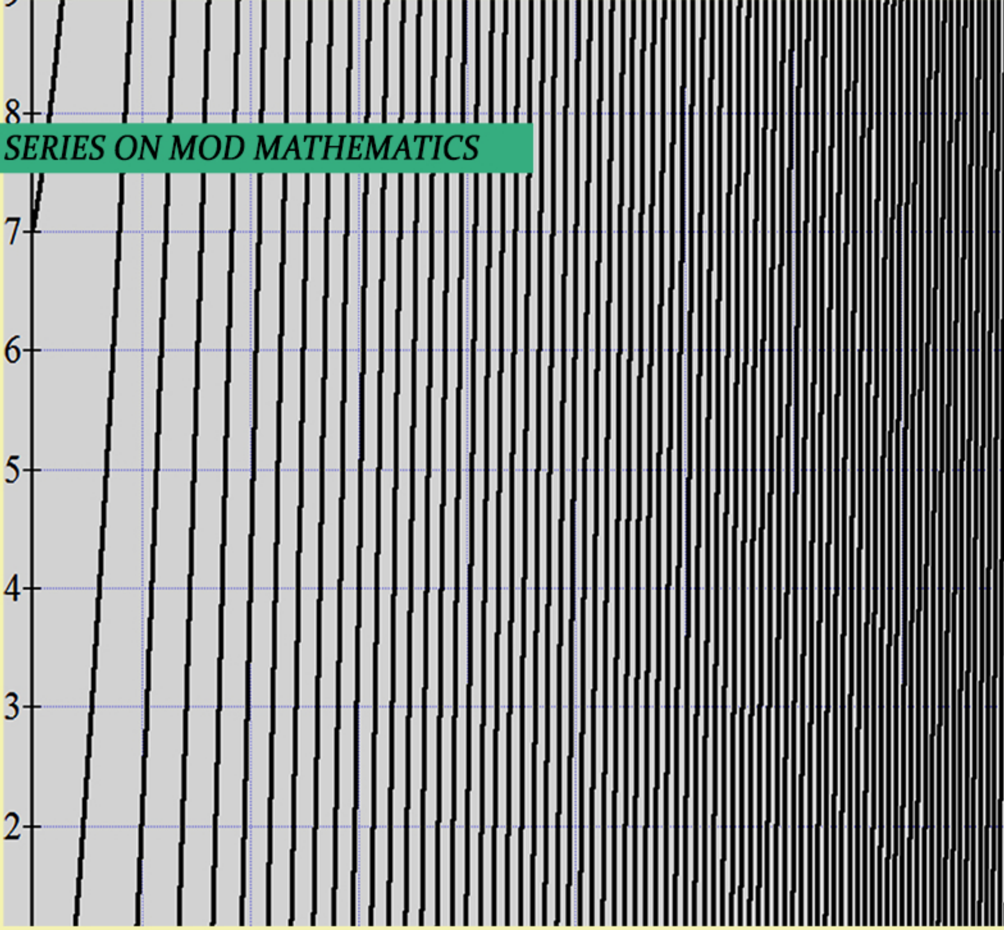
SERIES ON MOD MATHEMATICS



MOD FUNCTIONS A NEW APPROACH TO FUNCTION THEORY

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A New Approach to Function Theory

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CONTENTS

PREFACE

In this book the notion of MOD functions are defined on MOD planes. This new concept of MOD functions behaves in a very different way. Even very simple functions like y = nx has several zeros in MOD planes where as they are nice single line graphs with only (0, 0) as the only zero.

Further polynomials in MOD planes do not in general follows the usual or classical laws of differentiation or integration.

Even finding roots of MOD polynomials happens to be very difficult as they do not follow the fundamental theorem of algebra, viz a n^{th} degree polynomial p(x) in MOD plane or MOD intervals do not have n roots for + and × are defined on them, do not satisfy the distributive laws.

These drawbacks becomes a challenging issue. So study in this direction is open. In fact several open conjectures are proposed in this book. However this paradigm of shift will give new dimension to mathematics.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

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Chapter One

SPECIAL TYPE OF DECIMAL POLYNOMIALS

In this chapter we define several special types of decimal polynomials. The motivation for constructing decimal polynomials is that while doing integration or differentiation the power of x in the polynomial p(x) is either increased by 1 or decreased by 1 respectively.

We want to increase or decrease to any value inbetween 0 and 1. For this we have built decimal polynomial rings.

Further using these decimal polynomial rings we will be in a position to define, describe and develop the notion of MOD calculus.

DEFINITION 1.1: *Let $R[x^{0.1}]$ be the decimal polynomial ring generated by the decimal power of $x^{0.1}$.*

$$R[x^{0.1}] = \left\{ \sum_{i=0.1}^{\infty} a_i x^i \,\middle|\, a_i \text{ are reals} \right\}.$$

Clearly $R[x^{0.1}]$ is a ring for we see for any $p(x^{0.1})$ and $q(x^{0.1})$ in $R[x^{0.1}]$; $p(x^{0.1}) + q(x^{0.1}) \in R[x^{0.1}]$.

For instance if

$$p(x^{0.1}) = 3x^{0.3} + 4.7x^{0.6} + 8.3x^{0.9} - 13.7x^{8} + 3$$

and

$$q(x^{0.1}) = 7x^{0.1} - 9x^{0.6} + 14.7x^{0.9} - 3x^{8} - 10 \in R[x^{0.1}]$$

then

$$p(x^{0.1}) + q(x^{0.1}) = -7 + 7x^{0.1} + 3x^{0.3} - 4.3x^{0.6} + 23x^{0.9} - 16.7x^{8} \in R[x^{0.1}].$$

This is the way addition operation is performed on $R[x^{0.1}]$.

We see if

$$p(x^{0.1}) = 3 - 2x^{0.3} + 7x^{6.3}$$

and

$$q(x^{0.1}) = 2 + 4x^{0.5} + 10x^{0.7} \in R[x^{0.1}]$$

then

$$p(x^{0.1}) \times q(x^{0.1}) = (3 - 2x^{0.3} + 7x^{6.3})(2 + 4x^{0.5} + 10x^{0.7})$$

$$= 6 - 4x^{0.3} + 14x^{6.3} + 12x^{0.5} - 8x^{0.8} + 28x^{6.8} + 30x^{0.7} - 20x + 70x^{7} \in R[x^{0.1}].$$

This is the way + and × operation are performed. It can be easily verified $R[x^{0.1}]$ is a commutative ring of infinite order.

Now a natural question would be how to solve equations in $R[x^{0.1}]$.

The easiest way is $R[x^{0.1}]$ can be mapped isomorphically on to $R[x]$ by the map $\eta : R[x^{0.1}] \to R[x]$

$$(\text{or } \eta : R[x] \to R[x^{0.1}])$$

$$\text{by } \eta(r) = r \text{ if } r \in R$$

$$\eta(x^{0.1}) = x \text{ (or } \eta(r) = r \text{ for all } r \in R \text{ and } \eta(x) = x^{0.1})$$

$$\eta(x^{42}) = x^{4.2} \text{ and } \eta(x^{3}) = x^{0.3}.$$

So if $p(x) = x^2 - 4x + 4 \in R[x]$, then

$$p(x^{0.1}) = x^{0.2} - 4x^{0.1} + 4.$$

So solving p(x) is easy and from which we conclude the roots of $p(x^{0.1})$ are $x^{0.1} = 2, 2$.

On similar lines we can further lessen the power of x and define $R[x^{0.01}]$; this will also be a ring.

Clearly $R[x] \subseteq R[x^{0.1}] \subseteq R[x^{0.01}]$.

$R[x^{0.01}]$ is also a decimal polynomial ring generated by $x^{0.01}$.

We call these decimal polynomial rings as MOD polynomials or very small polynomials with real coefficients [5-10].

However we want to keep the power of x as $x^{0.1}$ or $x^{0.01}$ or $x^{0.001}$ or $x^{0.0001}$ and so on. This is the condition we impose for some easy working.

For $\eta : R[x^{0.001}] \to R[x]$ is obtained by dividing/multiplying the power of x by thousand.

Likewise $\eta : R[x] \to R[x^{0.001}]$ is got by dividing the power of x by 1000.

So η can also be realized as the homomorphism of a special type.

We make one special condition for the sake of complatability; we by no means take powers of $x^{0.1}$ as decimal powers; that is $\left(x^{0.1}\right)^n$, where n is always assumed to be a positive integer greater than or equal to 1.

So we have to take $\left(x^{0.1}\right)^{0.7}$ or any such sort. As far as possible we in this book define only; $\left(x^{0.1}\right)^{m} = x^{0.m}$ (m < 10) where m is an integer

$$\left(x^{0.1}\right)^{8} = x^{0.8},$$
$$\left(x^{0.1}\right)^{21} = x^{2.1},$$
$$\left(x^{0.1}\right)^{125} = x^{12.5}$$

and so on.

Likewise for $x^{0.01}$ also we do not raise to a fractional power of $x^{0.01}$. $\left(x^{0.1}\right)^{n}$ is defined if and only if $n \in N$.

Under these conditions and constraints only we work, that is why we call it as MOD polynomial real rings.

Study of MOD polynomial real rings can be done as a matter of routine.

Now we define MOD polynomial complex rings $C[x^{0.1}]$, $C[x^{0.01}]$ and $C[x^{0.001}]$ and so on.

We as in case of reals work with complex MOD polynomials.

Next we proceed onto describe MOD modulo integer polynomials $Z_n[x^{0.1}]$, $Z_n[x^{0.01}]$, $Z_n[x^{0.001}]$ and so on.

We can also have MOD interval modulo integer polynomials.

$[0, m)[x^{0.1}]$, $[0, m)[x^{0.01}]$, $[0, m)[x^{0.001}]$ and so on.

We will be using these MOD polynomial rings to build the MOD calculus.

Chapter Two

MOD POLYNOMIAL FUNCTIONS

The concept of MOD planes was introduced in [24]. Here we discuss about the MOD polynomial functions. Let [0, m) be the MOD interval ($m \geq 2$).

$p(x) \in [0, m)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, m) \right\}$ is defined as the MOD polynomials.

y = f (x) is called the MOD polynomial function in the independent variable x and y = f (x) is defined in the MOD real plane R_n (m).

The following facts are innovative and important.

(i) A polynomial defined over the MOD plane is continuous or otherwise depending on m of R_n (m).

(ii) For the same p (x) we have various types of associated graphs depending on R_n (m).

(iii) p (x) has infinite number of properties in contrast to $p(x) \in R[x]$ which is unique. This flexibility is enjoyed by MOD polynomial function which makes it not only interesting but useful in appropriate applications.

We will illustrate this situation by some examples.

Example 2.1: Let p (x) = x + 4 ∈ R[x] (R reals). The graph of x + 4 is as follows:

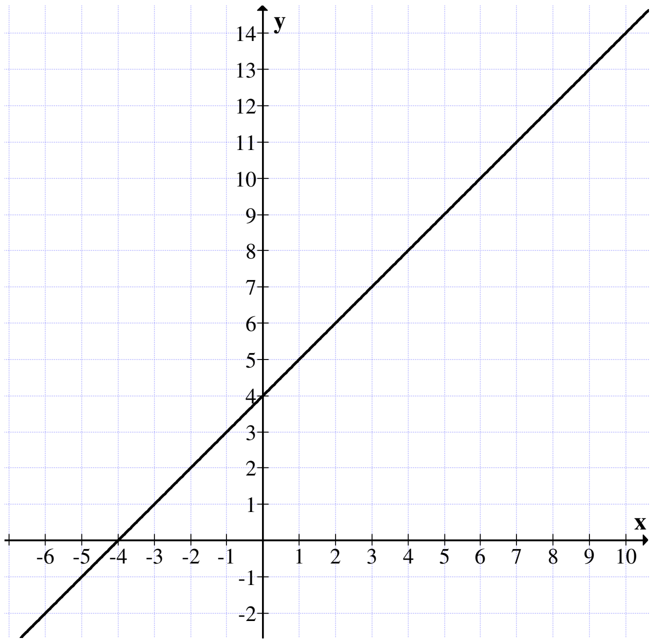

Figure 2.1

Clearly the function y = x + 4 is a continuous curve.

Now y = x + 4 is transformed into the MOD plane $R_n(2)$ as y = x. The graph of y = x in the MOD plane $R_n(2)$ is given in Figure 2.2.

Clearly the graph of y = x + 4 where the function is transformed as y = x is again a continuous curve.

Now let us find the curve of y = 4 + x in the MOD plane $R_n(3)$. The function y = x + 4 is transformed to y = x+1 in the MOD plane $R_n(3)$. The graph of y = x + 1 is given in Figure 2.3.

Clearly the function y = x + 1 is not a continuous function in the MOD plane $R_n(3)$.

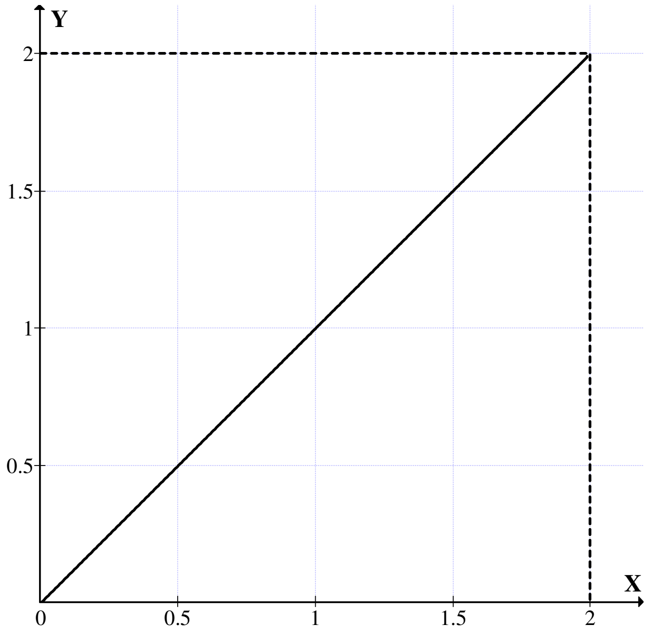

Figure 2.2

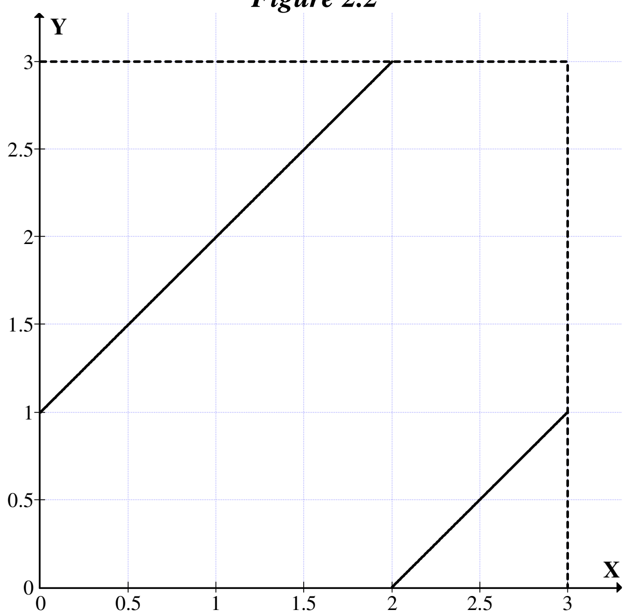

Figure 2.3

In fact two pieces of continues curves in the intervals [0, 2) and [2, 3).

Thus the function y = x + 1 is continuous in the interval [0, 2) then it is again continuous in the interval [2, 3).

At x = 2, y = 0. Thus the function increases from 1 to 1.999 in the interval [0, 2) and drops to 0 at x = 2 and again increases from 0 to 0.9999 …, in the interval [2, 3).

Next we consider the function y = x + 4 in the MOD plane $R_n(4)$. The graph of y = x in $R_n(4)$ is given in Figure 2.4.

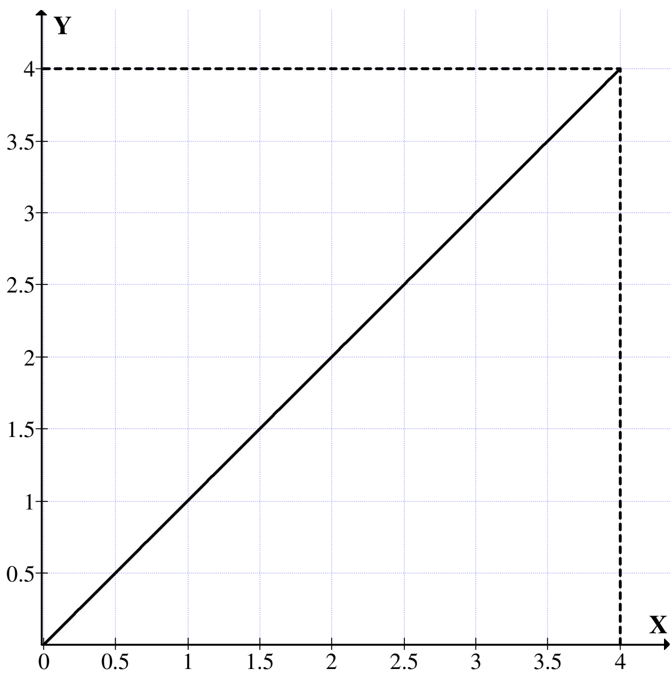

Figure 2.4

We see in case of the MOD plane $R_n(4)$; y = x + 4 that is y = x is a straight line similar to the function in the MOD plane $R_n(2)$. The function is continuous in $R_n(4)$.

Now we study the function y = x + 4 in the MOD plane $R_n(5)$. The associated graph is given in Figure 2.5.

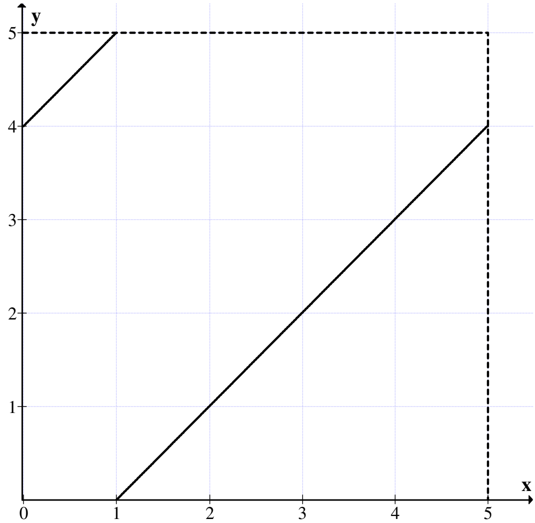

Figure 2.5

We see again the function y = x + 4 is not a continuous curve in $R_n(5)$. The function is continuous in [0, 1) and at 1 it drops to 0 and again y = x + 4 is continuous from [1, 5).

Thus the function is an increasing function in [0, 1) at 1 drops to 0 and again increasing in the interval [1, 5).

Now we define the function y = x + 4 in the MOD plane $R_n(6)$.

The graph of the function in the MOD plane $R_n(6)$ is given in Figure 2.6.

The function y = x + 4 is increasing in the interval [0, 2) and is continuous.

Further the function y = x + 4 is continuous in the interval [2, 6) and drops to 0 at x = 2.

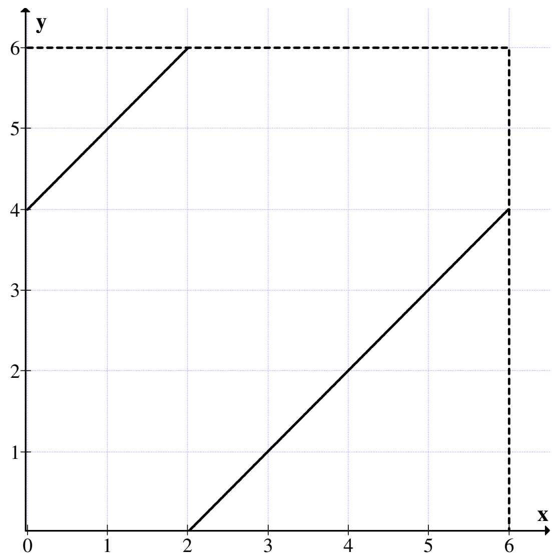

Figure 2.6

Consider the function y = x + 4 in the MOD plane $R_n(7)$. The graph of the function y = x + 4 is given in Figure 2.7.

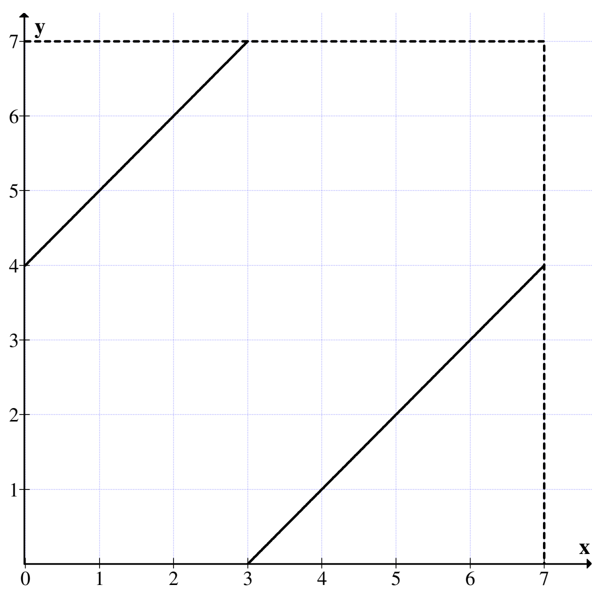

Figure 2.7

Here also the function is not continuous. At x = 3 it drops to zero. The function is increasing in the interval [0, 3) and drops to zero at 3 and increasing in the interval [3, 7).

The range values are [4, 7) and [0, 4).

Now we see the graph of the function y = x + 4 in the MOD plane $R_n(8)$.

The graph of the function y = x + 4 in the MOD plane $R_n(8)$ is as given as Figure 2.8.

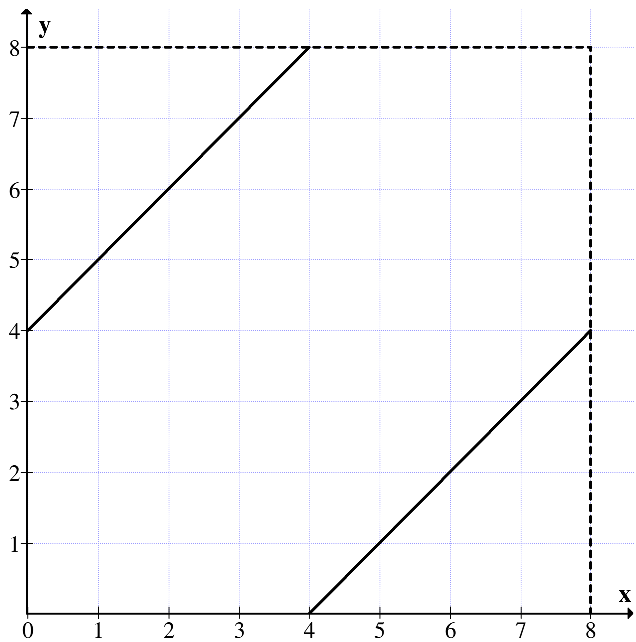

Figure 2.8

The function increases from 4 to 7.999 and drops to zero at x = 4 and again increases from 0 to 3.999.

Thus we see the function y = x + 4 in the MOD plane R_n (m) (m ≥ 4) is as follows:

$y = x + 4$ increases in the interval $[0, m–4)$ and drops at $m – 4$ to zero and again increases from $[m–4, m)$.

The graph of $y = x + 4$ in the MOD plane $R_n(m)$ is as follows:

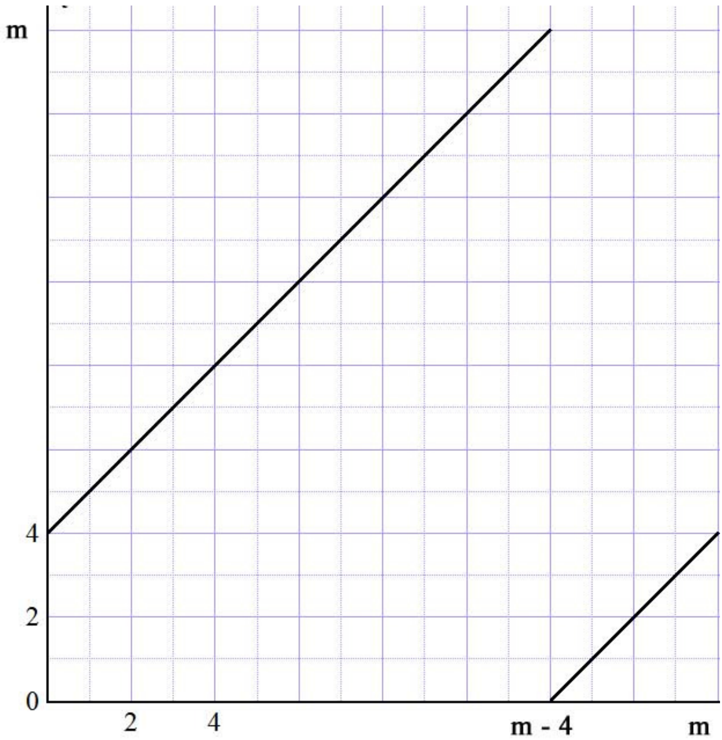

Figure 2.9

Example 2.2: Next we study the function $y = x^2 + 1$ in the real plane and then the graph of the function $y = x^2 + 1$ in the MOD plane $R_n(m)$. The graph of $y = x^2 + 1$ in the real plane R is given in Figure 2.10.

The function is a continuous curve.

Now we study the function $y_n = x^2 + 1$ in the MOD plane $R_n(2)$. At $x_1 = 1$ $y_n = 0$, at $x_2 = 1.7320508076$ $y = 0$. The associated graph is given in Figure 2.11.

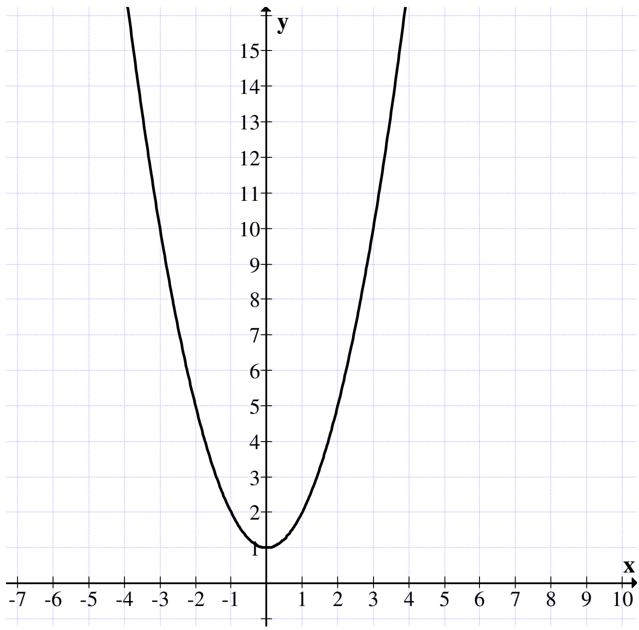

Figure 2.10

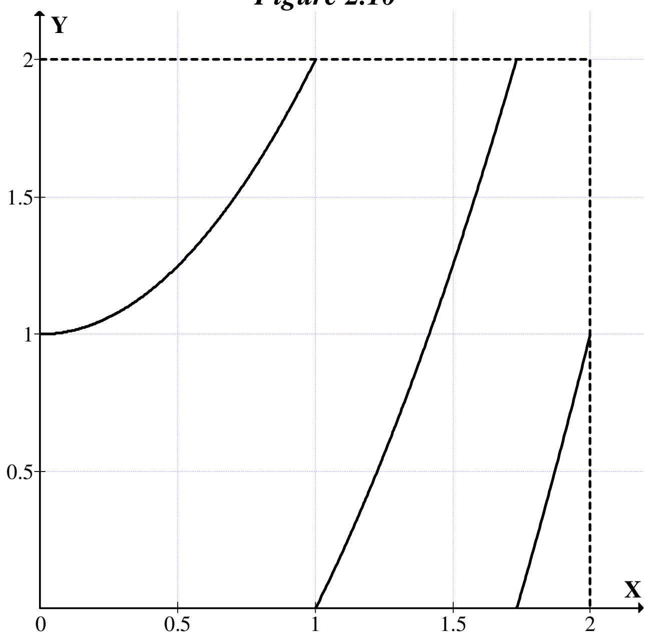

Figure 2.11

The function is not a continuous graph.

The function is continuous in the interval [0, 1), [1, 1.7320508074) and [1.7320508076, 2) and at x_1 and x_2 drops to zero.

Next we study the function $y = x^2 + 1$ in the MOD plane $R_n(3)$.

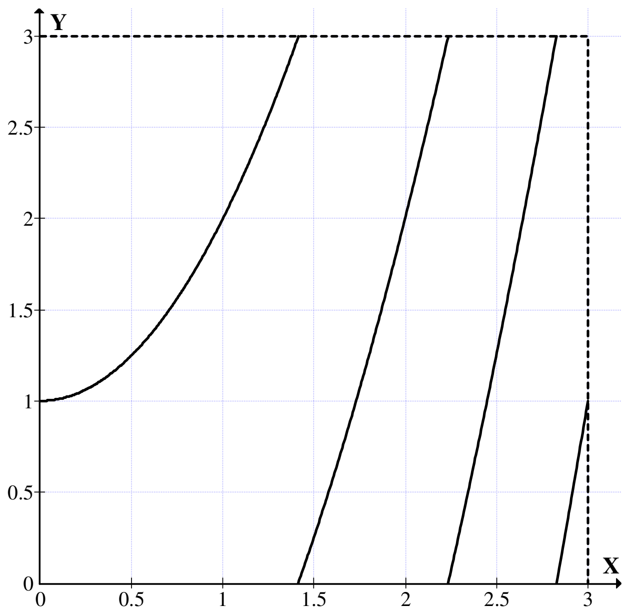

Figure 2.12

At x_1 = 1.4142135625, y = 0 in $R_n(3)$.

At x_2 = 2.2360679776, y = 0

At x_3 = 2.8284271248, y = 0

Thus $y_n = x^2 + 1$ in $R_n(3)$ has three zeros given by x_1, x_2 and x_3 we see $y_n = x^2 + 1$ in $R_n(2)$ also has only three zeros.

The function is continuous at [0.1, 1.4142135625…) and [1.4142135625, 3) drops to zero at, 1.4142135625.

At x = 2.2360679776, y = 0.

At x = 2.8284271248, y = 0.

Next we study the MOD function $y = x^2 + 1$ on the MOD plane $R_n(4)$. The graph of the function is as follows:

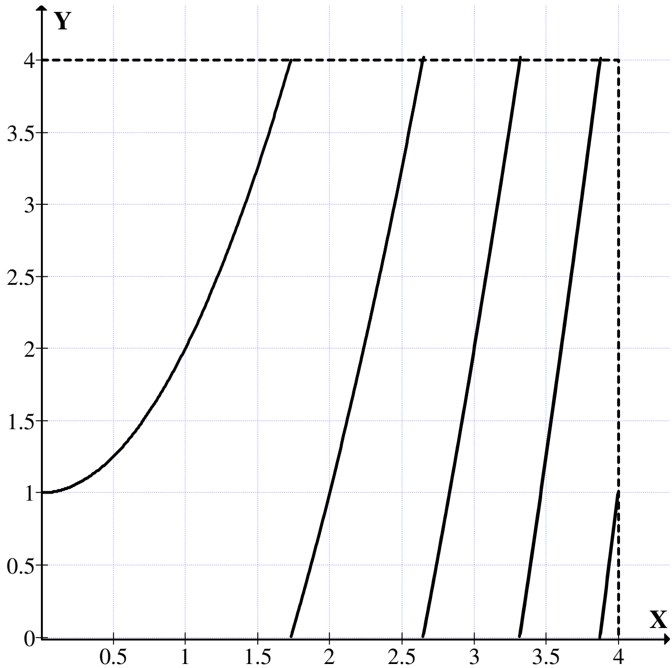

Figure 2.13

At x = 1.7320508076, y = 0.

At x = 2.645751311, y = 0.

At x = 3.31662479, y = 0.

Thus when $y = x^2 + 1$ is the function defined on R_n (m) $m \geq 2$.

The graph of the function in the plane R_n (m).

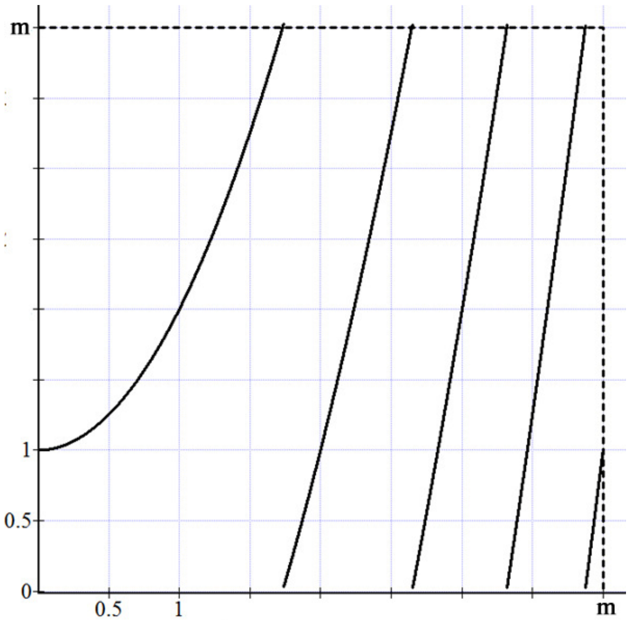

Figure 2.14

where for some t = s.9…. The function drops to zero.

The graph is not continuous has several branches and each branch is continuous in the interval [0, t…).

It is left as an open conjecture to find the zeros of $y = x^2 + 1$ in R_n (m).

(i) m prime.
(ii) m odd.
(iii) m even.

Example 2.3: Let us consider the function $y = x^3 + 1$ in the real plane and the MOD real planes R_n (m); m ≥ 2. The associated graph is given in Figure 2.15.

Now we consider the same function $y = x^3 + 1$ in the MOD plane $R_n(2)$ which is given in Figure 2.16.

