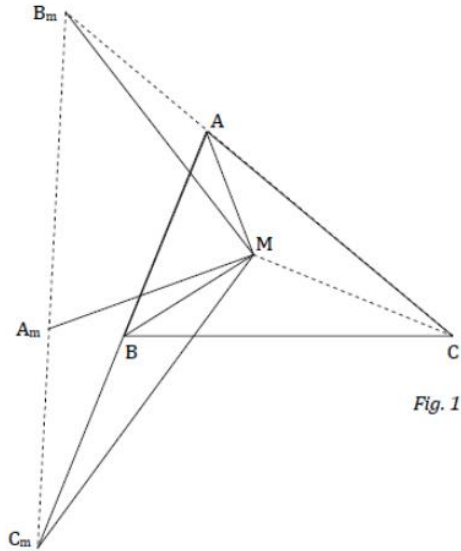


Fig. 1

Figure 1.

$$\text{Area } (BMAm) = \frac{1}{2} \cdot BM \cdot MA_m \cdot \sin(\widehat{BMAm}).$$

$$\text{Area } (CMAm) = \frac{1}{2} \cdot CM \cdot MA_m \cdot \sin(\widehat{CMAm}).$$

Since

$$m(\widehat{CMAm}) = \frac{3\pi}{2} - m(\widehat{AMC}),$$

it explains that

$$\sin(\widehat{CMAm}) = -\cos(\widehat{AMC});$$

$$\sin(\widehat{BMAm}) = \sin\left(\widehat{AMB} - \frac{\pi}{2}\right) = -\cos(\widehat{AMB}).$$

Therefore:

$$\frac{AmB}{AmC} = \frac{MB \cdot \cos(\widehat{AMB})}{MC \cdot \cos(\widehat{AMC})} \quad (1).$$

In the same way, we find that:

$$\frac{BmC}{BmA} = \frac{MC}{MA} \cdot \frac{\cos(\widehat{BMC})}{\cos(\widehat{AMB})} \quad (2);$$

$$\frac{CmA}{CmB} = \frac{MA}{MB} \cdot \frac{\cos(\widehat{AMC})}{\cos(\widehat{BMC})} \quad (3).$$

The relations (1), (2), (3), and the reciprocal Theorem of Menelaus lead to the collinearity of points Am, Bm, Cm .

Note.

Bobillier's Theorem can be obtained – by converting the duality with respect to a circle – from the theorem relative to the concurrency of the heights of a triangle.

1st Definition.

It is called Bobillier's transversal of a triangle ABC with respect to the point M the line containing the intersections of the perpendiculars taken in M on $AM, BM,$ and CM respectively, with sides BC, CA and AB .

Note.

The Bobillier's transversal is not defined for any point M in the plane of the triangle ABC , for example, where M is one of the vertices or the orthocenter H of the triangle.

2nd Definition.

If $ABCD$ is a convex quadrilateral and E, F are the intersections of the lines AB and CD , BC and AD respectively, we say that the figure $ABCDEF$ is a complete quadrilateral. The complete quadrilateral sides are AB, BC, CD, DA , and AC, BD and EF are diagonals.

2nd Theorem.

(Newton-Gauss)

The diagonals' middles of a complete quadrilateral are three collinear points. To prove 2nd theorem, refer to [1].

Note.

It is called *Newton-Gauss Line* of a quadrilateral the line to which the diagonals' middles of a complete quadrilateral belong.

3rd Theorem.

(Aubert)

If $ABCDEF$ is a complete quadrilateral, then the orthocenters H_1, H_2, H_3, H_4 of the triangles ABF , AED , BCE , and CDF respectively, are collinear points.

Proof.

