The Impossible is Possible! Squaring the Circle and Doubling the Cube in Euclidean Space-Time*

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Abstract

Squaring the Circle is a famous geometry problem going all the way back to the ancient Greeks. It is the great quest of constructing a square with the same area as a circle using a compass and straightedge in a finite number of steps. Since it was proved that π was a transcendental number in 1882, the task of Squaring the Circle has been considered impossible. Here, we will show it is possible to Square the Circle in Euclidean space-time. It is not possible to Square the Circle in Euclidean space alone, but it is fully possible in Euclidean space-time, and after all we live in a world with not only space, but also time. By drawing the circle from one reference frame and drawing the square from another reference frame, we can indeed Square the Circle. By taking into account space-time rather than just space the Impossible is possible! However, it is not enough simply to understand math in order to Square the Circle, one must understand some "basic" space-time physics as well. As a bonus we have added a solution to the impossibility of Doubling the Cube. As a double bonus we also have also boxed the sphere! As one will see one can claim we simply have bent the rules and moved a problem from one place to another. One of the main essences of this paper is that we can move challenging space problems out from space and into time, and vice versa.

Key words: Squaring the circle, Einstein special relativity theory, Euclidian space-time, length contraction, length transformation, relativity of simultaneity.

1 Introduction

Before Lindemann (1882) proved that π was a transcendental number[1] there was a long series of attempts to square the circle. Hobson (1913) did a thorough job in reviewing and describing the long and interesting history of squaring the circle, and he concluded:

> *It has thus been proved that π is a transcendental number...the impossibility of "squaring the circle" has been effectively established. – Ernest Hobson*

To get an idea of the impossibility of squaring the circle consider that we make a circle with radius $r = 1$, then the area of the circle must be $\pi r^2 = \pi$. To get a square with area π the length of each side must be $\sqrt{\pi}$. To construct a square with sides exactly $\sqrt{\pi}$ is impossible with only a compass and a straightedge in a finite number of steps.

In more recent times there have been a few claims of Squaring the Circle for certain non-Euclidean spaces such as the hyperbolic plane, also known as Bolyai–Lobachevskian geometry, see Jagy (1995) and

*This is version 1.5. Version 1.0 was made public available on www.ViXra.org Friday 19 February 2016. An extra solution was added in version 1.1. In version 1.2 I added a solution to Doubling the Cube. In version 1.3. I added a solution to Boxing the Sphere. In version 1.4 several smaller improvements and a section summarizing the essence of the paper was added. In version 1.5 the platonic solids and generalization introduced.

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[1] Hermes (1873) had just years before proved that e was a transcendental number.

Greenberg (2008). Still these "claims" have been overoptimistic. For example, there are no squares as such in the hyperbolic plane.

One cannot square the circle in Euclidean space alone, however as we will prove: *One can Square the Circle in Euclidean Space-Time* (at least hypothetically), and we are clearly living in Euclidean space-time and not only in space. Once we take into account how observations of time and distance are affected by motion then it surprisingly becomes possible to square the circle.

Before the late 19th century no one had figured out that the length of an object or the distance traveled or even time itself would be affected by how fast we moved. Interestingly, just a few years after it was proven that π was transcendental we got a break through in understanding that distances and time were affected by motion. FitzGerald (1889) was the first to suggest that the null result of the Michelson–Morley speed of light experiment possibly could be explained by assuming that the length of any material object (including the earth itself) contracts along the direction in which it is moving through the ether, or as explained in his own words:

> *I would suggest that almost the only hypothesis that can reconcile this opposition is that the length of the material bodies changes according as they are moving through the ether or across it, by an amount depending on the square of the ratio of their velocity to that of light.*
> – FitzGerald, May 1889

Lorentz (1892) mathematically formalizes length contraction, suggesting that objects and any type of matter that travels against the ether has to contract by $\sqrt{1-\frac{v^2}{c^2}}$, where v is the speed of the object against the ether and c is the well-known experimentally tested speed of light.[2]

Larmor (1900) added time-dilation[3] to the FitzGerald and Lorentz length contraction and was the first to develop a mathematical theory that is fully consistent with the null result of the Michelson-Morley result. Bear in mind that FitzGerald, Lorentz and Larmor all still assumed the ether existed and it was originally to "save" the ether that they introduced length contraction and time dilation. Even the famous mathematician and physicist Henry Poincaré believed in the presence of the ether. Still, in 1905 Poincaré concluded that it would be impossible to ever detect the earth's motion against the ether. Henry Poincaré therefore suggested to synchronize clocks a distance apart using the "assumption" that the one-way speed of light for synchronization purpose was the same as the well-tested round-trip speed of light. Einstein instead abandoned the ether and assumed that the true one-way speed of light was the same as the round-trip speed of light and used this assumption to synchronize his clocks.

Bear in mind that FitzGerald and original Lorentz length contraction is actually not mathematically the same as Einstein (1905, 1916) length contraction, this even if it looks mathematically identical at first sight. For example, Patheria (1974) points out correctly that there is a major difference between FitzGerald and Lorentz transformation on one side and Einstein length transformation on the other side:

> *It must be pointed out here that the contraction hypothesis, put forward by FitzGerald and Lorentz, was of entirely different character and must not be confused with the effect obtained here [Einstein length contraction]. That hypothesis did not refer to a mutually reciprocal effect; it is rather suggesting a contraction in the absolute sense, arising from the motion of an object with respect to the aether or, so to say, from 'absolute' motion. According to a relativistic standpoint, neither absolute motion nor any effect accruing therefrom has any physical meaning.*[4]

After Lorentz became heavily influenced by the view of Poincarè he seems to have changed his own view that motion against the ether likely not could be detected, so we should probably look at the speed v in the Poincaré (1905) adjusted Lorentz transformation[5] as the relative speed between frames rather than the speed against the ether, even though Lorentz not is very clear on this point in 1904. With this in mind, it is not incorrect to use the term *Lorentz contraction* in the Einstein theory for length contraction as many physicists do today. Personally I like to distinguish between FitzGerald and the (original) Lorentz contraction: on the one hand where the speed v represents the velocity against the ether, and on the other hand in the Einstein theory, where v represents the relative velocity as measured with Einstein-Poincarè synchronized clocks. Even if several different relativity theories exist we will concentrate on Squaring the Circle inside Einstein's special relativity theory here, that is inside what we can call Euclidian Einstein space-time.

[2] More precisely the well tested round-trip speed of light.

[3] Time dilation has been proven in a series of experiments, see for example Haefele (1970), Haefele and Keating (1971b,a) and Bailey and et al. (1977).

[4] See Patheria (1974) page 41.

[5] Which is the one referred to when physicists today talk about the Lorentz transformation.

2 Squaring the Circle in Space-Time

We will show that we can draw a square with area π without relying directly on π, but only indirectly on π. Assume a train platform (embankment) and a train. We first draw a circle on the embankment using only a compass, see figure 1 upper panel. We can call the radius of the circle one, that is $r = 1$. It could be one cm, one inch, one meter or whatever radius we prefer. Next we mark a straightedge with the compass so we have a length equal to the radius of the square. Next we build a perfect square based on this length. In other words, we have constructed a one by one square, also known as a unit square. This square should be built in a solid material that we can transport.

Next we will move the square on board a train. This train is currently at rest relative to the embankment. Then we accelerate this train to a very fast velocity relative to the embankment; we will get back to exactly what velocity later. Naturally, we could just as well have constructed the square on the train while the train was standing still relative to the embankment or later while the train was moving relative to the embankment. What is important is that we build the unit square (side length equal to the radius of the circle) in and from the frame it is at rest in.