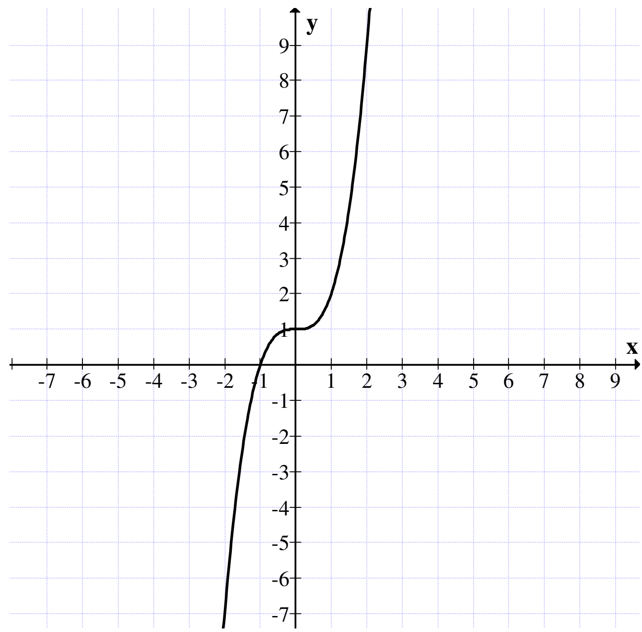

Figure 2.15

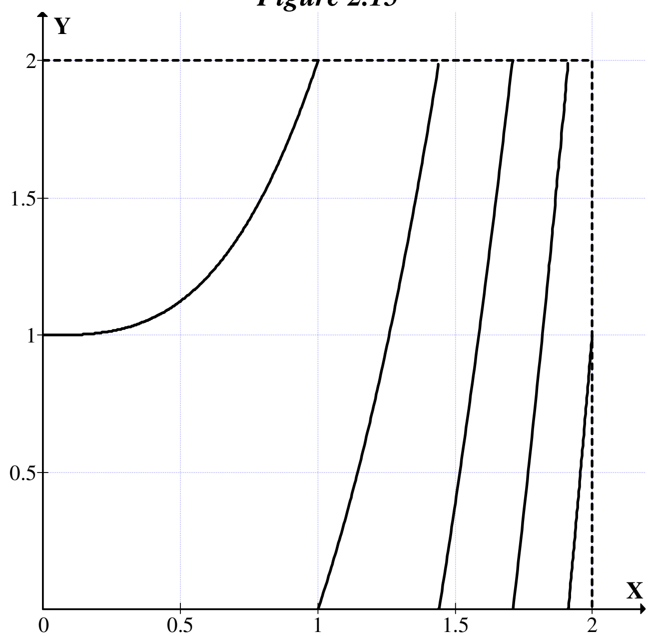

Figure 2.16

$y = x^3 + 1$ representation in the MOD plane $R_n(2)$.

The graph is not continuous, it is continuous and increasing in the interval [0, 1) and drops to zero at x = 1 and steady increases up to y = 1 in the interval [1, 2).

Next we study the function $y = x^3 + 1$ in $R_n(3)$.

For $x_1 = 1.2599210498$ the y value is zero.

There is a zero lying between 1.7 and 1.71.

At $x_2 = 1.7099759467$, y = 0.

At $x_3 = 2$, y = 0

Now at

$$x_4 = 2.2239800909, y = 0.$$

When

$$x_5 = 2.410142264 \text{ we get } y = 0.$$

When

$$x_6 = 2.5712815908; y = 0.$$

$$\text{At } x_7 = 2.7144176164 \text{ we get } y = 0.$$

$$\text{At } x_8 = 2.84386698 \text{ we get } y = 0.$$

$$\text{At } x_9 = 2.9624960684; y = 0.$$

Thus $y_n = x^3 + 1$ is not continuous and it has 9 zeros and there are 10 discontinuous curves.

Given by the Figure 2.17.

$(x_1)^3 + 1 = 3 \pmod 3 = 0,$
$(x_2)^3 + 1 = 6 \pmod 3 = 0,$
$(x_3)^3 + 1 = 9 \pmod 3 = 0,$
$(x_4)^3 + 1 = 12 \pmod 3 = 0,$
$(x_5)^3 + 1 = 15 \pmod 3 = 0,$

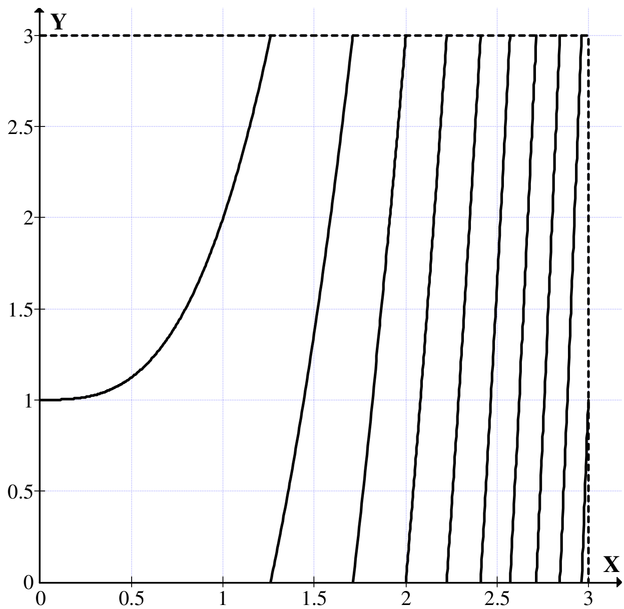

Figure 2.17

$(x_6)^3 + 1 = 18 \pmod 3 = 0,$
$(x_7)^3 + 1 = 21 \pmod 3 = 0,$
$(x_8)^3 + 1 = 24 \pmod 3 = 0$ and
$(x_9)^3 + 1 = 27 \pmod 3 = 0.$

Thus it is conjectured if $y_n = x^p + 1$ in the MOD plane $R_n(p)$; p a prime; will $y_n = x^p + 1$ have p^2 number of zeros says $x_1, \ldots, x_{p^2}$ with $(x_1)^p + 1 = p \pmod p = 0$ and so on.

$y = 0$ occurs when $x \in (1.2, 1.3)$ and the function is increasing upto $x = 1.7$, $y = 2.913$.

When $x = 2.8$, $y = 2.952$ when $x = 2.9$, $y = 1.38$.

The pattern of the function $y = x^3 + 1$ in the MOD plane $R_n(3)$ needs more investigation, thus the above figure gives most of the branches of the curve in the MOD plane $R_n(3)$.

Now we consider the function $y = x^3 + 1$ in the MOD plane $R_n(4)$.

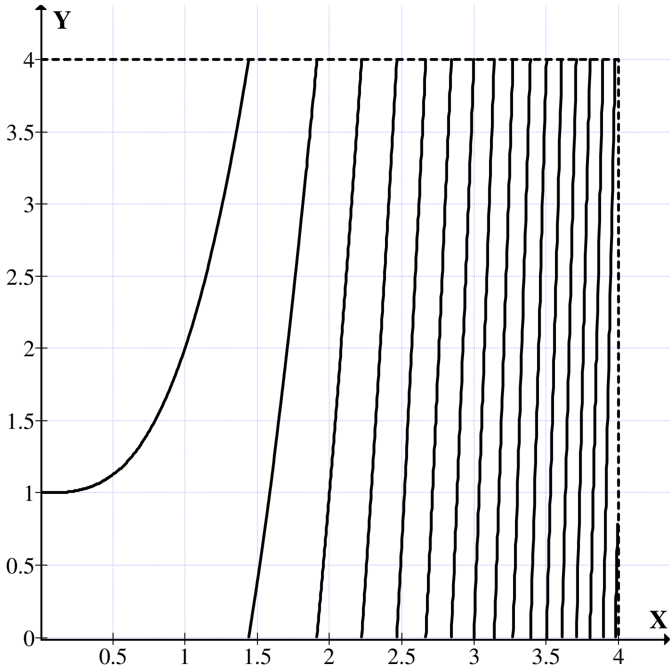

Figure 2.18

when x = 1.4	y = 3.744
when x = 1.42	y = 3.862
when x = 1.44	y = 3.985
when x = 1.49	y = 0.3079
when x = 1.48	y = 0.241792
when x = 1.46	y = 0.112136
when x = 1.45	y = 0.048625

So y = 0 for a point in the interval x = 1.44 and x = 1.45

When x = 1.448	y = 0.0360
When x = 1.445	y = 0.017196
When x = 1.446	y = 0.0234
When x = 1.444	y = 0.0109
When x = 1.443	y = 0.004685307
x = 1.4425	y = 0.0015
x = 1.4422	y = 3.9995

when x = 1.4423	y = 0.000315
when x = 1.44225	y = 0.0000027
when x = 1.442249	y = 3.9999964
when x = 1.4422499	y = 0.00000206
when x = 1.4422496	y = 0.000000185
when x = 1.44224958	y = 0.00000006
when x = 1.442249575	y = 0.000000029.

Thus y = 0 for a value of x in the interval
(1.442249575, 1.44224958)

The graph of the curve needs study for the graph is discontinuous.

Finding the number of branches of $y = x^3 + 1$ in $R_n(4)$ is left as an exercise to the reader.

Finally we study the graph of $y = x^3 + 1$ in the MOD plane $R_n(5)$.

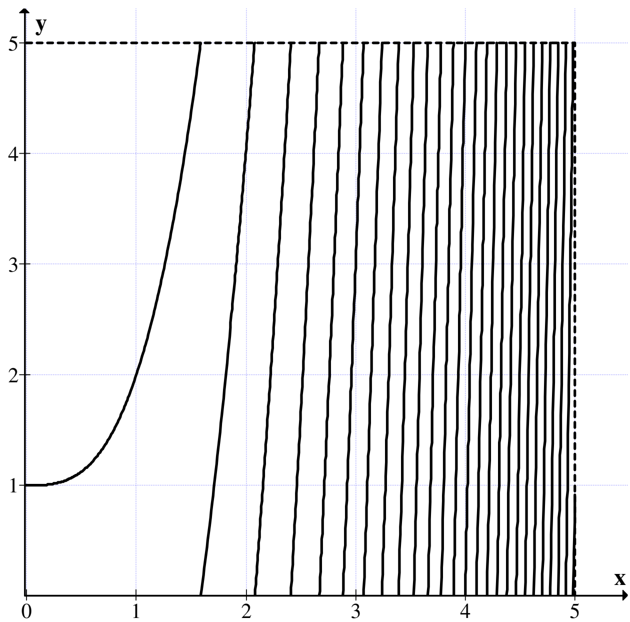

Figure 2.19

y = 0 at a point between 2.41 and 2.42.

Thus we get some four bits of the curve for the MOD equation $y = x^3 + 1$ in the MOD plane $R_n(5)$ [x].

However finding all the braches of the MOD equation $y = x^3 + 1$ in $R_n(5)$ is left as an exercise to the reader.

Now we study the same function $y = x^3 + 1$ in the MOD plane $R_n(6)$.

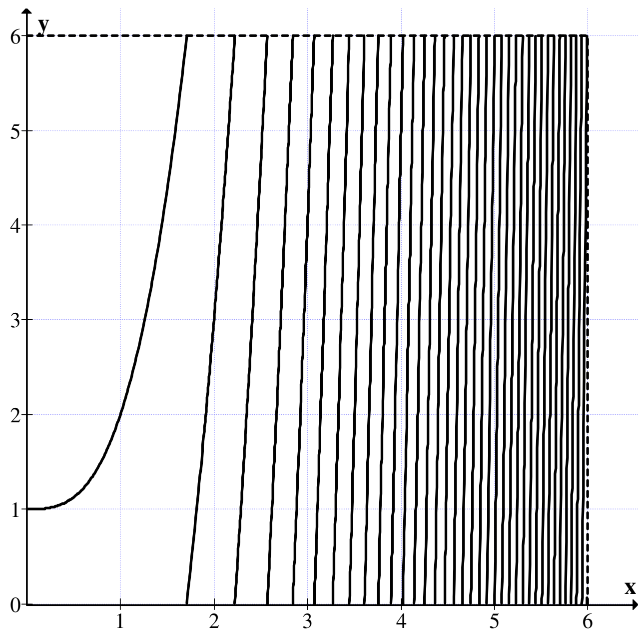

Figure 2.20

y = 0 for some x = (2.5, 2.6). For x = 3.05 we get y = 5.37 y = 0 for some $x \in$ (3.07, 3.08).

For y = 0 for some $x \in$ (4.02, 4.03).

This happens to be a open conjecture to study the number of zeros of $x^3 + 1 \in R_n$ (m); m = 2, 3, 4, …, m ($m < \infty$).

Thus when $y = x^3 + 1$ is the function defined on R_n (m), $m \geq 2$. The graph of the function in the plane R_n (m).

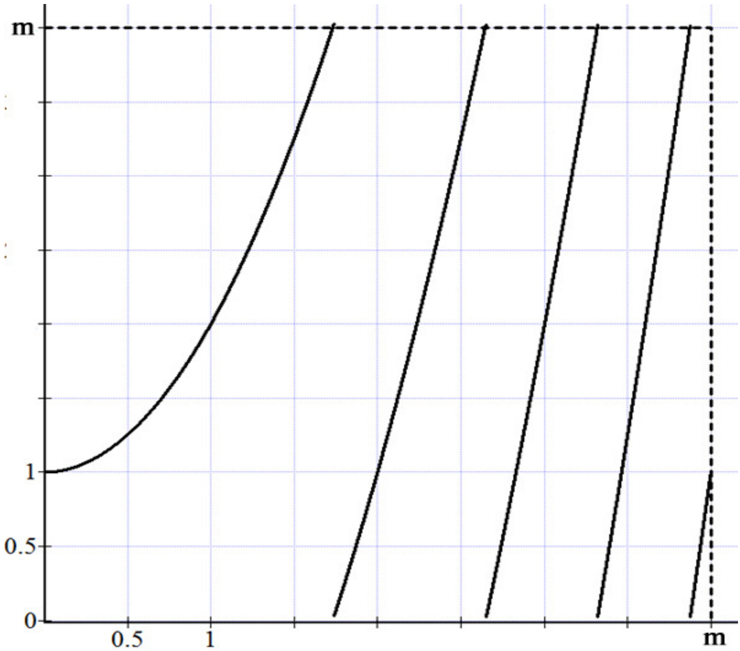

Figure 2.21

where for some t = s.9… The function drops to zero. The graph is not continuous one has to find all the branches are continuous in the interval [0, t…) and at t.9 drops to zero and again the function is a continuous increasing function in [t…m).

Even a simple function $x^3 + 1 = y$ which remains as it is in every MOD plane after transformation has very many different forms and none of them are continuous in the MOD plane.

Further it is another open conjecture to study how many bits of curves the function $y = x^3 + 1$ will be represented in the MOD plane R_n (m).

We mean by bits the number of continuous branches of the graph. Thus it is again dependent on the number of zeros a function $y = x^3 + 1$ has in R_n (m)[x].

This function $y = x^3 + 1$ is defined as the unchangable universal function as $y = x^3 + 1 \in R[x]$ remains the same on every MOD plane R_n (m)[x].

DEFINITION 2.1: *A function $y = f(x) \in R[x]$ which remains the same over R_n (m)[x] for every $2 \leq m < \infty$ is defined as the unchangeable universal MOD function.*

We will give examples of them.

Example 2.4: Let $y = x^2 + 1, x^3 + 1, x^5 + 1, x^4 + 1, \ldots, x^{t+1} + 1$, $(2 \leq t < \infty)$, $x^2 + x + 1, x^3 + x + 1, x^3 + x^2 + 1$ and so on.

$x^t + x^{t-1} + \ldots + 1$, $x^t + x^{t-r_1} + \ldots + 1$, $0 < r_1 < t$ are all unchangeable universal MOD functions.

DEFINITION 2.2: *Let $y = f(x) \in R[x]$. If $y = f(x)$ changes depending on R_n (m)[x] then we define $y = f(x)$ as a changeable universal MOD functions.*

We will give examples of them.

Example 2.5: Let $y = x^7 + 9x + 1 \in R[x]$ be the function. This is a changeable universal function in the MOD polynomial R_n (m)[x]. However for $m \geq 10$ this function $f(x) = x^7 + 9x + 1$ remains unchangeable. But for all $2 \leq m \leq 9$ the function is a changeable function. $y = x^7 + 9x + 1 = x^7 + x + 1$ in $R_n(2)$.

$y = x^7 + 9x + 1 = x^7 + 1$ in $R_n(3)$,

$x^7 + 9x + 1 = x^7 + x + 1$ in $R_n(4)$,

$x^7 + 9x + 1 = x^7 + 4x + 1$ in $R_n(5)$,

$x^7 + 9x + 1 = x^7 + 3x + 1$ in $R_n(6)$,

$x^7 + 9x + 1 = x^7 + 2x + 1$ in $R_n(7)$,

$x^7 + 9x + 1 = x^7 + x + 1$ in $R_n(8)$ and

$x^7 + 9x + 1 = x^7 + 1$ in $R_n(9)$.

Thus this function is a changeable one for $m \leq 9$.

Next we give an example with graphs.

Example 2.6: Let $y = x^2 + 9x + 1 \in R[x]$. Now
$y = x^2 + 9x + 1 = x^2 + x + 1 \in R_n(2)[x]$.

We just describe the graph of them. The graph of $y = x^2 + x + 1$ in the plane $R_n(2)$.

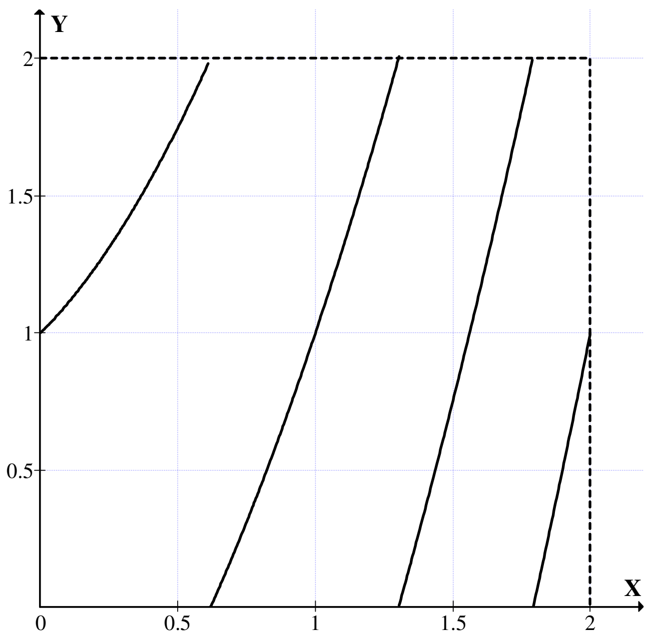

Figure 2.22

$y = 0$ for $x \in (0.6, 0.7)$

$y = 0$ for $x \in (1.3, 1.4)$

For

$$x = 1.79\ y = 1.99;\ y = 0 \text{ for } x \in (1.79, 1.799)$$

For

$$x = 1.98\ y = 0.9004.$$

So four bits of curves and the function attain zero in three points mentioned above.

Let $y = x^2 + 9x + 1 \in R[x]$ this function in $R_n(3)$ is $y = x^2 + 1$.

The graph of $x^2 + 1$ in the MOD plane $R_n(3)$ is as follows:

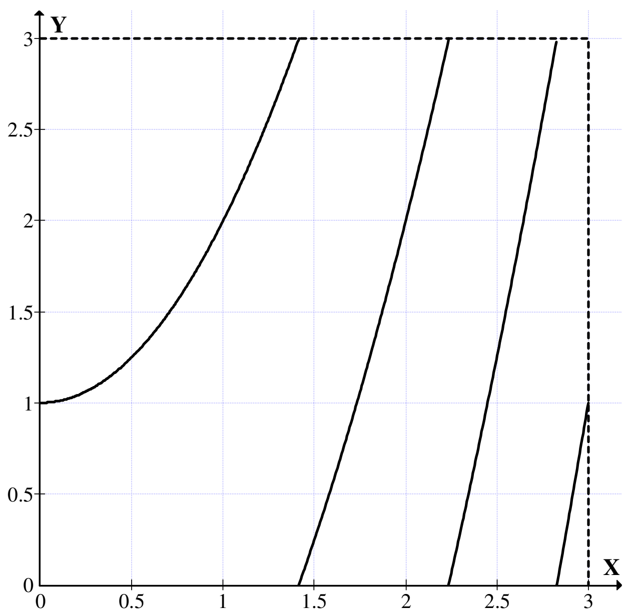

Figure 2.23

$y = 2.9881$ for $x = 1.41$ for $x = 1.415$ $y = 0.002225$ so for some $x \in (1.41, 1.415)$; $y = 0$.

For $x = 2.23$ $\quad\quad$ $y = 2.9729$

For x = 2.24 y = 0.0176

For x = 2.25 y = 0.0625.

Thus for some x ∈ (2.23, 2.24) y = 0.

For x = 2.9 y = 0.41. For x = 2.99; y = 0.9401.

Thus we get the above graph which has only two zero.

Next we consider the function $y = x^2 + 9x + 1 \in R[x]$ in the MOD plane $R_n(4)$.

The graph $y = x^2 + x + 1$ in the MOD plane $R_n(4)$ is as follows:

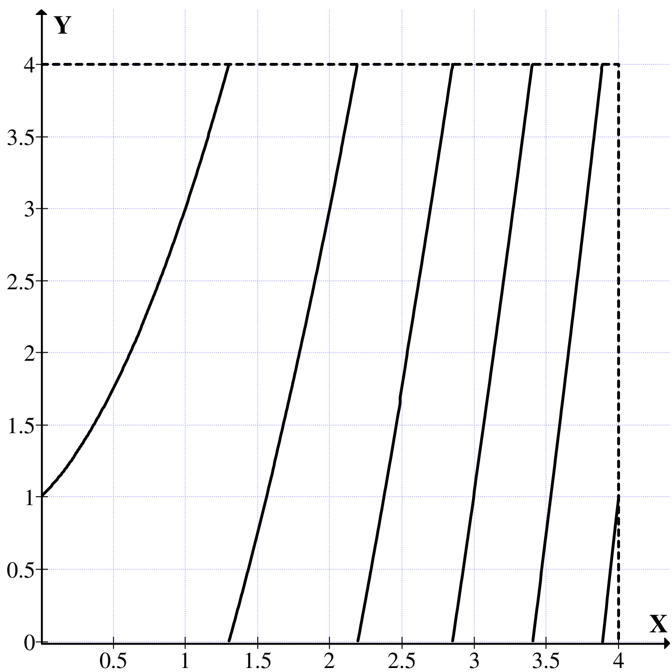

Figure 2.24

When x= 1.3, $y = 3.99$.

When $x = 1.5$, $y = 0.75$

$x = 1.4$, $y = 0.36$

$x = 1.35$, $y = 0.1725$

$x = 1.31$, $y = 0.026$

For some $x \in (1.3, 1.31)$ there a y such that $y = 0$.

For $x = 2.1$, $y = 3.51$

For some $x \in (2.18, 2.2)$, $y = 0$.

For $x = 3.4$ $y = 3.96$

For $x = 3.402$ $y = 3.97$

For $x = 3.405$ $y = 3.999$

For $x = 3.45$ $y = 0.3525$

Thus for some $x \in (3.405, 3.41)$, $y = 0$.

For $x = 3.9$; $y = 0.11$

For $x = 3.99$, $y = 0.9101$.

Thus $x^2 + x + 1$ has three zeros in the MOD plane $R_n(4)$.

Now the function $y = x^2 + 9x + 1$ in the MOD plane $R_n(5)$ is $y = x^2 + 4x + 1$.

The graph of $y = x^2 + 4x + 1$ in $R_n(5)$ is as follows:

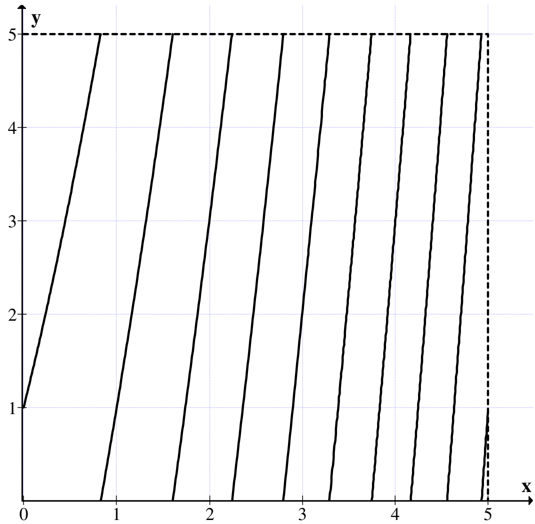

Figure 2.25

For some x ∈ (0.83, 0.84) y = 0.

For x = 1.6 we get y = 4.96 for x = 1.7, y = 0.69.

So for some x ∈ (1.6, 1.7) we have y = 0

For x = 2 we get y = 2.

For some x ∈ (2.2, 2.3) we have y = 0.

For some x ∈ (3.2, 3.3) we have y = 0.

For x = 4.1 y = 4.21 and for x = 4.2 y = 0.44

So for some x ∈ (4.1, 4.2) we have y = 0.

Thus the function $y = x^2 + 4x + 1$ has 5 zeros in the MOD plane $R_n(5)$.

The zeros lie in the intervals (0.83, 0.84), (1.6, 1.7), (2.2, 2.3), (2.75, 2.8), (3.2, 3.3), (4.1, 4.2), (3.7, 3.8), (4.5, 4.6) and (4.9, 4.95).

This MOD function $y = x^2 + 4x + 1 \in R_n(5)$ has 9 zeros and there are 10 bits of curves of which form the parts of the function.

Next we study the function $y = x^2 + 9x + 1$ in the MOD plane $R_n(6)$.

In $R_n(6)$ the function $y = x^2 + 3x + 1$.

Now we give the graph of $y = x^2 + 3x + 1$ in $R_n(6)$.

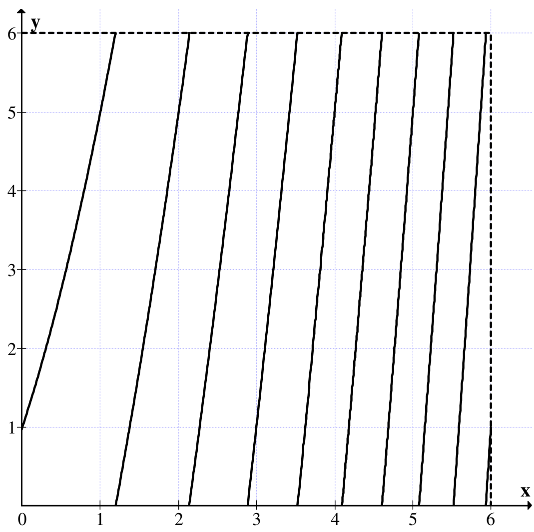

Figure 2.26

For some $x \in (1.1, 1.2)$ we have $y = 0$.

For $x = 1.3$, $y = 0.59$
For $x = 1.9$, $y = 4.31$
For $x = 2$, $y = 5$
For $x = 2.1$, $y = 5.71$
For $x = 2.14$ $y = 5.9996$
For $x = 2.2$, $y = 0.44$.

For $x = 2.15$; $y = 0.725$ so for some $x \in (2.14, 2.15)$ we get $y = 0$.

For $x = 2.6$ $y = 3.56$
For $x = 2.8$ $y = 5.24$
For $x = 2.9$ $y = 0.11$

So for some $x \in (2.8, 2.9)$ we have $y = 0$.

For $x = 3.5$ $y = 5.75$
For $x = 3$ $y = 1$
For $x = 3.6$ $y = 0.76$

So for some $x \in (3.5, 3.6)$, we have $y = 0$.

For $x = 4$ $y = 5$
For $x = 4.05$ $y = 5.55$
For $x = 4.06$ $y = 5.66$
For $x = 4.08$ $y = 5.88$
For $x = 4.09$ $y = 5.99$
For $x = 4.1$ $y = 0.11$

so for some $x \in (4.09, 4.1)$ we have $y = 0$. For some $x \in (4.6, 4.65)$ we have $y = 0$.

For $x = 5$, $y = 5$
For $x = 5.1$ $y = 0.31$
So for $x = 5.02$ $y = 5.26$
For $x = 5.03$ $y = 5.39$
For $x = 5.09$ $y = 0.178$
For $x = 5.08$ $y = 0.04$
For $x = 5.07$; $y = 5.9149$

so for some $x \in (5.07, 5.08)$ we have $y = 0$.

For $x = 5.5$ $y = 5.75$
For $x = 5.6$ $y = 1.16$
For $x = 5.55$ $y = 0.4525$
For $x = 5.54$ $y = 0.3116$
For $x = 5.52$ $y = 0.0304$
For $x = 5.51$ $y = 5.89$

so for some $x \in (5.51, 5.52)$ we have $y = 0$.

For $x = 5.9$, $y = 5.51$ and for some $x \in (5.9, 5.96)$ we have a $y = 0$.

We see $x^2 + 3x + 1$ in the MOD plane $R_n(6)$ has several zeros.

We have only given 8 disjoint bits of continuous curves.

Next we study $y = x^2 + 9x + 1 \in R[x]$ in the MOD plane $R_n(7)$.

We see in the MOD plane $x^2 + 9x + 1$ is $x^2 + 2x + 1$.

We now analyse the graph $y = x^2 + 2x + 1$ in the MOD plane $R_n(7)$ in the following.

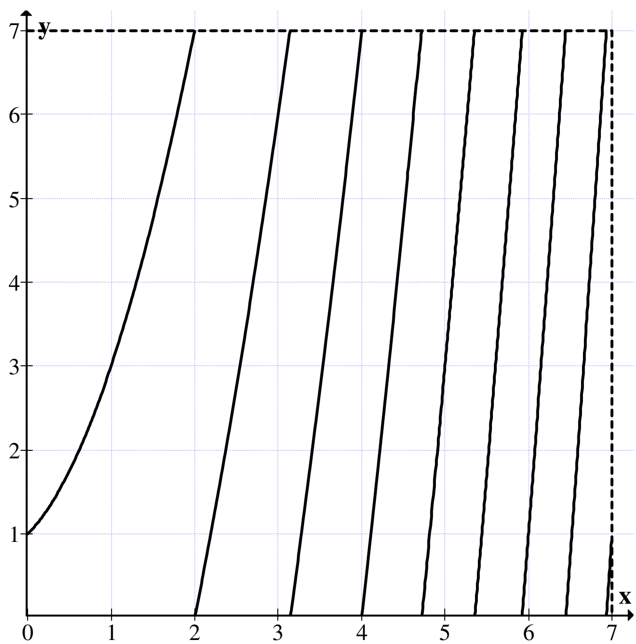

Figure 2.27

For $x \in (1.6, 1.7)$ we have $y = 0$.

For $x = 2.8$ $y = 0.44$ so for some $x \in (2.7, 2.8)$ we have $y = 0$.

For $x = 3.5$, $y = 6.25$
For $x = 3.6$; $y = 0.16$.

So for some $x \in (3.5, 3.6)$ we have $y = 0$.

For $x = 4$; $y = 4$
For $x = 4.1$ $y = 5.01$
For $x = 4.2$ $y = 6.04$
For $x = 4.3$ $y = 0.09$.

Thus for some $x \in (4.25, 4.3)$ we have $y = 0$.
For $x = 4.8$, $y = 5.64$
For $x = 4.9$ $y = 6.81$
For $x = 4.95$ $y = 0.4025$
For $x = 4.92$ $y = 0.0464$.

So for some $x \in (4.91, 4.92)$ we have $y = 0$.

For $x = 5.4$ $y = 5.96$
Further at $x = 6$ $y = 0$.

For $x \in (6.4, 6.5)$ we have $y = 0$.

For some $x \in (6.93, 6.94)$ we have $y = 0$.

Thus the curve has several bits of continuous curves.

We have the MOD function $y = x^2 + 2x + 1$ has several zeros in $R_n(7)$.

Next we study the function $y = x^2 + 9x + 1$ in the MOD plane $R_n(8)$.

The transformed MOD function is $x^2 + x + 1$.

The graph of $x^2 + x + 1$ in $R_n(8)$ is as follows:

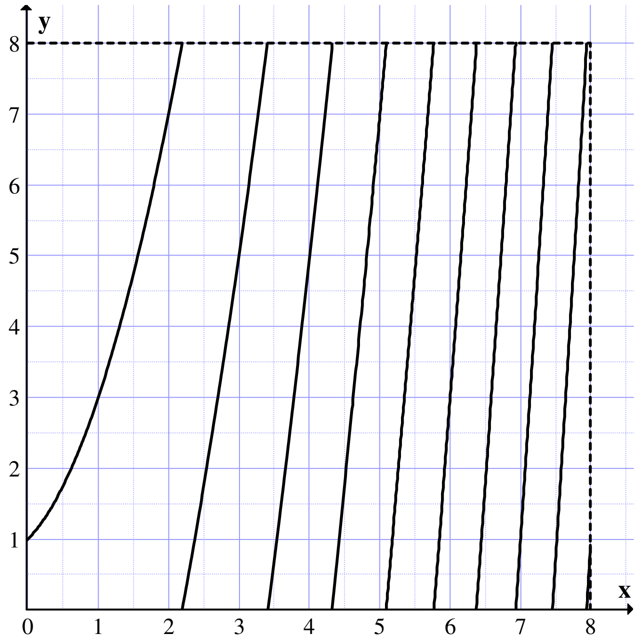

Figure 2.28

For x = 2.2 y = 0.04
For x = 2.1 y = 7.51
For x = 2.08 y = 7.4064
For x = 2.09 y = 7.4064
x = 2.15 y = 7.7725
x = 2.19 y = 7.9861
For x = 2.195 y = 0.013025

Thus for some x ∈ (2.19, 2.195) we have y = 0.

For x = 3.5 y = 0.75
For x = 3.4 y = 7.96

Thus for some x ∈ (3.4, 3.5), we have y = 0

For x = 4 y = 5
For x = 4.3 y = 7.79
For x = 4.4 y = 0.76

Thus for some x ∈ (4.3, 4.4) we have y = 0.

For x = 5 y = 7
For x = 5.1 y = 0.11

So for some x ∈ (5, 5.1), y = 0

For x = 5.7 y = 7.19
For x = 5.8 y = 0.44
For x = 5.9 y = 1.71

Thus for some x ∈ (5.7, 5.8) we have y = 0.

For x = 6 y = 3
For x = 6.2 y = 5.64
For x = 6.3 y = 6.99
For x = 6.4 y = 0.36
For x = 6.35 y = 7.6725

Thus for some x ∈ (6.35, 6.4) we have y = 0.

For x = 6.6 y = 3.16
For x = 6.7 y = 4.59
For x = 6.8 y = 0.04
For x = 6.85 y = 0.7725
For x = 6.84 y = 0.6256
For x = 6.82 y = 0.3324
For x = 6.81 y = 0.1861

Thus for some x ∈ (6.75, 6.8) we have a y = 0

For x = 7 y = 1
For x = 7.5 y = 0.75
For x = 7.4 y = 7.16

Thus for some x ∈ (7.4, 7.5), y = 0

For x = 7.8 y = 5.64
For x = 7.9 y = 7.31,

for some x ∈ (7.94, 7.95) we have a y = 0.

We have $x^2 + x + 1$ has several zeros in the MOD plane $R_n(8)$ only some of them are represented.

Next consider the function $y = x^2 + 9x + 1$ in the MOD plane $R_n(9)$ then $y = x^2 + 1$ we now give the graph of $y = x^2 + 1$ in the MOD plane $R_n(9)$.