Let A_1, B_1, F_1 be the feet of the heights of the triangle ABF and H_1 its orthocenter (see Fig. 2). Considering the power of the point H_1 relative to the circle circumscribed to the triangle ABF , and given the triangle orthocenter's property according to which its symmetric to the triangle sides belong to the circumscribed circle, we find that:

$$H_1A \cdot H_1A_1 = H_1B \cdot H_1B_1 = H_1F \cdot H_1F_1.$$

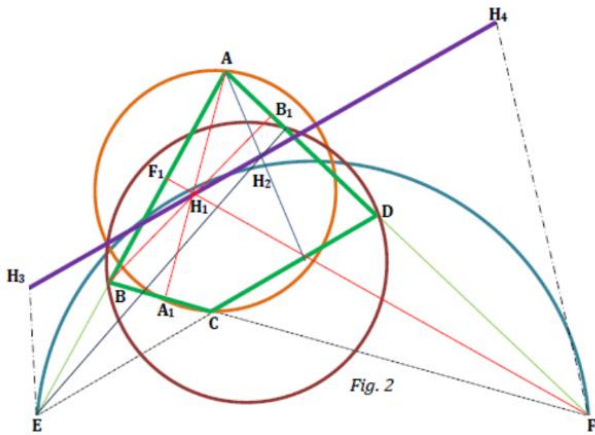


Figure 2.

This relationship shows that the orthocenter H_1 has equal power with respect to the circles of diameters $[AC], [BD], [EF]$. As well, we establish that the orthocenters H_2, H_3, H_4 have equal powers to these circles. Since the circles of diameters $[AC], [BD], [EF]$ have collinear centers (belonging to the Newton-

Gauss line of the $ABCDEF$ quadrilateral), it follows that the points H_1, H_2, H_3, H_4 belong to the radical axis of the circles, and they are, therefore, collinear points.

Notes.

1. It is called *the Aubert Line* or the line of the complete quadrilateral's orthocenters the line to which the orthocenters H_1, H_2, H_3, H_4 belong.

2. The Aubert Line is perpendicular on the Newton-Gauss line of the quadrilateral (radical axis of two circles is perpendicular to their centers' line).

4th Theorem.

(The Dual Theorem of the 3rd Theorem)

If $ABCD$ is a convex quadrilateral and M is a point in its plane for which there are the Bobillier's transversals of triangles ABC , BCD , CDA and DAB ; thereupon these transversals are concurrent.

Proof.

Let us transform the configuration in *Fig. 2*, by duality with respect to a circle of center M .

By the considered duality, the lines a, b, c, d, e și f correspond to the points A, B, C, D, E, F (their polars).

It is known that polars of collinear points are concurrent lines, therefore we have: $a \cap b \cap e = \{A'\}$,

$b \cap c \cap f = \{B'\}$, $c \cap d \cap e = \{C'\}$, $d \cap f \cap a = \{D'\}$, $a \cap c = \{E'\}$, $b \cap d = \{F'\}$.

Consequently, by applicable duality, the points A', B', C', D', E' and F' correspond to the straight lines AB, BC, CD, DA, AC, BD .

To the orthocenter H_1 of the triangle AED , it corresponds, by duality, its polar, which we denote $A'_1 - B'_1 - C'_1$, and which is the Bobillier's transversal of the triangle $A'C'D'$ in relation to the point M . Indeed, the point C' corresponds to the line ED by duality; to the height from A of the triangle AED , also by duality, it corresponds its pole, which is the point C'_1 located on $A'D'$ such that $m(\widehat{C'_1 M C'_1}) = 90^\circ$.

To the height from E of the triangle AED , it corresponds the point $B'_1 \in A'C'$ such that $m(\widehat{D' M B'_1}) = 90^\circ$.

Also, to the height from D , it corresponds $A'_1 \in C'D'$ on C such that $m(\widehat{A' M A'_1}) = 90^\circ$. To the orthocenter H_2 of the triangle ABF , it will correspond, by applicable duality, the Bobillier's transversal $A'_2 - B'_2 - C'_2$ in the triangle $A'B'D'$ relative to the point M . To the orthocenter H_3 of the triangle BCE , it will correspond the Bobillier's transversal $A'_3 - B'_3 - C'_3$ in the triangle $A'B'C'$ relative to the point M , and to the orthocenter H_4 of the triangle CDF , it will correspond the transversal $A'_4 - B'_4 - C'_4$ in the triangle $C'D'B'$ relative to the point M .

