A Gauge Theory of Gravity in Curved Phase-Spaces *

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Abstract

After a cursory introduction of the basic ideas behind Born's Reciprocal Relativity theory, the geometry of the cotangent bundle of spacetime is studied via the introduction of nonlinear connections associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. A novel gauge theory of gravity in the $8D$ cotangent bundle T^*M of spacetime is explicitly constructed and based on the gauge group $SO(6,2) \times_s R^8$ which acts on the tangent space to the cotangent bundle $T_{(\mathbf{x},\mathbf{p})}T^*M$ at each point $(\mathbf{x},\mathbf{p})$. Several gravitational actions involving curvature and torsion tensors and associated with the geometry of curved phase spaces are presented. A brief discussion of the vacuum field equations is provided in the conclusion.

Keywords : Gravity, Finsler Geometry, Born Reciprocity, Phase Space.

1 Born's Reciprocal Relativity in Phase Space

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality). The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where $X^i, T, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by

*Dedicated to the loving memory of Blanca Castro Ramirez

[2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

The $U(1,3) = SU(1,3) \otimes U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dp_0 + \delta_{ij} dx^i \wedge dp^j;\ i,j = 1,2,3$ and also the following Born-Green line interval in the $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\sigma)^2 \ = \ (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2} \left((dE)^2 \ - \ (dp_x)^2 \ - \ (dp_y)^2 \ - \ (dp_z)^2 \right) \tag{1.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details.

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \otimes U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \ = \ (dT)^2 \ - \ (dX)^2 \ + \ \frac{(dE)^2 \ - \ (dP)^2}{b^2} \ =$$

$$(d\tau)^2 \ \left(1 \ + \ \frac{(dE/d\tau)^2 \ - \ (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right) \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(2.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t \ = \ \sqrt{\frac{\hbar}{bc}}; \ \ \lambda_l \ = \ \sqrt{\frac{\hbar \ c}{b}}; \ \ \lambda_p \ = \ \sqrt{\frac{\hbar \ b}{c}}; \ \ \lambda_e \ = \ \sqrt{\hbar \ b \ c} \tag{1.3}$$

The gravitational constant can be written as $G = \alpha_G \ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales (2.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (2.2) invariant are [2]

$$T' \ = \ T \ cosh\xi \ + \ (\frac{\xi_v \ X}{c^2} \ + \ \frac{\xi_a \ P}{b^2}) \ \frac{sinh\xi}{\xi} \tag{1.4a}$$

$$E' \ = \ E \ cosh\xi \ + \ (-\xi_a \ X \ + \ \xi_v P) \ \frac{sinh\xi}{\xi} \tag{1.4b}$$

$$X' \;=\; X\; cosh\xi \;+\; (\xi_v\; T \;-\; \frac{\xi_a\; E}{b^2})\; \frac{sinh\xi}{\xi} \tag{1.4c}$$

$$P' \;=\; P\; cosh\xi \;+\; (\frac{\xi_v\; E}{c^2} \;+\; \xi_a\; T)\; \frac{sinh\xi}{\xi} \tag{1.4d}$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) \;=\; \frac{F}{F_{max}} \tag{1.5}$$

It is straightforwad to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the spacetime proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\sigma)^2 \;=\; (d\tau)^2 \;\left(\; 1 \;-\; \frac{F^2}{F^2_{max}}\;\right) \tag{1.6}$$

is truly left invariant under force (acceleration) boosts (1.4). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\; dT \wedge E + dX \wedge dP$.

One can verify also that the transformations eqs-(1.4) are invariant under the discrete transformations

$$(T, X) \rightarrow (E, P); \quad (E, P) \rightarrow (-T, -X), \quad b \rightarrow \frac{1}{b} \tag{1.7}$$

we argued [16] that the latter transformation $b \rightarrow \frac{1}{b}$ is a manifestation of the large/small tension T-duality symmetry in string theory. In natural units of $\hbar = c = 1$, the maximal proper force $\mathbf{b}$ has the same dimensions as a string tension (energy per unit length) $(mass)^2$. Novel physical consequences of Born's Reciprocal Relativity can be found in [5].

