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A New Type of Group Action Through the Applications of Fuzzy Sets and Neutrosophic Sets

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Abstract: Fuzzy sets are the most significant tools to handle uncertain data while neutrosophic sets are the generalizations of fuzzy sets in the sense to handle uncertain, incomplete, inconsistent, indeterminate, false data. In this paper, we introduced fuzzy subspaces and neutrosophic subspaces (generalization of fuzzy subspaces) by applying group actions.Further, we define fuzzy transitivity and neutrosophic transitivty in this paper. Fuzzy orbits and neutrosophic orbits are introduced as well. We also studied some basic properties of fuzzy subspaces as well as neutrosophic subspaces.

Key Words: Fuzzy set, neutrosophic set, group action, G-space, fuzzy subspace, neutrosophic subspace.

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§1. Introduction

The theory of fuzzy set was first proposed by Zadeh in the seminal paper [22] in 1965. The concept of fuzzy set is used successfully to modelling uncertain information in several areas of real life. A fuzzy set is defined by a membership function μ with the range in unit interval $[0,1]$. The theory and applications of fuzzy sets and logics have been studied extensively in several aspects in the last few decades such as control, reasoning, pattern recognition, and computer vision etc. The mathematical framework of fuzzy sets become an important area for the research in several phenomenon such as medical diagnosis, engineering, social sciences etc. Literature on fuzzy sets can be seen in a wide range in $[7,24,25,26]$.

The degree of membership of an element in a fuzzy set is single value between 0 and 1. Thus it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there is some kind of hesitation degree. Therefore, in 1986, Atanassov [1] introduced an extension of fuzzy sets called intuitionistic fuzzy set. An intuitionistic fuzzy sets incorporate the hesitation degree called hesitation margin and

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this hesitation margin is defining as 1 minus the sum of membership and non-membership degree. Therefore the intuitionistic fuzzy set is defined by a membership degree μ as well as a non-membership function υ with same range $[0,1]$. The concpet of Intuitionistic fuzzy sets have been applied successfully in several fields such as medical diagnosis, sale analysis, product marketing, financial services, psychological investigations, pattern recognition, machine learning decision making etc.

Smarandache [14] in 1980, introduced a new theory called Neutrosophy, which is basically a branch of philosophy that focus on the origin, nature, and scope of neutralities and their interactions with different ideational spectra. On the basis of neutrosophy, he proposed the concept of neutrosophic set which is characterized by a degree of truth membership T , a degree of indeterminacy membership I and a degree falsehood membership F. A neutrosophic set is powerful mathematical tool which generalizes the concept of classical sets, fuzzy sets [22], intuitionistic fuzzy sets [2], interval valued fuzzy sets [15], paraconsistent sets [14], dialetheist sets [14], paradoxist sets [14], and tautological sets [14]. Neutrosophic sets can handle the indeterminate, imprecise and inconsistent information that exists around our daily life. Wang et al. [17] introduced single valued neutrosophic sets in order to use them easily in scientific and engineering areas that gives an extra possibility to represent uncertain, incomplete, imprecise, and inconsistent information. Hanafy *et.al* further studied the correlation coefficient of neutrosophic sets [5, 6]. Ye [18] defined the correlation coefficient for single valued neutrosophic sets. Broumi and Smaradache conducted study on the correlation coefficient of interval neutrosophic set in [2]. Salama et al. [12] focused on neutrosophic sets and netrosophic topological spaces. Some more literature about neutrosophic set is presented in [4, 8, 10, 11, 13, 16, 19, 20, 23].

The notions of a G-spaces [3] were introduced as a consequence of an action of a group on an ordinary set under certain rulers and conditions. Over the passed history of Mathematics and Algebra, the theory of group action [3] has proven to be an applicable and effective mathematical framework for the study of several types of structures to make connection among them. The applications of group action can be found in different areas of science such as physics, chemistry, biology, computer science, game theory, cryptography etc which has been worked out very well. The abstraction provided by group actions is an important one, because it allows geometrical ideas to be applied to more abstract objects. Several objects and things have found in mathematics which have natural group actions defined on them. Specifically, groups can act on other groups, or even on themselves. Despite this important generalization, the theory of group actions comprise a wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several other fields.