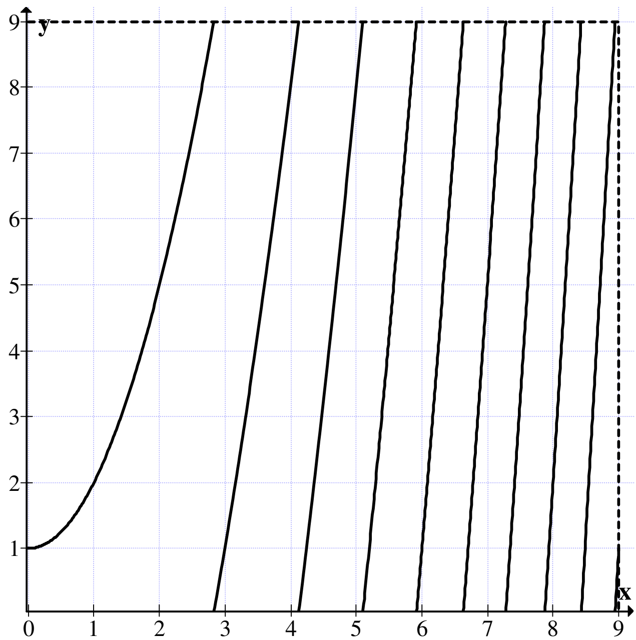

Figure 2.29

When x = 0, y = 1.

When x = 1, y = 2.

When x = 2, y = 5.

When x = 2.8, y = 8.84.

When x = 2.9; y = 0.41.

When x = 2.85 y = 0.12.

When x = 2.83 y = 0.0089.

So for some x ∈ (2.82, 2.83) we have y = 0.

For x = 4	y = 8
For x = 4.2	y = 0.64
For x = 4.1	y = 8.81
For x = 4.15	y = 0.225
For x = 4.14	y = 0.1396
For x = 4.13	y = 0.0569
For x = 4.12	y = 8.9744

Thus for some x ∈ (4.12, 4.13) we have y = 0.

For x = 4.3	y = 1.49
For x = 4.5	y = 3.25
x = 4.7	y = 5.09
x = 4.8	y = 6.04
For x = 5	y = 8.

For x = 5.1	y = 0.01
For x = 5.05	y = 8.5025
For x = 5.06	y = 8.6036
For x = 5.07	y = 8.7049
For x = 5.08	y = 8.8064
For x = 5.09	y = 8.9081

So some x ∈ (5.09, 5.10) we have y = 0.

x = 5.5	y = 4.25
x = 5.9	y = 8.81
x = 5.91	y = 8.9281
x = 5.915	y = 8.9872
x = 5.92	y = 0.0464
x = 5.95	y = 0.4025

Thus for some x ∈ (5.915, 5.92) we have a y = 0.

For $x = 6$	$y = 1$
For $x = 6.4$	$y = 5.96$
For $x = 6.5$	$y = 7.25$
For $x = 6.6$	$y = 8.56$
For $x = 6.7$	$y = 0.89$
For $x = 6.65$	$y = 0.2225$
For $x = 6.62$	$y = 8.82$
For $x = 6.64$	$y = 0.0896$
For $x = 6.63$	$y = 8.9569$

Thus for $x = (6.63, 6.64)$ we have y such that $y = 0$.

For $x = 7$	$y = 5$
For $x = 7.2$	$y = 7.84$
For $x = 7.3$	$y = 0.29$

So for $x \in (7.25, 7.3)$, $y = 0$.

For $x = 7.7$	$y = 4.29$
For $x = 7.8$	$y = 7.84$
$x = 7.9$	$y = 0.41$
$x = 7.85$	$y = 8.6225$

So for $x \in (7.85, 7.88)$ we have a $y = 0$.

For $x = 8.2$	$y = 5.24$
For $x = 8.3$	$y = 6.89$
For $x = 8.4$	$y = 8.44$
For $x = 8.5$	$y = 1.25$

Thus for $x \in (8.4, 8.45)$ we have a $y = 0$.

For $x \in (8.94, 8.95)$ there is $y = 0$.

We set $x^2 + 1$ ($x^2 + 9x + 1 \in R[x]$) in $R_n(9)[x]$ has several zeros only some of them are given.

Finally we see the graph $x^2 + 9x + 1 \in R[x]$ is the same in $R_n(10)$.

The graph of $x^2 + 9x + 1$ is as follows:

$$y = x^2 + 9x + 1$$

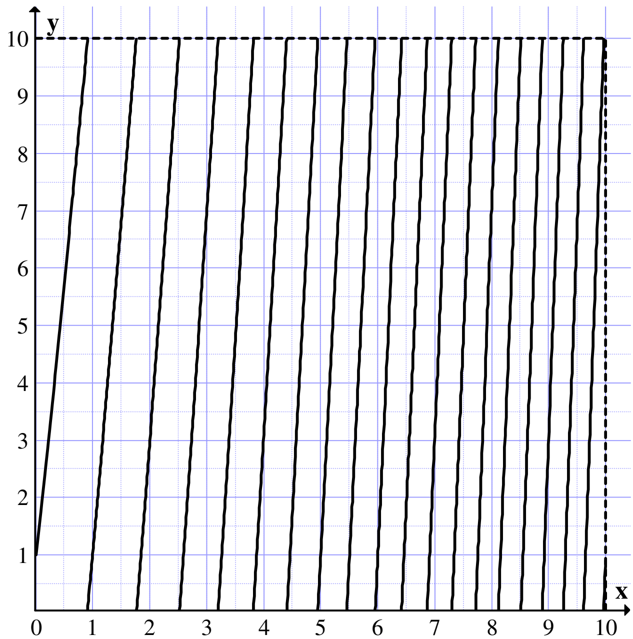

Figure 2.30

For $x \in (0.9, 0.91)$ we have a $y = 0$.

For x = 1,	y = 1
For x = 1.5	y = 6.75
For x = 1.8	y = 0.44
For x = 1.7	y = 9.19
For x = 1.75	y = 9.8125
For x = 1.76	y = 9.9376
For x = 1.77	y = 0.0629

Thus for x ∈ (1.76, 1.77) we a y such that y = 0.

For x = 2 y = 3
For x = 2.5 y = 9.75
For x = 2.52 y = 0.0304
For x = 2.51 y = 9.8901

Thus for some x ∈ (2.51, 2.52) we have y = 0.

For x = 3 y = 7
For x = 3.1 y = 8.51
For x = 3.18 y = 9.7324
For x = 3.19 y = 9.8861
For x = 3.2 y = 0.04

Thus for some x ∈ (3.19, 3.2) we have a y = 0.

Let x = 3.5 y = 4.75
Let x = 3.8 y = 9.64
For x = 3.82 y = 9.97
For x = 3.83 y = 0.4725

Thus for some x ∈ (3.82, 3.85) we have a y = 0.

Let x = 4 y = 3
Let x = 4.2 y = 6.44
For x = 4.4 y = 9.96
For x = 4.5 y = 1.75

Thus for x = 4.45 y = 0.8525

For x = 4.42 y = 0.3164
For x = 4.41 y = 0.1381

Thus for some x ∈ (4.4, 4.41) we have y = 0.

For x = 4.9 y = 9.11
For x = 4.98 y = 0.6204
For x = 4.95 we get y = 0.525

For $x = 4.94$ $y = 9.8636$

So for some $x \in (4.94, 4.95)$ we have a y such that $y = 0$.

For $x = 5$ $y = 1$
For $x = 5.45$ $y = 9.7525$
For $x = 5.4$ $y = 8.76$
For $x = 5.5$ $y = 0.75$
For $x = 5.48$ $y = 0.35$
For $x = 5.46$ $y = 9.9516$
For $x = 5.47$ $y = 0.1509$

For some $x \in (5.46, 5.47)$ we have $y = 0$.

For $x = 5.8$ $y = 6.84$
For $x = 5.9$ $y = 8.91$
For $x = 5.95$ $y = 9.95$

For some $x \in (5.95, 5.956)$ we get $y = 0$.

For $x = 6$ $y = 1$
For $x = 6.4$ $y = 9.56$
For $x = 6.5$ $y = 1.75$
For $x = 6.45$ $y = 0.6525$
For $x = 6.44$ $y = 0.4336$
For $x = 6.42$ $y = 9.9964$
For $x = 6.43$ $y = 0.2149$

Thus for some $x \in (6.42, 6.43)$ we have a $y = 0$.

For $x = 6.8$ $y = 8.44$
For $x = 6.85$ $y = 9.5725$
For $x = 6.87$ $y = 0.0269$

Thus for $x \in (6.86, 6.87)$ we have a $y = 0$.

$x = 7$ $y = 3$
For $x = 7.1$ $y = 5.31$
For $x = 7.3$ $y = 9.99$

For x = 7.35	y = 1.1725
For x = 7.33	y = 0.6989

Thus for some x ∈ (7.3, 7.33) we have a y = 0.

For x = 7.5	y = 4.75
For x = 7.6	y = 7.16
For x = 7.7	y = 9.59
For x = 7.71	y = 9.8341
For x = 7.75	y = 0.8125
For x = 7.73	y = 0.3229
if x = 7.72	y = 0.0784

Thus for x ∈ (7.71, 7.72) we see there exist a y = 0.

For x = 8	y = 7
For x = 7.9	y = 4.51
For x = 7.98	y = 6.5
For x = 8.5	y = 9.75
For x = 8.6	y = 2.36
For x = 8.55	y = 1.0525
For x = 8.54	y = 0.7916
For x = 8.52	y = 0.2704
For x = 8.51	y = 0.0101
For x = 8.505	y = 9.88

Thus for some x ∈ (8.505, 8.51) we have a y =0.

For x = 8.9	y = 0.31
For x = 8.8	y = 7.64
For x= 8.85	y = 8.9725
For x = 8.88	y = 9.77

So for some x ∈ (8.88, 8.9) we have a y = 0.

For x = 9	y = 3
For x = 9.1	y = 5.71
For x = 9.3	y = 1.19
For x = 9.5	y = 6.75

For x = 9.2	y = 8.44
For x = 9.25	y = 9.8125
For x = 9.27	y = 0.3629
For x = 9.26	y = 0.0876

Thus for some $x \in (9.25, 9.26)$ $y = 0$.

For x = 9.5	y = 6.75
For x = 9.6	y = 9.56
For x = 9.61	y = 9.84
For x = 9.62	y = 0.1244
For x = 9.65	y = 0.9725
For x = 9.7	y = 2.39
For x = 9.8	y = 5.24
For x = 9.615	y = 9.832

For some $x \in (9.615, 9.62)$, we have $y = 0$. For some $x \in (9.96, 9.97)$ we have a $y = 0$.

Thus for function $y = x_2 + 9x + 1$ in $R_n(10)[x]$ has atleast 18 zero.

So a second degree MOD equation in $R_n(10)[x]$ has over 18 roots or 18 zeros.

Now we see the function $y = x^2 + 9x + 1 \in R[x]$ remains the same in all MOD planes R_n (m); $m \geq 10$.

Now let $y = x^2 + 9x + 1$ be the function in the MOD plane $R_n(11)[x]$ to find the graph and zeros of $x^2 + 9x + 1$ in $R_n(11)$. The associated graph is given in Figure 2.31.

When

x = 0	y = 1
x = 0.5	y = 5.75
x = 0.7	y = 7.79
x = 0.9	y = 9.91

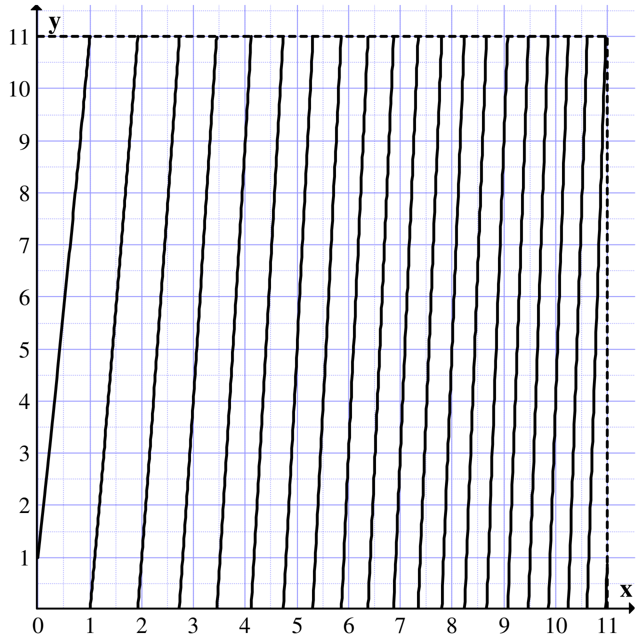

Figure 2.31

x = 1	y = 0
For x = 1.2	y = 3.24
For x = 1.1	y = 1.11
For x = 1.4	y = 4.56

Thus for x = 1.8, y = 0.44.

So a zero lies between 1.7 and 1.8.

For x = 1.7	y = 8.19
For x = 1.8	y = 9.44
For x = 1.9	y = 10.71
For x = 1.95	y = 0.3525
For x = 2	y = 1
For x = 2.2	y = 3.64

So a zero lies between (1.9, 1.95).

x = 2.5 y = 7.75

x = 2.6 y = 9.16

For x = 2.7 we get y = 10.59
For x = 2.8 wet get y = 1.04

So a zero lies roughly between 2.7 and 2.8.

For x = 2.75 we get y = 0.3125
For x = 2.73 we get y = 0.0229
For x = 2.72 we get y = 10.8784

So for some x ∈ (2.72, 2.73) there is a y = 0.

For x = 3	y = 4
For x = 3.5	y = 0.75
For x = 3.4	y = 10.16
For x = 3.48	y = 0.4304

For some x ∈ (3.45, 3.48) we have a y = 0.

For x = 3.8	y = 5.64
For x = 3.9	y = 6.31

For x = 4	y = 9
For x = 4.3	y = 3.19
For x = 4.6	y = 8.56
For x = 4.7	y = 10.39
For x = 4.2	y = 1.44
For x = 4.1	y = 10.71
For x = 4.15	y = 0.5725
For x = 4.13	y = 0.2269
For x = 4.11	y = 10.8821
For x = 4.12	y = 0.0544

Thus for some x ∈ (4.11, 4.12) there is a y = 0.

For x = 4.9	y = 3.11
For x = 4.8	y = 1.24
For x = 4.75	y = 0.3125

For x = 4.72 y = 10.7584
For x = 4.73 y = 10.9429
For x = 4.755 y = 0.35225

So for some x ∈ (4.73, 4.735) there exist a y = 0.

For x = 4.9 y = 3.11
For x = 5 y = 5
For x = 5.3 y = 10.29
For x = 5.4 y = 1.76
For x = 5.35 y = 0.7725
For x = 5.33 y = 0.3789
For x = 5.32 y = 0.1824
For x = 5.31 y = 10.9861

For some x ∈ (5.31, 5.32) we have a y = 0.

For x = 5.5 y = 3.75
For x = 5.6 y = 5.76
For x = 6 y = 3
For x = 5.8 y = 9.84
For x = 5.9 y = 0.91
For x = 5.83 y = 10.4585
For x = 5.84 y = 10.6656
For x = 5.85 y = 10.8725
For x = 5.86 y = 0.0796

Thus for some x ∈ (5.85, 5.86) we have a y = 0.

For x = 6.3 y = 9.39
For x = 6.4 y = 0.56
For x = 6.5 y = 10.472
For x = 6.36 y = 10.6896
For x = 6.37 y = 10.9069
For x = 6.38 y = 0.1244

Thus for some x ∈ (6.37, 6.38) we have a y = 0.

For x = 6.6 y = 4.96

For x = 6.8	y = 9.44
For x = 6.9	y = 0.71
For x = 6.85	y = 10.5725
For x = 6.87	y = 0.0269
For x = 6.86	y = 10.7996

Thus for some x ∈ (6.86, 6.87) we have a y = 0.

For x = 7	y = 3
For x = 7.3	y = 9.99
For x = 7.35	y = 0.1725
For x = 7.34	y = 10.9356

For x ∈ (7.34, 7.35) we have a y = 0.

For x = 7.5	y = 3.75
For x = 7.6	y = 6.16
For x = 7.7	y = 8.59
For x = 7.8	y = 0.04
For x = 7.75	y = 9.8125
For x = 7.77	y = 10.3029
For x = 7.78	y = 10.5484
For x = 7.79	y = 10.7941
For x = 7.795	y = 10.91702

So for some x ∈ (7.795, 7.8) we have a y = 0.

For x = 8	y = 5
For x = 7.9	y = 2.51
For x = 8.2	y = 10.04
For x = 8.23	y = 10.8029
For x = 8.24	y = 0.0576
For x = 8.3	y = 1.59

So for some x ∈ (8.23, 8.24) we have a y = 0.

For x = 8.5	y = 6.75
For x = 8.6	y = 9.36
For x = 8.7	y = 0.99

For x = 8.65	y = 10.6725
For x = 8.68	y = 0.4624
For x = 8.66	y = 10.9356
For x = 8.67	y = 0.1989

So for some x ∈ (8.66, 8.67) we have a y = 0.

For x = 9	y = 9
For x = 8.8	y = 3.64
For x = 8.9	y = 6.31
For x = 9.1	y = 0.71
For x = 9.05	y = 10.35
For x = 9.09	y = 0.4381
For x = 9.08	y = 0.1664
For x = 9.07	y = 10.8949

Thus for some x ∈ (9.07, 9.08) we have a y = 0.

For x = 9.3	y = 6.19
For x = 9.4	y = 8.96
For x = 9.5	y = 0.75
For x = 9.45	y = 10.3525
For x = 9.47	y = 10.9109
For x = 9.48	y = 0.1904

Thus for some x ∈ (9.47, 9.48) we have a y = 0.

For x = 9.7	y = 6.39
For x = 9.6	y = 3.56
For x = 9.8	y = 9.24
For x = 9.85	y = 10.67
For x = 9.855	y = 10.81
For x = 9.859	y = 10.93
For x = 9.88	y = 0.5344
For x = 9.86	y = 10.95
For x = 9.87	y = 0.2469

Thus for some x ∈ (9.86, 9.87) we have a y = 0.

For x = 10	y = 4
For x = 10.2	y = 9.84
For x = 10.25	y = 0.3125
For x = 10.22	y = 10.4284
For x = 10.24	y = 0.0176
For x = 10.23	y = 10.7229.

For some x ∈ (10.23, 10.24) we have a y = 0.

For x = 10.5	y = 7.75
For x = 10.3	y = 1.79
For x = 10.7	y = 2.79
For x = 10.4	y = 4.76
For x = 10.6	y = 10.76
For x = 10.65	y = 1.2725
For x = 10.63	y = 0.6669
For x = 10.62	y = 0.3644
For x = 10.61	y = 0.0621
For x = 10.605	y = 10.911
For x = 10.609	y = 0.031.

Thus for some x ∈ (10.605, 10.609) we have a y = 0.

For x = 10.8	y = 5.84
For x = 10.9	y = 8.99
For x = 10.99	y = 0.6901
For x = 10.95	y = 10.4525
For x = 10.97	y = 0.0709
For x = 10.96	y = 10.7616
For x = 10.966	y = 10.94
For x = 10.968	y = 0.09024.

For some x ∈ (10.966, 10.668) we have a y = 0.

Now for x = 10.999 y = 0.969 for x = 10.9999 y = 0.9969.

We have shown $x^2 + 9x + 1 \in R_n(11)[x]$ has atleast 20 zeros.

Finally we study the function $y = x^2 + 9x + 1$ in the MOD plane $R_n(12)$.

$y = x^2 + 9x + 1$ in $R_n(12)[x]$ is as follows:

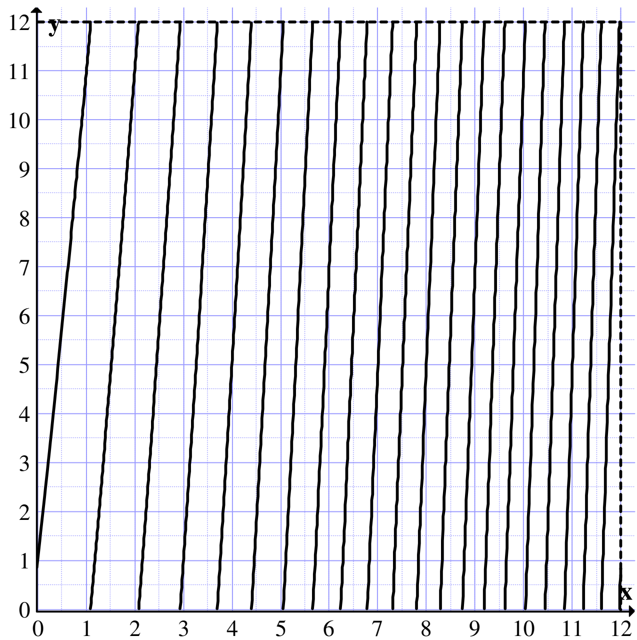

Figure 2.32

x = 0.5	y = 5.75
x = 0.9	y = 9.91
x = 1	y = 11
x = 1.1	y = 0.11
x = 1.05	y = 11.5525

For $x \in (1.05, 1.1)$ we have a $y = 0$.

For x = 1.4	y = 3.56
x = 1.7	y = 9.19

x = 1.6	y = 4.75
x = 1.9	y = 9.71
x = 2	y = 11
x = 2.1	y = 0.31
x = 2.6	y = 7.16
x = 2.06	y = 11.7836
x = 2.08	y = 0.0464
x = 2.07	y = 11.9149

Thus for x ∈ (2.07, 2.08) we have a y = 0.

Now for x = 2.5	y = 5.75
For x = 3	y = 1.

So we have a zero before x = 3.

Consider x = 2.9	y = 11.51
For x = 2.92	y = 11.80
For x = 2.94	y = 0.1036
For x = 2.93	y = 11.95

So for some x ∈ (2.93, 2.94) we have a y = 0.

For x = 3.3	y = 5.54
For x = 3.6	y = 10.36
For x = 3.7	y = 11.99
For x = 3.75	y = 0.8125
For x = 3.74	y = 0.6476
For x = 3.72	y = 0.3184
For x = 3.71	y = 0.1541

Thus for some x ∈ (3.7, 3.71) we have a y = 0.

For x = 3.9	y = 3.31
For x = 4	y = 5
For x = 4.4	y = 11.96
For x = 4.45	y = 0.8525
For x = 4.43	y = 0.4949
For x = 4.41	y = 0.1381

For x = 4.405 y = 0.049025
For x = 4.403 y = 0.013409
For x = 4.401 y = 11.977801

Thus for some x ∈ (4.401, 4.403) we have a y = 0.

For x = 4.8 y = 7.24
For x = 4.9 y = 9.11
For x = 5 y = 11
For x = 5.1 y = 0.91
For x = 5.65 y = 11.9525
For x = 5.06 y = 0.1436

Thus for some x ∈ (5.05, 5.06) we have a y = 0.

For x = 5.5 y = 8.75
For x = 5.8 y = 2.84
For x = 5.6 y = 10.76
For x = 5.7 y = 0.79
For x = 5.68 y = 0.3824
For x = 5.65 y = 11.7725
For x = 5.67 y = 0.1789

Thus for some x ∈ (5.66, 5.67) we have a y = 0.

For x = 6.2 y = 11.24
For x = 6.25 y = 0.3125
For x = 6.23 y = 11.8829
For x = 6.24 y = 0.0976

Thus for some x ∈ (6.23, 6.24) we have a y = 0.

For x = 6.5 y = 5.75
For x = 6.7 y = 10.18
For x = 6.8 y = 0.44

Thus for some x ∈ (6.7, 6.8) we have a y = 0.

For x = 6.9 y = 2.71

For x = 7	y = 5
For x = 7.3	y = 11.99
For x = 7.32	y = 0.4624
For x = 7.31	y = 0.2261
For x = 7.305	y = 0.1080
For x = 7.301	y = 0.013601

Thus for some x ∈ (7.3, 7.301) we have a y = 0.

For x = 7.6	y = 7.16
For x = 8	y = 5
For x = 7.9	y = 2.51
For x = 7.7	y = 9.59
For x = 7.8	y = 0.04
For x = 7.75	y = 10.8125
For x = 7.79	y = 11.7941
For x = 7.795	y = 11.9170
For x = 7.719	y = 0.015401

Thus for some x ∈ (7.795, 7.799) we have a y = 0.

For x = 8.2	y = 10.04
For x = 8.3	y = 0.59
For x = 8.27	y = 11.3125
For x = 8.29	y = 0.3341
For x = 8.28	y = 0.0784
For x = 8.275	y = 11.9506

Thus for some x ∈ (8.275, 8.28) we have a y = 0.

For x = 8.5	y = 5.75
For x = 8.6	y = 3.16
For x = 8.4	y = 10.99
For x = 8.8	y = 1.64
For x = 8.75	y = 0.3125
For x = 8.72	y = 11.5184
For x = 8.73	y = 11.7829
For x = 8.74	y = 0.0476
For x = 8.735	y = 11.9152

Thus for some x ∈ (8.735, 8.74) we have a y = 0.

For x = 9	y = 7
For x = 8.9	y = 4.31
For x = 9.1	y = 9.71
For x = 9.2	y = 0.44
For x = 9.18	y = 11.8924
For x = 9.185	y = 0.29225

Thus for some x ∈ (9.18, 9.185) we have a y = 0.

For x = 9.3	y = 3.19
For x = 9.5	y = 8.75
For x = 9.6	y = 11.56
For x = 9.62	y = 0.1244
For x = 9.615	y = 11.98322

Thus for some x ∈ (9.615, 9.62) we have a y = 0.

For x = 9.8	y = 5.24
For x = 10	y = 11
For x = 10.09	y = 1.6181
For x = 10.1	y = 1.91
For x = 10.04	y = 0.1616
For x = 10.05	y = 0.4525
For x = 10.2	y = 4.84
For x = 10.15	y = 3.3725
For x = 10.03	y = 11.8709

Thus for some x ∈ (10.03, 10.04) we have a y = 0.

For x = 10.3	y = 7.79
For x = 10.4	y = 10.76
For x = 10.5	y = 1.75
For x = 10.45	y = 0.2525
For x = 10.42	y = 11.3564

Thus for some x ∈ (10.4, 10.45) we have a y = 0.

For x = 10.8	y = 10.84
For x = 10.85	y = 0.3725
For x = 10.9	y = 1.91
For x = 10.83	y = 11.7589
For x = 10.84	y = 0.0656

Thus for some x ∈ (10.83, 10.84) we have a y = 0.

For x = 11	y = 5
For x = 11.2	y = 11.24
For x = 11.21	y = 11.5541
For x = 11.3	y = 4.36
For x = 11.4	y = 1.56
For x = 11.23	y = 0.1829

Thus for some x ∈ (11.21, 11.23) we have a y = 0.

For x = 11.6	y = 11.96
For x = 11.7	y = 3.19
For x = 11.9	y = 9.71
For x = 11.65	y = 1.5925
For x = 11.62	y = 0.6044
For x = 11.61	y = 0.2821

Thus for some x ∈ (11.6, 11.61) we have a y = 0. For some x ∈ (11.95, 11.96) there is a y which takes the value 0.

There is atleast 21 zeros given by the function

$$y = x^2 + 9x + 1$$

in the MOD plane $R_n(12)$.

After x takes the value 5 there are more zeros. Zeros becomes dense as x moves closer and closer to 12.

Thus in (10, 11) there are three zeros lying in the intervals (10.03, 10.04), (10.44, 10.45) and (10.83, 10.84).

This property infact is unique and is enjoyed solely by the functions in the MOD planes.

So instead of getting two zeros a second degree equation can have 21 zeros so seeking solutions in the MOD plane can be interesting and this will find appropriate applications.

However the function $y = x^2 + 9x + 1$ has no solution in the ring of integers.

Infact $x^2 + 9x + 1$ has no solution in the rational field.

However this has solution over the real field as

$$x = \frac{-9 \pm \sqrt{9^2 - 4}}{2}$$

$$= \frac{-9 \pm \sqrt{77}}{2} = \frac{-9 \pm 8.775}{2}$$

The real roots are

$\left(\frac{-9 + 8.775}{2}\right)$ and $\frac{-9 - 8.775}{2}$ that is –0.1125 and

–8.8875 both the roots are negative.

However if one has to find solution which is non negative can go to MOD planes and solve them.

Further as this has more than two roots a feasible root can be obtained or a cluster of roots can be taken as a solution.

Now this equation

$y = x^2 + 9x + 1$ will be changed in the MOD planes only for the coefficient of x the other two will remain the same.

Further the change takes place in the MOD plane $R_n(m)$, $m \leq 9$. For all $m \geq 10$ the equation $y = x^2 + 9x + 1$ remains the same.

Now generalize this equation as follows:

If $y = x^2 + m_1x + 1$ then this equation can maximum have one change and that two for the coefficient of the x term and in $R_n(m)$, $m \leq m_1$ will have change and for all $m > m_1$ no change takes places.

Consider the equation $y = x^2 + x + 6$.

We study this equation in the MOD planes.

The roots of this equation is

$$x = \frac{-1 \pm \sqrt{1 - 4 \times 6}}{2}$$

$$= \frac{-1 \pm i\sqrt{23}}{2}.$$

So the solution does not exist in R the reals as they are imaginary.

Now $y = x^2 + x + 6$ in $R_n(2)$ is as follows, $x^2 + x = y$.

The roots of this equation are given by the following graphs in the MOD plane $R_n(2)$.

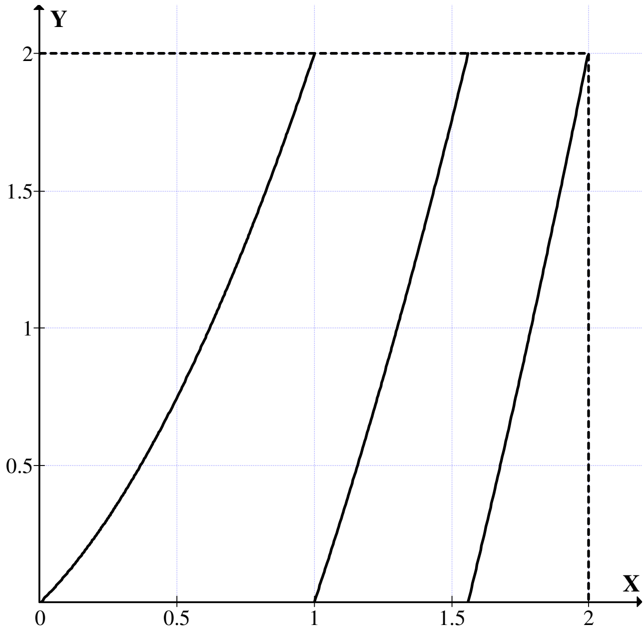

Figure 2.33

For x = 0.2	y = 0.24
For x = 1	y = 0
For x = 1.58	y = 0.0764
For x = 1.57	y = 0.0349
For x = 1.55	y = 1.9525
For x = 1.56	y = 1.9936

Thus for some x ∈ (1.56, 1.57) we have a y = 0.

For x = 1.6	y = 0.16
For x = 1.8	y = 1.04
For x = 1.9	y = 1.51
For x = 1.99	y = 1.9501.

Thus for the equation $x^2 + x = 0$. The three roots are x = 0, x = 1 and x some x ∈ (1.56, 1.57); y = 0.

For x = 1.562	y = 0.001844
For x = 1.5617	y = 000602

For x = 1.5616   y = 0.0001945
For x = 1.56159   y = 0.00015338
For x = 1.56158   y = 0.0000112096
For x = 1.56156   y = 0.000029634
For x = 1.561559   y = 0.00002551
For x = 1.561557   y = 0.000017264
For x = 1.561556   y = 0.000013141
For x = 1.561555   y = 0.000009018
For x = 1.561553   y = 0.000000772

Thus for some $x \in (1.561552, 1.561553)$ we have a $y = 0$.

Thus the MOD equation $y = x^2 + x$ in $R_n(2)$ has atleast three zeros.

Next we find $y = x^2 + x + 6$ in $R_n(3)$ to be $x^2 + x$.

The graph of $y = x^2 + x$ in $R_n(3)$ is as follows:

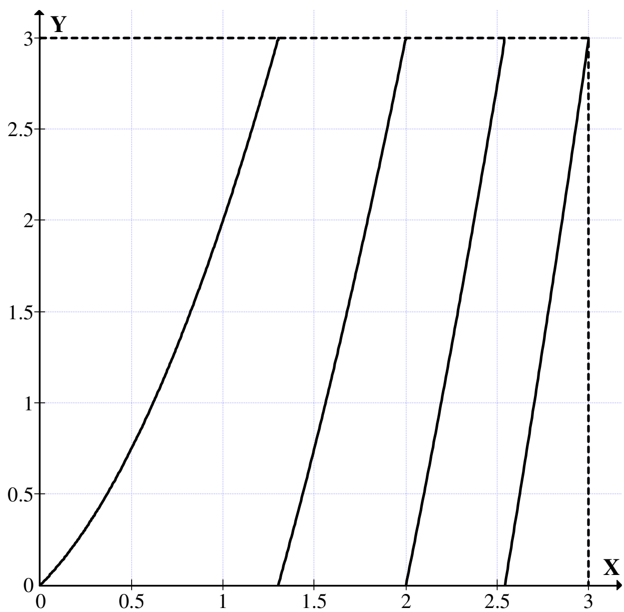

Figure 2.34

For $x = 0$	$y = 0$
For $x = 1$	$y = 2$
For $x = 0.5$	$y = 0.75$
For $x = 1.3$	$y = 2.99$
For $x = 1.4$	$y = 0.36$
For $x = 1.35$	$y = 0.1725$

So for some $x \in (1.3, 1.31)$ we have $y = 0$.

For $x = 1.38$	$y = 0.2844$
For $x = 1.34$	$y = 0.1356$
For $x = 1.4$	$y = 0.36$
For $x = 1.7$	$y = 1.59$
For $x = 2$	$y = 0$

For $x = 1.99$	$y = 2.9501$
For $x = 2.1$	$y = 0.51$
For $x = 2.5$	$y = 2.75$
For $x = 2.6$	$y = 0.36$
For $x = 2.56$	$y = 0.1136$
For $x = 2.52$	$y = 2.8704$
For $x = 2.53$	$y = 2.9309$
For $x = 2.54$	$y = 2.9916$

For some $x \in (2.54, 2.55)$ we have $y = 0$.

For $x = 2.55$	$y = 0.0525$
For $x = 2.7$	$y = 0.99$
For $x = 2.8$	$y = 1.64$
For $x = 2.9$	$y = 2.31$
For $x = 2.98$	$y = 2.8604$

We see $x^2 + x$ in $R_n(3)[x]$ has atleast 3 zeros.

Consider $y = x^2 + x + 6$ in $R_n(4)$. The equation changes to $y = x^2 + x + 2$ in $R_n(4)$.