To understand the *invariant* meaning of the interval in phase space $d\sigma$, and to show the consistency of eqs-(1.4,1.5,1.6), let us describe the following scenario. A massive free particle does not experience any force, thus the momentum is conserved so that $\frac{dp_a}{d\tau} = 0$ and the flat phase space interval is $(d\sigma)^2 = (d\tau)^2$. In an accelerated frame of reference the massive particle experiences a pseudo-force which implies that $\frac{dp'_a}{d\tau'} \neq 0$. Upon choosing an infinite rapidity parameter $\xi_a = \infty$ in eqs-(1.5), the value of the pseudo-force reaches its maximal proper value $F_{max} = \mathbf{b}$. Also, $(d\tau')^2 = \infty$ when the acceleration rapidity parameter is ∞, as one can verify from eqs-(1.4) by simple inspection. Since the interval in flat phase space $(d\sigma)^2$ (1.6), in an inertial frame and accelerated frame of reference, respectively, remains invariant under the transformations (1.4) one has that $(d\sigma)^2 = (d\tau)^2 = (d\tau')^2(1 - F^2/F^2_{max}) = \infty \times 0 \neq 0$. The latter product cannot be zero, because if $(d\tau)^2$ were zero, in the inertial non-accelerated frame of reference , this would mean that the massive free particle would have followed a null geodesic, which it cannot do since only massless photons can.

We explored in [5] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, *six* specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

A discussion of Mach's principle within the context of Born Reciprocal Gravity in Phase Spaces was described in [16]. The Machian postulate states that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. It is also consistent with equating the maximal proper force $m_{Planck}(c^2/L_{Planck})$ to $M_{Universe}(c^2/R_{Hubble})$ and reflecting a maximal/minimal acceleration duality. By invoking Born's reciprocity between coordinates and momenta, a minimal Planck scale should correspond to a minimum momentum, and consequently to an upper scale given by the Hubble radius. Further details can be found in [16].

The purpose of this work is to analyze the *curved* phase-space scenario in more detail and the geometry of the cotangent bundle of spacetime via the introduction of nonlinear connections associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. In the case of the cotangent space of a d-dim manifold T^*M_d the metric components can be equivalently rewritten in the *block* diagonal form [10] such that the line element is given by

$$(ds)^2 \ = \ g_{ij}(x^k, p_a)\ dx^i d\ x^j \ + \ h^{ab}(x^k, p_c)\ \delta p_a\ \delta p_b,$$

$$i, j, k = 1, 2, ..., d, \quad a, b, c = 1, 2, ..., d \tag{1.8}$$

if instead of using the standard coordinate basis frames one introduces the following *nonholonomic* frames (non-coordinate basis)

$$\delta_i \ = \ \delta/\delta x^i \ = \ \partial_{x^i} \ + \ N_{ia}\ \partial^a \ = \ \partial_{x^i} \ + \ N_{ia}\ \partial_{p_a}; \ \ \partial^a \ \equiv \ \partial_{p_a} \ = \ \frac{\partial}{\partial p_a} \tag{1.9}$$

One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \ \ \delta p_a \ = \ dp_a \ - \ N_{ja}\ dx^j \tag{1.10}$$

where the $N_{ja}(x,p)$-coefficients define a *nonlinear* connection. When $N_{ia} = 0$ and $h^{ab} = g^{ab}/b^2$, the interval in eq-(1.8) reduces to the Born-Green interval in eq-(1.1). In the very special case such that $N_{ja}(x,p) = \Gamma^k_{ja}(x) p_k$, the N-connection becomes linear in the momentum with $\Gamma^k_{ja}(x)$ being the underlying spacetime connection. The N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some non-holonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$, with nontrivial nonholonomy coefficients $W^\gamma_{\alpha\beta}$ given in terms of derivatives of N_{ia} [9], [10]. The indices α, β, γ comprise both base and fiber coordinate indices.