So far we have "only" worked in space, now we also need to work a little in time. At each corner of the square we mount a clock. Next we will synchronize these clocks using the Einstein clock synchronizing procedure. That is to say we are synchronizing the clocks with light signals assuming that the one-way speed of light is isotropic and the same as the well-tested round-trip speed of light.

Next we hang the square with the clocks out on the side of the train. At each clock we have connected a laser. At a given point in time, each clock will simultaneously trigger the lasers. Bear in mind the lasers are fired simultaneously as observed from the train. Einstein's relativity of simultaneity means that the lasers on the train are not fired simultaneously as observed from the embankment. The embankment is covered with photosensitive paper. The lasers (photons) will hit the embankment and make a mark for each corner. See figure 1 middle and lower panels.

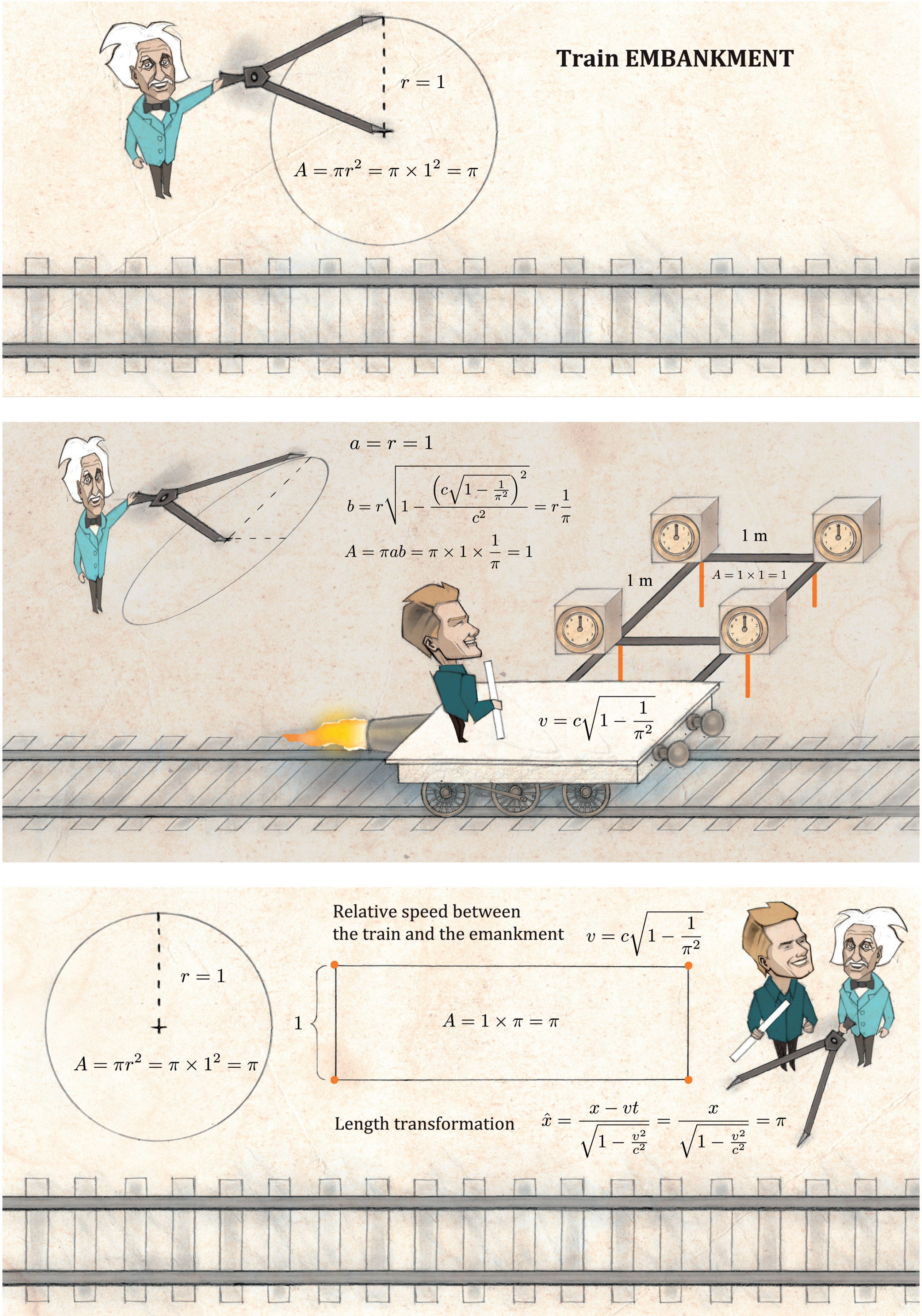
There is no length contraction in the transverse direction, so the distance between the two laser marks on the embankment in the transverse direction on the embankment must be one (for example 1 meter). However in the parallel direction we will have length contraction. The distance between the marks on the embankment will appear contracted from the train. Remember we are going to try to Square the Circle that is to make the area of the square equal to the circle. The area of the circle is π. The transverse length of the square is one. What is the speed of a train (second frame) that will give the sides of π in the embankment, but only 1 from the frame we draw the sides from (the train)? We must have a length contraction factor β equal to $\frac{1}{\pi}$ to accomplish this since $\pi \times \frac{1}{\pi} = 1$. This means we get the following equation to solve based on special relativity theory. Assume that $\bar{v}$ is the percentage of the speed of light c that the two frames are traveling relative to each other. That is $\bar{v} = \frac{v}{c}$. Further assume $y = \bar{v}^2$, then we must have

$$\begin{aligned}
\frac{1}{\pi} &= \beta \\
\frac{1}{\pi} &= \sqrt{1 - \frac{v^2}{c^2}} \\
\frac{1}{\pi} &= \sqrt{1 - \frac{\frac{v^2}{c^2}c^2}{c^2}} \\
\frac{1}{\pi} &= \sqrt{1 - \bar{v}^2} \\
\frac{1}{\pi} &= \sqrt{1 - \bar{v}^2} \\
\frac{1}{\pi} &= \sqrt{1 - y} \\
\left(\frac{1}{\pi}\right)^4 &= y^2 - 2y + 1 \\
\frac{1}{\pi^4} &= y^2 - 2y + 1 \\
y^2 - 2y + 1 - \frac{1}{\pi^4} &= 0. \qquad (1)
\end{aligned}$$

This is a quadratic equation with the following solution

$$y = \frac{2 \pm \sqrt{2^2 - 4\left(1 - \frac{1}{\pi^4}\right)}}{2}$$

Figure 1: Squaring the Circle.



$$\begin{aligned} y &= \frac{2 \pm \sqrt{4 - 4\left(1 - \frac{1}{\pi^4}\right)}}{2} \\ y &= 1 \pm \sqrt{\frac{1}{\pi^4}} \\ y &= 1 \pm \frac{1}{\pi^2} \end{aligned} \tag{2}$$

Remember $y = \bar{v}^2$, this means

$$\begin{aligned} \bar{v} = &= \sqrt{y} \\ \bar{v} = &= \sqrt{1 \pm \frac{1}{\pi^2}} \end{aligned} \tag{3}$$

as we must have $\bar{v} \leq 1$ only the negative solution makes sense and we get

$$\bar{v} = \sqrt{1 - \frac{1}{\pi^2}} \tag{4}$$

and this means we must travel at the following speed relative to the frame where the circle and the square is at rest when drawing (or observing) the square to Square the Circle[6]

$$v = c\sqrt{1 - \frac{1}{\pi^2}} \tag{5}$$

and we have

$$\begin{aligned} 1 &= \pi\beta \\ 1 &= \pi \times \sqrt{1 - \frac{\left(c\sqrt{1 - \frac{1}{\pi^2}}\right)^2}{c^2}} \\ 1 &= \pi\frac{1}{\pi} \end{aligned} \tag{6}$$

This means that when we have two frames traveling at a relative speed[7] of $v = c\sqrt{1 - \frac{1}{\pi^2}}$ we can indeed Square the Circle. There is nothing wrong with drawing the square in one frame and then "transferring" it to another frame. Some people might claim that this is bending the rules. This is partly true, but there were no rules about how fast we could move the pen, or if we should calculate the area as observed from the square itself, or from the moving pen (train) that is drawing it. We will return to some self-criticism a bit later in this paper. And as we soon will see this solution contains two solutions, including a simpler solution that we will discuss.