The graph of $y = x^2 + x + 2$ is

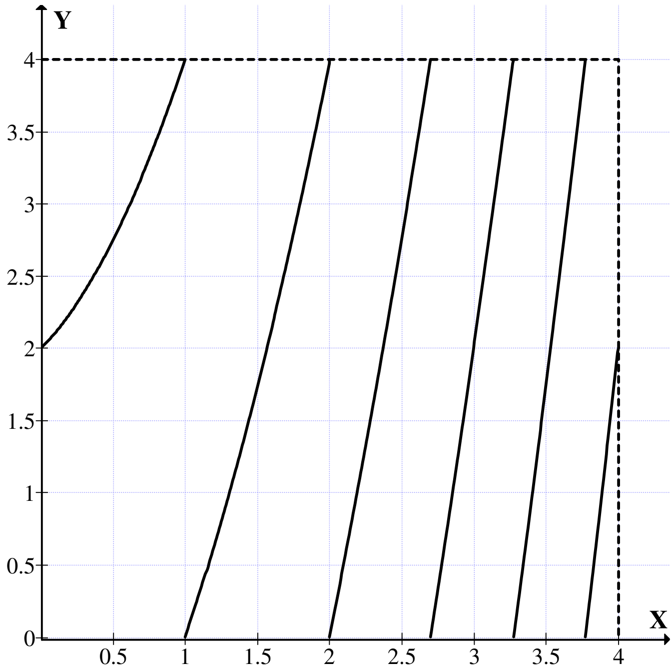

Figure 2.35

For x = 1	y = 0
For x = 1.3	y = 0.99
For x = 1.2	y = 0.64
For x = 1.1	y = 0.31
For x = 1.5	y = 0.75
For x = 1.7	y = 2.59
For x = 1.8	y = 3.04
For x = 2	y = 0
For x = 2.2	y = 1.04
For x = 2.4	y = 2.16
For x = 2.7	y = 3.99
For x = 2.75	y = 0.3125
For x = 2.74	y = 0.2476
For x = 2.71	y = 0.0541

For some $x \in (2.7, 2.71)$ we have a y = 0.

For x = 2.8	y = 0.64

For $x = 2.9$	$y = 1.31$
For $x = 3$	$y = 2$
For $x = 3.2$	$y = 3.44$
For $x = 3.3$	$y = 0.19$
For $x = 3.22$	$y = 3.5884$
For $x = 3.23$	$y = 3.6629$
For $x = 3.25$	$y = 3.8125$
For $x = 3.27$	$y = 3.9629$
For $x = 3.28$	$y = 0.0384$

For some $x \in (3.27, 3.28)$ we have a $y = 0$.

For $x = 3.9$	$y = 1.11$
For $x = 3.95$	$y = 1.5525$.

Clearly $y = x^2 + x + 2$ has more than three zeros in $R_n(4)$.

Now the equation $y = x^2 + x + 6$ in R[x] takes the form $y = x^2 + x + 1$ in $R_n(5)$. The graph of y in the $R_n(5)$ plane is as follows:

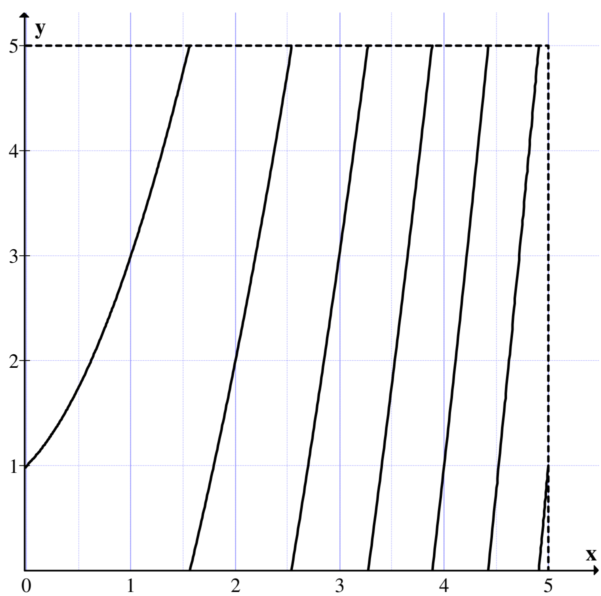

Figure 2.36

For $x = 0.8$	$y = 1.512$
For $x = 0.9$	$y = 2.71$
For $x = 1$	$y = 3$
For $x = 1.2$	$y = 3.64$
For $x = 1.4$	$y = 4.36$
For $x = 1.5$	$y = 4.75$
For $x = 1.54$	$y = 4.9116$
For $x = 1.55$	$y = 4.9525$
For $x = 1.59$	$y = 0.1181$
For $x = 1.56$	$y = 4.9936$
For $x = 1.57$	$y = 0.0349$

For some $x \in (1.56, 1.57)$ we have $y = 0$.

For $x = 3$	$y = 3$
For $x = 3.4$	$y = 0.96$
For $x = 3.3$	$y = 0.19$
For $x = 3.2$	$y = 4.44$
For $x = 3.25$	$y = 4.8125$

For some $x \in (3.25, 3.3)$ we have $y = 0$.

For $x = 3.7$	$y = 3.39$
For $x = 3.9$	$y = 0.11$
For $x = 3.8$	$y = 4.24$

For some $x \in (3.85, 3.9)$ we have $y = 0$.

For $x = 4$	$y = 1$
For $x = 4.5$	$y = 0.75$
For $x = 4.3$	$y = 3.79$
For $x = 4.4$	$y = 4.76$
For $x = 4.45$	$y = 0.2525$
For $x = 4.44$	$y = 0.1536$
For $x = 4.43$	$y = 0.0549$
For $x = 4.42$	$y = 4.9564$

Thus for some $x \in (4.42, 4.43)$ we have $y = 0$.
This equation $x^2 + x + 1 = y$ in $R_n(5)$ has atleast 5 zeros.

Now we consider the MOD equation of $y = x^2 + x + 6$ in the MOD plane $R_n(6)$. $y = x^2 + x$

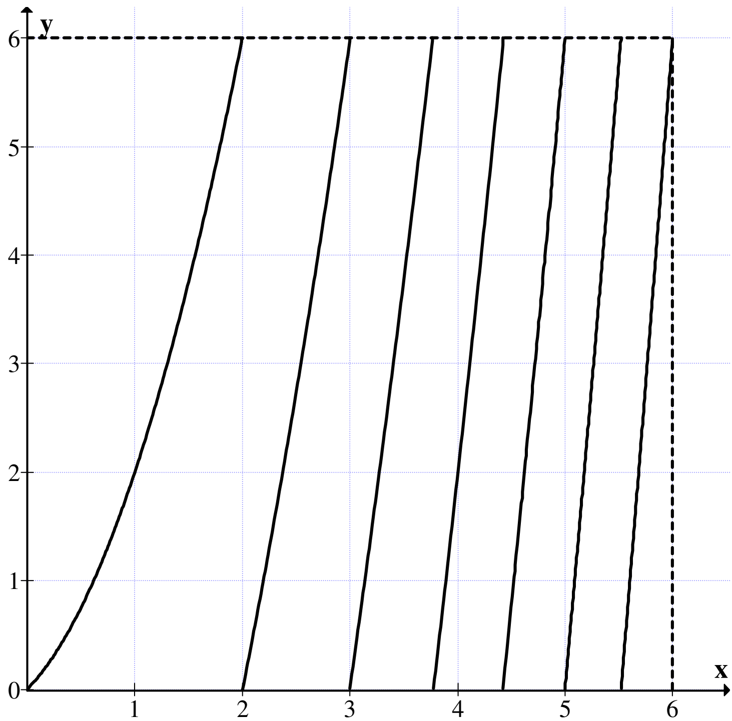

Figure 2.37

This MOD equation $x^2 + x = y$ has atleast seven zeros.

Next we study the equation $y = x^2 + x + 6 \in R[x]$. This remains the same in the MOD plane $R_n(7)$.

The graph of $y = x^2 + x + 6$ in $R_n(7)$ is given Figure 2.38. For $x \in (0.6, 0.7)$ we have $y = 0$.

For $x = 1$	$y = 1$
For $x = 1.5$	$y = 2.75$
For $x = 1.8$	$y = 4.04$

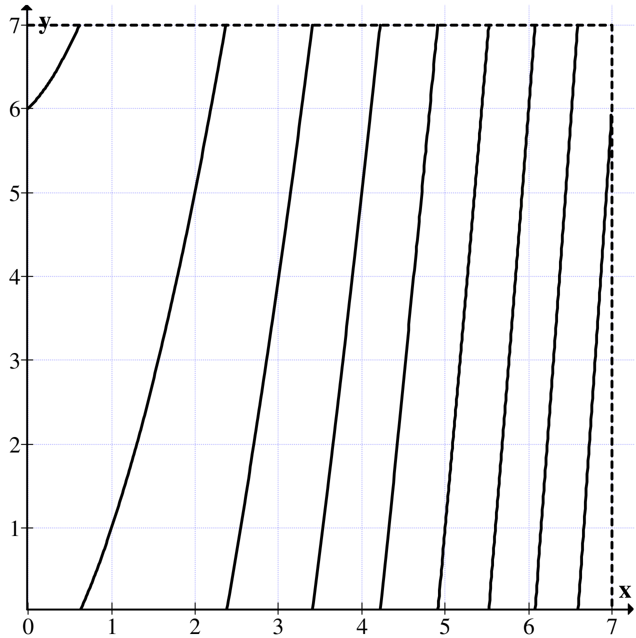

Figure 2.38

For x = 1.9	y = 4.51
For x = 2	y = 5
For x = 2.3	y = 6.59

For x = (2.3, 2.4) there is a y = 0.

For x = 2.5	y = 0.75
For x = 2.4	y = 0.16
For x = 2.8	y = 2.64
For x = 3	y = 4
For x = 2.7	y = 1.99
For x = 2.9	y = 3.31
For x = 3.2	y = 5.44
For x = 3.4	y = 6.96

For x = (3.4, 3.5) there is a y = 0.

For x = 3.5	y = 0.75
For x = 3.9	y = 4.11
For x = 4	y = 5

For $x = 4.2$	$y = 6.84$
For $x = 4.3$	$y = 0.79$
For $x = 4.6$	$y = 3.76$
For $x = 5$	$y = 1$
For $x = 4.47$	$y = 4.79$
For $x = 4.8$	$y = 5.84$
For $x = 4.9$	$y = 6.91$

For $x = (4.2, 4.3)$ there is a $y = 0$.

For $x = 4.92$	$y = 0.1264$
For $x = 4.91$	$y = 0.0184$

Thus for some $x \in (4.9, 4.91)$ we have $y = 0$.

For $x = 5$	$y = 1$
For $x = 5.3$	$y = 4.39$
For $x = 5.5$	$y = 6.75$
For $x = 5.7$	$y = 2.19$
For $x = 5.6$	$y = 0.96$

Thus for some $x \in (5.5, 5.6)$ we have $y = 0$.

For $x = 6$	$y = 0$
For $x = 6.1$	$y = 0.31$
For $x = 6.2$	$y = 1.64$

For $x \in (6.55, 6.6)$ we have a $y = 0$.

Thus $x^2 + x + 6$ has atleast seven zeros in $R_n(7)$.

Consider the MOD function $y = x^2 + x + 6$ in the MOD plane $R_n(8)$.

The associated graph is given in Figure 2.39.

Let $x = 0$	$y = 6$
Let $x = 0.5$	$y = 6.75$
Let $x = 0.7$	$y = 7.19$
Let $x = 0.8$	$y = 7.44$
Let $x = 0.9$	$y = 7.71$

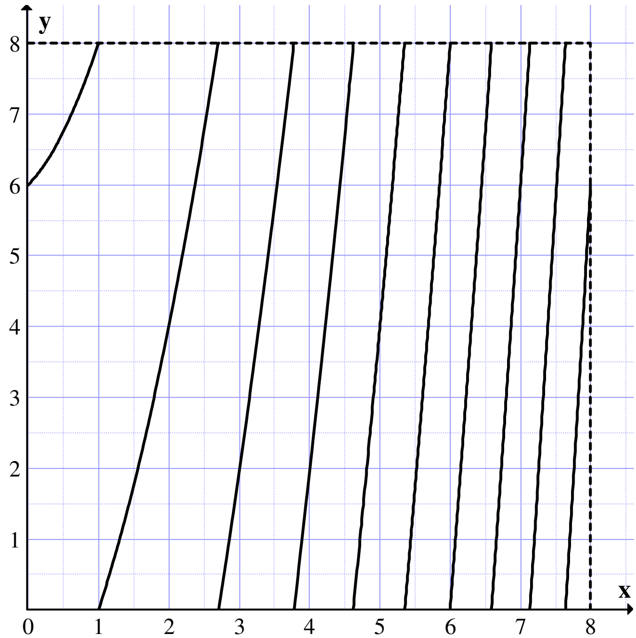

Figure 2.39

At x = 1	y = 0
For x = 1.3	y = 0.99
For x = 1.5	y = 1.75
For x = 2	y = 4
For x = 2.4	y = 6.16
For x = 2.7	y = 7.99
For x = 2.5	y = 6.75
For x = 2.6	y = 7.36
For x = 2.8	y = 0.64

Thus for some $x \in (2.7, 2.8)$ we have we y = 0

For x = 3	y = 2
For x = 3.5	y = 5.75
For x = 3.4	y = 2.96
For x = 3.6	y = 6.56
For x = 3.7	y = 7.39
For x = 3.8	y = 0.24

For x = 3.75 y = 7.8125

For some x ∈ (3.75, 3.8) we have a y = 0.

For x = 3.9 y = 1.11
For x = 4 y = 2
For x = 4.3 y = 4.79
For x = 4.6 y = 7.76
For x = 4.65 y = 0.2725
For x = 4.61 y = 7.8621
For x = 4.62 y = 7.9644
For x = 4.63 y = 0.0669

Thus for some x ∈ (4.62, 4.63) we have a y = 0.

For x = 5 y = 4
For x = 5.6 y = 2.96
For x = 5.3 y = 7.39
For x = 5.4 y = 0.56

For some x ∈ (5.3, 5.4) there exists a y = 0.

For x = 5.8 y = 5.44
For x = 5.6 y = 2.96
For x = 6 y = 0
For x = 6.3 y = 3.99
For x = 6.5 y = 6.75
For x = 6.6 y = 0.16

For some x ∈ (6.5, 6.6) we have a y = 0

For some x = 6.7 y = 1.59
For some x = 6.8 y = 3.04
For some x = 7 y = 6
For some x = 7.2 y = 1.04
For some x = 7.1 y = 8.51

Thus for some x ∈ (7.1, 7.2) we have a y = 0.

Let x = 7.3   y = 2.59
Let x = 7.5   y = 5.75
For x = 7.6   y = 9.36
For x = 7.7   y = 0.99
For x = 7.65   y = 0.1725

For some $x \in (7.6, 7.65)$ we have a y = 0.

For x = 7.7   y = 0.99
For x = 7.8   y = 2.64
For x = 7.9   y = 4.31
For x = 7.95   y = 5.1525

Thus the maximum y in that interval in 0.59999.

Thus the function $x^2 + x + 6$ has atleast 9 zero divisors.

Consider $y = x^2 + x + 6$ in the MOD plane $R_n(9)$. The graph of $y = x^2 + x + 6$ is as follows:

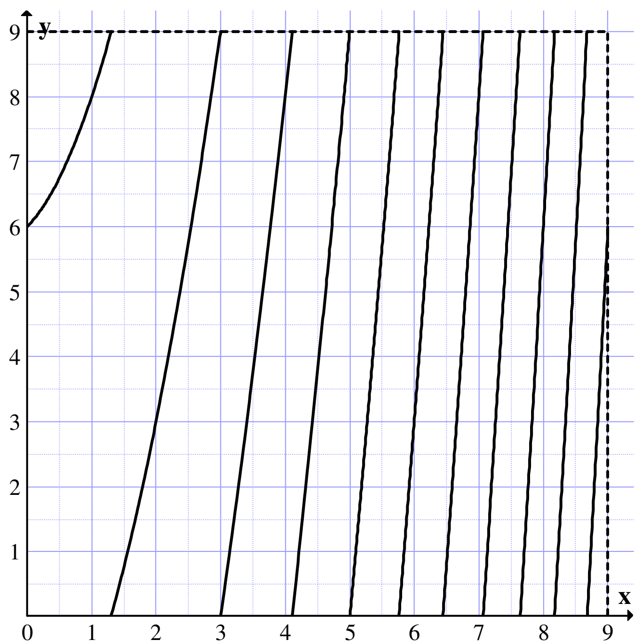

Figure 2.40

For $x = 0$	$y = 6$
For $x = 1$	$y = 8$
For $x = 1.2$	$y = 8.6$
For $x = 1.3$	$y = 8.99$
For $x = 1.35$	$y = 0.1725$

Thus for some $x \in (1.3, 1.35)$ we have a $y = 0$.

Let $x = 1.5$	$y = 0.75$
Let $x = 2$	$y = 3$
Let $x = 2.5$	$y = 5.75$
Let $x = 3$	$y = 0$
Let $x = 3.4$	$y = 2.96$
Let $x = 3.7$	$y = 5.39$
Let $x = 4$	$y = 8$
Let $x = 4.2$	$y = 0.84$
Let $x = 4.1$	$y = 8.91$

So for some $x \in (4.1, 4.2)$ we have a $y = 0$.

Let $x = 4.4$	$y = 2.76$
Let $x = 4.8$	$y = 5.84$
Let $x = 5$	$y = 0$
Let $x = 5.2$	$y = 2.24$
Let $x = 5.4$	$y = 4.56$
Let $x = 5.6$	$y = 6.96$
Let $x = 6$	$y = 3$
Let $x = 5.8$	$y = 0.44$
Let $x = 5.75$	$y = 8.8125$

Thus for some $x \in (5.75, 5.8)$ we have a $y = 0$.

Let $x = 5.9$	$y = 1.71$
Let $x = 6$	$y = 3$
Let $x = 6.3$	$y = 6.99$
Let $x = 6.5$	$y = 0.75$
Let $x = 6.4$	$y = 8.36$
Let $x = 6.45$	$y = 0.0525$

For some $x \in (6.4, 6.45)$ we have a $y = 0$.

Let $x = 6.6$	$y = 2.16$
Let $x = 6.8$	$y = 5.04$
Let $x = 7$	$y = 8$
Let $x = 7.1$	$y = 0.51$

For some $x \in (7, 7.1)$ we have a $y = 0$.

Let $x = 7.3$	$y = 3.59$
Let $x = 7.6$	$y = 8.36$

For some $x \in (7.6, 7.65)$ there a $y = 0$.

Let $x = 7.65$	$y = 0.1725$
Let $x = 7.8$	$y = 2.64$
Let $x = 8$	$y = 6$
Let $x = 8.4$	$y = 3.96$
Let $x = 8.2$	$y = 0.44$
Let $x = 8.15$	$y = 8.5725$

For some $x \in (8.15, 8.2)$ we have a $y = 0$.

For some $x = 8.6$, we have $y = 7.56$
For some $x = 8.7$ $y = 0.39$
For some $x \in (8.6, 8.7)$, we have a $y = 0$.
For some $x = 8.9$ $y = 4.11$

The maximum value y can take for $x = 8.99\ldots9$ is 5.9999.

We have at least 10 zeros for this second degree MOD function $y = x^2 + x + 6$ in $R_n(9)$ [x].

Now consider the function $y = x^2 + 7x + 4 \in R[x]$ we will study the structure of $y = f(x)$ in all the MOD planes.

The function $y = x^2 + 7x + 4$ in the MOD plane $R_n(2)$ is $y = x^2 + x$.

The graph of $y = x^2 + x$ in $R_n(2)$ is as follows:

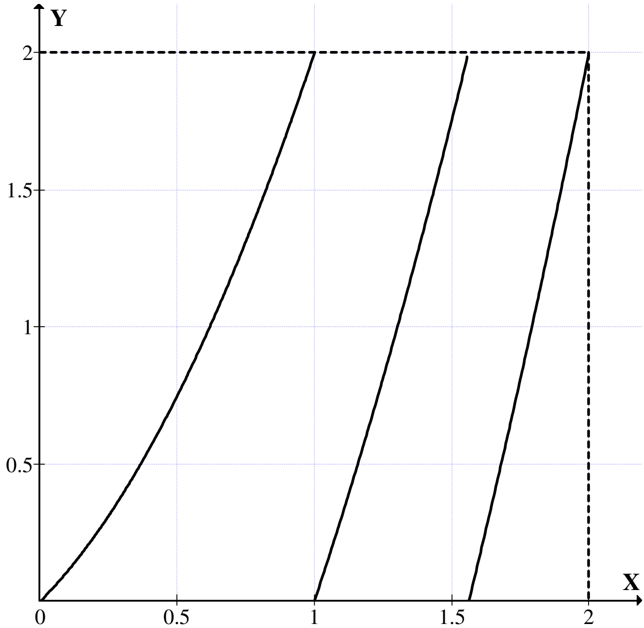

Figure 2.41

For x = 0.2	y = 0.24
For x = 0.5	y = 0.75
For x = 0.7	y = 1.19
For x = 1	y = 0
For x = 1.8	y = 1.04
For x = 0.8	y = 1.44
For x = 0.9	y = 1.71
For x = 1.9	y = 1.51
For x = 1.5	y = 1.75
For x = 1.6	y = 0.16
For x = 1.55	y = 1.9525

For some $x \in (1.55, 1.6)$ we have a y = 0.

For x = 1.95	y = 1.7525
For x = 1.99	y = 1.95

Thus $x^2 + 2$ has 3 zeros in the $R_n(2)$ MOD plane. But for $x^2 + 7x + 4 \in R[x]$ the roots are

$$\frac{-7 \pm \sqrt{49 - 4 \times 4}}{2}$$

$$= \frac{-7 \pm \sqrt{33}}{2} = \frac{-7 \pm 5.7445}{2}.$$

The roots are

$$\frac{-12.7445}{2}, \frac{-1.2555}{2}.$$

Both the roots are negative in R. However the function $y = x^2 + 7x + 4$ in the MOD plane $R_n(2)$ has at least 3 real roots in (0, 2).

Consider $y = x^2 + 7x + 4$ in the MOD plane $R_n(3)$. In $R_n(3)$, $y = x^2 + x + 1$. The graph of $y = x^2 + x + 1$ in the MOD plane $R_n(3)$ is as follows:

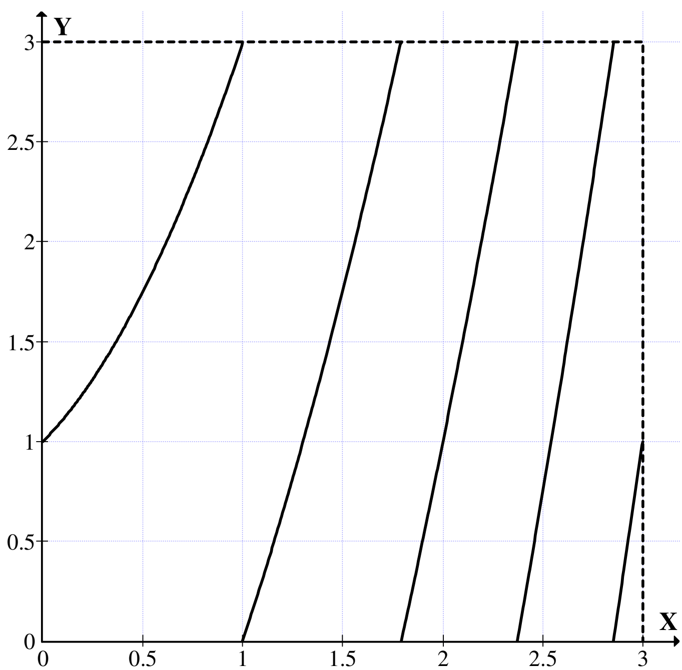

Figure 2.42

For $x = 0.5$ $y = 1.75$
For $x = 0.99$ $y = 2.9701$

When $x = 1$, $y = 0$.
For $x = 1.2$ $y = 0.64$
For $x = 1.6$ $y = 2.16$
For $x = 1.8$ $y = 0.04$
For $x = 1.7$ $y = 2.59$

So for some $x \in (1.7, 1.8)$ we have a $x = 0$.
For $x = 2$ $y = 1$
For $x = 2.3$ $y = 2.59$
For $x = 2.4$ $y = 0.16$

Thus for some $x \in (2.3, 2.4)$ we have $y = 0$.
For $x = 2.5$ $y = 0.75$
For $x = 2.7$ $y = 1.99$
For $x = 2.8$ $y = 2.64$
For $x = 2.85$ $y = 2.9725$
For $x = 2.88$ $y = 0.1744$
For $x = 2.86$ $y = 0.0396$

Thus for some $x \in (2.85, 2.86)$ we have a $y = 0$.
For $x = 2.999$ $y = 0.9999$
For $x = 2.9$ $y = 0.31$

Thus in the MOD plane $R_n(3)$ $x^2 + x + 1$ has at least four zeros.

Consider $y = x^2 + 7x + 4$ in the MOD plane $R_n(4)$. The equation or the function is $y = x^2 + 3x$.

The graph of the function of the function $y = x^2 + 3x$ in $R_n(4)$ is given in Figure 2.43.

$x = 0$ $y = 0$
$x = 0.3$ $y = 0.99$
$x = 0.5$ $y = 1.75$
$x = 0.9$ $y = 3.51$

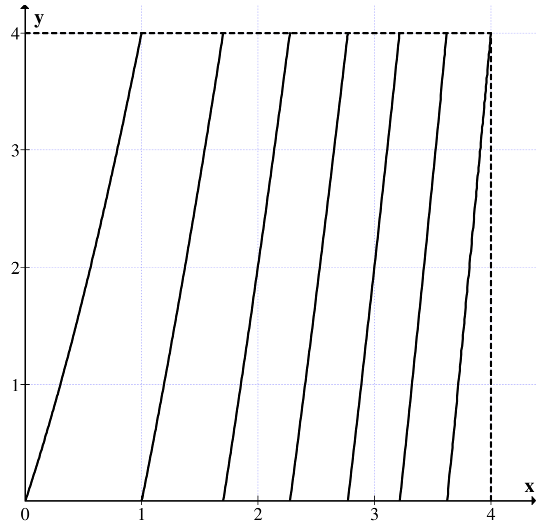

Figure 2.43

x = 1 y = 0
x = 1.2 y = 1.04
For x = 1.4 y = 2.16
For x = 1.35 y = 1.8725
For x = 1.5 y = 2.75
For x = 1.6 y = 3.36
For x = 1.7 y = 3.99

For some x ∈ (1.7, 1.75) we have y = 0.

For x = 2 y = 2
For x = 2.4 y = 0.96
For x = 2.35 y = 0.572
For x = 2.3 y = 0.19
For x = 2.2 y = 3.44
For x = 3 y = 2

For some x ∈ (2.2, 2.3) we have y = 0

For x = 2.4 y = 0.96
For x = 2.7 y = 3.39
For x = 2.8 y = 0.24

For some x ∈ (2.7, 2.8) we have a y = 0

For x = 3 y = 2
For x = 3.2 y = 3.84
For x = 3.3 y = 0.79
Thus for some x ∈ (3.2, 3.3) y = 0

For x = 3.5 y = 2.75
For x = 3.6 y = 3.76
For x = 3.7 y = 0.79
For x = 3.65 y = 0.2725
For some x ∈ (3.6, 3.65) we have y = 0

Thus $x^2 + 3x$ in the MOD plane $R_n(4)$ has atleast seven zeros.

Next we study $y = x^2 + 7x + 4$ in the MOD plane $R_n(5)$. The function changes to $y = x^2 + 2x + 4$ we now give the graph of $y = x^2 + 2x + 4$ in the MOD plane $R_n(5)$ in Figure 2.44.

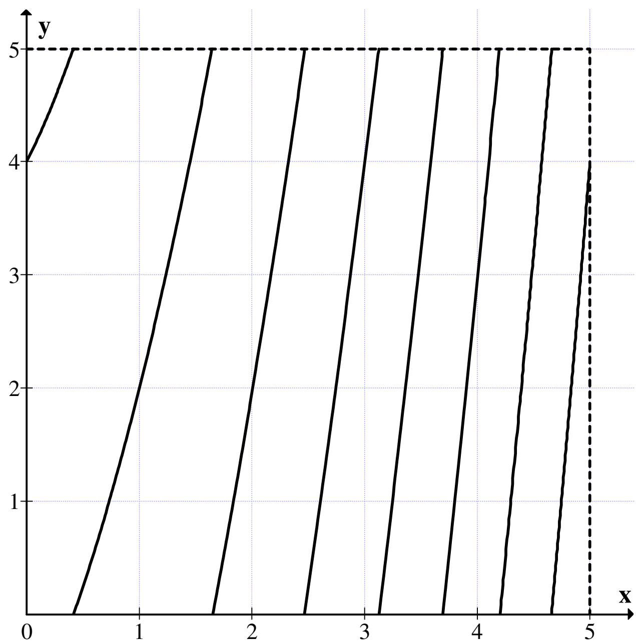

Figure 2.44

x = 0.2 y = 4.44

$x = 0.3$ $y = 4.69$
$x = 0.4$ $y = 4.96$

For some $x \in (0.4, 0.5)$ there is a $y = 0$.

$x = 0.5$ $y = 0.25$
$x = 0.8$ $y = 1.24$
$x = 1$ $y = 2$
$x = 1.3$ $y = 3.29$
$x = 1.5$ $y = 4.25$
$x = 1.6$ $y = 4.76$
$x = 1.7$ $y = 0.29$
$x = 1.65$ $y = 0.0225$

So for some $x \in (1.6, 1.65)$ we have a $y = 0$.

$x = 2$ $y = 2$
for $x = 2.4$ $y = 4.56$
for $x = 2.5$ $y = 0.25$
so for $x \in (2.4, 2.5)$ we have $y = 0$

Let $x = 2.7$ $y = 1.69$
Let $x = 2.6$ $y = 0.96$
Let $x = 2.9$ $y = 3.21$
Let $x = 3$ $y = 4$
For $x = 3.2$ $y = 0.64$
For $x = 3.1$ $y = 4.81$
For $x = 3.15$ $y = 0.2225$

Thus for some $x \in (3.1, 3.15)$ we have $y = 0$

For $x = 3.4$ $y = 2.36$
For $x = 3.3$ $y = 1.49$
For $x = 3.6$ $y = 4.16$
For $x = 3.65$ $y = 4.6225$
For $x = 3.7$, $y = 0.9$

So for $x \in (3.65, 3.7)$ we have $y = 0$.

For $x = 3.8$ $y = 1.04$
For $x = 4$ $y = 3$
For $x = 4.2$ $y = 0.4$
Thus for $x = 4.1$ $y = 4.01$

So for some x ∈ (4.1, 4.2) we have y = 0.

For x = 4.4 y = 2.16
For x = 4.6 y = 4.36
For x = 4.65 y = 4.9225
For x = 4.66 y = 0.0356
Thus for some x in (4.65, 4.66) is 3.9999.

Thus $y = x^2 + 2x + 4$ has atleast seven zeros in the MOD plane $R_n(5)$.

Now we study $y = x^2 + 7x + 4$ in the MOD plane $R_n(6)$.

The MOD function in $R_n(6)$ is $y = x^2 + x + 4$.

We give the graph of the MOD function $y = x^2 + x + 4$ in the MOD plane $R_n(6)[x]$.

$$y = x^2 + x + 4 \in R_n(6)[x]$$

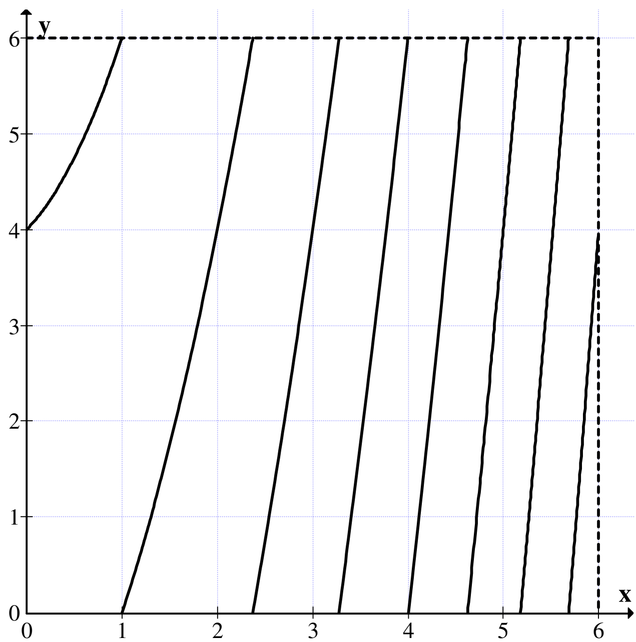

Figure 2.45

For $x = 1.5$	$y = 1.75$
For $x = 1.8$	$y = 3.04$
For $x = 2$	$y = 4$
For $x = 2.3$	$y = 5.59$
For $x = 2.4$	$y = 0.16$
For $x = 2.35$	$y = 5.8725$

So for some $x \in (2.35, 2.4)$ we have a $y = 0$.

For $x = 2.7$	$y = 1.99$
For $x = 2.5$	$y = 0.75$
For $x = 2.6$	$y = 1.36$
For $x = 2.8$	$y = 2.64$
For $x = 3$	$y = 4$
For $x = 3.2$	$y = 5.44$
For $x = 3.3$	$y = 0.19$
For $x = 3.25$	$y = 5.8125$

Thus for some $x \in (3.25, 3.3)$ we have $y = 0$.

For $x = 3.5$	$y = 1.75$
For $x = 3.4$	$y = 0.96$
For $x = 3.8$	$y = 4.24$
For $x = 4$	$y = 0$
For $x = 4.3$	$y = 2.79$
For $x = 4.5$	$y = 4.75$
For $x = 4.6$	$y = 5.16$
For $x = 4.7$	$y = 0.79$
For $x = 4.65$	$y = 0.2725$

For some $x \in (4.6, 4.65)$ we have a $y = 0$.

Now for $x = 4.8$	$y = 1.84$
For $x = 5$	$y = 4$
For $x = 5.2$	$y = 0.24$
For $x = 5.1$	$y = 5.11$

Thus for some $x \in (5.15, 5.2)$ we have a $y = 0$.

For $x = 5.3$	$y = 1.39$
For $x = 5.5$	$y = 3.75$
For $x = 5.7$	$y = 0.19$
For $x = 5.6$	$y = 4.96$

For x = 5.65 y = 5.5725
For some x ∈ (5.65, 5.7) we have a y = 0

For x = 5.9 y = 2.71
For x = 5.99 y = 3.8701

Thus for this interval x ∈ (5.7, 6) the maximum value y can take in 3.9999. We see $x^2 + x + 4$ has atleast seven zeros in the MOD plane $R_n(6)$.

Next we study the function $y = x^2 + 7x + 4 \in R[x]$ in the MOD plane $R_n(7)$.