An N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form [10], [9]

$$D_{\delta_j}(\delta_i) \;=\; H^k_{ij}\;\delta_k;\;\; D_{\delta_j}(\partial^a) \;=\; -\;H^a_{bj}\;\partial^b; \tag{1.11a}$$

$$D_{\partial^a}(\delta_i) \;=\; C^{ka}_i\;\delta_k;\;\; D_{\partial^a}(\partial^b) \;=\; -\;C^{ba}_c\;\partial^c \tag{1.11b}$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i \;=\; \frac{1}{2}T^i_{jk}\;dx^j\wedge dx^k \;+\; C^{ia}_j\;dx^j\wedge\delta p_a \tag{1.12a}$$

$$\Omega_a \;=\; \frac{1}{2}R_{jka}\;dx^j\wedge dx^k \;+\; P^b_{aj}\;dx^j\wedge\delta p_b \;+\; \frac{1}{2}S^{bc}_a\;\delta p_b\wedge\delta p_c \tag{1.12b}$$

and the curvature 2-forms are

$$\Omega^i_j \;=\; \frac{1}{2}R^i_{jkm}\;dx^k\wedge dx^m \;+\; P^{ia}_{jk}\;dx^k\wedge\delta p_a \;+\; \frac{1}{2}S^{iab}_j\;\delta p_a\wedge\delta p_b \tag{1.13}$$

$$\Omega^a_b \;=\; \frac{1}{2}R^a_{bkm}\;dx^k\wedge dx^m \;+\; P^{ac}_{bk}\;dx^k\wedge\delta p_c \;+\; \frac{1}{2}S^{acd}_b\;\delta p_c\wedge\delta p_d \tag{1.14}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i,\ \partial^a = \partial/\partial p_a$ is given by $dx^i,\ \delta p_a = dp_a - N_{ja}dx^j$. The explicit expressions for the terms

$$T^i_{jk},\;\; C^{ia}_j,\;\; R_{jka},\;\; P^b_{aj},\;\; S^{bc}_a,\;\; R^i_{jkm},\;\; P^{ia}_{jk},\;\; S^{iab}_j,\;\; R^a_{bkm},\;\; P^{ac}_{bk},\;\; S^{acd}_b \tag{1.15}$$

in eqs-(1.12-1.14) are given explicitly in terms of the coefficients of eq-(1.1) and the nonlinear connection and nonholonomy coefficients as shown in [10], [9]. The expressions are rather lengthy, for this reason we refer to [10], [9] for detailed calculations.

The Hamilton geometry of the phase space of particles whose motion is characterized by generalized dispersion relations was recently studied by [6]. In this framework, spacetime and momentum space are naturally *curved* and *intertwined*, allowing for a simultaneous description of both spacetime curvature and non-trivial momentum space geometry. The interplay between spacetime curvature and non-trivial momentum space effects was essential in the notion of "relative locality" and in the deepening of the relativity principle [7].

In the cotangent space description one has covariance under a more *restricted* set of coordinate transformations of the form [10]

$$x'^i \;=\; x'^i(x^j),\;\; p'_i \;=\; p_j\;\frac{\partial x^j}{\partial x'^i} \tag{1.16}$$

such that there is an *entanglement* of spacetime and momentum variables in the transformed momentum fiber coordinates. However, Quaplectic transformations in flat phase space have a different form $x'^i = x'^i(x^j, p_j)$ and $p'_i = p'_i(x^j, p_j)$. Thus one cannot accommodate the Quaplectic transformations in eqs-(1.4) to curved phase spaces (the cotangent bundle T^*M) in the manner described by eq-(1.16). This problem is beyond the scope of this work. A plausible solution is to $complexify$ the spacetime cotangent bundle by introducing complex coordinates $z^\mu = x^\mu + ip_\mu/b$, and whose complex conjugate momenta are π_μ, along with the transformations $z'^\mu = z'^\mu(z^\nu), \pi'_\mu = \pi_\nu \frac{\partial z^\nu}{\partial z'^\mu}$. This would lead to a mixing of x^μ and p_μ encoded in the transformations of the base coordinates $z'^\mu = z'^\mu(z^\nu)$.