In Einstein's special relativity theory, length contraction is reciprocal. So we could just as well have drawn the circle on board of the train and then we could draw the square[8] in the train from the embankment. The situation would be symmetrical. The velocity of the train relative to the embankment is the same as the velocity of the embankment relative to the train as long as they are measured with Einstein-Poincarè synchronized clocks.

3 Checking the area

Again the circle is drawn on the embankment (or on the train) and the square is drawn on the train and then transferred to the embankment from the train, or visa versa. The circle is drawn with radius $r = 1$ as observed from the embankment frame, and the square is initially drawn with sides 1 by 1 from the frame it is at rest in.

The observer on the embankment sees a circle with radius one and a rectangle with sides 1 by π, both at rest on the embankment. The area of the circle is π and the area of the rectangle is π. The observer on the train can check the areas. The train observer sees a perfect square (a unit square) with area 1, the square is at rest with respect to the train. The circle (that is at rest on the embankment) is observed as an ellipse by the train observer. Figure 2 illustrates how the two different frames observe the circle (ellipse) and the square (rectangle).

[6]The square is not a square from the rest frame, but from the observer frame. We are constructing from the observer frame.
[7]As measured with Einstein-Poincaré synchronized clocks.
[8]Or at least the side lines for the square.

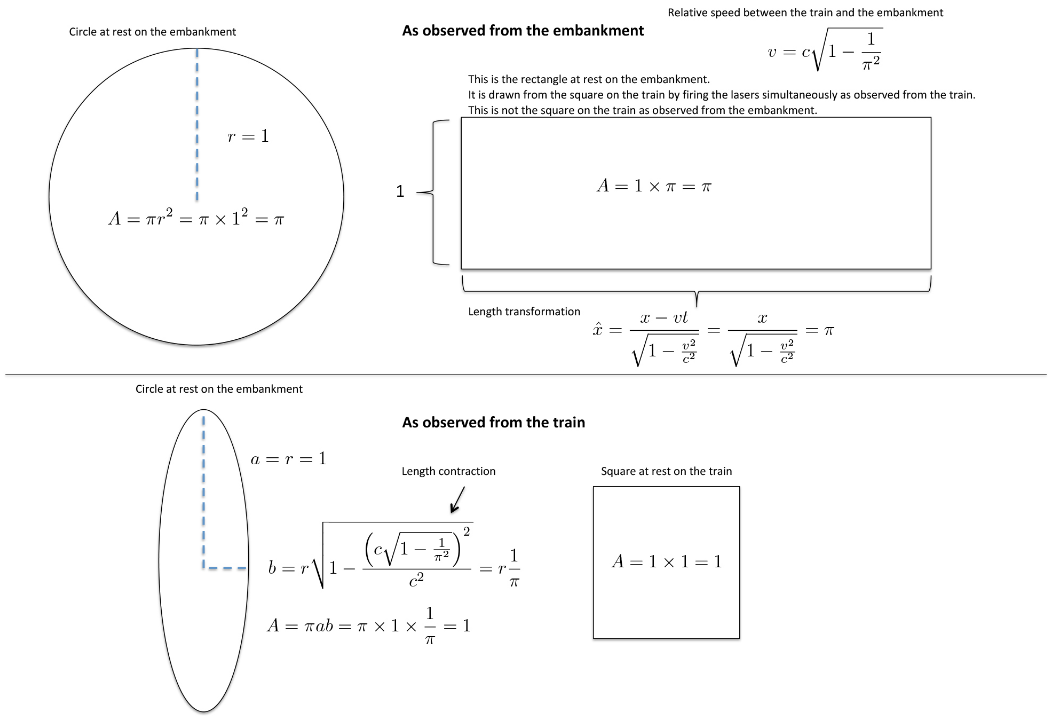


Figure 2: Comparison between embankment frame and the train frame.

As there is no length contraction in the transverse direction, the ellipse semi-major as observed from the train is the same as the radius of the circle as observed from the embankment, that is r. The semi-minor axis on the ellipse is contracted as observed from the train and must be

$$r\sqrt{1 - \frac{\left(c\sqrt{1 - \frac{1}{\pi^2}}\right)^2}{c^2}} = r\frac{1}{\pi}. \tag{7}$$

And if we have chosen radius $r = 1$ this gives a semi-minor axis of $\frac{1}{\pi}$. The area of an ellipse is given by $A_{ellipse} = \pi ab$, where a and b are the semi-major and the semi-minor axes respectively. This gives us an area of the ellipse of $A_{ellipse} = \pi \times 1 \times \frac{1}{\pi} = 1$. This is the same as the area of the square on board the train as observed from the train.

Bear in mind that in the solution as observed from the train, we do not need to have any clocks mounted to the square to make marks on the embankment. Here we simply draw a unit circle on the embankment (in one frame) as observed from the embankment and then we draw a unit square on board the train from the train, for example on the floor of the train. The square can be drawn on board the train while we are standing still relative to the embankment, or later while we are moving relative to the embankment. Now we simply observe the circle on the embankment from the train. The circle will appear as an ellipse and the area of the ellipse and the square are the same: they are one. A drawn unit circle has the same area as a drawn unit square, quite remarkable indeed.

Both the train observer and the embankment observer can agree that the area of the circle and the square are identical. However, the embankment observer will claim that the square is a rectangle and the train observer will claim that the circle is an ellipse. Still, the circle was drawn and observed as a circle by the drawer and the square was drawn and observed as a square by the drawer. We have indeed Squared the Circle! And this seems to be the “only“ way to square a circle in Euclidian Einstein space-time.

4 How can this be?

Einstein‘s special relativity theory predicts reciprocal length contraction. Assume two identical meter sticks are made in the same reference frame $L = 1$. Next one of the meter sticks is carried on board the train. The train is accelerated to a speed of $c\sqrt{1 - \frac{1}{\pi^2}}$. Now from the train the meter stick at rest on the embankment will be observed to have length contracted and have a length of: $1 \times \sqrt{1 - \frac{\left(c\sqrt{1 - \frac{1}{\pi^2}}\right)^2}{c^2}} \approx$

0.3183. At the same time the meter stick at rest in the train will be observed to have length ≈ 0.3183 as observed from the embankment. In Einstein‘s special relativity theory, length contraction is reciprocal. Still, in the section above we claimed that the laser signals sent out simultaneously (simultaneously as observed from the train) will make marks on the embankment that have a distance between them of π meters. How can a one meter stick in the train that is observed to be contracted from the embankment actually make marks on the embankment that are π meters apart?