§2. Literature Review and Basic Concepts

Definition 2.1([22]) *Let X be a space of points and let $x \in X$. A fuzzy set A in X is characterized by a membership function μ which is defined by a mapping $\mu : X \to [0,1]$. The fuzzy set can be represented as*

$$A = \{\langle x, \mu(x)\rangle : x \in X\}.$$

Definition 2.2([14]) *Let X be a space of points and let $x \in X$. A neutrosophic set A in X is characterized by a truth membership.*

function T , an indeterminacy membership function I , and a falsity membership function F. T, I, F are real standard or non-standard subsets of $]0^-, 1^+[$, and $T, I, F : X \to]0^-, 1^+[$. The neutrosophic set can be represented as

$$A = \{\langle x, T(x), I(x), F(x)\rangle : x \in X\}.$$

There is no restriction on the sum of T, I, F, so $0^- \leq T + I + F \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+[$. Thus it is necessary to take the interval $[0, 1]$ instead of $]0^-, 1^+[$ for technical applications. It is difficult to apply $]0^-, 1^+[$ in the real life applications such as engineering and scientific problems.

Definition 2.3([3]) *Let Ω be a non empty set and G be a group. Let $\upsilon : \Omega \times G \longrightarrow \Omega$ be a mapping. Then υ is called an action of G on Ω if for all $\omega \in \Omega$ and $g, h \in G$, there are*

(1) $\upsilon(\upsilon(\omega, g), h) = \upsilon(\omega, gh)$

(2) $\upsilon(\omega, 1) = \omega$, *where* 1 *is the identity element in* G.

Usually we write ω^g instead of $\upsilon(\omega, g)$. Therefore (1) and (2) becomes as

(1) $(\omega^g)^h = (\omega)^{gh}$. For all $\omega \in \Omega$ and $g, h \in G$.

(2) $\omega^1 = \omega$.

A set Ω with an action of some group G on it is called a G-space or a G-set. It basically means a triplet (Ω, G, υ).

Definition 2.4([3]) *Let Ω be a G-space and $\Omega_1 \neq \phi$ be a subset of Ω. Then Ω_1 is called a G-subspace of Ω if $\omega^g \in \Omega_1$ for all $\omega \in \Omega_1$ and $g \in G$.*

Definition 2.5([3]) *Let Ω be a G-space. We say that Ω is transitive G-space if for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$.*

§3. Fuzzy Subspace

Definition 3.1 *Let Ω be a G-space. Let $\mu : \Omega \to [0, 1]$ be a mapping. Then μ is called a fuzzy subspace of Ω if $\mu(\omega^g) \geq \mu(\omega)$ and $\mu\left(\omega^{g^{-1}}\right) \leq \mu(\omega)$ for all $\omega \in \Omega$ and $g \in G$.*

Example 3.1 Let $\Omega = (\mathbb{Z}_4, +)$ and $G = \{0, 2\} \leq \mathbb{Z}_4$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. Then Ω is a G-space. We define $\mu : \Omega \to [0, 1]$ by

$$\mu(0) = \frac{1}{2} \text{ and } \mu(1) = \mu(2) = \mu(3) = 1$$

Then clearly μ is a fuzzy subspace of Ω.

Definition 3.2 *Let Ω_μ be a fuzzy subspace of the G-space Ω. Then μ is called transitive fuzzy subspace if for any α, β from Ω, there exist $g \in G$ such that $\mu(\alpha^g) = \mu(\beta)$.*

Example 3.2 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. We define $\mu : \Omega \to [0, 1]$ by

$$\mu(0) = \frac{1}{2} \text{ and } \mu(1) = \mu(2) = \mu(3) = 1$$

Then clearly μ is a transitive fuzzy subspace of Ω.

Theorem 3.1 *If Ω is transitive G-space, then μ is also transitive fuzzy subspace.*

Proof Suppose that Ω is transitive G-space. Then for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$. This by taking μ on both sides, we get $\mu(\alpha^g) = \mu(\beta)$ for all $\alpha, \beta \in \Omega$. Hence by definition μ is a transitive fuzzy subspace of Ω. □

Definition 3.3 *A transitive fuzzy subspace of Ω is called fuzzy orbit.*

Example 3.3 Consider above Example, clearly μ is a fuzzy orbit of Ω.

Theorem 3.2 *Every fuzzy orbit is trivially a fuzzy subspace but the converse may not be true.*

For converse, see the following Example.