To finalize this section, we remark that in this letter we are following another approach than the one based on Hamilton geometry in investigating curved phase spaces. In the next section, a novel gauge theory of gravity in the $8D$ cotangent bundle T^*M of four-dimensional spacetime is constructed and based on the gauge group $SO(6,2) \times_s R^8$. Several gravitational actions associated with the geometry of curved phase spaces are presented. The geometry of the $8D$ tangent bundle of $4D$ spacetime and the physics of a limiting value of the proper acceleration in spacetime [4] has been studied by Brandt [3] . Generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the $infinite$ acceleration limit. We must emphasize that the results described in the next section are quite different than those obtained earlier by us in [14] and by [10], [9], [3], [6] among others.

2 Gauge Theories of Gravity in the Cotangent Bundle

In this section we will construct a novel gauge theory of gravity in the $8D$ cotangent bundle T^*M based on the gauge group given by the semidirect product $SO(6,2) \times_s R^8$. Let us begin with a Lie group $\mathcal{G}$; its associated Lie algebra is spanned by the generators $\mathcal{L}_A, A = 1, 2, \ldots,$ dim $\mathcal{G}$, and whose structure constants are f^C_{AB}. The Lie algebra commutator is $[\mathcal{L}_A, \mathcal{L}_B] = f^C_{AB}\mathcal{L}_C$. The components of the gauge field strength in the $8D$ cotangent bundle T^*M, and corresponding to the Lie-algebra valued gauge fields $\mathcal{A}^A_i\mathcal{L}_A, \mathcal{A}^A_a\mathcal{L}_A$, are

$$\mathcal{F}^A_{ij} = \delta_i\mathcal{A}^A_j - \delta_j\mathcal{A}^A_i + [\ \mathcal{A}_i,\ \mathcal{A}_j\ ]^A =$$

$$(\frac{\partial}{\partial x^i} + N_{ib}\frac{\partial}{\partial p_b})\ \mathcal{A}^A_j - (\frac{\partial}{\partial x^j} + N_{jb}\frac{\partial}{\partial p_b})\ \mathcal{A}^A_i +$$

$$\mathcal{A}^B_i\ \mathcal{A}^C_j\ f^A_{BC} \tag{2.1}$$

$$\mathcal{F}^A_{ab} = \frac{\partial}{\partial p^a}\mathcal{A}^A_b - \frac{\partial}{\partial p^b}\mathcal{A}^A_a + \mathcal{A}^B_a\ \mathcal{A}^C_b\ f^A_{BC} \tag{2.2}$$

$$\mathcal{F}^A_{ia} \;=\; \delta_i \mathcal{A}^A_a \;-\; \partial_a \mathcal{A}^A_i \;+\; \mathcal{A}^B_i \; \mathcal{A}^C_a \; f^A_{BC} \tag{2.3}$$

$$\mathcal{F}^A_{ai} \;=\; \partial_a \mathcal{A}^A_i \;-\; \delta_i \mathcal{A}^A_a \;+\; \mathcal{A}^B_a \; \mathcal{A}^C_i \; f^A_{BC} \tag{2.4}$$

there is anti-symmetry in the indices $\mathcal{F}^A_{ia} = -\mathcal{F}^A_{ai}$ and the particular Lie-algebra-valued two-form field strength is $\mathcal{F}^A_{ia} dx^i \wedge \delta p^a$ where $dx^i \wedge \delta p^a \;= -\; \delta p^a \wedge dx^i$.