The signals sent out simultaneously from each end of the rod (the rods making up the unit square) at rest in the train will not be observed to be sent out simultaneously from the embankment. Events happening simultaneously on the train as observed with Einstein synchronized clocks will from the embankment have an observed time difference of

$$\frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}}. \tag{8}$$

This is well-known from Einstein‘s theory as the relativity of simultaneity.[9] Where L is the length of the rod as observed from the frame it is at rest in, for example 1 meter, the distance between the marks on the embankment from the lasers sent out from the two ends of rod simultaneously as measured from the train will make marks at the embankment with the following distance apart as measured from the embankment

$$L\sqrt{1-\frac{v^2}{c^2}}+\frac{Lv}{c^2\sqrt{1-\frac{v^2}{c^2}}}v=\frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \tag{9}$$

and since $v=c\sqrt{1-\frac{1}{\pi^2}}$ we get

$$\frac{L}{\sqrt{1-\frac{c^2\left(1-\frac{1}{\pi^2}\right)}{c^2}}}=L\pi \tag{10}$$

and if $L=1$ meter then the length between the two marks on the embankment between the square sides parallel to the railroad will be π meters. This is just another way to check that our results in the previous section are correct and consistent with Einstein‘s special relativity theory. That we can Square the Circle in Euclidian Einstein space-time is also strongly related to how clocks are synchronized in special relativity theory: the clocks are Einstein (Poincarè) synchronized. We could also have found this directly from the Lorentz transformation:

$$\hat{x}=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}.$$

In the train frame the lasers (actually 4 if one includes each corner) are fired simultaneously as observed from Einstein synchronized clocks in the frame where the lasers-clocks are at rest relative to each other. This means the time between the clocks fired as observed from this frame must be $t=0$. When the relative speed between the two frames as measured with Einstein synchronized clocks is $v=c\sqrt{1-\frac{1}{\pi^2}}$ we get the following length transformation:

$$\hat{x}=\frac{x-c\sqrt{1-\frac{1}{\pi^2}}\times 0}{\sqrt{1-\frac{\left(c\sqrt{1-\frac{1}{\pi^2}}\right)^2}{c^2}}}=x\pi \tag{11}$$

where x is the distance between two events in the rest frame, in this case the distance between the lasers on the straightened on the train, that is $x=L$. Further $\hat{x}$ is the distance between these points plus the distance the train traveled in the time difference between these two lasers firing as measured from the other frame. The length transformation takes into account length contraction and relativity of simultaneity.

We could also have Squared the Circle using other relativity theories, such as the ether theory of Joseph Larmor from 1900. One of the main differences would be that the Squaring of the Circle would not be reciprocal between the frames then, see Haug (2014) for detailed discussion on this topic. In this article we will limit ourselves to squaring the circle under Einstein‘s special relativity theory, which involves Einstein synchronization of clocks. However, this is more than just a theory; when using Einstein synchronized clocks these predictions are actually aligned with how we would observe the world in relation to squaring the circle.

[9]Equation 8 is well known from the literature, see for example Comstock (1910), Carmichael (1913), Dingle (1940), Bohem (1965) and Krane (2012). The formula follows directly from the Lorentz transformation.

5 Summary of procedure

In this section we shortly summarize the procedure for Squaring the Circle:

Solution one:

1. We are drawing a unit circle on the embankment (reference frame one) with a compass.
2. Without changing the compass, we construct a square where each side has a radius equal to the radius of the circle. The square could be drawn on the floor of the train. The train is currently at rest relative to the embankment.
3. Accelerate the train to a speed relative to the embankment of $v = c\sqrt{1 - \frac{1}{\pi^2}}$ as measured from both the embankment and the train.[10] The relative speed between the reference frames is in Einstein's special relativity theory reciprocal.
4. The circle on the embankment will now be observed as an ellipse with the same area as the square from the train. In this case, both the ellipse (that was drawn as a circle) and the square would both have area 1 (rather than π) as observed from the train. In other words we have squared the circle.

Solution two:

1. Draw a circle on the embankment (reference frame one) with a compass. Choose any radius and call this radius one (one meter, one foot, one cm or any other length). The radius is the distance between the two points of the compass.
2. Without changing the compass, construct a square where each side has a radius equal to the radius of the circle. The square should be made of a robust material so we can move the square.
3. Move the square on board a train that is currently at rest relative to the embankment; alternatively we could have constructed the square directly on board the train while the train is standing still or while it is moving.
4. Accelerate the train to a velocity relative to the embankment of $v = c\sqrt{1 - \frac{1}{\pi^2}}$ as measured from both the embankment and the train.[11] The relative speed between the reference frames is in Einstein's special relativity theory reciprocal.
5. Mount a clock to each corner of the square. Synchronize these clocks using Einstein-Poincaré synchronization while the train is moving at velocity $v = c\sqrt{1 - \frac{1}{\pi^2}}$ relative to the embankment.
6. Simultaneously, as measured by the clocks in each corner of the perfect square on the train, fire the lasers. These laser signals will burn four dots on the ground. These dots will mark the corners of a rectangle as observed from the embankment. The length of the sides of the rectangle parallel to the railroad will be related to length transformation rather then length contraction. If the signals arrived simultaneously as observed from the embankment, then it would be length contraction rather than length transformation. The square on the train is indeed observed as length-contracted from the embankment, but not the transferred square (rectangle)
7. Measure the area of the circle and the square and they will have the same area. From the train the circle is observed to have an elliptical shape and the square is a unit square. From the embankment the circle is a circle and the square is a rectangle. The areas of the circle and the square, both as observed from the train, are the same, namely one. The area of the circle and the transferred square (the rectangle on the embankment) have area π as observed from the embankment. In other words we have squared the circle.

Solution two also contains solution one. Solution one is the simplest as it does not need the clocks and the lasers in each corner of the square. We could also have done this the other way around. That is to draw the unit circle onboard the train and to draw the unit square on the embankment. The result would be the same as above.

6 More general solution

The solution above only holds between a unit circle and a unit square. Here we will see if there is a more general solution. Again assume that we have drawn a unit circle, then what is the limitation we have on the length of the sides of the square and what is the velocity we need to travel at when transferring this

[10] This speed is as measured with Einstein synchronized clocks.
[11] This speed is as measured with Einstein synchronized clocks.

square to the other frame? By transferring I mean when using the square that we move to fire the lasers simultaneously to mark the embankment. This time $\bar{v} = \frac{v}{c}$ and $y = \bar{v}^2$. We get the following equation to solve,

$$\begin{aligned}
\frac{L^2}{\pi} &= \sqrt{1 - \frac{v^2}{c^2}} \\
\frac{L^2}{\pi} &= \sqrt{1 - \frac{\frac{v^2}{c^2}c^2}{c^2}} \\
\frac{L^2}{\pi} &= \sqrt{1 - \bar{v}^2} \\
\frac{L^2}{\pi} &= \sqrt{1 - \bar{v}^2} \\
\frac{L^2}{\pi} &= \sqrt{1 - y} \\
\left(\frac{L^2}{\pi}\right)^4 &= y^2 - 2y + 1 \\
\frac{L^8}{\pi^4} &= y^2 - 2y + 1 \\
y^2 - 2y + 1 - \frac{L^8}{\pi^4} &= 0. \qquad (12)
\end{aligned}$$