The Bobillier's transversals $A'_i - B'_i - C'_i$, $i = \overline{1,4}$ correspond to the collinear points H_i , $i = \overline{1,4}$.

These transversals are concurrent in the pole of the line of the orthocenters towards the considered duality.

It results that, given the quadrilateral $A'B'C'D'$, the Bobillier's transversals of the triangles $A'C'D'$, $A'B'D'$, $A'B'C'$ and $C'D'B'$ relative to the point M are concurrent.

References.

- [1] Florentin Smarandache, Ion Patrascu: *The Geometry of Homological Triangles*. The Education Publisher Inc., Columbus, Ohio, USA - 2012.
- [2] Ion Patrascu, Florentin Smarandache: *Variance on Topics of Plane Geometry*. The Education Publisher Inc., Columbus, Ohio, USA - 2013.

Bibliography

- [1] C. Barbu: Teoreme fundamentale din geometria triunghiului [Fundamental Theorems of Triangle Geometry]. Bacau, Romania: Editura Unique, 2008.
- [2] N. Blaha: Asupra unor cercuri care au ca centre două puncte inverse [On some circles that have as centers two inverse points], in “Gazeta Matematica”, vol. XXXIII, 1927.
- [3] D. Brânzei, M. Miculița. Lucas circles and spheres. In „The Didactics of Mathematics”, volume 9/1994, Cluj-Napoca (Romania), p. 73-80.
- [4] C. Coșniță: Coordonées baricentrique [Barycentric Coordinates]. Bucarest – Paris: Librairie Vuibert, 1941.
- [5] C. Coșniță: Teoreme și probleme alese de matematică [Theorems and Problems], București: Editura de Stat Didactică și Pedagogică, 1958.
- [6] D. Efremov: Noua geometrie a triunghiului [The New Geometry of the Triangle], translation from Russian into Romanian by Mihai Miculița. Zalău, Romania: Cril Publishing House, 2010.

- [7] Roger A. Johnson: Advanced Euclidean Geometry. New York: Dover Publications, 2007.
- [8] T. Lalescu: Geometria triunghiului [The Geometry of the Triangle]. Craiova: Editura Apollo, 1993.
- [9] N. N. Mihaileanu: Lecții complementare de geometrie [Complementary Lessons of Geometry], Editura Didactică și Pedagogică, București, 1976.
- [10] Gh. Mihailescu: Geometria elementelor remarcabile [The Geometry of the Outstanding Elements]. Bucharest: Tehnica Publishing House, 1957.
- [11] Ion Patrascu: Probleme de geometrie plană [Planar Geometry Problems]. Craiova: Editura Cardinal, 1996.
- [12] I. Patrascu: Axe și centre radicale ale cercului adjuncte unui triunghi [Axis and radical centers of the adjoin circle of a triangle], in “Recreații matematice”, year XII, no. 1, 2010.
- [13] Ion Patrascu, Gh. Margineanu: Cercul lui Tucker [Tucker’s Circle], in “Alpha journal”, year XV, no. 2/2009.
- [14] I. Patrascu, F. Smarandache, Variance on Topics of Plane Geometry, Educational Publisher, Ohio, USA, 2013.

- [15] F. Smarandache, I. Patrascu, The Geometry of Homological Triangles. The Education Publisher, Ohio, USA, 2012.
- [16] V. Gh. Voda: Triunghiul – ringul cu trei colțuri [The Triangle-The Ring with Three Corners]. Bucharest: Editura Albatros, 1979.
- [17] Eric W. Weisstein: First Droz-Farny's circle. From Math World – A Wolfram WEB Resurse, <http://mathworld.wolfram.com/>.
- [18] P. Yiu, A. P. Hatzipolakis. The Lucas Circles of a Triangle. In “Amer. Math. Monthly”, no. 108/2001, pp. 444-446. <http://www.math.fau.edu/yiu/monthly437-448.pdf>.

We approach several themes of classical geometry of the circle and complete them with some original results, showing that not everything in traditional math is revealed, and that it still has an open character.

The topics were chosen according to authors' aspiration and attraction, as a poet writes lyrics about spring according to his emotions.

ISBN 978-1-59973-465-1