Example 3.4 Let $\Omega = S_3 = \{e, y, x, x^2, xy, x^2y\}$ and $G = \{e, y\} \leq S_3$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\rho^\sigma = \rho\sigma$ for all $\rho \in \Omega$ and $\sigma \in G$. Then clearly Ω is a G-space. Let $\mu : \Omega \to [0, 1]$ be defined as $\mu(e) = \mu(y) = \mu(x) = \mu(x^2) = \mu(xy) = \mu(x^2y) = \frac{2}{5}$. Thus μ is a fuzzy subspace of Ω but μ is not a transitive fuzzy subspace of Ω as μ has the following fuzzy orbits:

$$\begin{aligned} \mu_1 &= \left\{\mu(e) = \mu(y) = \frac{2}{5}\right\}, \\ \mu_2 &= \left\{\mu(x) = \mu(x^2) = \frac{2}{5}\right\}, \\ \mu_3 &= \left\{\mu(xy) = \mu(x^2y) = \frac{2}{5}\right\}. \end{aligned}$$

Definition 3.4 *Let Ω be a G-space and Ω_μ be a fuzzy subspace. Let $\alpha \in \Omega$. The fuzzy stabilizer is denoted by $G_{\mu(\alpha)}$ and is defined to be $G_{\mu(\alpha)} = \{g \in G : \mu(\alpha^g) = \mu(\alpha)\}$.*

Example 3.5 Consider the above Example. Then

$$G_{\mu(e)} = G_{\mu(y)} = G_{\mu(x)} = G_{\mu(x^2)} = G_{\mu(xy)} = G_{\mu(x^2y)} = \{e\}.$$

Theorem 3.3 *If G_α is G-stabilizer, then $G_{\mu(\alpha)}$ is a fuzzy stabilizer.*

Theorem 3.4 *Let $G_{\mu(\alpha)}$ be a fuzzy stabilizer. Then $G_{\mu(\alpha)} \leq G_\alpha$.*

Remark 3.1 *Let $G_{\mu(\alpha)}$ be a fuzzy stabilizer. Then $G_{\mu(\alpha)} \leq G$.*

§4. Neutrosophic Subspaces

Definition 4.1 *Let Ω be a G-space. Let $A : \Omega \to [0,1]^3$ be a mapping. Then A is called a neutrosophic subspace of Ω if The following conditions are hold.*

(1) $T(\omega^g) \geq T(\omega)$ and $T\left(\omega^{g^{-1}}\right) \leq T(\omega)$,

(2) $I(\omega^g) \leq I(\omega)$ and $I\left(\omega^{g^{-1}}\right) \geq I(\omega)$ and

(3) $F(\omega^g) \leq F(\omega)$ and $F\left(\omega^{g^{-1}}\right) \geq F(\omega)$ for all $\omega \in \Omega$ and $g \in G$.

Example 4.1 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω which is defined by $\omega^g = \omega + g$. Then Ω is a G-space under this action of G. Let $A : \Omega \to [0,1]^3$ be a mapping which is defined by

$$T(0) = 0.5, T(1) = T(2) = T(3) = 1,$$

$$I(0) = 0.3 \text{ and } I(1) = I(2) = I(3) = 0.1,$$

and

$$F(0) = 0.4 \text{ and } F(1) = F(2) = F(3) = 0.2.$$

Thus clearly A is a neutrosophic subspace as A satisfies conditions (1), (2) and (3).

Theorem 4.1 *A neutrosophic subspace is trivially the generalization of fuzzy subspace.*

Definition 4.2 *Let A be a neutrosophic subspace of the G-space Ω. Then A is called fuzzy transitive subspace if for any α, β from Ω, there exist $g \in G$ such that*

$$\begin{aligned} F(\alpha^g) &= F(\beta), \\ F(\alpha^g) &= F(\beta), \\ F(\alpha^g) &= F(\beta). \end{aligned}$$

Example 4.2 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. We define $A : \Omega \to [0,1]^3$ by

$$\begin{aligned} T(0) &= \frac{1}{2} \text{ and } T(1) = T(2) = T(3) = 1, \\ I(0) &= \frac{1}{3} \text{ and } I(1) = I(2) = I(3) = 1, \\ F(0) &= \frac{1}{4} \text{ and } F(1) = F(2) = F(3) = 1. \end{aligned}$$

Then clearly A is a neutrosophic transitive subspace of Ω.