This is a quadratic equation with the following solution

$$\begin{aligned}
y &= \frac{2 \pm \sqrt{2^2 - 4\left(1 - \frac{L^8}{\pi^4}\right)}}{2} \\
y &= \frac{2 \pm \sqrt{4 - 4\left(1 - \frac{L^8}{\pi^4}\right)}}{2} \\
y &= 1 \pm \sqrt{\frac{L^8}{\pi^4}} \\
y &= 1 \pm \frac{L^4}{\pi^2}. \qquad (13)
\end{aligned}$$

Remember that $y = \bar{v}^2$, this means

$$\begin{aligned}
\bar{v} = &= \sqrt{y} \\
\bar{v} = &= \sqrt{1 \pm \frac{L^4}{\pi^2}} \qquad (14)
\end{aligned}$$

as we must have $\bar{v} \leq 1$ only the negative solution makes sense and we get

$$\bar{v} = \sqrt{1 - \frac{L^4}{\pi^2}} \qquad (15)$$

and this gives us the relative speed of the frames

$$v = c\sqrt{1 - \frac{L^4}{\pi^2}}. \qquad (16)$$

Further in equation 16 we must have $\frac{L^4}{\pi^2} < 1$. Solved with respect to L this gives us

$$\begin{aligned}
\frac{L^4}{\pi^2} &< 1 \\
L^4 &< \pi^2 \\
L &< \sqrt{\pi} \qquad (17)
\end{aligned}$$

when $L = \sqrt{\pi}$ then $v = 0$ and then it is not possible to square the circle, as we already know. In other words it is not possible to square the circle from only one reference frame, we need to use two reference frames to square the circle. The general solution is that we can draw any square with sides shorter than $\sqrt{\pi}$. I assume all lengths $0 > L < \sqrt{\pi}$ that are not transcendental can be used to square a unit circle. A square with length L is constructed on the ground or in the train and then moved onto the train. Then the train is accelerated to the following velocity: $v = c\sqrt{1 - \frac{L^4}{\pi^2}}$. Or we could have accelerated the train first and then constructed the square on the train while it was moving relative to the embankment, this makes no difference. Next accurate clocks in each corner of the square are synchronized while traveling and the lasers are fired simultaneously as observed from the train to mark the embankment. The area of the square as measured from the embankment or the train will have the same area as the circle. Again the square on the embankment drawn from the train will be observed as a rectangle from the embankment, but it was initially drawn as a square.

As the general solution holds for any velocity between $0 > v < c$ we do not need a super-fast futuristic train or space-rocket to square the circle. When the sides of the square are very close to $\sqrt{\pi}$, we do not need all the digits of π in the sides of the square, as long as $v > 0$. For example, we could theoretically construct a printer where the printer head moves at speed v relative to the paper. The printer head consist of a square with each sides lengths very close to $\sqrt{\pi}$ as observed from the printer head. Simply think of the paper as the embankment and the printer head as the train in the example above. The printer head is a perfect square with sides L as observed from the printer head. In each corner of the printer head is a clock that is Einstein synchronized while the printer head moves at velocity v relative to the paper. The corners of the square are simultaneously firing a laser as measured from the printer head clocks. The laser marks on the paper as observed from the paper will not be a perfect square, but a rectangle. The circle is drawn to be a perfect circle as observed from the rest frame of the paper. Again this is just a parallel to the train example, which is much more realistic in the way that the printer head just needs to move at a speed $v > 0$ and not at a speed close to that of light. Still we like the first non-general solution the best from a mathematical point of view. It is almost like magic to draw a unit circle and a unit square that both end up having area 1, or alternatively π from the embankment.

7 Additional Solution

Here we mention an additional solution[12]. Assume a train travels at velocity of $v = c\sqrt{1 - \frac{1}{\pi}}$ relative to the embankment. The train has a unit rod hanging out on the side. On each end of the rod we mount a clock with a laser. The clocks are Einstein synchronized while traveling at velocity $v = c\sqrt{1 - \frac{1}{\pi}}$. At a given point in time both of the lasers are fired simultaneously down towards the ground. That is simultaneously as observed from the train. The distance between the laser marks on the embankment we get from the Lorentz length contraction, and it must be:

$$\hat{x} = \frac{x - c\sqrt{1 - \frac{1}{\pi}} \times 0}{\sqrt{1 - \frac{\left(c\sqrt{1-\frac{1}{\pi}}\right)^2}{c^2}}} = x\sqrt{\pi}$$

Again $x = L$ and if we gave $L = 1$ then we have a length on the ground equal to $\sqrt{\pi}$. We make a straightedge out of this and construct a square. Next we draw a circle with area π on the embankment using a compass. We now have a perfect circle and a perfect square both with area π on the embankment as observed from the embankment. This is also reciprocal when using Einstein synchronized clocks. We could just as well have started out with the rod on the embankment and made the marks on the train. This solution is very nice since we then have a perfect circle and a perfect square in the same reference frame both with area π.

8 Doubling the Cube

Doubling the cube is another geometrical problem closely connected to Squaring the Circle. The quest of Doubling the Cube consists of making a cube with double the volume of another cube simply by using a compass and straightedge. The impossibility of doubling the cube in Euclidean space was proven by Wantzel in 1837. For example, if we have a unit cube with volume one then we need a line segment of $L = \sqrt[3]{2}$. The impossibility of doubling the cube is equivalent to the fact that $\sqrt[3]{2}$ is not a constructible figure using just a compass and straightedge. Still, this impossibility only holds in Euclidean space; in

[12]This solution was suggested by Mandark Astronominov on Twitter after I put a link to this paper there.

space-time we can Double the Cube using a compass and straightedge and Einstein synchronized clocks. Again we define $\bar{v} = \frac{v}{c}$ and $y = \bar{v}^2$. We get the following equation to solve,

$$\begin{aligned}
1 &= \sqrt[3]{2}\beta \\
\frac{1}{\sqrt[3]{2}} &= \beta \\
\frac{1}{\sqrt[3]{2}} &= \sqrt{1-\frac{v^2}{c^2}} \\
\frac{1}{\sqrt[3]{2}} &= \sqrt{1-\frac{\frac{v^2}{c^2}c^2}{c^2}} \\
\frac{1}{\sqrt[3]{2}} &= \sqrt{1-\bar{v}^2} \\
\frac{1}{\sqrt[3]{2}} &= \sqrt{1-\bar{v}^2} \\
\frac{1}{\sqrt[3]{2}} &= \sqrt{1-y} \\
\left(\frac{1}{\sqrt[3]{2}}\right)^4 &= y^2-2y+1 \\
\frac{1}{(\sqrt[3]{2})^4} &= y^2-2y+1 \\
y^2-2y+1-\frac{1}{(\sqrt[3]{2})^4} &= 0. \qquad (18)
\end{aligned}$$

This is a quadratic equation with the following solution

$$\begin{aligned}
y &= \frac{2 \pm \sqrt{2^2-4\left(1-\frac{1}{(\sqrt[3]{2})^4}\right)}}{2} \\
y &= \frac{2 \pm \sqrt{4-4\left(1-\frac{1}{(\sqrt[3]{2})^4}\right)}}{2} \\
y &= 1 \pm \sqrt{\frac{1}{(\sqrt[3]{2})^4}} \\
y &= 1 \pm \frac{1}{(\sqrt[3]{2})^2} \qquad (19)
\end{aligned}$$