We shall choose the gauge group to be the semidirect product $SO(6,2) \times_s R^8$ which is the extension of the $4D$ Poincare group $SO(3,1) \times_s R^4$ given by the semidirect product of the Lorentz group with the translations. The flat metric in the tangent space to the cotangent bundle $T_{(\mathbf{x},\mathbf{p})}T^*M$, at the point $(\mathbf{x},\mathbf{p})$, is $\eta_{AB} = diag\;(-,+,+,+,-,+,+,+)$. There are two timelike directions corresponding to the temporal coordinate x^0 and the energy p^0.

The $SO(6,2)$ Lie algebra generators $\mathcal{L}_{AB}$ obey the commutation relations

$$[\mathcal{L}_{AB},\; \mathcal{L}_{CD}] \;=\; (\eta_{BC}\mathcal{L}_{AD} - \eta_{AC}\mathcal{L}_{BD} - \eta_{BD}\mathcal{L}_{AC} + \eta_{AD}\mathcal{L}_{BC}). \tag{2.5}$$

The other commutators associated with the translation generators $\mathcal{P}_A$ are

$$[\mathcal{L}_{AB},\; \mathcal{P}_C] \;=\; (\; \eta_{BC}\; \mathcal{P}_A \;-\; \eta_{AC}\; \mathcal{P}_B\;); \quad [\mathcal{P}_A,\; \mathcal{P}_B] \;=\; 0 \tag{2.6}$$

The metric G_{MN} in the $8D$ cotangent bundle T^*M is given by

$$G_{MN} \;=\; G_{MN}(x,p) \;=$$

$$\begin{pmatrix} g_{ij}(x,p) \;+\; h_{ab}(x,p)\; N^a_i(x,p)\; N^b_j(x,p) & -\; N^a_i(x,p)\; h_{ab}(x,p) \\ -\; N^b_j(x,p)\; h_{ab}(x,p) & h_{ab}(x,p) \end{pmatrix} \tag{2.7}$$

The entries of G_{MN} have different units, one could introduce suitable factors of $\mathbf{b}$ in order to have the same units for all the entries of G_{MN} if one wishes. For simplicity we shall set $\mathbf{b} = 1$. One could also have complex (Hermitian) metrics of the form $G_{MN} = G_{(MN)} + iG_{[MN]}$ with an antisymmetric piece $G_{[MN]}$. We refer to [11] for a study of gauge theories of Born Reciprocal Gravity based on the Quaplectic group [2] given by the semidirect product of the (pseudo) unitary group with the Weyl-Heisenberg group.

The frame E^A_M fields are introduced such that

$$G_{MN} \;=\; E^A_M\; E^B_N\; \eta_{AB} \tag{2.8}$$

where $A, B = 1, 2, \ldots, 8$ are the indices of the tangent space to the $8D$ cotangent bundle $T_{(\mathbf{x},\mathbf{p})}T^*M$, at each point $(\mathbf{x},\mathbf{p})$. $M, N = 1, 2, \ldots, 8$ are the indices of the cotangent bundle T^*M of the $4D$ spacetime manifold M.

The Lie-algebra valued gauge field is

$$\mathbf{A}_M \;=\; \Omega^{AB}_M\; \mathcal{L}_{AB} \;+\; E^A_M\; \mathcal{P}_A \tag{2.9}$$

where Ω^{AB}_M (analog of the spin connection) is the field that gauges the $SO(6,2)$ symmetry. E^A_M gauges the (Abelian) local translations in $T_{(\mathbf{x},\mathbf{p})}T^*M$. Defining the derivative operators as

$$\hat{\partial}_M \equiv (\delta_i,\ \partial_a) = (\frac{\partial}{\partial x^i} + N_{ib}\frac{\partial}{\partial p_b},\ \frac{\partial}{\partial p_a}) \tag{2.10}$$

the Lie-algebra valued field strength is given by

$$\mathbf{F}_{MN} = \hat{\partial}_M \mathbf{A}_N - \hat{\partial}_N \mathbf{A}_M + [\mathbf{A}_M,\ \mathbf{A}_N] \tag{2.11}$$