$y = x^2 + 4$ in the MOD plane $R_n(7)$. The graph of $y = x^2 + 4$ in the MOD plane $R_n(7)$ is as follows:

$$y = x^2 + 4 \in R_n(7)[x]$$

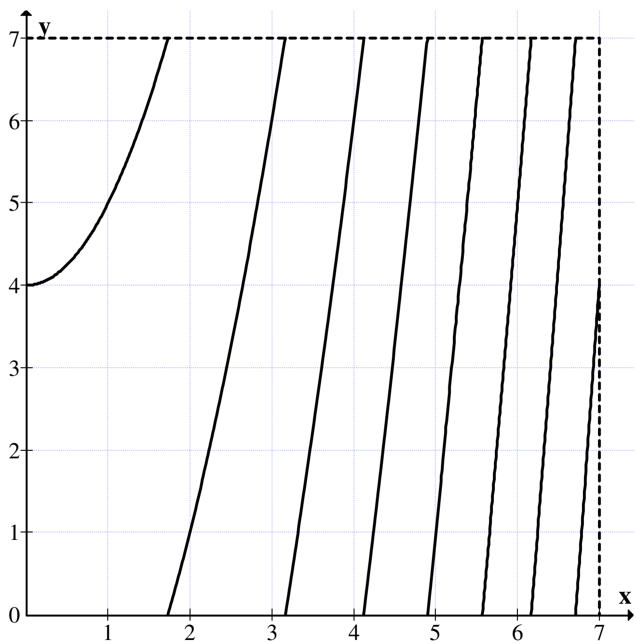

Figure 2.46

When x = 0	y = 4
When x = 1	y = 5
When x = 1.5	y = 6.25
For x = 1.2	y = 5.44
For x = 1.6	y = 6.56
For x = 1.7	y = 6.89
For x = 1.8	y = 0.24
For x = 1.75	y = 0.0625

For some x ∈ (1.7, 1.75) we have a y = 0

For x = 2	y = 1
For x = 2.5	y = 3.25
For x = 3	y = 6
For x = 3.2	y = 0.24
For x = 3.1	y = 6.61

For some x ∈ (3.1, 3.2) we have y = 0.

For x = 3.8	y = 4.44
For x = 4	y = 6
For x = 4.1	y = 6.81
For x = 4.2	y = 0.64
For x = 4.15	y = 0.2225
For x = 4.13	y = 0.0569

For x ∈ (4.1, 4.13) we have a y = 0.

For x = 4.4	y = 2.36
For x = 4.5	y = 3.25
For x = 4.7	y = 5.09
For x = 4.8	y = 6.04
For x = 4.9	y = 0.01
For x = 4.85	y = 6.5225

For some x ∈ (4.85, 4.9) there exist a y = 0.

For x = 5	y = 1
For x = 5.5	y = 6.25
For x = 5.6	y = 0.36
For x = 5.55	y = 6.8025

Thus for some x ∈ (5.55, 5.6) we have a y = 0.

For $x = 6$ $y = 5$
For $x = 6.2$ $y = 0.44$
For $x = 6.1$ $y = 6.21$
For $x = 6.15$ $y = 6.8225$
For some $x \in (6.15, 6.2)$ we must have $y = 0$.

For $x = 6.5$ $y = 4.25$
For $x = 6.7$ $y = 6.89$
For $x = 6.75$ $y = 0.5625$
Thus for some $x \in (6.7, 6.75)$ we have $y = 0$

For $x = 6.99$ $y = 3.86$
Thus in the interval (6.7, 7) y attains the maximum value of 3.9999 only.

Hence the MOD function $y = x^2 + 4$ has atleast 7 zeros.

Now in all other MOD planes $R_n(m)$, $y = x^2 + 7x + 4$ is the same for all $m \geq 8$.

We study the function $y = x^2 + 7x + 4$ in the MOD plane $R_n(8)[x]$.

The graph in the MOD plane is given in Figure 2.47.

For $x = 0$ $y = 4$
For $x = 0.5$ $y = 7.75$
For $x = 0.6$ $y = 0.56$
For $x = 0.55$ $y = 0.1525$
So for some $x \in (0.50, 0.55)$ we have a $y = 0$

For $x = 1.3$ $y = 6.79$
For $x = 1.4$ $y = 7.76$
For $x = 1.45$ $y = 0.2525$
For some $x \in (1.4, 1.45)$ we have a $y = 0$.

For $x = 1.6$ $y = 1.76$
For $x = 1.7$ $y = 2.79$
For $x = 1.8$ $y = 3.84$

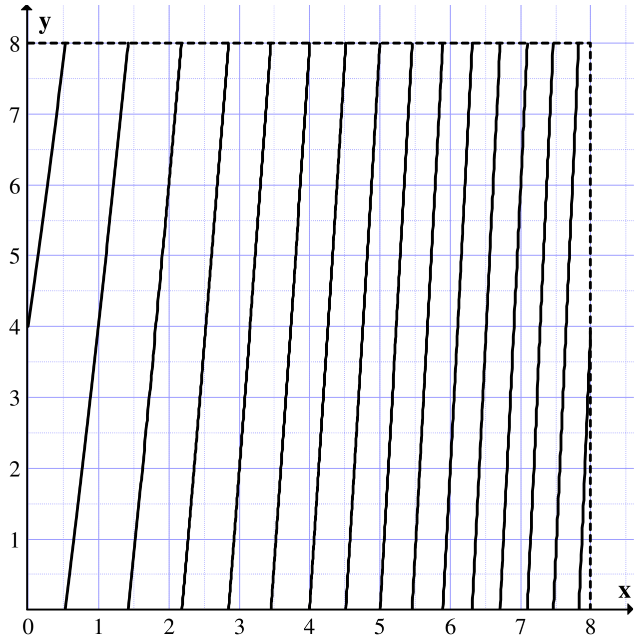

Figure 2.47

For x = 2 y = 6
For x = 2.3 y = 1.39
For x = 2.2 y = 0.24
For x = 2.15 y = 8.6725
Thus for some x ∈ (2.15, 2.2) we have a y = 0

For x = 2.4 y = 2.56
For x = 2.7 y = 6.19
For x = 2.5 y = 3.75
For x = 2.8 y = 7.44
For x = 2.85 y = 0.0725
Thus for some x ∈ (2.8, 2.85) we have a y = 0

For x = 3 y = 2
For x = 3.5 y = 0.75
For x = 3.4 y = 7.36
For x = 3.45 y = 8.0525
For some x ∈ (3.4, 3.45) we have a y = 0

For $x = 3.8$ $y = 5.04$
For $x = 4$ $y = 0$
For $x = 3.95$ $y = 7.2525$
For $x = 4.3$ $y = 4.59$
For $x = 4.5$ $y = 7.75$
For $x = 4.55$ $y = 0.5526$

For some $x \in (4.5, 4.55)$ we have a $y = 0$.

For $x = 5$, $y = 0$
For $x = 5.2$ $y = 3.44$
For $x = 5.4$ $y = 6.96$
For $x = 5.5$ $y = 0.75$
For $x = 5.45$ $y = 7.8525$

Thus for some $x \in (5.45, 5.5)$ we have a $y = 0$

For $x = 5.7$ $y = 4.39$
For $x = 5.8$ $y = 6.24$
For $x = 5.9$ $y = 0.11$
For $x = 5.85$ $y = 7.1725$

Thus for some $x \in (5.85, 5.9)$ we have a $y = 0$.

At $x = 6$ $y = 2$
For $x = 6.3$ $y = 7.79$

For $x = 6.35$ $y = 0.77$
For $x = 6.33$ $y = 0.3789$
For $x = 6.32$ $y = 0.1824$
For $x = 6.310$ $y = 7.9861$

Thus for some $x \in (6.31, 6.32)$ we have a $y = 0$.

For $x = 6.5$ $y = 3.75$
For $x = 6.6$ $y = 5.76$
For $x = 6.7$ $y = 7.79$

Thus for $x = 6.75$ $y = 0.8125$

For $x = 6.73$ $y = 0.4029$
For $x = 6.71$ $y = 8.9941$
For $x = 6.72$ $y = 0.1984$

Thus for some $x \in (6.71, 6.72)$ we have a $y = 0$

For $x = 7$ $\quad$ $y = 6$
For $x = 7.3$ $\quad$ $y = 4.39$
For $x = 7.1$ $\quad$ $y = 0.11$
For $x = 7.05$ $\quad$ $y = 7.0525$
Thus for some $x \in (7.05, 7.1)$ we have a $y = 0$.

For $x = 7.4$ $\quad$ $y = 6.56$
For $x = 7.5$ $\quad$ $y = 0.75$
For $x = 7.45$ $\quad$ $y = 7.6525$
Thus for some $x \in (7.45, 7.5)$ we have a $y = 0$.

For $x = 7.6$, $\quad$ $y = 2.96$
For $x = 7.8$ $\quad$ $y = 7.44$
For $x = 7.85$ $\quad$ $y = 0.5725$
Thus for some $x \in (7.8, 7.85)$ $y = 0$.

For $x = 7.9$ $\quad$ $y = 1.71$
For $x = 7.99$ $\quad$ $y = 3.77$
Thus for all $x \in (7.8, 7.85)$ $y = 0$

For $x = 7.9$ $\quad$ $y = 1.71$
For $x = 7.99$ $\quad$ $y = 3.77$

Thus for all $x \in (7.8, 8)$ we see the maximum value y can get is 3.999.

Further this equation $y = x^2 + 7x + 4$ in the plane $R_n(8)$ has fifteen zeros where as $x^2 + 7x + 4 \in R[x]$ has two negative value only.

Now we study functions of the form $y = 8x^2 + 5x + 7 \in R[x]$ the roots of the equation are

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 7 \times 8}}{16}$$

The roots are imaginary in R[x].

Now we study this function in R_n (m).

In $R_n(2)$ the function $y = 8x^2 + 5x + 7$ changes to x + 1.

Now the roots of x + 1 in $R_n(2)[x]$ and the graph of x + 1 in $R_n(2)$ is as follows:

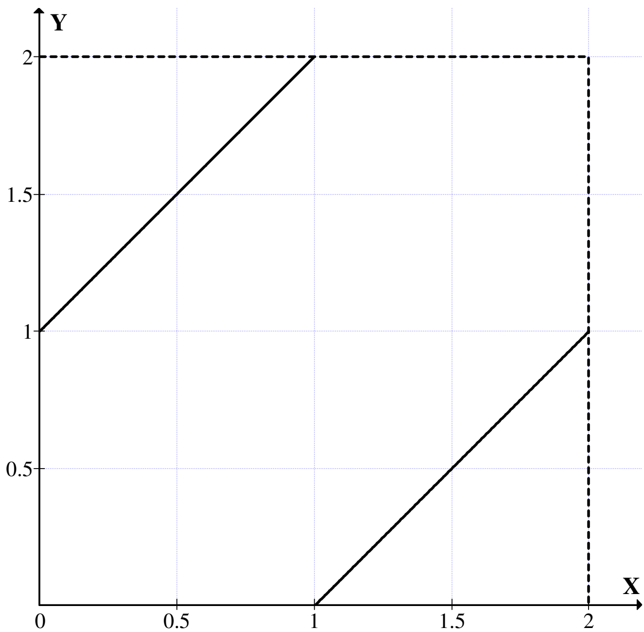

Figure 2.48

For x = 0.5	y = 1.5
For x = 1	y = 0
For x = 0.5	y = 1.5
For x = 1.2	y = 0.2
For x = 1.5	y = 0.5
For x = 1.9	y = 0.9.

The maximum value y can get in (1, 2) is 0.9999.

Thus in the MOD plane $R_n(2)$ the equation has one zero.

Next we study $y = 8x^2 + 5x + 7$ in $R_n(3)[x]$.

In $R_n(3)[x]$; $y = 2x^2 + 2x + 1$.

The graph of $y = 2x^2 + 2x + 1$ is as follows:

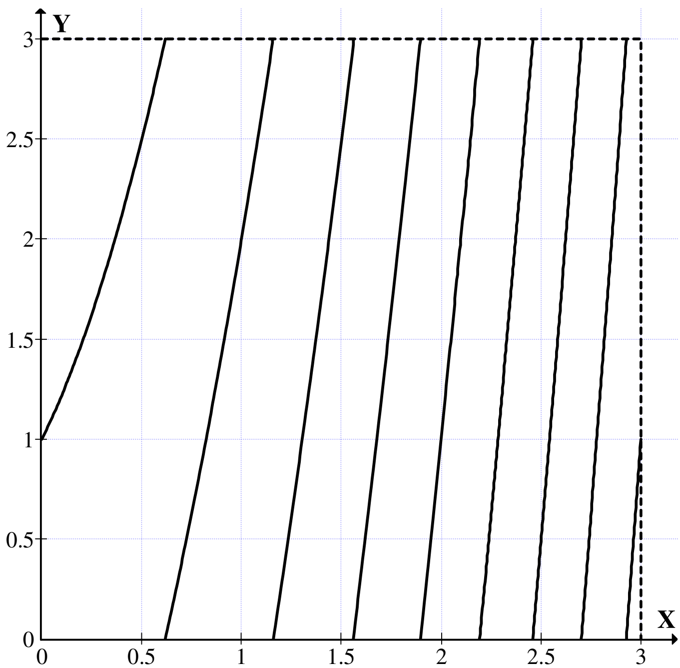

Figure 2.49

For x = 0.5 y = 2.5
For x = 0.6 y = 2.92
For x = 0.7 y = 0.38

Thus for some x ∈ [0.6, 0.7) we have a y = 0.

For x = 1 y = 2
For x = 1.2 y = 0.28
For x = 1.1 y = 2.62
For x = 1.4 y = 1.72
For x = 1.3 y = 0.98

For $x = 1.5$ $y = 2.5$
For $x = 1.55$ $y = 2.905$
For $x = 1.6$ $y = 0.32$
Thus the some $x \in (1.55, 1.6)$ we have a $y = 0$
For some $x \in (1.1, 1.2)$ we have a $y = 0$.

For $x = 1.7$ $y = 1.18$
For $x = 1.8$ $y = 2.08$
For $x = 1.9$ $y = 0.02$
So for some $x = (1.85, 1.9)$ we have a $y = 0$.

For $x = 2$ $y = 1$
For $x = 2.1$ $y = 2.02$
For $x = 2.2$ $y = 0.08$
For $x = 2.15$ $y = 2.545$
For some $x \in (2.15, 2.2)$ we have a $y = 0$.

For some $x = 2.3$ $y = 1.18$
For some $x = 2.4$ $y = 2.32$
For some $x = 2.45$ $y = 2.905$
$x = 2.5$ $y = 0.5$
thus for some $x \in (2.45, 2.5)$ we have a $y = 0$.

For $x = 2.6$ $y = 1.72$
For $x = 2.7$ $y = 2.98$
For $x = 2.75$ $y = 0.625$
For some $x \in (2.7, 2.75)$ we have a $y = 0$.

For $x = 2.8$ $y = 1.28$
For $x = 2.9$ $y = 2.62$
For $x = 2.99$ $y = 0.86$
For $x = 2.95$ $y = 0.5$
For $x = 2.92$ $y = 2.89$
For some $x \in (2.92, 2.93)$ there a $y = 0$.

For $x = 2.93$ $y = 0.298$
For $x = 2.999$ $y = 0.986002$

Thus for every $x \in (2.92, 3)$ we have y to take the maximum of 0.986002.

We have atleast eight 8 zeros for $y = 2x^2 + 2x + 1$ in $R_n(3)$.

Consider $y = 8x^2 + 5x + 7$ in $R_n(4)$. It takes the form $x + 3$. The graph of $x + 3$ in $R_n(4)$ is as follows:

$$y = x + 3 \in R_n(3)$$

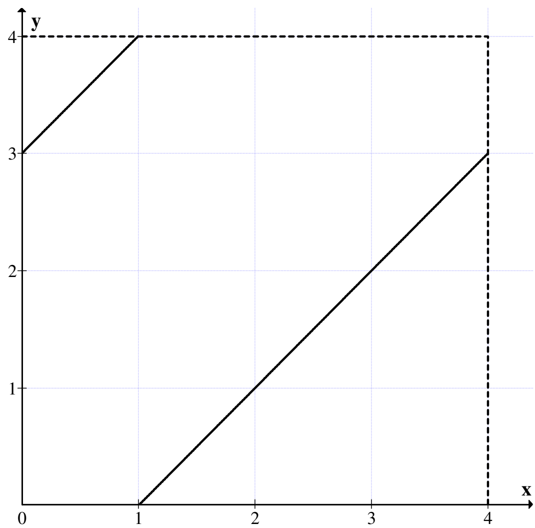

Figure 2.50

For $x = 0$	$y = 3$
For $x = 0.5$	$y = 3.5$
For $x = 0.99$	$y = 3.99$
For $x = 1$	$y = 0$
For $x = 1.2$	$y = 0.2$
For $x = 1.9$	$y = 0.9$

So for $x = 2$	$y = 1$
For $x = 3$	$y = 2$
For $x = 3.5$	$y = 2.5$
For $x = 3.99$	$y = 2.99$

The equation $y = x + 3$ has atleast one zero, given by $x = 1$.

Consider $y = 8x^2 + 5x + 7$, y takes the form $3x^2 + 2$ in $R_n(5)[x]$.

The graph of $y = 3x^2 + 2$ in $R_n(5)$ is as follows.

$y = 3x^2 + 2 \in R_n(5)$

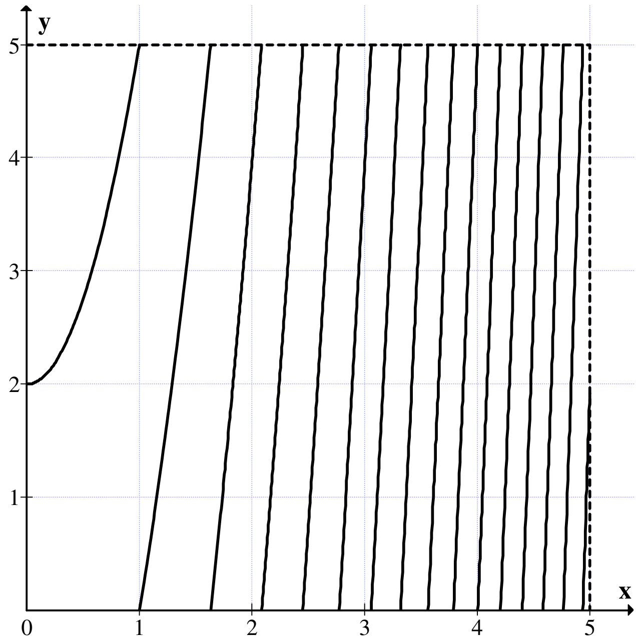

Figure 2.51

For $x = 0$	$y = 2$
For $x = 1$	$y = 0$
For $x = 0.9$	$y = 4.43$
For $x = 0.99$	$y = 4.9403$
For $x = 1.2$	$y = 1.32$
For $x = 1.4$	$y = 2.88$
For $x = 1.5$	$y = 3.75$
For $x = 1.6$	$y = 0.68$
For $x = 1.55$	$y = 4.2075$

For $x \in (1.55, 1.6)$ we have $y = 0$

For $x = 1.8$	$y = 1.72$
For $x = 2$	$y = 4$
For $x = 2.2$	$y = 1.52$

For x = 2.1 y = 0.23

For some x ∈ (2, 2.1) we have a y = 0.

For x = 2.3 y = 2.87
For x = 2.4 y = 4.28
For x = 2.5 y = 0.75
For x = 2.45 y = 0.0075

Thus for some x ∈ (2.45, 2.5) we have a y = 0.

For x = 2.6 y = 2.28
For x = 2.7 y = 3.87
For x = 2.8 y = 0.52
For x = 2.75 y = 4.6875

For some x ∈ (2.75, 2.8) we have a y = 0

For x = 3 y = 4
For x = 3.1 y = 0.83
For x = 3.05 y = 4.9075

Thus for some x ∈ (3.05, 3.1) we have a y = 0.

For x = 3.3 y = 4.67
For x = 3.35 y = 0.6675
For x = 3.32 y = 0.0672

Thus for some x ∈ (3.30, 332) we have y = 0.

For x = 3.7 y = 3.07
For x = 3.8 y = 0.32
For x = 3.75 y = 4.1875
For x = 3.78 y = 4.8652
For x = 3.79 y = 0.0923

Thus for some x ∈ (3.78, 3.79) we have a y = 0.

For x = 3.9 y = 2.63

For x = 4   y = 0
For x = 4.2   y = 4.92
For x = 4.1   y = 4.53
For x = 4.22   y = 0.4252
For x = 4.21   y = 0.1723

For x ∈ (4.2, 4.21) we have a y = 0.

For x = 4.4   y = 0.08
For x = 4.35   y = 3.7675
For x = 4.39   y = 4.8163
For x ∈ (4.39, 4.4) we have a y = 0.

For x = 4.6   y = 0.48
For x = 4.55   y = 4.1075
For x = 4.58   y = 4.9292

For some x ∈ (4.58, 4.6) we have a y = 0.

For x = 4.7   y = 3.27
For x = 4.8   y = 1.12
For x = 4.75   y = 4.6875

We see for x = 4.78   y = 0.54

Thus for some x ∈ (4.75, 4.78) we have a y = 0.

For x = 4.9   y = 4.03
For x = 4.99   y = 1.7003
For x = 4.92   y = 4.6192
For x = 4.93   y = 4.9147
For x = 4.94   y = 0.2108
For x = 4.935   y = 0.062675

For some x ∈ (4.93, 4.935) we have y = 0

For x = 4.999   y = 1.97

W see the equation $y = 3x^2 + 2$ has atleast 14 zeros in the MOD plane $R_n(5)$.

Next we consider the equation $y = 8x^2 + 5x + 7$ in the MOD plane $R_n(6)[x]$.

In $R_n(6)[x]$ the equation $y = 8x^2 + 5x + 7$ assumes the form $y = 2x^2 + 5x + 1$. The graph of $y = 2x^2 + 5x + 1$ in the MOD plane $R_n(6)[x]$ is as follows:

$$y = 2x^2 + 5x + 1 \in R_n(6)[x]$$

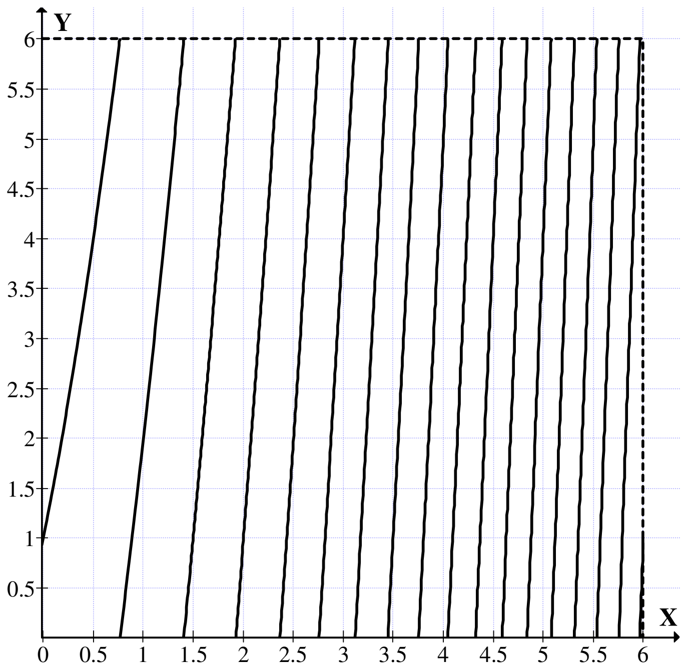

Figure 2.52

For $x = 0$	$y = 1$
For $x = 0.5$	$y = 4$
For $x = 1$	$y = 2$
For $x = 0.8$	$y = 0.28$
For $x = 0.7$	$y = 5.48$
For $x = 0.75$	$y = 5.875.$

For some $x_1 \in (0.75, 0.8)$ we have $y = 0$.

$x = 1.3$	$y = 4.88$
For $x = 1.4$	$y = 5.92$
For $x = 1.42$	$y = 0.1328$.

So for some $x_2 \in (1.4, 1.42)$ we have $y = 0$.

For $x = 1.6$	$y = 2.12$
For $x = 1.8$	$y = 4.48$
For $x = 1.9$	$y = 5.72$
For $x = 2$	$y = 1$.
For $x = 1.95$,	$y = 0.335$.

So for some $x_3 \in (1.9, 1.95)$ we have $y = 0$.

For $x = 2.4$	$y = 0.52$
For $x = 2.38$	$y = 0.2288$
For $x = 2.35$	$y = 5.795$.

For some $x_4 \in (2.35, 2.38)$ we have $y = 0$.

For $x = 2.6$	$y = 3.52$
For $x = 2.7$	$y = 5.08$
For $x = 2.75$	$y = 5.875$
For $x = 2.8$	$y = 0.68$.

Thus for some $x_5 \in (2.75, 2.8)$ we have a $y = 0$.

For $x = 3$	$y = 4$
For $x = 3.2$	$y = 1.48$
For $x = 3.1$	$y = 5.72$
For $x = 3.15$	$y = 0.595$
For $x = 3.14$	$y = 0.4192$
For $x = 3.12$	$y = 5.0688$.

Thus for some $x_6 \in (3.12, 1.14)$ we have $y = 0$.

For $x = 3.3$	$y = 3.28$
For $x = 3.5$	$y = 1.00$
For $x = 3.4$	$y = 5.12$
For $x = 3.45$	$y = 0.055$.

For some $x_7 \in (3.4, 3.45)$ we have $y = 0$.

For $x = 3.6$ $y = 2.92$
For $x = 3.8$ $y = 0.88$
For $x = 3.7$ $y = 4.88$
For $x = 3.78$ $y = 0.4768$
For $x = 3.75$ $y = 5.875$.

Thus for some $x_8 \in (3.75, 3.78)$ we have $y = 0$

For $x = 3.9$ $y = 2.92$
For $x = 4$ $y = 5$.

For some $x_9 \in (4, 4.06)$ there is a zero.

For $x = 4.2$ $y = 3.28$
For $x = 4.3$ $y = 5.48$
For $x = 4.5$ $y = 4$

For some $x_{10} \in (4.3, 4.35)$ we have a $y = 0$.

For $x = 4.7$ $y = 2.68$
For $x = 4.6$ $y = 0.32$
For $x = 4.58$ $y = 5.8528$

So for some $x_{11} \in (4.58, 4.6)$ we have $y = 0$.
For some $x_{12} \in (4.8, 4.85)$ we have a $y = 0$.

For $x = 5$ $y = 4$
For $x = 5.06$, $y = 5.5072$

For some $x_{13} \in (5.06, 5.08)$ we have a $y = 0$.

For $x = 5.2$ $y = 3$
For $x = 5.4$ $y = 2.32$
For $x = 5.3$ $y = 5.68$

So for some $x_{14} \in (5.3, 5.35)$ we have $y = 0$.

For some $x = 5.4$ $y = 2.32$
For $x = 5.5$ $y = 5$
For $x = 5.55$ $y = 0.105$

For some $x_{15} \in (5.5, 5.55)$, $y = 0$.

For $x = 5.6$ $y = 1.72$
For $x = 5.8$ $y = 1.28$

So for some $x_{16} \in (5.6, 5.76)$, $y = 0$.

For $x = 5.75$ $\quad$ $y = 5.875$
For $x = 5.9$ $\quad$ $y = 4.12$

$y = 2x^2 + 5x + 1$ has atleast 16 zeros in the MOD plane $R_n(6)$.

Next consider the function $y = 8x^2 + 5x + 7$ in the MOD plane $R_n(7)$. The graph of the function in the MOD plane is as follows.

The function takes the form $y = x^2 + 5x$ in $R_n(7)$.

$$y = x^2 + 5x \in R_n(7)[x]$$

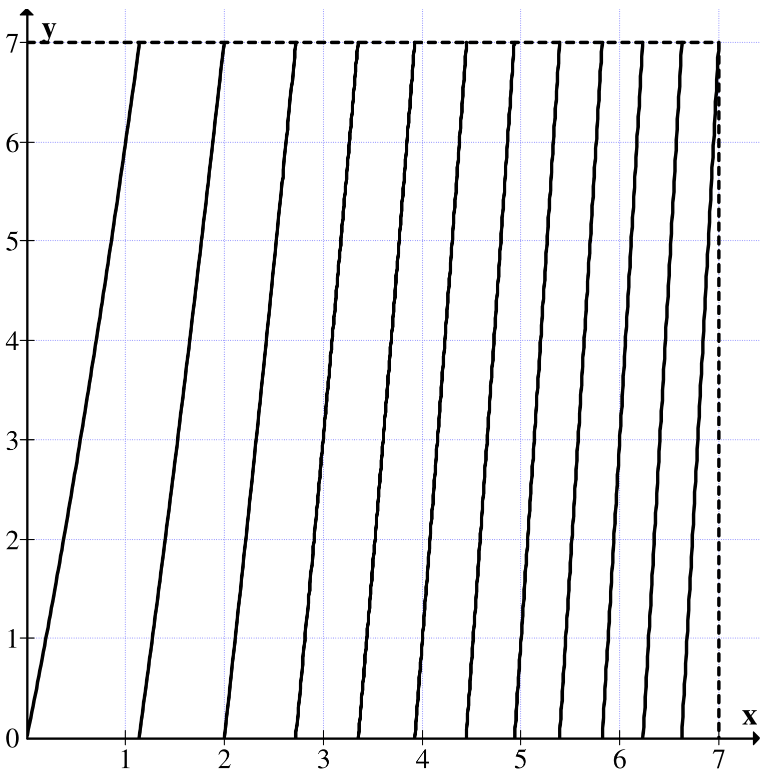

Figure 2.53

For $x = 1.1$ $y = 6.71$
For $x = 1.2$ $y = 7.44$
For $x = 1.15$ $y = 7.0975$
For some $x_1 \in (1.1, 1.15)$ we have $y = 0$.

For $x_2 = 2$ $y = 0$.
For $x = 2.7$ $y = 6.79$
For $x = 2.8$ $y = 0.84$
For $x = 2.75$ $y = 0.3125$
For $x = 2.72$ $y = 6.9984$
For some $x_3 \in (2.72, 2.73)$ we have $y = 0$

For $x = 2.73$ $y = 0.1029$
For $x = 2.9$ $y = 1.91$
For $x = 3$ $y = 3$
For $x = 3.5$ $y = 1.75$
For $x = 3.4$ $y = 0.56$
For $x = 3.3$ $y = 6.39$
For some $x_4 \in (3.3, 3.4)$ we have $y = 0$.

For $x = 3.5$ $y = 2.75$
For $x = 3.7$ $y = 5.19$
For $x = 3.9$ $y = 6.71$
For $x = 3.95$ $y = 0.3525$
For $x = 3.92$ $y = 6.9664$
For $x = 3.93$ $y = 0.949$
For some $x_5 \in (3.92, 3.93)$ we have $y = 0$

For $x = 4.2$ $y = 3.64$
For $x = 4.4$ $y = 6.36$
For $x = 4.5$ $y = 0.75$
For $x = 4.45$ $y = 0.0525$
For $x = 4.44$ $y = 6.7136$
Thus for some $x_6 \in (4.44, 4.45)$ we have $y = 0$

For $x = 4.8$ $y = 5.04$
For $x = 4.9$ $y = 6.51$
For $x = 5$ $y = 1$
For $x = 4.95$ $y = 0.2525$

For $x = 4.94$ $y = 0.1036$
For $x = 4.93$ $y = 6.9549$.
Thus for some $x_7 \in (4.93, 4.94)$ we have a $y = 0$.

For $x = 5.2$ we have $y = 4.04$ for $x = 5.5$ $y = 1.75$.
For $x = 5.4$ $y = 0.16$
For $x = 5.3$ $y = 5.59$
For $x = 5.35$ $y = 6.3725$.
Thus for some $x_8 \in (5.35, 5.4)$ we have a $y = 0$.

For $x = 5.6$ $y = 3.36$
For $x = 5.8$ $y = 6.64$
For $x = 5.83$ $y = 0.1389$
For $x = 5.82$ $y = 6.9724$.
Thus for some $x_9 \in (5.82, 5.83)$ we have $y = 0$.

For $x = 6$ we have $y = 3$
For $x = 6.3$ we have $y = 1.19$
For $x = 6.2$ $y = 6.44$
For $x = 6.25$ $y = 0.3125$.
Thus for some $x_{10} \in (6.21, 6.25)$ w have $y = 0$.

$x = 6.21$, $y = 6.6141$.

For $x = 6.4$ $y = 2.96$
For $x = 6.6$ $y = 6.56$
For $x = 6.65$ $y = 0.4725$
For $x = 6.63$ $y = 0.1069$
Thus for some $x_{11} \in (6.60, 6.63)$ we have a $y = 0$.

For $x = 6.8$; $y = 3.24$.
For $x = 6.9$ $y = 5.11$
For $x = 6.99$ $y = 6.8101$
Thus $x^2 + 5x = y$ in $R_n(7)[x]$ has atleast 11 zeros.

Let $y = 8x^2 + 5x + 7$ be in $R_n(8)$ then the transformed MOD equation is $y = 5x + 7$. The graph of $y = 5x + 7$ in the MOD plane $R_n(8)$ is as follows.

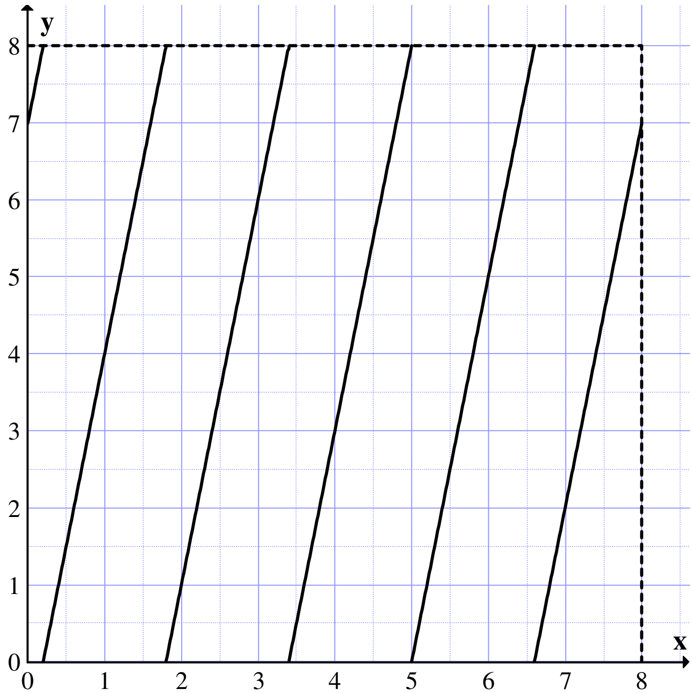

Figure 2.54

When	x = 0	y = 7
	For x_1 = 0.2	y = 0
	For x = 0.3	y = 0.5
	For x = 0.4	y = 1
	For x = 0.8	y = 3
	For x = 1	y = 4
	For x = 1.5	y = 6.5
	For x = 1.7	y = 7.5
	For x_2 = 1.8	y = 0
	For x = 2.4	y = 3
	For x = 2.9	y = 5.5
	For x = 3	y = 6
	For x = 3.2	y = 7
	For x = 3.3	y = 7.5
	For x = 3.5	y = 0.5

For $x_3 = 3.4$ $y = 0$
For $x = 3.8$ $y = 2$
For $x = 4$ $y = 3$
For $x = 4.6$ $y = 6$
For $x = 4.8$ $y = 7$
For $x = 4.9$ $y = 7.5$

For $x_4 = 5$ $y = 0$
For $x = 5.5$ $y = 2.5$
For $x = 6$ $y = 5$
For $x = 6.5$ $y = 7.5$
For $x = 6.7$ $y = 0.5$

For $x_5 = 6.6$ $y = 0$
For $x = 7$ $y = 2$
For $x = 6.8$ $y = 1$
For $x = 7.5$ $y = 4.5$
For $x = 7.9$ $y = 6.5$

In this MOD plane $5x + 7$ has atleast 5 zeros.