Remember $y = \bar{v}^2$, this means

$$\begin{aligned}
\bar{v} = &= \sqrt{y} \\
\bar{v} = &= \sqrt{1 \pm \frac{1}{(\sqrt[3]{2})^2}} \qquad (20)
\end{aligned}$$

as we must have $\bar{v} \le 1$ only the negative solution makes sense and we get

$$\bar{v} = \sqrt{1-\frac{1}{(\sqrt[3]{2})^2}} \qquad (21)$$

and this means we must travel at the following speed relative to the train platform

$$v = c\sqrt{1-\frac{1}{(\sqrt[3]{2})^2}} \qquad (22)$$

and we have

$$
\begin{aligned}
1 &= \sqrt[3]{2}\beta \\
1 &= \sqrt[3]{2} \times \sqrt{1 - \frac{\left(c\sqrt{1-\frac{1}{(\sqrt[3]{2})^2}}\right)^2}{c^2}} \\
1 &= \sqrt[3]{2}\frac{1}{\sqrt[3]{2}}
\end{aligned} \tag{23}
$$

So to Double the Cube we make a unit rod. We then make a unit cube with this rod. We next bring a unit rod to a train at rest relative to the embankment. We accelerate the train to velocity $v = c\sqrt{1-\frac{1}{(\sqrt[3]{2})^2}}$. Next we mount a clock with a time-release laser at each end of each rod. The clocks are Einstein synchornized while the train is traveling. We hang the rod out on the side of the train parallel to the embankment. Next we fire both lasers simultaneously as observed from the train. This gives two marks on the embankment with distance $\sqrt[3]{2}$ apart. We can use this to make another rod and then construct a new cubes with side length $\sqrt[3]{2}$ and volume $(\sqrt[3]{2})^3 = 2$. We have Doubled the Cube.

9 Boxing the Sphere

The volume of a sphere is $V = \frac{4}{3}\pi r^3$. If we set the radius to $r = 1$ we have a unit sphere. The volume of a unit sphere is $V = \frac{4}{3}\pi$. To make a cube with the same volume as the sphere (boxing the sphere), we would need a cube with side length $\sqrt[3]{\frac{4\pi}{3}}$. We cannot construct such a cube (just in space) with just a compass and a straightedge in a finite number of steps. However, we can box the sphere in Euclidean space-time.

At the embankment one first rotate the compass to construct a circle, then one rotates that circle to construct the sphere. A sphere can in this way be seen as meta-construction by a compass.[13] At the embankment we will use the compass to make a rod with a length equal to the radius of the circle. Next we bring this rod on board of a train. We mount a clock on each side if the rod. Each clock has a time-release laser. Next we accelerate the train to a velocity of

$$v = c\sqrt{1 - \frac{1}{(\sqrt[3]{\frac{4\pi}{3}})^2}} \tag{24}$$

While we are traveling at this velocity, we are Einstein synchronizing the clocks. The rod with the clocks is hanging out of the train in the parallel direction to the train track. Next we fire both the lasers simultaneously as observed from the train. This will make two marks on the embankment; from the embankment they will be $\sqrt[3]{\frac{4\pi}{3}}$ apart. We will use this distance to make a new rod. This rod we will use to make a cube. The volume of the cube (box) is $\left(\sqrt[3]{\frac{4\pi}{3}}\right)^3 = \frac{4}{3}\pi$. We have boxed the sphere! We could also have extended this solution to hold for any sphere. The general solution is

$$v = c\sqrt{1 - \frac{1}{(\sqrt[3]{\frac{4\pi}{3r}})^2}}. \tag{25}$$

9.1 Platonic Solids and the Sphere

We have already Boxed the Sphere and the box that forms the cube is one of the five Platonic solids. Here we will provide the solutions for the other Platonic Solids in relation to the Sphere.

Tetrahedron the Sphere

The next Platonic solid is the Tetrahedron. The volume of a Tetrahedron is $V = \frac{a^3}{6\sqrt{2}}$. To Tetrahedron the Sphere, we need to have $a = \sqrt[3]{\pi 8\sqrt{2}}$ and we need to travel at a velocity of

$$v = c\sqrt{1 - \frac{1}{\left(\sqrt[3]{\pi 8\sqrt{2}}\right)^2}} \tag{26}$$

[13] Thanks to Traden4Alpha at the www.wilmott.com forum for pointing out to me how to make a sphere with just a compass.

After this, we will follow the same procedures as we did for Boxing the Sphere. We have Tetrahedroned the Sphere.

Octahedron the Sphere

The volume of a Octahedron is $V = \frac{\sqrt{2}}{3}a^3$. To Octahedron the Sphere, we need to have $a = \sqrt[3]{\frac{4\pi}{\sqrt{2}}}$ and need to travel between the two frames at a velocity of

$$v = c\sqrt{1 - \frac{1}{\left(\sqrt[3]{\frac{4\pi}{\sqrt{2}}}\right)^2}} \tag{27}$$

Once again, we will follow the same procedures as we did for Boxing the Sphere. We have Octahedroned the Sphere.

Icosahedron the Sphere

The volume of an Icosahedron is $V = \frac{5}{12}(3+\sqrt{5})a^3$. To Icosahedron the Sphere, we need the side length of the Icosahedron to be $a = \sqrt[3]{\frac{16\pi}{5(3+\sqrt{5})}}$ and we need to travel at a velocity of

$$v = c\sqrt{1 - \frac{1}{\left(\sqrt[3]{\frac{16\pi}{5(3+\sqrt{5})}}\right)^2}} \tag{28}$$

Then we follow the same procedures as for Boxing the Sphere. We have Icosahedroned the Sphere.

Dodecahedron the Sphere

The volume of a Dodecahedron is $V = \frac{1}{4}(15+7\sqrt{5})a^3$. To Dodecahedron the Sphere we need the side length in the Dodecahedron to be $a = \sqrt[3]{\frac{16\pi}{3(15+7\sqrt{5})}}$ and we need to travel at a velocity of

$$v = c\sqrt{1 - \left(\sqrt[3]{\frac{16\pi}{3(15+7\sqrt{5})}}\right)^2} \tag{29}$$

However, this time we cannot send the signals from the rod simultaneously as measured from the train. Instead, we will need a length a shorter than the unit rod. We need to rely on length contraction rather than length transformation to construct it. To get length contraction, we have to send the signal simultaneously from the train as measured from the embankment. We can do this by Einstein synchronizing the two clocks on each end of the unit rod while the train is still at rest relative to the embankment. Next we move the rod with the clocks on board the train and accelerate the train to the velocity given in equation 29. Then both of the lasers on the train will fire simultaneously, but this will be simultaneously as observed from the embankment, not simultaneously as observed from the train. The mark between the two lasers on the embankment will be related to the length contraction of the rod l. The unit rod $L = 1$ has turned into a length of

$$L\beta = L\sqrt{1 - \frac{v^2}{c^2}} = L\sqrt{1 - \frac{\left(c\sqrt{1 - \left(\sqrt[3]{\frac{16\pi}{3(15+7\sqrt{5})}}\right)^2}\right)^2}{c^2}} = L\sqrt[3]{\frac{16\pi}{3(15+7\sqrt{5})}}. \tag{30}$$

From this new rod on the embankment, we will build a Dodecahedron. We have Dodecahedroned the Sphere.

10 Equilateral Triangle the Circle

The area of a unit circle is $A = \pi r^2 = \pi$. The area of an Equilateral Triangle is $A = \frac{\sqrt{3}}{4}a^2$, where a is the length of the side of the triangle. To make an Equilateral Triangle with the same area as the circle (Equilateral Triangle the Circle), we would need a circle with side length $a = \sqrt{\frac{4\pi}{\sqrt{3}}}$. We cannot construct such an Equilateral Triangle in space alone with only a compass and a straightedge in a finite number of steps. However, we can construct such a triangle in Euclidean space-time.