The curvature two-form associated with the spin connection $\Omega_M^{AB} = -\Omega_M^{BA}$ is

$$\mathcal{R}_{MN}^{AB} \equiv \mathcal{F}_{MN}^{AB} = \hat{\partial}_M \Omega_N^{AB} - \hat{\partial}_N \Omega_M^{AB} + \Omega_{[M}^{AC}\ \Omega_{N]}^{CB} \tag{2.12}$$

and whose explicit components are

$$\mathcal{R}_{ij}^{AB} \equiv \mathcal{F}_{ij}^{AB} = (\frac{\partial}{\partial x^i} + N_{ib}\ \frac{\partial}{\partial p_b})\ \Omega_j^{AB} - (\frac{\partial}{\partial x^j} + N_{jb}\ \frac{\partial}{\partial p_b})\ \Omega_i^{AB} + \Omega_{[i}^{AC}\ \Omega_{j]}^{CB} \tag{2.13}$$

$$\mathcal{R}_{ab}^{AB} \equiv \mathcal{F}_{ab}^{AB} = \frac{\partial}{\partial p^a}\ \Omega_b^{AB} - \frac{\partial}{\partial p^b}\ \Omega_a^{AB} + \Omega_{[a}^{AC}\ \Omega_{b]}^{CB} \tag{2.14}$$

$$\mathcal{R}_{ia}^{AB} \equiv \mathcal{F}_{ia}^{AB} = (\frac{\partial}{\partial x^i} + N_{ib}\ \frac{\partial}{\partial p_b})\ \Omega_a^{AB} - \frac{\partial}{\partial p^a}\ \Omega_i^{AB} + \Omega_{[i}^{AC}\ \Omega_{a]}^{CB} \tag{2.15}$$

and $\mathcal{F}_{ai}^{AB} = -\mathcal{F}_{ia}^{AB}$. A summation over the repeated indices is implied and $[MN]$ denotes the anti-symmetrization of indices with weight one.

The explicit components of the torsion two-form defined as

$$\mathcal{T}_{MN}^{A} \equiv \mathcal{F}_{MN}^{A} = \hat{\partial}_M E_N^A - \hat{\partial}_N E_M^A + \Omega_{[M}^{AC}\ E_{N]}^{C} \tag{2.16}$$

are

$$\mathcal{T}_{ij}^{A} \equiv \mathcal{F}_{ij}^{A} = (\frac{\partial}{\partial x^i} + N_{ib}\ \frac{\partial}{\partial p_b})\ E_j^A - (\frac{\partial}{\partial x^j} + N_{jb}\ \frac{\partial}{\partial p_b})\ E_i^A + \Omega_{[i}^{AC}\ E_{j]}^{C} \tag{2.17}$$

$$\mathcal{T}_{ab}^{A} \equiv \mathcal{F}_{ab}^{A} = \frac{\partial}{\partial p^a}\ E_b^A - \frac{\partial}{\partial p^b}\ E_a^A + \Omega_{[a}^{AC}\ E_{b]}^{C} \tag{2.18}$$

$$\mathcal{T}_{ia}^{A} \equiv \mathcal{F}_{ia}^{A} = (\frac{\partial}{\partial x^i} + N_{ib}\ \frac{\partial}{\partial p_b})\ E_a^A - \frac{\partial}{\partial p^a}\ E_i^A + \Omega_{[i}^{AC}\ E_{a]}^{C} \tag{2.19}$$

and $\mathcal{F}_{ai}^{A} = -\mathcal{F}_{ia}^{A}$.