Consider the equation $y = 8x^2 + 5x + 7$ in the MOD plane $R_n(9)$. The graph is given in Figure 2.55.

When $x = 0$, $y = 7$.

For $x = 0.1$ $y = 7.58$
For $x = 0.2$ $y = 8.32$
For $x = 0.3$ $y = 0.22$
For $x = 0.4$ $y = 1.28$

For $x = 0.36$ $y = 4.6368$
For $x = 0.22$ $y = 8.4872$
For $x = 0.25$ $y = 8.75$
For $x = 0.27$ $y = 8.9332$

Thus for some $x_1 \in (0.27, 0.28)$, $y = 0$.

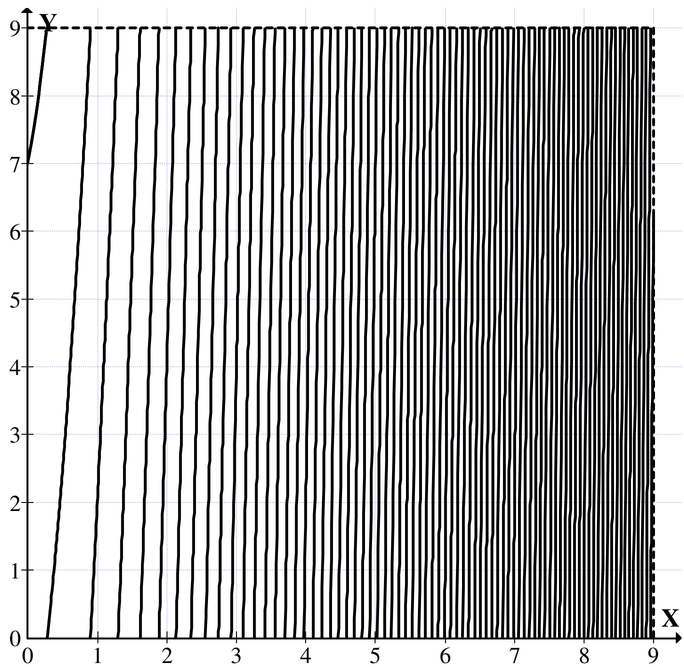

Figure 2.55

For x = 0.5 y = 2.5
For x = 0.6 y = 3.88
For x = 0.8 y = 7.12
For x = 0.9 y = 8.98
For x = 1 y = 2
For x = 0.95 y = 0.97

For some $x_2 \in (0.9, 0.97)$ we have y = 0.

For x = 1.2 y = 6.52
For x = 1.3 y = 0.02
For x = 1.29 y = 9.7625

Thus for $x_3 \in (1.2, 1.29)$ we have a y = 0.

For x = 1.33 y = 0.8012
For x = 1.4 y = 2.68
For x = 1.45 y = 4.07
For x = 1.5 y = 5.5
For x = 1.7 y = 2.62

For $x = 1.6$ $y = 9.48$
For $x = 1.64$ $y = 0.7168$
For $x = 1.62$ $y = 0.0952$
For $x = 1.61$ $y = 9.7868$

Thus for some $x_4 \in (1.61, 1.62)$ we have a $y = 0$.

For $x = 1.9$ $y = 2.62$
For $x = 1.9$ $y = 0.38$
For $x = 2$ $y = 4$
For $x = 1.85$ $y = 7.63$
For $x = 1.89$ $y = 0.0268$
For $x = 1.88$ $y = 9.6752$.

Thus for some $x_5 \in (1.87, 1.88)$ we have $y = 0$

For $x = 2$ $y = 4$
For $x = 2.3$ $y = 6.82$
For $x = 2.35$ $y = 8.93$
$x_6 \in (2.35, 2.36)$ we have $y = 0$.
For $x = 2.4$ $y = 2.08$
For $x = 2.37$ $y = 0.7852$
For $x = 2.38$ $y = 1.2152$
For $x = 2.39$ $y = 1.6468$
For $x = 3$ $y = 4$
For $x = 3.2$ $y = 5.92$
For $x = 3.4$ $y = 8.48$
For $x = 3.42$ $y = 0.6712$

So for $3.41 = 0.0748$

Thus for $x_7 \in (3.4, 3.41)$ we have a $y = 0$.

For $x = 3.6$ $y = 2.68$
For $x = 3.54$ $y = 7.9528$
For $x = 3.57$ $y = 0.8092$
For $x = 3.56$ $y = 0.1888$
For $x = 3.55$ $y = 8.57$

Thus for some $x_8 \in (3.55, 3.56)$ we have a $y = 0$

For $x = 2.43$ $y = 3.88$
For $x = 2.5$ $y = 6.5$

For $x = 2.55$ $y = 8.77$
For $x = 2.56$ $y = 0.2288$

For some $x_9 \in (2.5, 2.56)$ we have $y = 0$.

For $x = 2.6$ $y = 2.08$
For $x = 2.65$ $y = 4.43$
For $x = 2.7$ $y = 6.82$
For $x = 2.75$ $y = 8.25$
For $x = 2.76$ $y = 0.7404$

Thus for some $x_{10} \in (2.75, 2.76)$ we have a $y = 0$.

For $x = 2.8$ $y = 2.71$
For $x = 2.9$ $y = 7.98$
For $x = 2.93$ $y = 0.3292$

Thus for some $x_{11} \in (2.9, 2.93)$ $y = 0$

For $x = 3.5$ $y = 5.5$
For $x = 3.56$ $y = 0.1888$

For some $x_{12} \in (3.5, 3.56)$ we have $y = 0$

Consider $x = 3.6$ $y = 2.68$
For $x = 3.7$ $y = 0.2$
For $x = 3.67$ $y = 6.1012$
For $x = 3.68$ $y = 7.7392$
For $x = 3.69$ $y = 8.3788$

For some $x_{13} \in (3.69, 3.7)$ we have $y = 0$.

For $x = 3.8$ $y = 6.52$
For $x = 3.85$ $y = 0.83$
For $x = 3.83$ $y = 8.5012$

For some $x_{14} \in (3.83, 3.85)$ we have $y = 0$.

For $x = 3.9$ $y = 4.18$
For $x = 4$ $y = 2$
For $x = 3.95$ $y = 5.57$
For $x = 3.97$ $y = 8.9375$

For some $x_{15} \in (3.97, 4)$ we have $y = 0$.

For x = 4.5 y = 3.5
For x = 4.3 y = 5.42
For x = 4.4 y = 3.88
For x = 4.35 y = 0.13
For x = 4.34 y = 8.3848

For $x_{16} \in (4.34, 4.35)$ we have y = 0.

For x = 4.5 y = 3.5
For x = 4.45 y = 7.67
For x = 4.46 y = 8.4328
For x = 4.47 y = 0.1972

So for some $x_{17} \in (4.46, 4.47)$ we have y = 0

For x = 4.52 y = 4.0432
For x = 4.55 y = 6.37
For x = 4.56 y =7.1488
For x = 4.57 y = 7.9292
For x = 4.58 y = 8.7112
For x = 4.59 y = 0.4948

So for some $x_{18} \in (4.58, 4.59)$ we have y = 0.

For x = 4.6 y = 1.28
For x = 4.65 y = 5.23
For x = 4.7 y = 0.22
For x = 4.69 y = 8.4188

Thus for some $x_{19} \in (4.69, 4.7)$ we have a y = 0.

For x = 4.73 y = 2.6335
For x = 4.8 y = 8.32
For x = 4.9 y = 7.58
For x = 4.85 y = 3.43
For x = 4.83 y = 1.7812
For x = 4.81 y = 0.1388

Thus for some $x_{20} \in (4.8, 4.81)$ we have a y = 0.

For x = 5, y = 4 and so on. This has many zeros.

Zeros becomes dense as $x \to 4$ and so on.

We see by studying an equation in a MOD plane we get several solutions.

We leave open the following problems / conjectures.

Conjecture 2.1: Can a second degree equation $y = ax^2 + bx + c$ have only two roots in all MOD planes?

Conjecture 2.2: Characterize those second degree equations which satisfies conjecture 2.1 in at least one MOD plane.

Conjecture 2.3: Characterize those second degree equation and the MOD planes which has only less than 10 roots.

Conjecture 2.4: Can a second degree equation in a MOD plane have infinite number of roots?

Conjecture 2.5: Study conjectures 2.1 to 2.4 for any third degree equation.

Conjecture 2.6: Study the conjectures 2.1 to 2.4 for any nth degree equation ($n \geq 4$).

Conjecture 2.7: Show in MOD planes the polynomials behave in a very different way.

Conjecture 2.8: Obtain those polynomials and MOD planes in which the fundamental theorem of algebra is true for any nth degree polynomial.

In this book only study of polynomials of second degree alone is done.

However with appropriate modifications the results are true for all polynomials of different degrees.

Finally it is important to mention all MOD polynomials in MOD planes have roots further the same polynomial in different MOD planes have different sets of roots unlike the real plane

R[x] in which a polynomial of degree n only n roots which is unique in R.

Here in $R_n(m)$ as m is varied for p(x) roots also vary.

Certainly these special properties enjoyed by these MOD planes (small planes) will enable a variety of applications in case of MOD polynomials defined on them.

All polynomials defined on MOD planes (small planes) in this book will be termed as MOD polynomials.

Chapter Three

MOD COMPLEX FUNCTIONS IN THE MOD COMPLEX PLANE $C_n(m)$

In this chapter a study of polynomial functions are made in small or MOD complex planes, $C_n(m)$. Let $p(x) \in C_n(m)[x]$; p(x) has the coefficients from the MOD complex plane $C_n(m)$.

Any polynomial $p(x) \in C[x]$ where $C = \{a + bi \mid a, b \in R, i^2 = -1\}$ can be made into MOD complex polynomial in $C_n(m)[x]$.

We will first illustrate this situation by some examples.

Example 3.1: Let

$$p(x) = (8 + 5i)x^3 + (-7 + 4i)x^2 + (6 - 6i)x + (9 - 10i) \in C[x].$$

The representation of p(x) in the MOD complex plane $C_n(2)$ is $i_F x^3 + x^2 + 1$ where $i_F^2 = 1$.

The representation of p(x) in the complex MOD plane $C_n(3)$ is $(2 + 2i_F)x^3 + (2 + i_F)x^2 + 2i_F$; $i_F^2 = 2$.

The representation of p(x) in the complex MOD plane $C_n(4)$ is

$i_Fx^3 + x^2 + (2 + 2i_F)x + (1 + 2i_F)$; $i_F^2 = 3$.

The representation of p(x) in the complex MOD plane $C_n(5)$.

$3x^3 + (3 + 4i_F)x^2 + (1 + 4i_F)x + 4$ where $i_F^2 = 4$.

The representation of p(x) in complex MOD plane $C_n(6)$ is as follows:

$p(x) = (2 + 5i_F)x^3 + (5 + 4i_F)x^2 + (3 + 2i_F)$ where $i_F^2 = 5$.

Now the representation of p(x) in the complex MOD (small) plane $C_n(7)$ is as follows:

$p(x) = (1 + 5i_F)x^3 + 4i_Fx^2 + (6 + i_F)x + (2 + 4i_F) \in C_n(7)$; $i_F^2 = 6$.

The representation of p(x) in the MOD complex plane $C_n(8)$ is as follows:

$5i_Fx^3 + (1 + 4i_F)x^2 + (6 + 2i_F)x + (1 + 6i_F)$; $i_F^2 = 7$.

The representation of p(x) in the complex MOD plane $C_n(9)$ is $(8 + 5i_F)x^3 + (2 + 4i_F)x^2 + (6 + 3i_F)x + 8i_F$; $i_F^2 = 8$.

We see all the eight equations are different from $p(x) \in C[x]$.

Now p(x) will not be p(x) in general in any MOD complex plane $C_n(m)$ for any value of m.

Now the graphs using which zeros of the MOD complex functions are studied by examples.

Example 3.2: Let $f(x) = (3 + 7i)x + (5 – 2i) \in C[x]$.

Now f(x) takes the following form in the complex MOD plane $C_n(2)$.

$(1 + i_F)x + 1 \in C_n(2)[x]$ ($i_F^2 = 1$).

Now value of f(x) in C[x] is $(3 + 7i)\, x + 5 – 2i = 0$;

$(3 + 7i)x = –5 + 2i$ that is $x = \frac{-5+2i}{3+7i}$

$$x = \frac{(-5+2i)(3-7i)}{(3+7i)(3-7i)}$$

$$= \frac{-15+6i+35i+14}{9+49}$$

$$x = \frac{41i-1}{58} \in C.$$

Now we find the value of x in $C_n(2)$.

$(1 + i_F)\, x + 1 = 0$ so that

$$x = \frac{1}{1+i_F} \text{ hence } x = \frac{1+i_F}{(1+i_F)^2}$$

$= \frac{1+i_F}{0}$, an indeterminate so x has no value in $C_n(2)$.

Thus a linear equation in the MOD plane may or may not be solvable or the solution may not exist.

Now f(x) = (3 + 7i)x + 5 – 2i takes the form

f(x) = $i_F x + 2 + i_F$ in the complex MOD plane $C_n(3)$; $i_F^2 = 2$.

Let $i_F x + 2 + i_F = 0$.

Thus $x = \frac{1+2i_F}{i_F} = \frac{(1+2i_F)2i_F}{i_F \times 2i_F} = \frac{2i_F+2}{1}$.

So x takes the value $2 + 2i_F$ in the complex MOD plane $C_n(3)$.

Next we study the same f(x) in the MOD complex plane $C_n(4)$.

f(x) = (3 + 7i) x + 5 – 2i takes the value $(3 + 3i_F)x + 1 + 2i_F$ in $C_n(4)[x]$ where $i_F^2 = 3$.

Thus $(3 + 3i_F)x + 1 + 2i_F = 0$ gives $x = \frac{3+2i_F}{3+3i_F}$

$$= \frac{(3+2i_F)(3+i_F)}{(3+3i_F)(3+i_F)} = \frac{9+6i_F+3i_F+2\times 3}{9+9i_F+3i_F+3\times 3} = \frac{3+i_F}{2} \text{ in } C_n(4).$$

But $\frac{1}{2}$ does not exist as it is not defined in Z_4 more so in $C_n(4)$.

Thus x is undefined in $C_n(4)$ and $C_n(2)$.

Next consider f(x) = (3 + 7i)x + 5 – 2i in $C_n(5)[x]$;

$f(x) = (3 + 2i_F)x + 3i_F$; $i_F^2 = 4$.

$$(3 + 2i_F)x + 3i_F = 0.$$

$$\text{So } x = \frac{2i_F}{3+2i_F}$$

$$= \frac{2i_F \times 3 + 3i_F}{(3+2i_F)(3+3i_F)} = \frac{i_F + 24}{9 + 6i_F + 9i_F + 6\times 4}$$

$$= \frac{4+i_F}{3} = 3 + 2i_F.$$

$x = 3 + 2i_F$ is the root of f(x) in the MOD complex plane $C_n(5)$.

Now consider f(x) = (3 + 7i)x + 5–2i in the MOD complex plane $C_n(6)$.

$$f(x) = (3 + i_F)x + 5 + 4i_F$$

$$i_F^2 = 5. \ (3 + i_F)x + 5 + 4i_F = 0$$

$$x = \frac{-(5+4i_F)}{3+i_F} = \frac{1+2i_F}{3+i_F}$$

$$= \frac{(1+2i_F)(3+5i_F)}{(3+i_F)(3+5i_F)} = \frac{3+6i_F+5+10\times 5}{9+3i_F+15i_F+25}$$

$$= \frac{4}{4} \text{ is an indeterminate in } C_n(6).$$

Hence has no root.

Consider $f(x) = (3 + 7i_F)x + 5 - 2i_F$ in $C_n(7)$;

$$f(x) = 3x + 5 + 5i_F \in C_n(7)[x];$$

$$f(x) = 0 \text{ implies } x = \frac{(5+5i_F)}{3}$$

$$= \frac{+2+2i_F\times 5}{3\times 5}.$$

$x = 3 + 3i_F$ is the root.

Again if we consider f(x) in the MOD plane $C_n(8)$ we get $f(x) = (3 + 7i_F)\, x + 5 + 6i_F$.

Thus f(x) = 0 gives

$$x = \frac{3+2i_F}{3+7i_F}$$

$$= \frac{(3+2i_F)(3+i_F)}{(3+7i_F)(3+i_F)}$$

$$= \frac{9+6i_F+3i_F+2\times 7}{9+21i_F+3i_F+7\times 7} = \frac{i_F+7}{2}$$

is an indeterminate as $\frac{1}{2}$ is not defined in $C_n(8)$.

Let us consider f(x) as the transformed function.

$$(3 + 7i)x + (5–2i) \text{ in } C_n(9).$$

$$f(x) = (3 + 7i_F)x + 5 + 7i_F;\ i_F^2 = 8.$$

If f(x) = 0 then

$$x = \frac{4+2i_F}{3+7i_F}$$

$$= \frac{(4+2i_F)(3+2i_F)}{(3+7i_F)(3+2i_F)}$$

$$= \frac{12+6i_F+8i_F+4\times 8}{9+21i_F+6i_F+14\times 8}$$

$$= \frac{44+14i_F}{27i_F+121}$$

$$= \frac{8+5i_F}{4} = \frac{8+5i_F\times 7}{4\times 7} = 2 + 8i_F.$$

Thus the function is solvable.

Consider f(x) = (3 + 7i)x + (5–2i) in $C_n(10)$ which is as follows:

$$f(x) = (3 + 7i_F)x + 5 + 8i_F;\ i_F^2 = 9$$
$$(3 + 7i_F)x + 5 + 8i_F = 0.$$

$(3 + 7i_F)x + 5 + 8i_F = 0$ multiply by $(3 + 3i_F)$ the conjugate of $3 + 7i_F$.

$(3 + 7i_F)(3 + 3i_F)x + (5 + 8i_F)(3 + 3i_F) = 0$

$(9 + 21i_F + 9i_F + 21 \times 9)x + 15 + 24i_F + 15i_F + 24 \times 9 = 0$

$8x + 9i_F + 1 = 0.$

As coefficient of x is a zero divisor in Z_{10} this equation is not solvable for x.

Thus in the MOD complex plane even linear equations in the variable x may not be solvable for x.

Thus in view of all these examples we can have the following theorem.

THEOREM 3.1: *Let $C_n(m)[x]$ be the MOD complex plane. p(x) = ax + b, a, b ∈ $C_n(m)$ is solvable if and only if a is not a zero divisor but a unit in $C_n(m)$.*

Proof follows from simple calculations.

Corollary 3.1: Let p(x) ∈ $C_n(m)[x]$ be a MOD complex polynomial of degree s (s ≥ 1), p(x) has solution only if coefficient of the highest degree of x is a unit in $C_n(m)$.

Proof is left as an exercise to the reader.

Next we study other types of properties on the complex MOD plane.

The concept of derivatives, limits happens to very different in case of functions in the complex MOD plane $C_n(m)$.

Example 3.3: Let $f(z) = u(x, y) + i_F v(x, y)$; $z \in C_n(3)$ be a MOD complex function defined in some neighbourhood of $z_0 = x_0 + i_F y_0$.

$$\text{To find } \lim_{z \to 1+i_F} (z^2 + z + 1)$$

Let $f(z) = z^2 + z + 1 = (x + i_F y)^2 + x + i_F y + 1$

$$= u(x, y) + i_F v(x, y)$$

$$= x^2 + 2y^2 + 2i_F xy + i_F y + x + 1$$

$$= x^2 + 2y^2 + x + 1 + i_F (2xy + y)$$

$$\lim_{(x,y)\to(1,1)} = 1 + 2 + 1 + 1 = 2$$

$$\lim_{(x,y)\to(1,1)} v(x,y) = 2 + 1 = 0.$$

Thus $\lim_{z\to(1+i_F)} f(z) = 2$.

Example 3.4: Let $f(z) = u(x, y) + i_F v(x, y)$; $z \in C_n(5)$ be the function in the MOD complex plane $C_n(5)$.

To find $\lim_{z\to 1+i_F} (f(x)) = (z^2 + 3z + 1)$.

Let $f(z) = z^2 + 3z + 1$

$$= x^2 + 4y^2 + 3x + 1 + i_F (3y + 2xy).$$

Calculating the limits for u and v we obtain

$$\lim_{(x,y)\to(1,1)} u(x,y) = 1 + 4 + 3 + 1 = 4.$$

$$\lim_{(x,y)\to(1,1)} v(x,y) = 3 + 2 \equiv 0 \pmod 5.$$

Thus $\lim_{z\to 1+i} f(z) = 4$

Now let $f(z) = z^2 + 1 \in C_n(2)$;
$z^2 + 1 = x^2 + y^2 + 1 + 0.i_F$.

$$\lim_{z\to(1,1)} f(z) = 1 + i.$$

$$\lim_{(x,y)\to(1,1)} u(x,y) = 1 + 1 + 1 = 1.$$

$$\lim_{(x,y)\to(1,1)} v(x,y) = 0.$$

Thus $\lim_{z\to 1+i_F} f(z) = 1.$

Let $f(z) = z^2 + 4z + 1 \in C_n(6)$.

$$u(x, y) + i_F v(x, y) = x^2 + 5y^2 + 1 + 4x + i_F (2xy + 4y)$$

$\lim_{z\to 1+i} f(z)$, find

$$\lim_{(x,y)\to(1,1)} u(x,y) = 1 + 5 + 1 + 4 = 5.$$

$$\lim_{(x,y)\to(1,1)} v(x,y) = 2 + 4 = 0.$$

Thus $\lim_{z \to 1+i} f(z) = 5$.

The main observation from these examples the same function has different values in the complex MOD planes $C_n(m)$ as –1 is different for each $C(Z_m)$.

Next we consider a different function f(z) = $\frac{z^2 - 2i}{z^2 - 2z + 2}$

lim as $z \to 1 + i$ in f(z).

Now f(z) in the complex MOD plane $C_n(2)$ is as follows:

$$f(z) = \frac{z^2}{z^2}$$

$$\lim_{z \to 1+i_F} \frac{z^2}{z^2} = \frac{(1+i_F)^2}{(1+i_F)^2} = \frac{1+2i_F+1}{1+2i_F+1} = \frac{0}{0}.$$

It is difficult to solve this at this stage.

Suppose the function is in $C_n(3)$.

Then f(z) = $\frac{z^2 - 2i}{z^2 - 2z + 2}$ in C will be transformed into a MOD function in MOD plane $C_n(3)$ as follows:

$$f(z) = \frac{z^2 + i_F}{z^2 + z + 2} \;;$$

$z \to 1 + i_F$

$$\lim_{z \to 1+i_F} \frac{z^2 + i_F}{z^2 + z + 2} = \frac{0}{0}.$$

But this can be factored as follows:

$$\lim_{z \to 1+i_F} = \frac{z^2 + i_F}{z^2 + z + 2} = \lim_{z \to 1+i_F} \left[\frac{(z+2+2i_F)(z+1+i_F)}{(z=2+2i_F)(z+2+i_F)} \right]$$

as

$$z^2 + i_F = (z + 2 + 2i_F)\,(z + 1 + i_F)$$

$$= z^2 + 2z + 2zi_F + z + 2 + 2i_F + zi_F + 2i_F + 4$$
$$= z^2 + i_F.$$

Further

$$z^2 + z + 2 = (z + 2 + 2i_F)\,(z + 2 + i_F)$$

$$= z^2 + 2z + 2zi_F + zi_F + 2i_F + 4 + 2z + 4 + 4i_F$$

$$= z^2 + z + 2.$$

$$\lim_{z \to 1+i_F} = \frac{z+1+i_F}{z+2+i_F} = \frac{1+i_F+1+i_F}{1+i_F+2+i_F}$$

$$= \frac{2+i_F}{2i_F} = \frac{1+i_F}{i_F}$$

$$= \frac{i_F+2}{2} = 2i_F + 1.$$

So the f(z) in the MOD complex plane $C_n(3)$ has a limit $1 + 2i_F$.

Consider f(z) in the MOD complex plane $C_n(4)$,

$\lim_{z \to 1+i} \left(\frac{z^2 - 2i}{z^2 - 2z + 2} \right)$ in the MOD plane $C_n(4)$ is

$$\lim_{z \to 1+i_F} \left[\frac{z^2 + 2i_F}{z^2 + 2z + 2} = \frac{P(z)}{Q(z)} \right] = \frac{0}{0}.$$

Now factorize P(z) and Q(z) as

$$P(z) = (z + 3 + 3i_F)\ (z + 1 + i_F)$$

and

$$Q(z) = (z + 3 + 3i_F)\ (z + 3 + i_F)$$

$$\lim_{z \to 1+i_F} \left(\frac{z^2 + 2i_F}{z^2 + 2z + 2} \right)$$

$$= \lim_{z \to 1+i_F} \left[\frac{(z+3+3i_F)(z+1+i_F)}{(z+3+3i_F)(z+3+i_F)} \right]$$

$$= \lim_{z \to 1+i_F} \left(\frac{(z+1+i_F)}{(z+3+i_F)} \right)$$

$$= \frac{1+i_F+1+i_F}{1+i_F+3+i_F} = \frac{2+2i_F}{2i_F} \text{ as } 2^2 \equiv 0 \text{ (mod 4)};$$

The solution does not exist.

So in the MOD complex plane $C_n(4)$ the limit does not exist for the function $f(z) = \dfrac{z^2 + 2i_F}{z^2 + 2z + 2}$.

Consider the function $f(z) = \dfrac{z^2 - 2i}{z^2 - 2z + 2}$ in the MOD complex plane $C_n(5)$.

$$f(z) = \frac{z^2 + 3i_F}{z^2 + 3z + 2} = \frac{P(z)}{Q(z)} \text{ as}$$

$$z \to 1 + i_F,$$

$$f(z) = \frac{1 + 2i_F + 4 + 3i_F}{1 + 2i_F + 4 + 3 + 3i_F + 2} = \frac{0}{0}.$$

Let us factorize P(z) and Q(z) in $C_n(5)$.

$$P(z) = (z + 4 + 4i_F)\,(z + 1 + i_F)$$

and

$$Q(z) = (z + 4 + 4i_F)\,(z + 4 + i_F)$$

$$\lim_{z \to 1+i_F} \left(\frac{P(z)}{Q(z)}\right) = \lim_{z \to 1+i_F} \left[\frac{(z+4+4i_F)(z+1+i_F)}{(z+4+4i_F)(z+4+i_F)}\right]$$

$$= \lim_{z \to 1+i_F} \left(\frac{z+1+i_F}{(z+4+i_F)}\right)$$

$$= \frac{2+2i_F}{2i_F} = \frac{1+i_F}{i_F} = \frac{i_F+4}{4} = 4i_F + 1.$$

Thus in $C_n(5)$ the limit of the function exist.

Let us consider the function

$$f(z) = \frac{z^2 - 2i}{z^2 + 2 - 2z}$$

in the MOD complex plane $C_n(6)$.

$$f(z) = \frac{z^2 + 4i_F}{z^2 + 2 + 4z} = \frac{P(z)}{Q(z)}$$

$$\frac{P(z)}{Q(z)} = \frac{1 + 2i_F + 5 + 4i_F}{1 + 2i_F + 5 + 2 + 4 + 4i_F} = \frac{0}{0}.$$

$$P(z) = (z + 5 + 5i_F)\,(z + 1 + i_F)$$

and

$$Q(z) = (z + 5 + 5i_F)\,(z + 5 + i_F).$$

$$\lim_{z \to 1+i_F} \frac{P(z)}{Q(z)} = \lim_{z \to 1+i_F} \left[\frac{(z+5+5i_F)(z+1+i_F)}{(z+5+5i_F)(z+5+i_F)}\right]$$

$$= \lim_{z \to 1+i_F} \left(\frac{z+1+i_F}{(z+5+i_F)}\right) = \frac{1+i_F+1+i_F}{1+i_F+5+i_F}$$

$$= \frac{2+2i_F}{2i_F}$$

as 2 is a zero divisor in 6 this value is not defined for

$$\frac{2+2i_F}{2i_F} = \frac{2i_F+4}{4}.$$

As 4 is a zero divisor in Z_6 the value $\frac{2i_F+4}{4}$ is not defined.

Thus

$$f(z) = \frac{z^2-2i}{z^2-2z+2}$$

as $z \to 1+i$ has no limit to exist in $C_n(2m)$, $2 \leq m < \infty$.

Consider

$$f(z) = \frac{z^2-2i}{z^2-2z+2}$$

in the MOD plane $C_n(9)$.

$$f(z) = \frac{z^2+7i_F}{z^2+7z+2} = \frac{P(z)}{Q(z)}$$

$$= \frac{1+2i_F+8+7i_F}{1+2i_F+8+7+7+i_F+2} = \frac{0}{0}.$$

Factorize P(z) and Q(z) in $C_n(9)$

$$P(z) = (z + 8 + 8i_F)\ (z + 1 + i_F)$$

and

$$Q(z) = (z + 8 + 8i_F)\ (z + 8 + i_F)$$

$$\lim_{z \to 1+i_F} \frac{P(z)}{Q(z)} = \lim_{z \to 1+i_F} \left[\frac{(z+8+8i_F)(z+1+i_F)}{(z+8+8i_F)(z+8+i_F)}\right]$$

$$= \lim_{z \to 1+i_F} \left[\frac{(z+1+i_F)}{(z+8+i_F)}\right]$$

$$= \left[\frac{1+i_F+1+i_F}{1+i_F+8+i_F}\right]$$

$$= \frac{2+2i_F}{2i_F}$$

$$= \frac{1+i_F}{i_F} = \frac{i_F+8}{8} = 1 + 8i_F.$$

Thus limit of f(z) exists as $z \to 1 + i_F$ in the MOD complex plane $C_n(9)$.

In all other MOD complex planes $C_n(m)$; m a odd integer the limit of the function, f(z) exists as $z \to 1+ i_F$.

For instance consider $C_n(15)$

$$f(z) = \frac{z^2 - 2i}{z^2 - 2z + 2} \text{ where } z \to 1+ i$$

$$\lim_{z \to 1+i} \left(\frac{z^2 - 2i}{z^2 - 2z + 2}\right) = 1 - i$$

in the complex plane C.

f(z) in the MOD complex plane $C_n(15)$ is

$$f(z) = \frac{z^2 + 13i_F}{z^2 + 13z + 2} = \frac{P(z)}{Q(z)} \text{ as } z \to 1 + i_F$$

$$f(z) = \frac{0}{0} \text{ as}$$

$$\lim_{z \to 1+i_F} \left(\frac{(1+i_F)^2 + 13i_F}{(1+i_F)^2 + 13(1+i_F) + 2} \right)$$

$$= \frac{1 + 2i_F + 14 + 13i_F}{1 + 2i_F + 14 + 13 + 13i_F + 2} = \frac{0}{0}.$$

Factorize P(z) and Q(z)

$$\begin{aligned} P(z) &= (z + 14 + 14i_F)(z + 1 + i_F) \\ &= z^2 + 14z + 14i_Fz + z + 14 + 14i_F + zi_F + 14i_F \\ &\qquad + 14 \times 14 \\ &= z^2 + 13i_F. \end{aligned}$$

Thus P(z) can be factored.

Consider

$$\begin{aligned} Q(z) &= (z + 14 + 14i_F)(z + 14 + i_F) \\ &= z^2 + 14z + 14zi_F + 14z + 196 + 196i_F + zi_F \\ &\qquad + 14i_F + 196 \\ &= z^2 + 13z + 2. \end{aligned}$$

Hence Q(z) can also be factored.

$$\lim_{z \to 1+i_F} \left(\frac{z^2 + 13i_F}{z^2 + 13z + 2} \right) = \lim_{z \to 1+i_F} \left(\frac{(z + 14i_F + 14)(z + i_F + 1)}{(z + 14i_F + 14)(z + 14 + i_F)} \right)$$

$$= \lim_{z \to 1+i_F} \left(\frac{z + i_F + 1}{z + 14 + i_F} \right)$$

$$= \frac{1 + i_F + 1 + i_F}{1 + i_F + 14 + i_F} = \frac{2 + 2i_F}{2i_F}$$

$$= \frac{2 + 2i_F}{2i_F} \times \frac{i_F}{i_F}$$

$$= \frac{2i_F + 2 \times 14}{2 \times 14}$$

$$= \frac{2i_F + 13}{13}$$

$$= \frac{13^3 (2i_F + 13)}{13 \times 13^3}$$

$$= 2 \times 13^3 i_F + 1$$

$$= 14i_F + 1.$$

Thus the limit exists for f(z) in case of the MOD complex plane $C_n(15)$.

However the limit of f(z) does not exist in the complex MOD plane $C_n(24)$.

So from this analysis it is easily said if f(x) is the complex MOD function in the MOD complex plane $C_n(2m+1)$ then

$$\lim_{z \to i_F + 1} f(z) = 1 + 2mi_F.$$

If f(x) is in $C_n(2m)$ the limit $z \to i_F + 1$ of f(z) does not exist.

However the function plays in a very different way for each of the MOD complex planes $C_n(m)$.

Further throughout this book any $f(z) = f(x, y)$ as $z = x + yi_F$ so working with this function is realized as a matter of convenience.

For instance if $f(z) = z^3$ where $z \in C_n([0, 3))$ then does the differential of f(z) exist in the MOD complex plane $C_n([0, 3))$?

If f(z) a continuous function?

Such study is carried out in the following

$$f(z) = f(x, y) = z^3$$

$$= (x + yi_F)^3$$

$$= x^3 + 2y^3 i_F \text{ in } C_n([0,3))$$

$f(0) = f(0,0) = 0.$

$f(1, 1) = f(z = 1 + i_F)) = 1 + 2i_F.$

$f(0.5, 0.5) = f(z = 0.5 + 0.5i_F) = 0.125 + 0.25i_F.$

$f(2, 0) = 2 \pmod 3.$

Several results in this direction can be done.

Chapter Four

TRIGONOMETRIC, LOGARITHMIC AND EXPONENTIAL MOD FUNCTIONS

In this chapter for the first time the notion of MOD trigonometric functions, MOD exponential functions and MOD logarithmic functions are introduced and defined.

Clearly in a MOD plane $R_n(m)$ if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points then $PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ is defined; as the MOD distance and the maximum bound for the distance between two points in the MOD plane $R_n(m)$ is less than or equal to $2m\sqrt{2}$.