At the embankment one first rotates the compass to construct a circle. Staying at the embankment, we will use the compass to make a rod with a length equal to the radius of the circle. Next we will bring this rod on board the train. We mount a clock on each side of the rod. Each clock has a time-release laser. Next we will accelerate the train to a velocity of

$$v = c\sqrt{1 - \frac{1}{\sqrt{\frac{4\pi}{\sqrt{3}}}}} \tag{31}$$

While we are traveling at this velocity, we will Einstein synchronize the clocks. Then we will hang the rod with the clocks out of the train in the direction parallel to the train track. Then we fire both of the lasers simultaneously as observed from the train. This will make two marks on the embankment; from the embankment they will be $\sqrt{\frac{4\pi}{\sqrt{3}}}$ apart. We will take this distance to make a new rod that we will use to construct the Equilateral Triangle. The area of the triangle is $A = \frac{\sqrt{3}}{4}\left(\sqrt{\frac{4\pi}{\sqrt{3}}}\right)^2 = \pi$. We have Equilateral Triangled the Circle!

11 Table summary of solutions and further discussion

Below are two tables summarizing the solutions we have provided:

Table 1: This table shows the length of the sides needed to Platonic Solid the Sphere and the relative velocity needed to do that.

Various solutions:	Length needed a	v
Squaring the Circle solution 1	π	$c\sqrt{1-\frac{1}{\pi^2}}$
Squaring the Circle solution 2	$\sqrt{\pi}$	$c\sqrt{1-\frac{1}{\pi}}$
Triangle the Sphere	$\sqrt{\frac{4\pi}{\sqrt{3}}}$	$c\sqrt{1-\frac{1}{\sqrt{\frac{4\pi}{\sqrt{3}}}}}$
Doubling the Cube	$\sqrt[3]{2}$	$c\sqrt{1-\frac{1}{(\sqrt[3]{2})^2}}$

Table 2: This table shows the length of the sides needed to Platonic Solid the Sphere and the relative velocity needed to do that.

Platonic Solids	Volume	Length needed a	v
Boxing the Sphere	a^3	$\sqrt[3]{\frac{4\pi}{3}}$	$c\sqrt{1-\frac{1}{(\sqrt[3]{\frac{4\pi}{3}})^2}}$
Tetrahedron the Sphere	$\frac{a^3}{6\sqrt{2}}$	$\sqrt[3]{\pi 8\sqrt{2}}$	$c\sqrt{1-\frac{1}{\left(\sqrt[3]{\pi 8\sqrt{2}}\right)^2}}$
Octahedron the Sphere	$\frac{\sqrt{2}}{3}a^3$	$\sqrt[3]{\frac{4\pi}{\sqrt{2}}}$	$c\sqrt{1-\frac{1}{\left(\sqrt[3]{\frac{4\pi}{\sqrt{2}}}\right)^2}}$
Dodecahedron the Sphere	$\frac{1}{4}(15+7\sqrt{5})a^3$	$\sqrt[3]{\frac{16\pi}{3(15+7\sqrt{5})}}$	$c\sqrt{1-\left(\sqrt[3]{\frac{16\pi}{3(15+7\sqrt{5})}}\right)^2}$
Icosahedron the Sphere	$\frac{5}{12}(3+\sqrt{5})a^3$	$\sqrt[3]{\frac{16\pi}{5(3+\sqrt{5})}}$	$c\sqrt{1-\frac{1}{\left(\sqrt[3]{\frac{16\pi}{5(3+\sqrt{5})}}\right)^2}}$

So we actually have two general very solutions. If we need to utilize length transformation to create the length a, then we have the following general solution for the velocity

$$v = c\sqrt{1 - \frac{1}{a^2}} \tag{32}$$

If we need a rod longer than our unit rod (the intial rod) then we need to mount two clocks on this rod and synchronize the clocks on board of the train

On the other hand, if we need to utilize length contraction[14] to created the needed length a and then we have the following general solution for the velocity

[14] Among our solutions only the Dodecahedron needs this solution; this is because we need an $a < L$.

$$v = c\sqrt{1-a^2} \tag{33}$$

So if we need a length a, we must first find out if this length is shorter or longer than our unit rod. If it is longer than the rod, then we need to utilize length transformation and if it is shorter, then we need to utilize length contraction. We can move any troublesome constants out of space and into the velocity, and use the velocity to make our needed troublesome length in the other reference frame. As we soon will explain, when the troublesome constant first is moved into the velocity we can decide if we want to move it into time or space.

12 A critical look at the solution

Have I really squared the circle? It is not the first time someone has claimed to have squared the circle. One of the longest and most intense intellectual disputes of all time was between philosopher Thomas Hobbes who claimed that he had squared the circle and the mathematician John Wallis who claimed Hobbes not had squared the circle, see Jesseph (2008). The conclusion was that Hobbes not had squared the circle.

One could claim the solution to squaring the circle in this paper simply has moved the problem into the velocity between the two reference frames. The velocity needed to square the circle is indeed a function of π. Still there was never mentioned any restriction on the velocity of the observer in the quest for squaring the circle. One could also argue that I am bending the rules by using clocks in addition to compass and a straightedge. Even in solution one, where we simply draw a circle in one frame and a square in another frame, we would need two Einstein-Poincaré synchronized clocks to measure the one-way velocity of the train. So any solution requires clocks, as we are working in space-time rather than just space, that is we take into account motion. Or one could naturally try to argue that we simply by coincidence could be traveling at velocity $v = c\sqrt{1-\frac{L^4}{\pi^2}}$ or $v = c\sqrt{1-\frac{1}{\pi}}$ and therefore not would need clocks to find this velocity.

There are indeed several reasons to claim I have bent the rules of squaring the circle slightly. Still one could just as well argue that the claimed impossibility of squaring the circle is rooted historically in that the quest premises were outlined before we had developed good space-time theories. In my defense, one has to keep track of time when working in space-time, and again we do not only live in space, we live in space and time. Previous solution attempts have not taken time into account, nor have they considered that time and space are affected by motion. Furthermore, previous solution attempts have not been clear on from what reference frame the circle and square are drawn from and what reference frame they are observed from. It is indeed possible to square the circle if one takes into account space and time and how space and time measurements are affected by motion.

In practice it would be close to impossible to get a velocity of exactly $v = c\sqrt{1-\frac{L^4}{\pi^2}}$ or $v = c\sqrt{1-\frac{1}{\pi}}$. Or at least this would require infinite precision in our measuring devices. However, this is more of a measuring problem than a Squaring the Circle quest problem (?). One could argue that Squaring the Circle in space-time is a question of clock accuracy. The more accurate the clocks the more precisely we can measure the velocity. Ultimately we would need continuous time clocks to Square the Circle in space-time.

13 Moving problems from space to time

One of the main results (and possibly the very essence of the paper) is that we can move the necessary space measurements related to challenging constants like π, $\sqrt[3]{2}$ and $\sqrt[3]{\frac{4\pi}{3}}$ out from space and into time or out from time and back into space.

To illustrate how we can move the challenge from space to time completely, lets revisit our initial squaring of the circle solution. To square the circle in our first solution, we needed an exact velocity of $v = c\sqrt{1-\frac{1}{\pi^2}}$. This velocity contains π and some people may argue that we have simply moved the problem of squaring the circle into the velocity between the frames. This is true. However, velocity consists of a measure or given distance divided by the measured time interval it took to travel that distance. Because of this structure, we can decide if we want to move π into its distance component (space measurement) or into its time component (time measurement).