The frame fields allow us to construct the curvature tensor on the cotangent bundle T^*M as follows

$$\mathcal{R}^Q_{MNP} \equiv \mathcal{R}^{AB}_{MN} \; E^Q_A \; E_{BP} = \mathcal{F}^{AB}_{MN} \; E^Q_A \; E_{BP} \tag{2.20}$$

where the explicit components $\mathcal{F}^{AB}_{MN}$ are obtained in eqs- (2.13-2.15) . E^M_A is the inverse frame field such that $E^M_A \; E^B_M = \delta^B_A$ and $E_{AM} \; E^M_B = \eta_{AB}$. The contraction of indices yields the Ricci-like tensors.

$$\mathcal{R}_{MP} = \delta^N_Q \; \mathcal{R}^Q_{MNP} \tag{2.21a}$$

A further contraction yields the generalized Ricci scalar

$$\mathcal{R} = G^{MP} \; \mathcal{R}_{MP} \tag{2.21b}$$

The Torsion tensors are

$$\mathcal{T}_{MNQ} = \mathcal{F}^A_{MN} \; E_{AQ}, \quad \mathcal{T}^Q_{MN} = \mathcal{F}^A_{MN} \; E^Q_A, \quad \mathcal{T}_M = \delta^N_Q \; \mathcal{T}^Q_{MN} \tag{2.22}$$

A Lagrangian, linear in the curvature scalar and quadratic in torsion, can be chosen to be

$$\mathcal{L} = c_1 \; \mathcal{R} + c_2 \; \mathcal{T}_{MNQ} \; \mathcal{T}^{MNQ} + c_3 \; \mathcal{T}_M \; \mathcal{T}^M. \tag{2.23}$$

where c_1, c_2, c_3 are numerical coefficients. The action is

$$S = \frac{1}{2\kappa^2} \int_{\Omega_8} d^8Y \; \sqrt{| \; det \; G_{MN} \; |} \; \mathcal{L} \tag{2.24}$$

where κ^2 is the analog of the gravitational coupling constant and the $8D$ measure of integration involves

$$d^8Y \equiv dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge \delta p_1 \wedge \delta p_2 \wedge \delta p_3 \wedge \delta p_4 \tag{2.25}$$

with

$$\delta p_a = dp_a - N_{ai} \; dx^i \tag{2.26}$$

Other measures besides $\sqrt{|detG_{MN}|}$ in eq-(2.24) can be used in tangent/cotangent bundles. For example, see the discussion on the Busemann-Hausdorff and Holmes-Thompson measure in [12]. For simplicity we shall retain the ordinary measure in eq-(2.24).

The curvature (2.13-2.15) depends on the geometric quantities g_{ij}, h_{ab}, N_{ia} that describe the metric (2.7) and Ω^{AB}_M. The number of degrees of freedom $d(2d+1)$ associated with g_{ij}, h_{ab}, N_{ia} is the same as the number of degrees of freedom of a metric G_{MN} in $2d$ dimensions. Furthermore, if the torsion (2.16) is set to zero one can solve Ω^{AB}_M in terms of E^A_M. To sum up, in the absence of torsion, the action (2.24) represents effectively a Poincare-like gauge theory of gravity in 8 dimensions, written in a *nonholonomic* coordinate basis, and where the gauge group is $SO(6,2) \times_s R^8$.

Bars [13] has proposed a gauge symmetry in phase space. One of the consequences of this gauge symmetry is a new formulation of physics in spacetime. Instead of one time there must be *two* times, while phenomena described by one-time physics in $3+1$ dimensions appear as various shadows of the same phenomena that occur in $4+2$ dimensions with one extra space and one extra time dimensions (more generally, $d+2$). Problems of ghosts and causality are resolved automatically by the $Sp(2,R)$ gauge symmetry in phase space.