Such is the distance and certainly distance between two points exist while measuring distance in reals as only positive value is taken so also in case of distance in MOD planes $R_n(m)$ the distance can be m when m occurs and we do not mark it as zero however the distance is naturally bound by $\leq 2m\sqrt{2}$. However $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are different.

We can thus define this as MOD distance.

We will first illustrate it by some examples.

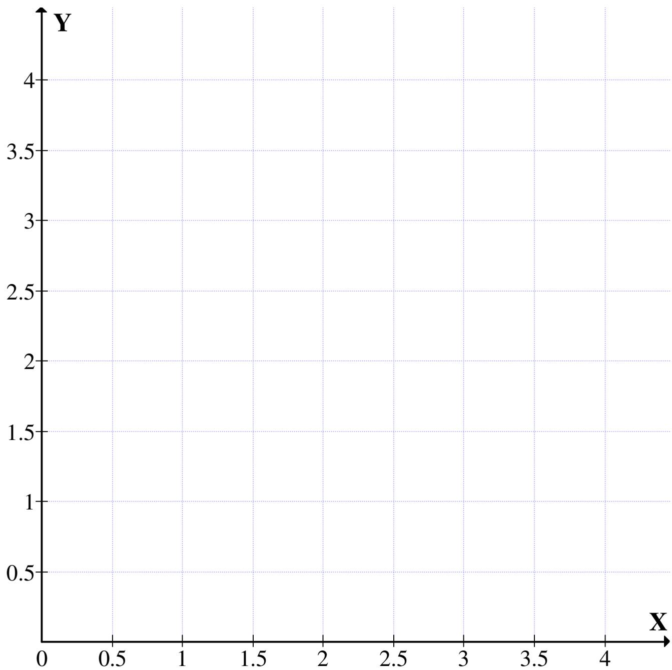

Figure 4.1

Let P = (3, 0) and Q = (0, 3) $\in R_n(4)$.

To find

$$\left(\overrightarrow{PQ}\right)^2 = (3-0)^2 + (0-3)^2$$
$$= 9 + 1 = 2.$$

$$\left(\overrightarrow{QP}\right)^2 = (0-3) + (3-0) = 2.$$

Here $\overrightarrow{PQ}$ and $\overrightarrow{QP}$; but in general this equality may not be true in case of MOD distance as well as modulo distance.

For take A = (3.2, 1.5) and B = (2.8, 2.5) $\in R_n(4)$;

$$\left(\overrightarrow{AB}\right)^2 = (3.2 - 2.8)^2 + (1.5 - 2.5)^2$$
$$= (3.2 + 1.2)^2 + (1.5 + 1.5)^2$$
$$= 0.16 + 1$$
$$= 1.16 \qquad \ldots \qquad I$$

$$\left(\overrightarrow{BA}\right)^2 = (2.8 - 3.2)^2 + (2.5 - 1.5)^2$$
$$= (2.8 + 1.8)^2 + (2.5 + 2.5)^2$$
$$= 0.36 + 1$$
$$= 1.36 \qquad \ldots \qquad II$$

Clearly I and II are different. So for MOD distance as well as the modulo distance one has to keep in mind the direction in which measurement is to be taken.

Further in case of MOD distance $\left(\overrightarrow{PQ_n}\right)^2$ is never zero. The maximum value it can take is $2m\sqrt{2}$ where P, Q $\in$ $R_n(m)$ and the MOD distance $\left(\overrightarrow{PQ_n}\right)^2$ and $\left(\overrightarrow{QP_n}\right)^2$ is never zero.

However modulo distance is zero.

Thus by defining the MOD distance we have solved the conjecture proposed in [24].

DEFINITION 4.1: *Let $R_n(m)$ be the real MOD plane. Let P, Q $\in$ $R_n(m)$; $\left(\overrightarrow{PQ_n}\right)^2$ and $\left(\overrightarrow{QP_n}\right)^2$ is defined as the MOD distance and is bounded by the maximum value $2m\sqrt{2}$.*

Note m $\neq$ 0. In case of modulo distance in the MOD plane m = 0 so $2m\sqrt{2} = 0$ (mod m).

Further it is important to observe $\left(\overrightarrow{PQ_n}\right)^2 \neq \left(\overrightarrow{QP_n}\right)^2$ in general so always in case of both MOD distance as well as modulo distance the length depends on the direction.

We will give some more examples of them.

Example 4.1: Let $R_n(5)$ be the MOD plane on the interval [0, 5). Let P (0.5, 3.2) and Q (3.5,m 2) $\in R_n(5)$.

$$\left(\overrightarrow{PQ_n}\right)^2 = (0.5, 3.5)^2 + (3.2 - 2)^2$$

$$= 22 + (6.2)^2$$

$$= 4 + 38.44 = 42.44.$$

The MOD distance is $\left(\overrightarrow{PQ_n}\right)^2 = \sqrt{42.44}$.

$\overrightarrow{PQ_n} = 6.5145 < 14.14.$

The modulo distance is

$$\left(\overrightarrow{PQ}\right)^2 = (0.5 - 3.5)^2 + (3.2 - 2)^2$$

$$= (0.5 + 1.5)^2 + (3.2 + 3)$$

$$= 22 + (1.2)^2$$

$$= 4 + 1.44 = 0.44$$

$$PQ = 0.663.$$

Thus the modulo distance is not the same as MOD distance. Further

$$\left(\overrightarrow{QP_n}\right)^2 = (3.5 - 0.5)^2 + (2 - 3.2)^2$$

$$= (3.5 + 4.5)^2 + (2 + 1.8)^2$$

$$= 64 + 14.44$$

$$= \quad 78.44.$$

$$\overrightarrow{QP_n} \quad = \quad \sqrt{78.44} \text{ (mod m)}$$

$$= \quad 8.8566 < 14.14.$$

It is very important to note that we find the value of $\left(\overrightarrow{QP_n}\right)^2$ and find its root. In no place modulo m is used while calculating the MOD distance, for MOD distance is always less than $2m\sqrt{2}$.

While calculating the modulo distance, $\left(\overrightarrow{PQ}\right)^2$ (mod m) is taken and its root is the modulo distance.

Example 4.2: let $R_n(6)$ be the MOD plane.

Let P (3, 2.9) and Q(5, 0.3) $\in$ $R_n(6)$.

Let us find the MOD distance $\overrightarrow{PQ_n}$ and the modulo distance;

$$\overrightarrow{PQ_n}^2 \quad = \quad (3-5)^2 + (2.9-0.3)^2$$

$$= \quad (3+1)^2 + (2.9+5.7)^2$$

$$= \quad 16 + 73.96$$

$$= \quad 89.96.$$

$$\sqrt{\overrightarrow{PQ_n}^2} = \quad 9.485.$$

$$\overrightarrow{PQ_n}^2 \quad = \quad (3+1)^2 + (2.9+5.7)^2$$

$$= \quad 4^2 + (2.6)^2$$

$= 16 + 6.76 = 4.76.$

$\sqrt{\overrightarrow{PQ}^2} = 2.1817.$

Clearly $\sqrt{\overrightarrow{PQ}^2} \neq \sqrt{\overrightarrow{PQ_n}^2}$.

$$\text{Now } \overrightarrow{QP_n}^2 = (5–3)^2 + (0.3 – 2.9)^2$$

$$= 8^2 + 3.4^2$$

$$= 64 + 11.56 = 75.56.$$

$$\overrightarrow{QP_n} = 8.6925.$$

Clearly $\overrightarrow{QP_n} \neq \overrightarrow{PQ_n}$.

$$\text{Now } \overrightarrow{QP}^2 = (5–3)^2 + (0.3 – 2.9)^2$$

$$= 2^2 + 3.4^2$$

$$= 4 + 11.56$$

$$= 3.56.$$

$$\overrightarrow{QP}^2 = 1.8868.$$

For modulo distance $\overrightarrow{QP} \neq \overrightarrow{PQ}$.

Now we give one more example of MOD distance and modulo distance.

Example 4.3: Let $R_n(10)$ be the MOD plane.

Let P (8, 2) and Q (1, 9) $\in R_n(10)$

$$\overrightarrow{PQ_n}^2 = (8–1)^2 + (2 + 1)^2$$

$$= (8 + 9)^2 + (2 + 1)^2$$

$$= 172 + 9$$

$$= 289 + 9 = 298.$$

$$\overrightarrow{PQ_n}^2 = 17.26.$$

$$\overrightarrow{QP_n}^2 = (1–8)^2 + (9–2)^2$$

$$= (1 + 2)^2 + (9 + 8)^2$$

$$= 9 + 172 = 298.$$

$$\overrightarrow{PQ_n}^2 = 17.26.$$

$$\overrightarrow{QP_n}^2 = (1–8)^2 + (9–2)^2$$

$$= (1+2)^2 + (9 + 8)^2$$

$$= 9 + 172 = 298.$$

$$\left(\overrightarrow{QP_n}\right) = 17.26 < 28.2843.$$

$$\text{Now } \overrightarrow{PQ}^2 = (8–1)^2 + (2–9)^2$$

$$= 7^2 + 3^2 = 49 + 9 = 58$$

$$\sqrt{\overrightarrow{PQ}^2} = 7.6158.$$

Here $\overrightarrow{PQ_n} = \overrightarrow{QP_n}$

Consider P = (8.3, 4.5) and Q = (7.5, 6.5) $\in R_n(10)$

The MOD distance $\overrightarrow{PQ_n}$ and $\overrightarrow{QP_n}$ is as follows;

$$\overrightarrow{PQ_n}^2 = (8.3 - 7.5)^2 + (4.5 - 6.5)^2$$

$$= (8.3 + 2.5)^2 + (4.5 + 3.5)^2$$

$$= 10.8^2 + 8^2$$

$$= 116.64 + 64$$

$$= 180.64.$$

$$\overrightarrow{PQ_n}^2 = 13.44 = \overrightarrow{PQ_n}$$

$$\overrightarrow{QP_n}^2 = (7.5 - 8.3)^2 + (6.5 - 4.5)^2$$

$$= (7.5 + 1.7)^2 + (6.5 + 5.5)^2$$

$$= 9.2^2 + 12^2$$

$$= 84.64 + 144$$

$$= 228.64$$

$$= 15.1208 = \overrightarrow{QP_n}.$$

$$\overrightarrow{PQ}^2 = (8.3 - 7.5)^2 + (6.5 - 4.5)^2$$

$$= (9.2)^2 - 2^2$$

$$= 84.64 + 4$$

$$= 8.64 + 4 = 2.64.$$

$\overrightarrow{QP} \quad = \quad 1.62.$

Both are different.

This will pave way to find methods to define MOD trigonometric functions.

The way MOD distance was defined according to the authors needed some change for $\overrightarrow{PQ_n} \neq \overrightarrow{QP_n}$ in general.

We over come this possibility by the following methods.

While calculating the MOD distance in no place the modulo property was used so why should one use the modulo property in finding $(x_1 - x_2)^2 + (y_1 - y_2)^2$ so here also if the difference was taken and not taking $-x_2 = m - x_2$ in $R_n(m)$ we could get $\overrightarrow{PQ_n} = \overrightarrow{QP_n}$ so the direction need not exist.

Thus by this technique which is a natural extension of the distance concept with no difficulty MOD distance by also means behave as the usual distance.

So from now on ward by MOD distance we mean only this and the MOD distance which exploits modulo property from now on wards will be know as the pseudo MOD distance.

Clearly in case of pseudo MOD distance $\overrightarrow{PQ_n} \neq \overrightarrow{QP_n}$ but in MOD distance $PQ_n = QP_n$.

We will first illustrate this situation by an example or two.

Example 4.4: Let $R_n(9)$ be the real MOD plane.

Let P = (7.5, 8) and Q = (0.5, 3) $\in R_n(9)$.

The MOD distance

$$PQ_n = \sqrt{(7.5-0.5)^2+(8-3)^2}$$

$$= \sqrt{49+25}$$

$$= \sqrt{74} = 8.6.$$

The pseudo MOD distance

$$\overrightarrow{PQ_n} = \sqrt{(7.5-0.5)^2+(8-3)^2}$$

$$= \sqrt{(7.5+8.5)^2+(8+6)^2}$$

$$= \sqrt{16^2+14^2} = \sqrt{256+196}$$

$$= \sqrt{452} = 21.26 < 25.46.$$

Clearly the pseudo MOD distance is different from the MOD distance.

Consider the modulo distance

$$\overrightarrow{PQ} = \sqrt{(7.5-0.5)^2+(8-3)^2}$$

$$= \sqrt{(7.5+8.5)^2+(8+6)^2}$$

$$= \sqrt{7^2+5^2} = \sqrt{49+25}$$

$$= \sqrt{74} = \sqrt{2}\,.$$

$$\overrightarrow{QP_n} = \sqrt{(0.5-7.5)^2+(3-8)^2}$$

$$= \sqrt{2^2+4^2}$$

$$= \sqrt{20} = 4.47.$$

$$\overrightarrow{QP_n} = \sqrt{(0.5-7.5)^2+(3-8)^2}$$

$$= \sqrt{7^2+5^2}$$

$$= \sqrt{49+25}$$

$$= \sqrt{74} = 8.6.$$

Thus $PQ_n = QP_n$ when the distance is the MOD distance. It is easily verified for in case of pseudo MOD distance and modulo distance both the distances are not equal.

Now let P (5, –3) and Q = (7, 8) $\in$ R $\times$ R. The distance

$$PQ = \sqrt{(5-7)^2+(-3-8)^2}$$

$$= \sqrt{2^2+11^2}$$

$$= \sqrt{125} = 11.1803.$$

Consider P, Q in the MOD plane $R_n(8)$. P(5, –3) is transformed to $P_1(5, 5)$ in the MOD plane $R_n(8)$; Q(7, 8) is transformed to (7, 0) in the MOD plane $R_n(8)$

The MOD distance is $\sqrt{(5-7)^2+5^2}$

$$= \sqrt{2^2 + 25}$$
$$= \sqrt{29}$$
$$= 5.385.$$

So for any value of P, Q in R × R one can find the P_n, Q_n in any of the MOD planes $R_n(m)$ by executing the MOD real transformation [24].

We can also find the MOD distance of P_n and Q_n in $R_n(m)$. Now having defined MOD distance we can define the trigonometric functions.

In the real plane for Q any real number. Construct the angle whose measure in θ radians, with vertex at the origin of a rectangular coordinate system P(x, y) by any point on the side P of the angle θ.

The possible position in the real plane R as follows:

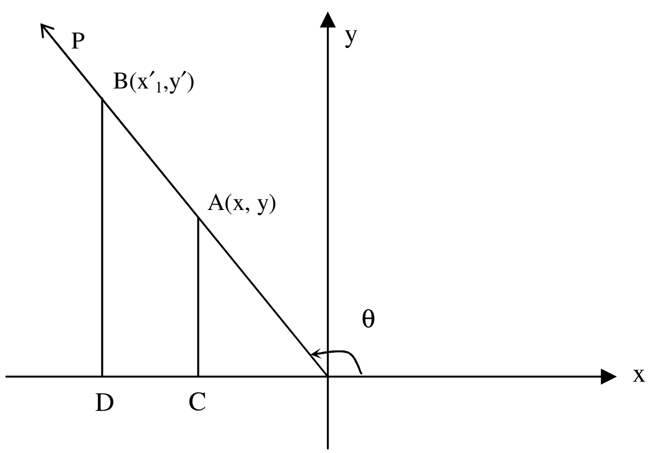

Figure 4.2

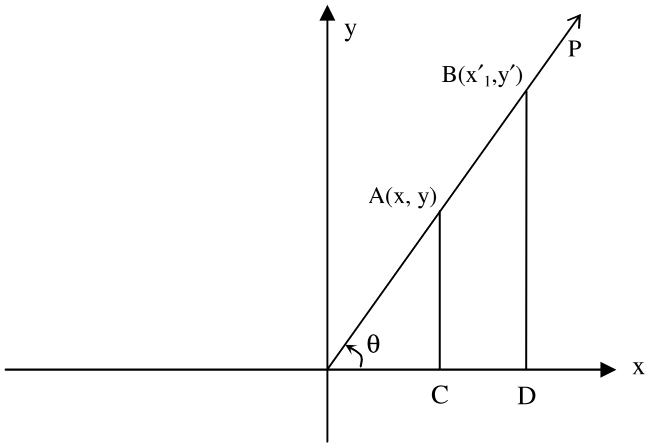

Figure 4.3

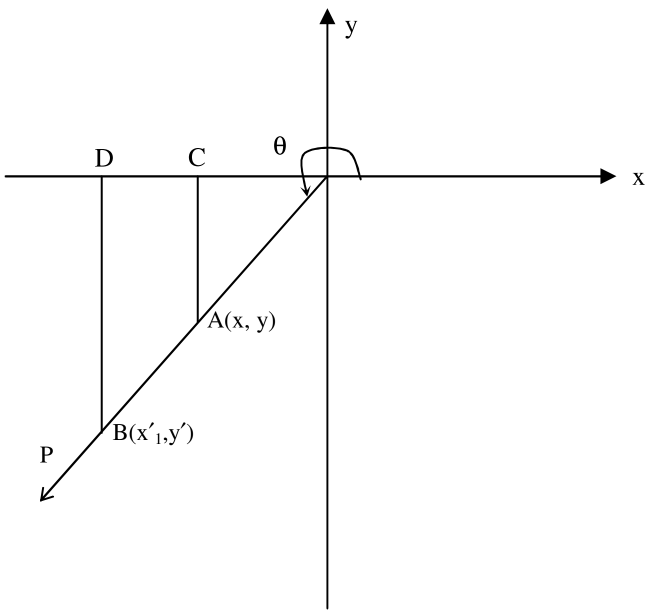

Figure 4.4

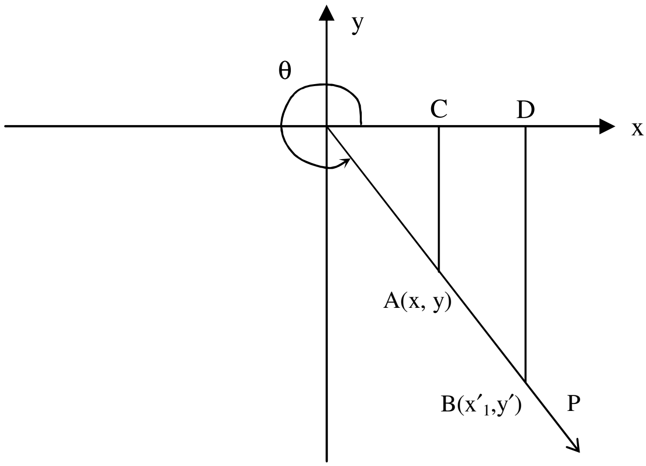

Figure 4.5

Let OA = r.

(i) $\frac{y}{r}$ is called the sine of θ and is written as sinθ.

(ii) $\frac{x}{r}$ is called the cosine of θ and is written as cosθ.

(iii) $\frac{y}{x}$ is called the tangent of θ and is written as tanθ provided θ is not an odd multiple of π/2.

(iv) $\frac{x}{y}$ is called the cos tangent of θ and is written as cotθ provided θ is not an even multiple of π/2.

(v) $\frac{r}{x}$ is called the secant of θ and is written as secθ provided θ is not an odd multiple of π/2.

(vi) $\frac{r}{x}$ is called the cosecant of θ and is written as cosecθ provided θ is not an odd multiple of π/2.

The functions sinθ, cosθ, secθ, cosecθ, cotθ and tanθ are called trigonometric function. It is important to note that these trigonometric functions are well defined and their values depends only on the value of θ and not on the position of point P on the terminal side.

The MOD trigonometric functions are defined on the MOD plane which has only first quadrant. However with the notion of MOD distance certainly all the six trigonometric MOD functions are well defined. The MOD representation of the trigonometric functions in the MOD real plane $R_n(m)$, $2 \leq m < \infty$ is as follows:

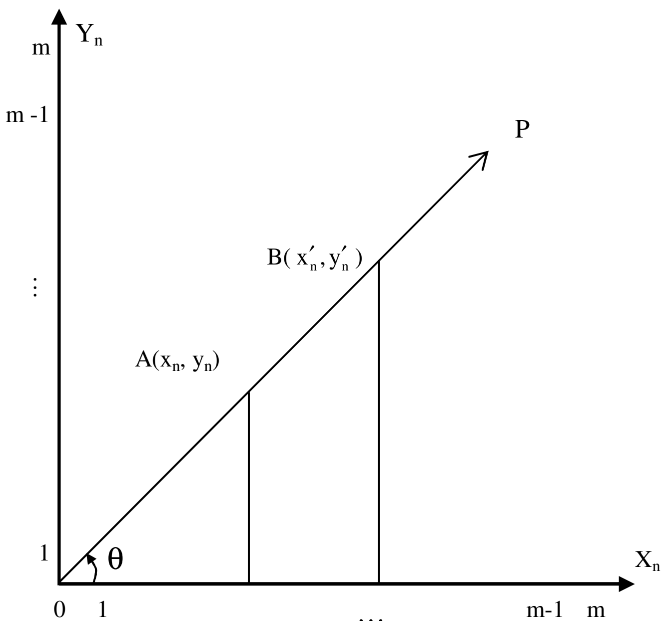

Figure 4.6

Here OA = r_n

$\frac{y_n}{r_n}$ is called the MOD sine of θ written as nsinθ.

$\frac{x_n}{r_n}$ is called the MOD sine of θ written as ncosθ.

$\frac{y_n}{x_n}$ is called the MOD tangent of θ written as ntanθ.

$\frac{x_n}{y_n}$ is called the MOD cotangent of θ written as ncotθ.

$\frac{y_n}{x_n}$ is called the MOD secant of θ written as nsecθ.

and $\frac{r_n}{y_n}$ is called the MOD cosecant of θ written as ncosecθ.

B be any other point on OA then $\frac{y_n'}{r_n'} = \frac{y_n}{r_n}$.

If OP along X_n axis then $\frac{y_n}{r_n} = \frac{0}{r_n} = 0$ for even A (x_n, y_n) on OP where OA = r_n.

If OP along Y_n axis then $\frac{y_n}{r_n} = \frac{r_n}{r_n} = 1$ for every A(x_n, y_n) where OA = r_n.

The ratio $\frac{y_n}{r_n}$ is only dependent on the value of θ and not on the position of A.

The definition of nsinθ is well defined.

All the basis trigonometric identities are true as we use only MOD distance in case of MOD trigonometric functions also.

Here the concept of signs of MOD trigonometric functions has no meaning unlike usual trigonometric function.

We have one MOD plane for a given integer m that is also restricted to the first quadrant.

Now the domain and range of the MOD trigonometric functions are

for OA = r_n,

$$n\sin\theta = \frac{y_n}{r_n},\ 0 \leq y_n \leq r_n \text{ and } 0 \leq \sin\theta \leq 1.$$

Domain of nsinθ = $0 \leq y \leq 2\sqrt{2}$ m

Range of nsinθ is [0, 1].

For ncosθ domain is $0 \leq y \leq 2\sqrt{2}$ m and

range of ncosθ is [0, 1].

For ntanθ range is [0, m).

Domain of ntanθ is [0, m) – (2p+1) π/2; p ∈ Z

Domain of ncot is [0, m) – (nπ / n ∈ Z).

Range of ncotθ is [0, m).

Domain of nsecθ is [0, m) – (2n + 1) π/2, n ∈ Z.

Range of nsecθ is [0, m) \ (0, 1).

Domain of ncosecθ [0, m) – nπ; n ∈ Z.

Range of ncosecθ [0, m) – (0, 1).

Almost all other properties can be derived for MOD trigonometric functions with appropriate modifications.

The value of MOD ratios for θ = 0, π/6, π/4, π/3 and π/2.

	θ = 0°	π/6 = 30°	π/4 = 45°	π/3 = 60°	π/2 = 90°
nsinθ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
ncosθ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
ntanθ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞ = m
ncotθ	∞ = m	$\sqrt{3}$	1	$1/\sqrt{3}$	0
nsecθ	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞ = m
ncosecθ	∞ = m	2	$\sqrt{2}$	$2/\sqrt{3}$	1

MOD trigonometric functions are defined only in the first quadrant as MOD planes $R_n(m)$ have only one quadrant.

Further as it is a semiopen square ∞ is m.

Further we try to obtain the graph of trigonometric function in the MOD planes $R_n(15)$.

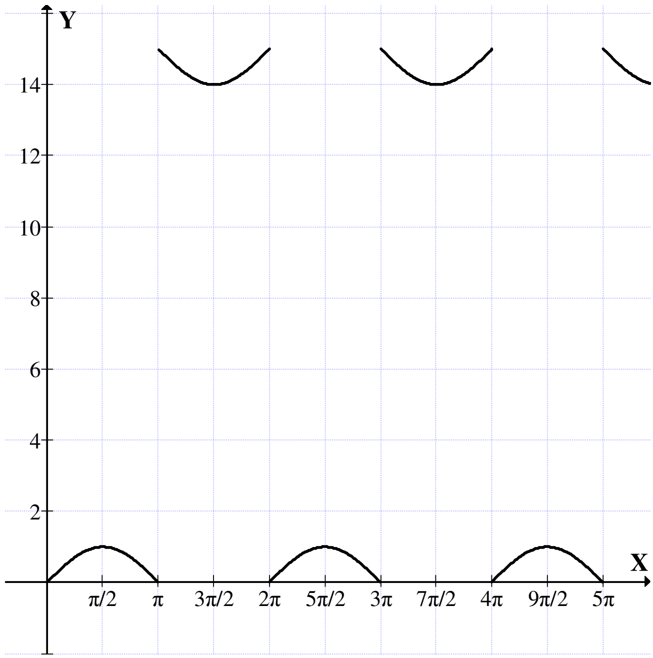

Figure 4.7: $y_n = n\sin x$

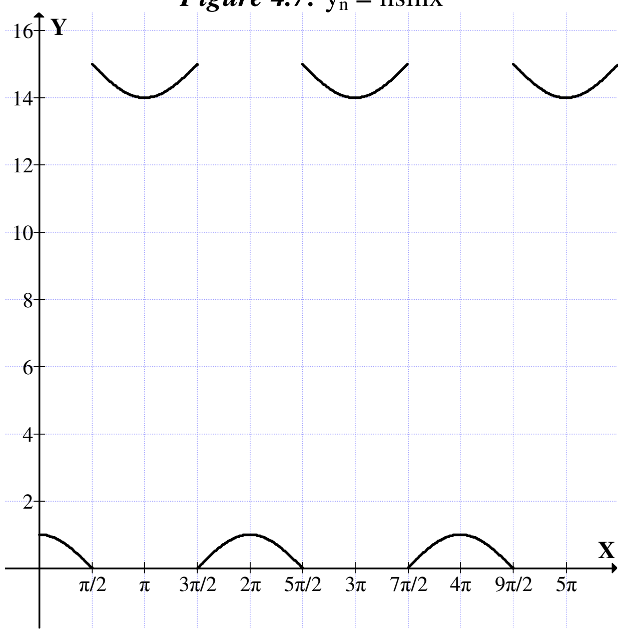

Figure 4.8: $y_n = n\cos x$

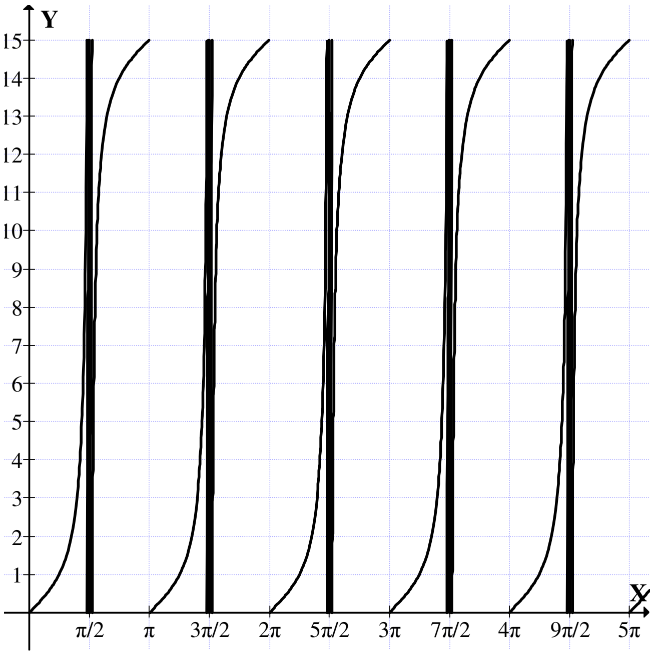

Figure 4.9: y_n = ntanx

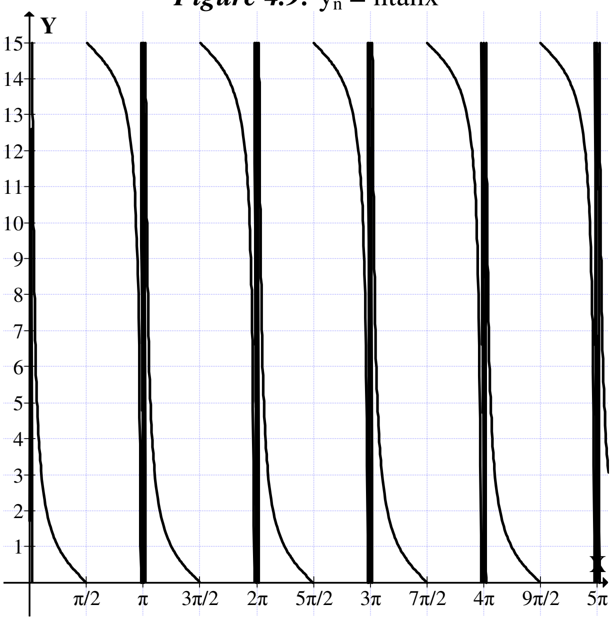

Figure 4.10: y_n = ncotx

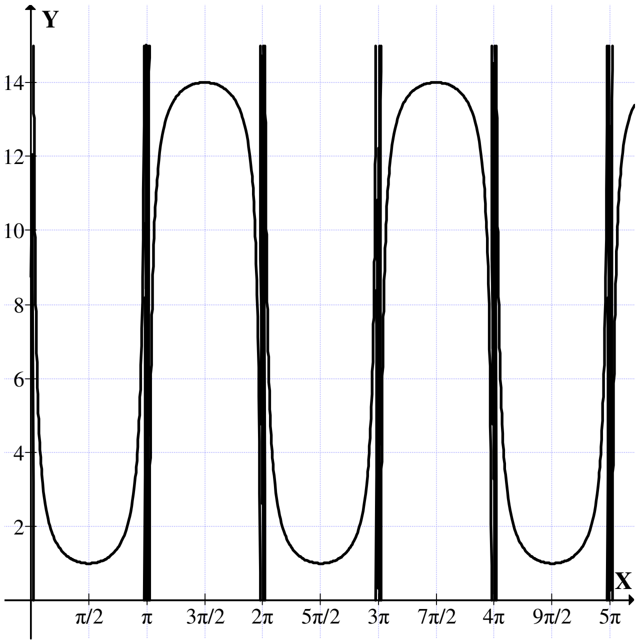

Figure 4.11: y_n = ncosec x

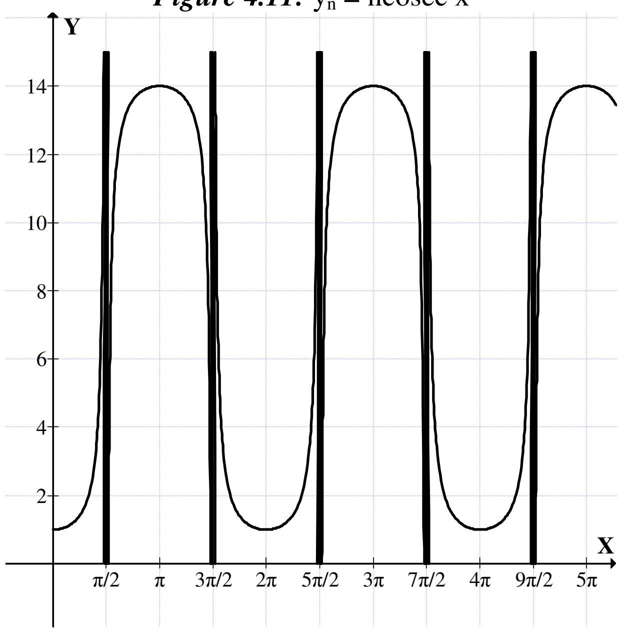

Figure 4.12: y = nsecx

However the relation between angles and derivation of the identities can be done with simple appropriate modifications.

On similar lines MOD inverse trigonometric functions are defined and related results can be obtained as a matter of routine.

Of course $n\sin^2\theta + 2\cos^2\theta = 1$ is true even in case of MOD functions.

However for all practicalities n trigonometric functions are defined only for $[0, \pi/2]$ however for higher values it repeats appropriately.

Keeping only this in mind the graphs of the MOD trigonometric functions where drawn.

For in the MOD plane negative values have no meaning for all the three other quadrants are mapped by the MOD transformation function into the MOD plane which occupies the first quadrant and is bounded by the product of the intervals $[0, m) \times [0, m)$. So $n \sin(-\theta) = n \sin \theta$. The main demand is $0 \leq \theta \leq \pi/2$.

Only based on the MOD transformation one is forced to work with $0 \leq \theta \leq \pi/2$ for all other values are by MOD transformations mapped appropriately.

However a few examples of them will be provided.

Having defined the MOD distance clearly the following table is given for guidance and working in case of inverse MOD trigonometric functions.

MOD trigonometric inverse function	**Domain**	**Range**
nsine	[0,m)	[0,1]
ncosine	[0,m)	[0,1]
ntangent	[0,m) – {(2n+1)π/2,n $\in Z_m$}	[0,m)
ncotangent	[0,m) = {nπ, n $\in Z_m$}	[0,m)
nsecant	[0,m)={(2n+1)π/2,n $\in Z_m$}	[0,m) – (0,1)
ncosecant	[0,m) – {nπ, n $\in Z_m$}	[0,m) – (0,1)

The MOD graph of the MOD inverse trigonometric function inverse MOD sine function.

y_n = nsinx. Domain = [0,n)

Range = [0,1], $y_n = \text{nsin}^{-1}x$. In the following figures the MOD plane is taken as $R_n(6)$.