In the quest to square the circle, the standard length unit we decided to use was the radius of the circle that we first drew with the compass on the embankment. This radius is what we used to make a rod and the rod became our unit length. This is our fixed measure unit in space; it is known, and it is simply the rod in its rest frame. We did not need to know anything about π to construct this rod.

Next we brought the rod on board the train. Next we mount a laser receiver clock on each end of the rod. Assume further that we would have a light source on the embankment going in the transverse direction of the train track towards the train. To get the velocity needed to square the circle, we need to measure the time interval it takes for the rod on board of the train to pass the light source on the ground. This time interval we must get exactly to

$$t = \frac{L}{c\sqrt{1 - \frac{1}{\pi^2}}} \tag{34}$$

Since this time interval contains π, we would need clocks with infinite precision, as well as an infinite number of measurements and adjustments in the velocity, to reach this velocity exactly. Further, the two clocks on the rod must be Einstein synchronized every time we change the velocity of the train. Indeed, it would require, an infinite number of time measurements to get to the time interval in equation 34. So this means that we have basically moved the problem from space (measurements) to time (measurements).

So certainly one can claim that we have not Squared the Circle. However, our method has changed the quest completely. The original Square the Circle quest is about the impossibility of constructing certain measurements like $\sqrt{\pi}$ in space, while we have moved the quest into the measurement of a time interval that is connected to π. This is in our view quite remarkable. Challenges in space measurements and spatial constructions can be transformed into challenges in time measurements.

We have not seen the possibility of moving troublesome constants from a space framework to a time framework discussed in this way in the literature before. With an optimistic view, this can potentially open up new possibilities in geometry and other scientific fields, at least from an interpretation standpoint. The challenge of measuring something in space can be shifted to a challenge of measuring something in time and vice versa. We can swap space challenges with time challenges and see where that leads us, particularly with regard to some classic "impossible" problems.

Squaring the circle, and doubling the cube have parallels to the Gordian Knot. The Gordian Knot is an 'impossible' knot that can only be solved by thinking outside the box (I mean outside the sphere). An oracle prophesied that the one who united the Gordian Knot would become the king of Asia. According to one fable, Alexander the Great sliced the knot with a sword stroke and thereby 'solved" the problem. Possibly some would claim that I have not Squared the Circle, but using the sword of time I have sliced the Squaring of the Circle, the Doubling of the Cube, the Boxing of the Sphere, and the Equilateral Triangling of the Circle.

If the prophecy is true, then I should become the King of the Circle!

14 Is length contraction for real?

A question that often comes up when someone mentions length contraction is if length contraction is for real. This is an important question that we not will resolve here, but that we will mention briefly. We have to be very careful with what we mean about "real". We will claim that Einstein length contraction is real in the sense that this is what we will observe with Einstein synchronized clocks. Both length contraction and length transformation require a minimum of two clocks in the cases discussed here.

We need two clocks as we also need to know the relative speed between the frames. Part of the length contraction and length transformation has to do with the synchronization of clocks and relativity of simultaneity. After studying the subject carefully for years, we are convinced that our conclusion above holds as long as we use Einstein synchronized clocks and special relativity theory is based on Einstein synchronized clocks.

One should think this question was fully resolved, and possibly it is, but reading through a series of university text books covering special relativity theory one can at least see there are still slightly different views among physicists on whether length contraction is real or not. For example Shadowitz (1969) claims

> *If the measurements are optical then, to avoid an incorrect result, the light photons must leave the two points at the same time, as measured by the observers: they must leave simultaneously. It is clear that the process of length measurement is different from the process of seeing. Amazingly, this distinction was not noticed until 1959, when it was first pointed out by James Terrell.*[15]

Further Lawden (1975) claims

> *The contraction is not to be thought of as the physical reaction of the rod to its motion and as belonging to the same category of physical effects as the contraction of a metal rod when*

[15]See Shadowitz (1969) page 61.

cooled. It is due to changed relationship between the rod and the instruments measuring its length.[16]

While for example Rindler (2001) claims

> *This length contraction is no illusion, no mere accident of measurement or convention. It is real in every sense. A moving rod is really short! It could really be pushed into a hole at rest in the lab into which it would not fit if it were moving and shrunk.*[17]

and Freedman and Young (2016) claims

> *Length contraction is real! This is not an optical illusion! The ruler really is shorter in the reference frame S than it is in S'.*[18]

and Harris (2008) claims

> *It is a grave mistake to dismiss length contraction as an optical illusion caused by delays in light traveling to the observer from a moving object. This effect is real.*[19]

On the other hand Krane (2012) seems to partly claim something different

> *Length contraction suggests that the objects in motion are measured to have shorter length than they do at rest. The objects do not actually shrink; there is merely a difference in the length measured by different observers. For example, to observers on Earth a high-speed rocket ship would appear to be contracted along its direction of motion, but to an observer on the ship it is the passing Earth that appears to be contracted.*[20]

Einstein length contraction will be observed as described also in this paper as long as we use Einstein synchronized clocks. Einstein length contraction is reciprocal between frames, while for example the FitzGerald and Larmor use of length contraction is not reciprocal, because in ether theories one has a preferred reference frame. In special relativity theory any frame watching an object in another frame it will appear contracted. A frame making marks in the other frame will, on the other hand, seem expanded. However this length expansion is due to length transformation as well as length contraction. Length contraction and length transformations are linked, but they are not the same.[21]

In our case we are actually making a circle in the rest frame of the circle so here there should be no disputes. The square we are first making on the train, but we are transferring it to the embankment by lasers in solution two. These laser pens are fired simultaneously as observed from the pen. The pen is the train or even a printer head that travels at speed $c\sqrt{1-\frac{L^4}{\pi^2}}$ relative to the paper. The lasers are not fired simultaneously as observed from the embankment and this is the reason that we get a rectangle on the embankment. Our theory is fully consistent with Einsteins special relativity theory, and could at least hypothetically be performed in practice with the expected result we have described above.

15 Conclusion

It is possible to Square the Circle by constructing the circle and the square from two different reference frames traveling at speed $v = c\sqrt{1-\frac{L^4}{\pi^2}}$ or $v = c\sqrt{1-\frac{1}{\pi}}$ relative to each other.[22] More precisely it is not possible to Square the Circle in Euclidean space using only a compass and straightedge, but it is possible to Square the Circle in Euclidean space-time using compass, straightedge and Einstein synchronized clocks. We could argue that this is bending the rules and moving the problem of transcendental π into a transcendental velocity between the reference frames rather than directly into the construction of the Circle and the Square. Still, one could just as well argue that the previous attempts to square the circle have not taken into account that observations of space and time are affected by motion and that space and time are closely connected.

Have I really Squared the Circle? Have I made the Impossible Possible? Only space-time can tell if this paper leads to celebration, silence, death, or an intellectual War similar to that between Hobbes and Wallis. Before you shoot the messenger make sure you have studied length contraction, length transformation and relativity of simultaneity rigorously.

[16] See Lawden (1975) page 12.
[17] See Rindler (2001) page 62.
[18] Freedman and Young (2016) page 1229.
[19] Harris (2008) page 11.
[20] Krane (2012) page 35.
[21] See also Haug (2014).
[22] As measured with Einstein synchronized clocks.

More important than if we have Squared the Circle or not is that we have shown that any troublesome constant like π, $\sqrt{\pi}$, $\sqrt{2}$ can be moved from the space dimension and into the time dimension. The very essence of the paper is that space challenges can be replaced by time challenges or vice versa.

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