The ordinary $4D$ Einstein-Hilbert action can be written in terms of the vielbeins e_i^a and spin connection ω_i^{ab} as

$$S \;=\; \frac{1}{16\pi G} \int \; e_i^a \wedge e_j^b \wedge R_{kl}^{cd}(\omega_i^{ab}, e_i^a)\; \epsilon_{abcd}\; \epsilon^{ijkl} \tag{2.27}$$

The natural extension of (2.27) to the $8D$ cotangent bundle T^*M is

$$\frac{1}{2\kappa^2} \int \; E_{M_1}^{A_1} \wedge E_{M_2}^{A_2} \wedge E_{M_3}^{A_3} \wedge E_{M_4}^{A_4} \wedge E_{M_5}^{A_5} \wedge E_{M_6}^{A_6} \wedge \mathcal{R}_{M_7 M_8}^{A_7 A_8}\; \epsilon_{A_1 A_2 \ldots A_8}\; \epsilon^{M_1 M_2 \ldots M_8} \tag{2.28}$$

One could also introduce Lanczos-Lovelock-like Lagrangians in D-dimensions, written in terms of the generalized Kronecker deltas,

$$\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} \;=\; \frac{1}{n!}\; \delta^{\mu_1\nu_1}_{[\alpha_1\beta_1}\; \delta^{\mu_2\nu_2}_{\alpha_2\beta_2}\; \ldots\; \delta^{\mu_n\nu_n}_{\alpha_n\beta_n]} \tag{2.29}$$

as

$$\mathcal{L} \;=\; \sum_{n=0}^{|D/2|} a_n\; \mathcal{R}^{(n)},\;\; \mathcal{R}^{(n)} \;=\; \frac{1}{2^n}\; \delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n}\; \prod\; \mathcal{R}^{\alpha_r\beta_r}_{\mu_r\nu_r} \tag{2.30}$$

where $|D/2|$ is the integer part of $D/2$; a_n are coupling constants of dimensions $(length)^{2n-D}$. In the $8D$ cotangent bundle case T^*M the range of indices is $\alpha,\beta = 1,2,\ldots,8$; $\mu,\nu,\ldots,8$. The first four indices correspond to the four-dim spacetime, and the last four indices to the momentum space. Despite the product of curvatures, the advantage of Lanczos-Lovelock Lagrangians is that they lead to field equations containing only derivatives of the metric up to *second* order, and in arbitrary number of dimensions.

The field equations associated with the above actions $\mathbf{S}$ are obtained via an Euler variation with respect to the independent fields appearing in the description of the metric of the cotangent bundle G_{MN} displayed in eq-(2.7)

$$\frac{\delta \mathbf{S}}{\delta g_{ij}} = 0,\;\; \frac{\delta \mathbf{S}}{\delta h_{ab}} = 0,\;\; \frac{\delta \mathbf{S}}{\delta N_i^a} = 0 \tag{2.31}$$

it is beyond the scope of this letter to find solutions to these very complicated set of differential equations. One could also follow a different approach to gravity in curved phase spaces described in section **1**. By recurring to eqs-(1.11-1.15), and writing the metric in *block* diagonal form which allows to factorize the determinant of the metric as $(det g_{ij})(det h_{ab})$, one could study the analog of the

Einstein vacuum field equations

$$\mathcal{R}_{ij} \; - \; \frac{1}{2} (\; \mathcal{R} + \mathcal{S} \;) \; g_{ij} \; = 0 \; ; \quad S_{ab} \; - \; \frac{1}{2} (\; \mathcal{R} + \mathcal{S} \;) \; h_{ab} \; = 0 \tag{2.32}$$

and supplemented by the equations

$$\frac{\delta \mathcal{R}}{\delta N_i^a} \; + \; \frac{\delta \mathcal{S}}{\delta N_i^a} \; = \; 0 \tag{2.33}$$

where the spacetime and internal space scalar curvatures are, respectively,

$$\mathcal{R} \; = \; \delta_i^j \; R^i_{kjl} \; g^{kl}; \;\; \mathcal{S} \; = \; \delta_b^d \; S_d^{abc} \; h_{ac} \tag{2.34}$$

These type of equations were studied by Vacaru [9] and some solutions were found in some special cases. We leave the study of the field equations described by eqs-(2.31) for future work.

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