Theorem 4.2 *If Ω is transitive G-space, then A is also neutrosophic transitive subspace.*

Proof Suppose that Ω is transitive G-space. Then for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$. This by taking T on both sides, we get $T(\alpha^g) = T(\beta)$ for all $\alpha, \beta \in \Omega$. Similarly, we can prove it for the other two components I and F. Hence by definition A is a neutrosophic transitive subspace of Ω. □

Definition 4.3 *A neutrosophic transitive subspace of* Ω *is called neutrosophic orbit.*

Example 4.3 Consider above Example 4.2, clearly A is a neutrosophic orbit of Ω.

Theorem 4.3 *All neutrosophic orbits are trivially the generalization of fuzzy orbits.*

Theorem 4.4 *Every neutrosophic orbit is trivially a neutrosophic subspace but the converse may not be true.*

For converse, see the following Example.

Example 4.4 Let $\Omega = S_3 = \{e, y, x, x^2, xy, x^2y\}$ and $G = \{e, y\} \leq S_3$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\rho^\sigma = \rho\sigma$ for all $\rho \in \Omega$ and $\sigma \in G$. Then clearly Ω is a G-space. Let $A : \Omega \to [0, 1]$ be defined as

$$\begin{aligned} T(e) &= T(y) = T(x) = T\left(x^2\right) = T(xy) = T\left(x^2y\right) = \frac{2}{5}, \\ I(e) &= I(y) = I(x) = I\left(x^2\right) = I(xy) = I\left(x^2y\right) = \frac{3}{7}, \\ F(e) &= F(y) = F(x) = F\left(x^2\right) = F(xy) = F\left(x^2y\right) = \frac{4}{9}. \end{aligned}$$

Thus A is a neutrosophic subspace of Ω but A is not a neutrosophic transitive subspace of Ω as A has the following neutrosophic orbits:

$$\begin{aligned} T_1 &= \left\{\mu(e) = \mu(y) = \frac{2}{5}\right\}, I_1 = \left\{I(e) = I(y) = \frac{3}{7}\right\}, F_1 = \left\{F(e) = F(y) = \frac{4}{9}\right\}, \\ \mu_2 &= \left\{\mu(x) = \mu\left(x^2\right) = \frac{2}{5}\right\}, I_2 = \left\{I(x) = I\left(x^2\right) = \frac{3}{7}\right\}, F_2 = \left\{F(x) = F\left(x^2\right) = \frac{4}{9}\right\}, \\ \mu_3 &= \left\{\mu(xy) = \mu\left(x^2y\right) = \frac{2}{5}\right\}, I_3 = \left\{I(xy) = I\left(x^2y\right) = \frac{3}{7}\right\}, F_3 = \left\{F(xy) = F\left(x^2y\right) = \frac{4}{9}\right\}. \end{aligned}$$

Definition 4.4 *Let* Ω *be a* G*-space and* A *be a neutrosophic subspace. Let* $\alpha \in \Omega$. *The neutrosophic stablizer is denoted by* $G_{A(\alpha)}$ *and is defined to be*

$$G_{A(\alpha)} = \{g \in G : T(\alpha^g) = T(\alpha), I(\alpha^g) = I(\alpha), F(\alpha^g) = F(\alpha)\}.$$

Example 4.5 Consider the above Example 4.4. Then

$$G_{A(e)} = G_{A(y)} = G_{A(x)} = G_{A(x^2)} = G_{A(xy)} = G_{A(x^2y)} = \{e\}.$$

Theorem 4.5 *If* G_α *is* G*-stabilizer, then* $G_{A(\alpha)}$ *is a neutrosophic stabilizer.*

Theorem 4.6 *Every neutrosophic stabilizer is a generalization of fuzzy stabilizer.*

Theorem 4.7 *Let $G_{A(\alpha)}$ be a neutrosophic stabilizer. Then $G_{A(\alpha)} \leq G_{\alpha}$.*

Remark 4.1 *Let $G_{A(\alpha)}$ be a neutrosophic stabilizer. Then $G_{A(\alpha)} \leq G$.*

§5. Conclusion

In this paper, we introduced fuzzy subspaces and neutrosophic subspaces (generalization of fuzzy subspaces) by applying group actions.Further, we define fuzzy transitivity and neutrosophic transitivity in this paper. Fuzzy orbits and neutrosophic orbits are introduced as well. We also studied some basic properties of fuzzy subspaces as well as neutrosophic subspaces. In the near future, we are applying these concepts in the field of physics, chemistry and other related fields to find the uncertainty in symmetries.

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