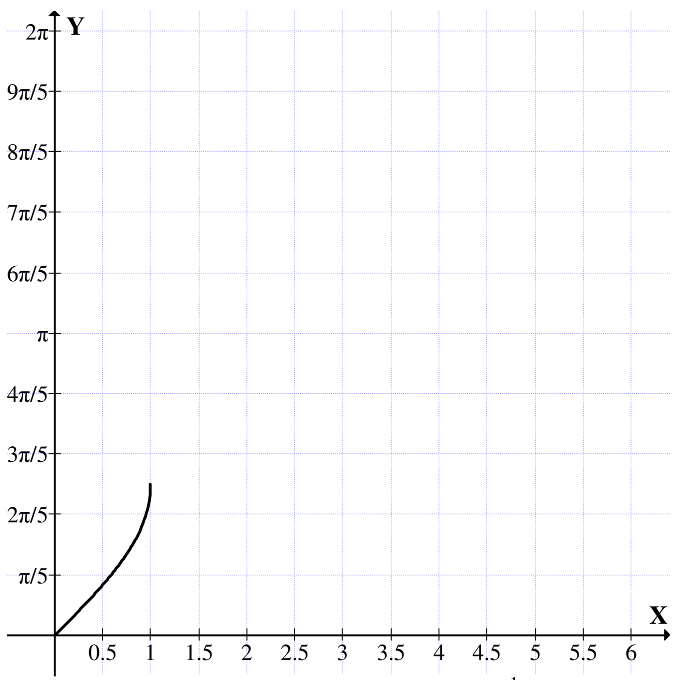

Figure 4.13: $y_n = \text{nsin}^{-1}x$

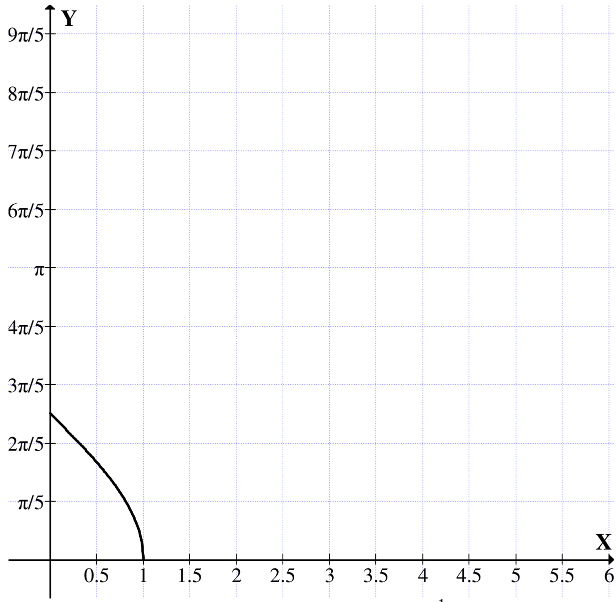

Figure 4.14: $y_n = n\cos^{-1}x$

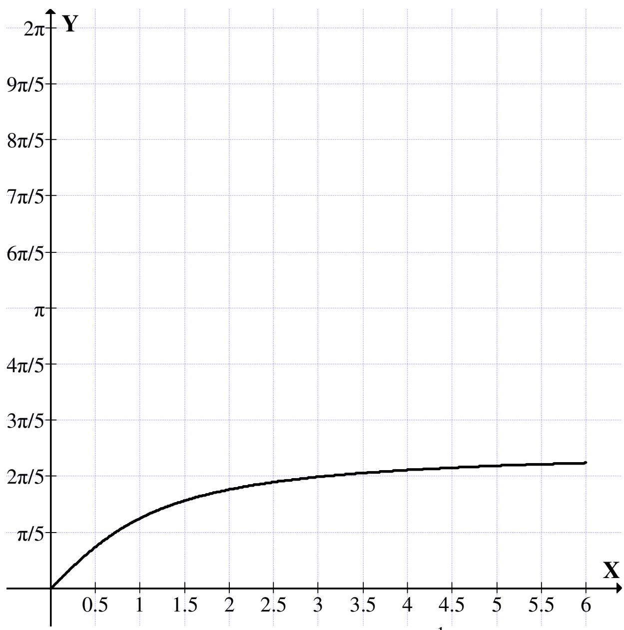

Figure 4.15: $y_n = n\tan^{-1} x$

Similarly other MOD trigonometric function graphs are obtained in the MOD plane.

Now having seen how MOD trigonometric functions are defined we make the following observations.

In the first place in a MOD plane we have only one quadrant. Of course all quadrants of the real plane using the MOD transformation can be mapped onto the MOD plane.

Secondly several types of distance are defined between elements in a MOD plane. Only MOD distance acts like the usual distance. MOD distance alone can help in the defining of the MOD trigonometric functions as other distance pseudo MOD distance cannot be used for in general; $\overrightarrow{PQ_n} \neq \overrightarrow{QP_n}$.

The very question of defining modulo distance in a MOD plane is ruled out as the distance between two distinct points in a MOD plane can be zero.

Thus this study is innovative and the MOD trigonometric functions can take only positive values. Further depending on the MOD plane over which the MOD trigonometric function is defined the graph also changes.

Several open conjectures are suggested.

(1) Enumerate all those trigonometric identities which are true in case of MOD trigonometric functions.

(2) Enumerate all those trigonometric identities which are not true in case of MOD trigonometric functions.

(3) Study of conjectures (1) and (2) in case of MOD trigonometric inverse functions.

It is important to note if $n\cos^2\theta + n\sin^2\theta = 1$ in the MOD planes $R_n(m)$, then $n\cos^2\theta = 1 + (m-1)\ n\sin^2\theta$ and $n\sin^2\theta = 1 + (m-1)\ n\cos^2\theta$.

Keeping this in mind while working with representation in the MOD complex plane $C_n(m)$.

We see $z = x + yi_F = n \cos\theta + i_F n\sin\theta$, however working with them should be carefully and appropriately done using the above identities. Further $i_F^2 = (m-1)$.

Thus using these notions properties can be derived.

Next the notion of MOD exponential functions is defined as follows:

DEFINITION 4.2: *Let $R_n(m)$ ($m \geq 3$) be the MOD plane. ne^x denotes the MOD exponential function where $x \in [0, t)$. Clearly the notion of e^{-x} has no meaning for $ne^{-x} = ne^{(m-1)}x$.*

Thus the MOD graph of the MOD exponential function ne^x in the MOD plane $R_n(m)$ is as follows:

Let us consider the MOD exponential function ne^x in the MOD exponential plane $R_n(7)$.

The graph of this ne^x is given in Figure 4.16.

In the first plane we have to determine that t.
Always if m = 7, t < m, clearly t < 2 for when x = 2, $e^2 = 7.389$.

Thus for ne^x, $x \in [0, 2]$ then we have the smallest plane in which this MOD exponential plane can be accommodated.

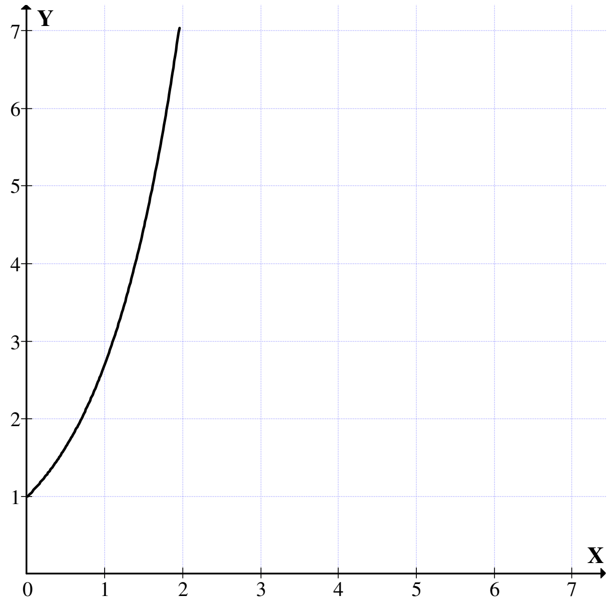

Figure 4.16

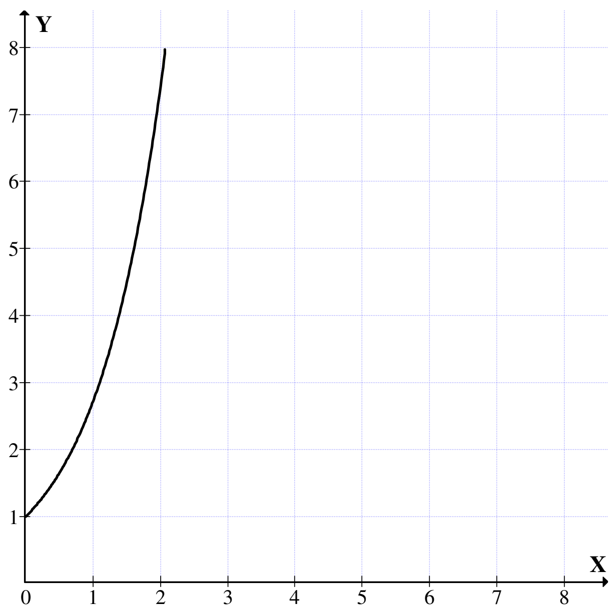

Figure 4.17

So for the MOD exponential function e^x; x = 2 we have the smallest MOD plane to be $R_n(8)$ and the all MOD planes in which x = 2 and $0 \leq x < 3$ accommodated is $R_n(20)$.

So ne^x; $0 \leq x \leq 3$ can fit in the MOD plane $R_n(21)$. The MOD exponential graph of the function ne^x; $0 \leq x \leq 3$ is as follows.

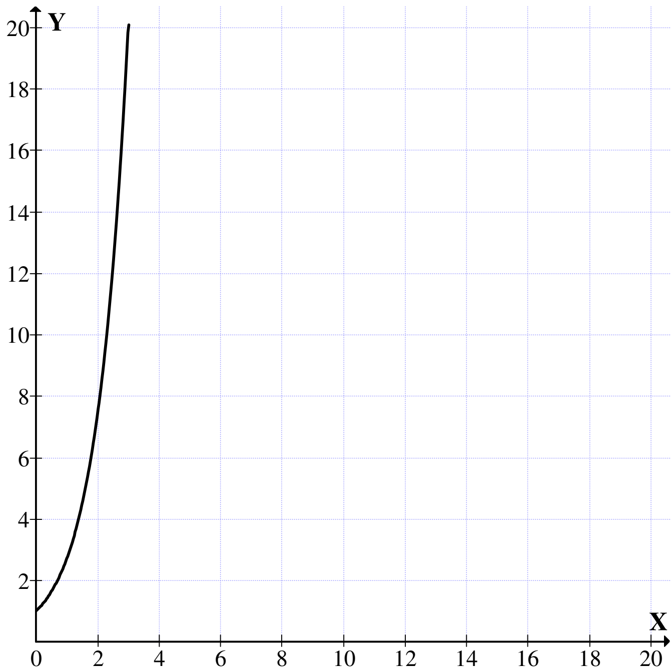

Figure 4.18

So the smallest MOD plane which can accommodate the MOD exponential function ne^x; $0 \leq x \leq 3$ in $R_n(21)$.

All MOD planes $R_n(m)$, $3 \leq m \leq 54$ can accommodate only MOD exponential function ne^x; $0 \leq x \leq 3$ where x = 4 that is for MOD exponential functions ne^x; $0 \leq x \leq 4$ the smallest MOD plane which can accommodate is $R_n(55)$.

All MOD planes $R_n(m)$ $3 \leq m \leq 148$ can accommodate only MOD exponential functions ne^x; $0 \leq x \leq 4$.

It is pertinent to keep on record that for a very small or considerably small x of the MOD exponential function ne^x one needs a very large MOD plane for it to be defined.

For instance for x = 9, that is $0 \leq x \leq 9$ of the MOD exponential function ne^x; one needs the smallest MOD plane to be $R_n(8104)$.

For $0 \leq x \leq 8$ of the MOD exponential function ne^x one needs the smallest MOD plane to be $R_n(2981)$.

One can see for a very small variation of 1 in the value of x nearly there is an increase of 5000 in m in the MOD plane $R_n(m)$.

For x = 10 the smallest MOD plane in which ne^x is defined is $R_n(22027)$.

So for the change of 1 from 9 to 10 of x in ne^x the change in m is 22027 – 8104 = 13,923.

Thus the way ne^x grows or to be more precise e^x grows is seen from the study of finding the smallest MOD planes.

Having seen the definition of MOD exponential function now we proceed onto define the notion of MOD hyperbolic function.

The MOD hyperbolic trigonometric function in the MOD plane $R_n(m)$; $3 \leq m < \infty$ is in

$$\text{sinhx} = \frac{e^x + (m-1)e^{(m-1)x}}{2}$$

$$\text{ncoshx} = \frac{e^x + e^{(m-1)x}}{2}$$

$$\text{ntanhx} = \frac{e^x + (m-1)e^{(m-1)x}}{e^x + e^{(m-1)x}}$$

$$\text{ncothx} = \frac{e^x + e^{(m-1)x}}{e^x + (m-1)e^{(m-1)x}}$$

$$\text{nsechx} = \frac{2}{e^x + e^{(m-1)x}}$$

and

$$\text{ncosechx} = \frac{2}{e^x + (m-1)e^{(m-1)x}}.$$

It is important at this juncture to keep on record it is not an easy task to trance these MOD hyperbolic trigonometric functions.

In fact research in this direction left open.

Next here we just introduce the notion of MOD logarithmic functions.

Since for every $x \in R\backslash\{0\}$ log x is defined. Here we define mainly for the base 10 the MOD logarithmic function.

In the first place if $R_n(m)$ is the MOD plane and R a real plane we have a MOD transformation from R to $R_n(m)$.

Now the graph of usual $\log_{10}$ base function has the following graph in reals is given in Figure 4.19.

However in the MOD plane $R_n(60)$ the MOD $\log_{10}x$ denoted by nlogx is given in Figure 4.20.

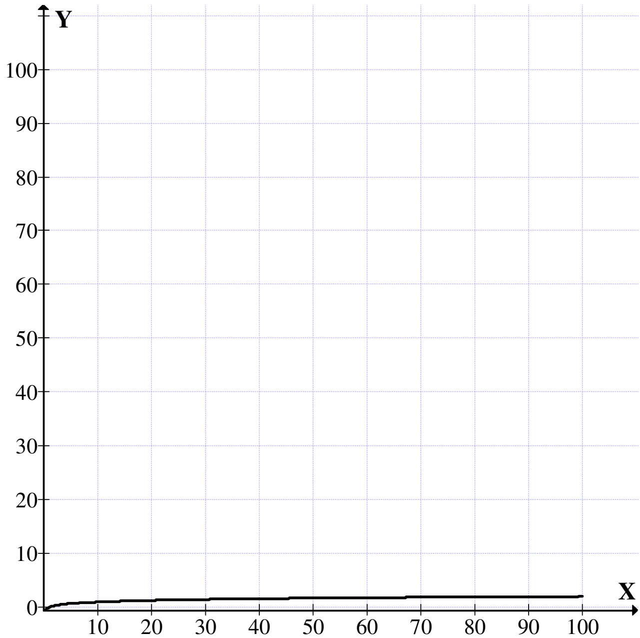

Figure 4.19

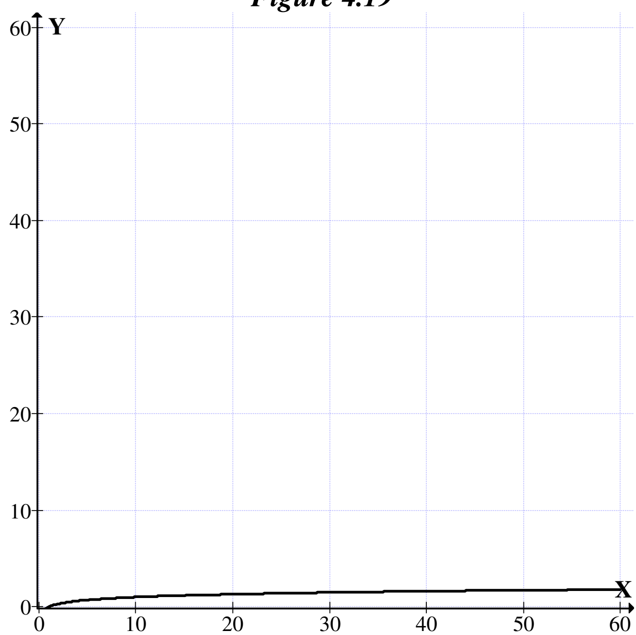

Figure 4.20

So the MOD logarithmic graph is not an infinite curve depending on m of the $R_n(m)$ it will be long or short however certainly there is no change in the curve so the first quadrant of real plane can be sliced depending on m and the same curve will serve the purpose as $t \in [0, m)$ for any MOD real plane $R_n(m)$.

Authors leave open the problems related with MOD logarithmic functions. Here only study or definition is made with respect to base 10.

However it is not difficult to obtain new logarithmic functions for any suitable base.

This work is left as an exercise to the reader.

Chapter Five

NEUTROSOPHIC MOD FUNCTIONS AND OTHER MOD FUNCTIONS

In this chapter we define neutrosophic MOD functions (or small neutrosophic functions) using neutrosophic MOD planes, neutrosophic MOD complex modulo integer functions, MOD dual number functions, MOD special dual like number functions and MOD quasi dual like number functions using the MOD neutrosophic complex modulo integer plane $C_n^I(m)$, MOD dual number plane $R_n(m)(g)$ with $g^2 = 0$, MOD special dual like number plane $R_n(m)h$, $h^2 = h$ and MOD special quasi dual number plane $R_n(m)k$; $k^2 = (m - 1)k$ respectively.

These situations will be described by examples as the definitions can be easily done analogous to functions in chapter II and III.

Example 5.1: Let $R_n^I(9) = \{a + bI \mid a, b \in [0, 9); I^2 = I\}$ be the MOD neutrosophic plane.

Let

$$R_n^I(9)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n^I(9) \right\}$$

be the MOD neutrosophic polynomial. Every polynomial p(x) ∈ $R_n^I(9)[x]$ is a MOD neutrosophic function or small neutrosophic function.

p(x) = $3.5Ix^3 + (4 + 2.5I)x + (7 + 3I)$ is a MOD neutrosophic polynomial function.

p(x) can be differentiated as well as integrated.

Some polynomial functions may not be integrable or derivable.

$$\frac{dp(x)}{dx} = 3\,(3.5)Ix^2 + (4 + 2.5I)$$
$$= 1.5Ix^2 + (4 + 2.5I).$$

$$\frac{d^2p(x)}{dx^2} = 3Ix + 0$$

$$\frac{d^3p(x)}{dx^3} = 3I.$$

Thus as in case of usual polynomials over reals this MOD neutrosophic polynomial has all the derivatives.

Example 5.2: Let $R_n^I(6)[x]$ be the set of all MOD neutrosophic polynomials. Every p(x) ∈ $R_n^I(6)[x]$ is a MOD polynomial neutrosophic functions.

Let

$$p(x) = 3x^4 + (4 + 2I)x^3 + (2.3 + 0.75I) \in R_n^I(6).$$

$$p'(x) = \frac{dp(x)}{dx} = 0 \text{ (mod 6)}.$$

So natural form of differentiation is true for this function.

Another importance about these functions is the roots of the MOD neutrosophic polynomials.

Consider

$$p(x) = (x + 2 + 4I) \times (x + 3.5I) \times (x + 0.8) \in R_n^I(6)[x].$$

Clearly if x = 4 + 2I then p(4 + 2I) = 0.

If x = 2.5I then also consider p(2.5I) = 0.

If x = 5.2 then p(x) = p(5.2) = 0.

Now consider

$$(x + 2 + 4I)\ (x + 3.5I)\ (x + 0.8)$$

$$= [x^2 + (2 + 4I)x + 3.5Ix + (2 + 4I)\ (3.5I)]\ (x + 0.8)$$

$$= (x^2 + (2 + 1.5I)x + 3I)\ (x + 0.8)$$

$$= x^3 + (2 + 1.5I)x^2 + 3Ix + 0.8x^2 + (1.6 + 1.2I)x + 2.4I$$

$$= x^3 + (2.8 + 1.5I)x^2 + (1.6 + 4.2I)x + 2.4I = q(x).$$

q(4 + 2I)

$$= (4 + 2I)^3 + (2.8 + 1.5I)\ (4 + 2I)^2 + (1.6 + 4.2I) + 2.4I$$

$$= (16 + 4I + 16I)\ (4 + 2I) + (2.8 + 1.5I)\ (2I + 4) + (6.4 + 4.8I + 3.2I + 2.4I) + 2.4I$$

$$= 4 + 2I + (5.2 + 3I + 5.6I) + 0.4 + 0.8I$$

$$= 3.6 + 5.4I$$

$$\neq 0.$$

In the first place

$$p(x) \neq q(x).$$

When the polynomial is expanded 4 + 2I fails to be a root of q(x).

Now

$$q(2.5I) = (2.5I)^3 + (2.8 + 1.5I)\ 2.5I + (1.6 + 4.2I)\ (2.5I) + 2.4I$$

$$= 0.625I + I + 3.75I + 4.00I + 10.50I + 2.4I \neq 0.$$

So 2.5I is not a root of q(x).

Finally consider

$$q(5.2) = (5.2)^3 + (2.8 + 1.5I)(5.2)^2 + (1.6 + 4.2I)\, 5.2 + 2.4I$$

$$= 2.608 + 2.512 + 4.56I + 2.32 + 3.84I + 2.4I \neq 0.$$

Hence 5.2 is also not a root of q(x).

Thus we see all the 3 roots of p(x) are not roots of q(x) though q(x) is got only by multiplying the three linear polynomials.

Thus it remains as a open conjecture finding the roots of the MOD neutrosophic functions.

The problem is doubled as the MOD neutrosophic polynomials is only a pseudo ring.

Example 5.3: Let $R_n^I(5)[x]$ be the MOD neutrosophic polynomial pseudo ring.

$$p(x) = (x + 2I + 3)(x + 4I) \in R_n^I(5)[x].$$

$$p(x) = (x + 2I + 3)(x + 4I)$$

so the roots of p(x) are 3I + 2 and I.

$$p(3I + 2) = 0 \text{ and } p(I) = 0.$$

Let

$$q(x) = (x + 2I + 3)(x + 4I)$$

$$= x^2 + 2Ix + 3x + 4Ix + 8I + 12I$$

$$= x^2 + (I + 3)x.$$

$$q(I) = I^2 + I + 3I = 0.$$

So I is a root of q(x).

$q(3I + 2) = (3I + 2)^2 + (I + 3)(3I + 2)$

$= 9I + 6I + 6I + 4 + 3I + 9I + 2I + 6$

$= 0.$

So 3I + 2 is also a root and p(x) = q(x).

The main reason for this is $p(x) \in \langle Z_5 \cup I \rangle[x]$ so is neutrosophic ring.

Consider $p(x) = (x + 0.3 + 0.7I)(x + 0.8) \times (x + 0.3I + 2)$.

Clearly x = 4.7 + 4.3I, x = 5.2 and x = 3 + 4.7I are roots of p(x).

For p(4.7 + 4.3I) = 0,

p(5.2) = 0 and p(3 + 4.7I) = 0

But
$(x + 0.3 + 0.7I)(x + 0.8)(x + 2 + 0.3I)$

$= (x^2 + 0.3x + 0.7Ix + 0.8x + 0.24 + 0.56I)(x + 2 + 0.3I)$

$= x^3 + 0.3x^2 + 0.7Ix^2 + 0.8x^2 + 0.24x + 0.56I + 2x^2 + 0.6x + 1.4Ix + 1.6x + 0.48 + 1.12I + 0.3Ix^2 + 0.09Ix + 0.21Ix + 0.24Ix + 0.072I + 0.168I$

$= x^3 + (0.3 + 0.7I + 0.8 + 2 + 0.3I)x^2 + (0.24 + 0.6 + 1.4I + 1.6 + 0.09I + 0.21I + 0.24I)x + 0.56I + 0.48 + 1.12I + 0.168I + 0.072I$

$= x^3 + (3.1 + I)x^2 + (1.9 + 1.94I)x + (0.48 + 1.92I) = q(x).$

$q(4.7 + 4.3I) = (4.7 + 4.3I)^3 + (3.1 + I)(4.7 + 4.3I)^2 + (1.9 + 1.94I)(4.7 + 4.3I) + (0.48 + 1.92I)$

$= (2.09 + 3.49I + 0.42I)\ (4.7 + 4.3I) + (2.09 + 3.49I + 0.42I)\ (I + 3.1) + (3.93 + 4.118I + 3.17I + 3.342I) + 0.48 + 1.92I$

$= (4.823 + 3.987I + 3.377I + 1.813I) + (2.09I + 3.91I + 1.479 + 2.121I) + 4.41 + 2.550I$

$= 0.742 + 4.848I \neq 0.$

So 4.7 + 4.3I is not a root of q(x).

$q(5.2) = (5.2)^3 + (3.1 + I)\ (5.2)^2 + (1.9 + 1.94I)\ (5.2) + (0.48 + 1.92I)$

$= 0.608 + 2.04I + 1.324 + 4.88 + 0.88I + 0.48 + 1.92I$

$= 2.292 + 4.84I \neq 0.$

So x = 5.2 is not a root of q(x).

$q(3 + 4.7I) = (3 + 4.7I)^3 + (3.1 + I)\ (3 + 4.7I)^2 + (1.9 + 1.94I)\ (3 + 4.7I) + (0.48 + 1.92I)$

$= 2 + (4.7I)^3 + 2 \times 4.7I + 4 \times (4.7I)^2 + 3.1 \times 4 + 4I + 3.1 \times (4.7)^2I + (4.7)^2I + 4.7I + 3.1 \times 4.7I + 0.7 + 0.82I + 1.9 \times 4.7I + 1.94 \times 4.7I + 0.48 + 1.92I$

$= (2 + 2.4 + 0.7 + 0.48) + (4.7I + 3.823I + 4.4I + 3.36I + 4I + 3.479I + 2.09I + 4.57I + 0.82I + 3.93I + 4.118I + 1.92I)$

$= 0.58 + 1.21I \neq 0.$

Thus 3 + 4.7I is also not a root of q(x).

None of the roots of p(x) is a root of q(x) as the distributive law is not true in $R_n^I(5)[x]$.

Thus solving for roots happens to be a very difficult task in this case.

So in case of MOD neutrosophic functions the problem of solving equations is as difficult as that of solving MOD real functions.

Next we proceed on to work with functions on MOD complex neutrosophic modulo integers; $C_n^I(m)[x]$.

$C_n^I(m) = \{a_0 + a_1I + a_2i_F + a_3i_FI \mid a_0, a_1, a_2, a_3 \in [0, m)\}$ ($i_F^2 = m-1$, $I^2 = I$).

$$C_n^I(m)[x] = \left\{\sum_{i=0}^{\infty} \alpha_i x^i \;\middle|\; \alpha_i \in C_n^I(m)\right\}$$

is the MOD neutrosophic complex modulo integer pseudo polynomial ring [24].

Example 5.4: Let $x^2 + 3i_F + 4.3I \in C_n^I(10)[x]$. Solve for x.

$x^2 = 7i_F + 5.7I$, $x = \sqrt{7i_F + 5.7I}$.

Finding roots of these values happen to be a open conjecture.

However as $I^2 = I$ one can say $\sqrt{I} = I$.

Apart from this finding root of $\sqrt{7i_F + 5.7I}$ remains a challenging problem.

Let $p(x) = (3 + 0.7i_F + 5i_FI)x^3 + (7.2I + 4i_F + 2)x + 4 + 5Ii_F + 8.9i_F + 4I$ be a function in $C_n^I(10)[x]$.

$$\frac{dp(x)}{dx} = (3\,(3 + 0.7i_F + 5i_FI)x^2 + (7.2 + 4i_F + 2)$$

$$\frac{d^2p(x)}{dx^2} = 2\,(9 + 2.1i_F + 5i_FI)x + 0$$

$$= (8 + 4.2i_F)x.$$

$$\frac{d^3p(x)}{dx^3} = 8 + 4.2i_F.$$

Now for the integral

$$\int p(x)\,dx = \frac{(3+0.7i_F+5i_FI)x^4}{4} + \frac{(7.2I+4i_F+2)x^2}{2} + (4 + 5Ii_F + 8.9i_F + 4I)\,x + C$$

However the first two terms are not defined as 4 and 2 are zero divisors in $C_n^I(10)$. So one cannot say given a function in $C_n^I(10)[x]$ its integral will exist.

It may exist or at times may not be defined.

Example 5.5: Let

$$p(x) = x^7 + 0.3x^2 + 1.4Ix + (2.5 + 0.7i_F) \in C_n^I(7)[x]$$

be a MOD polynomial function in $C_n^I(7)[x]$.

Finding roots of p(x) is a challenging problem.

$$\frac{dp(x)}{dx} = 0.6x + 1.4I.$$

So a seventh degree MOD polynomial in $C_n^I(7)[x]$ has its first derivative to be a polynomial in x.

Thus the usual laws of derivatives are not in general true for the MOD polynomials.

Let $p(x) = 3.5Ix^2 + (1.7I + 2.3i_F) \in C_n^I(7)[x]$.

The derivative $\frac{dp(x)}{dx} = 0$.

So its derivative is zero however p(x) is not a constant MOD polynomial.

$$\text{Now} \int p(x)\,dx = \frac{3.5Ix^3}{3} + (1.7I + 2.3i_F)x + C$$

$$= 3.5Ix^3 + (1.7I + 2.3i_F)x + C.$$

Thus the integral of this MOD polynomial function is well defined and the integral exist.

Let $a(x) = (3.5 + 4.2i_F + 3I + 2.1i_FI)x^6 \in C_n^I(7)[x]$.

$$\text{Clearly} \int a(x)\,dx = \frac{(3.5 + 4.2i_F + 3I + 2.1Ii_F)x^7}{7} + C.$$

Thus the integral is not defined as $\frac{1}{7}$ is undefined in $C_n^I(7)[x]$.

$$\text{However } \frac{da(x)}{dx} = 6(3.5 + 4.2i_F + 3I + 2.1i_FI)x^5$$

$= (4.2i_F + 4I + 5.6Ii_F)x^5$ exist and is well defined.

$$\frac{d^2a(x)}{dx^2} = 5(4.2i_F + 4I + 5.6i_FI)x^4$$

$$= 6I\,x^4.$$

$$\frac{d^3a(x)}{dx^3} = 6I \times 4x^3$$

$$= 3x^3.$$

$$\frac{d^4a(x)}{dx^4} = 2x^2$$

$$\frac{d^5a(x)}{dx^5} = 4x$$

and $$\frac{d^6a(x)}{dx^5} = 4 \text{ a constant.}$$

However we started with a coefficient from $C_n^I(7)$ and it has all the 4 terms real complex modulo factor i_F, neutrosophic factor I and the combination of both.

However the 6[th] derivative gave only a constant which is a real integer 4.

Example 5.6: Let $s(x) = 3i_Fx^4 + (4 + 0.5I + 2i_F)x^3 + (4.2 + 3i_FI)x + (2.3 + 4.8i_F + 0.3i_FI + 5.12I) \in C_n^I(6)[x]$ be a MOD polynomial function with neutrosophic modulo complex number coefficients.

$$\frac{ds(x)}{dx} = 4 \times 3i_Fx^3 + 3\,(4 + 0.5I + 2i_F)x^2 + (4.2 + 3i_FI)$$

$$= (1.5I)x^2 + (4.2 + 3i_F)\,I.$$

$$\frac{d^2s(x)}{dx^2} = 3Ix + 0$$

$$\frac{d^3 s(x)}{dx^3} = 3I \text{ and } \frac{d^4 s(x)}{dx^4} = 0.$$

However this MOD function is a forth degree polynomial so the forth derivative must be a constant under natural or usual condition.

Now

$$\int s(x)\,dx = \frac{3i_F x^5}{5} + \frac{(4+0.5I+2i_F)x^4}{4} + \frac{(4.2+3i_F)x^2}{2}$$
$$+ (2.3 + 4.8i_F + 0.3i_F I + 5.12I)x + C.$$

But $\int s(x)\,dx$ is not defined as $\frac{1}{4}$ and $\frac{1}{2}$ are not defined in $C_n^I(6)[x]$.

Thus the integral does not exist.

So MOD polynomial neutrosophic complex modulo integer functions behave in a very different way from the usual polynomial functions.

It is left as an open conjecture to solve equations in the MOD complex neutrosophic functions.

The problem occurs mainly because the collection $C_n^I(m)[x]$ is only a pseudo polynomial ring.

So $(x + a_0 + a_1I + a_2i_F + a_3Ii_F) \times (x + b_0 + b_1I + b_2i_F + b_3Ii_F) \times (x + c_0 + c_1I + c_2i_F + c_3Ii_F) \neq x^3 + (\alpha_1 + \alpha_2 + \alpha_3)x^2 + \ldots + \alpha_1 \alpha_2 \alpha_3$;

where

$$\alpha_1 = a_0 + a_1I + a_2i_F + a_3Ii_F$$
$$\alpha_2 = b_0 + b_1I + b_2i_F + b_3Ii_F$$

and $$\alpha_3 = c_0 + c_1I + c_2i_F + c_3Ii_F.$$

Let $p(x) = (x + 0.3 + 4i_F + 5.1i_FI) \times (x + 7 + 0.8I + 4.2i_F) \in C_n^I(8)[x]$.

$$p(x) = x^2 + [(0.3 + 4i_F + 5.1i_FI) + (7 + 0.8I + 4.2i_F)] + (0.3 + 4i_F + 5.1i_FI)(7 + 0.8I + 4.2i_F)$$

$$= x^2 + (7.3 + 0.2i_F + 0.8I + 5.1i_FI)x + (2.1 + 4i_F + 3.7i_FI + 0.24I + 0.32Ii_F + 4.08i_FI + 5.94I + 4.6i_F + 5.6)$$

$$= x^2 + (7.3 + 0.2i_F + 0.8I + 5.1i_FI)x + (7.7 + 0.6i_F + 6.18I + 0.10i_FI)$$

$$= q(x).$$

The roots of p(x) are

$$x = 7.7 + 4i_F + 2.9i_FI$$

and

$$x = 1 + 7.2I + 3.8i_F.$$

$$p(7.7 + 4i_F + 2.9i_FI) = 0$$

and

$$q(7.7 + 4i_F + 2.9i_FI)$$

$$= (7.7 + 4i_F + 2.9i_FI)^2 + (7.3 + 0.2i_F + 0.8I + 5.1Ii_F) \times (7.7 + 4i_F + 2.9i_FI) + (7.7 + 0.6i_F + 6.18I + 0.1i_FI)$$

$$= (5.9 + 2.87I + 4.66Ii_F) + (0.21 + 1.54i_F + 6.16I + 7.27Ii_F + 5.2i_F + 5.6 + 3.2Ii_F + 4.4I + 5.17i_FI + 2.32I + 2.32i_FI + 7.53I) + (7.7 + 0.6i_F + 6.18I + 0.1i_FI)$$

$$= (3.41 + 5.46I + 7.34i_F + 6.72Ii_F)$$
$$\neq 0.$$

So p(x) ≠ q(x) and the root of p(x) is not the same as that of q(x) and vice versa.

Next we proceed onto define the notion of MOD real dual number functions.

$R_n(m)(g) = \{a + bg \mid g^2 = 0, a, b \in [0, m)\}$ is defined as the MOD real dual number plane.

We will just for the sake of completeness describe the plane.

Let $R_n(5)g$ be the MOD real dual number plane.

The point 3 + 2.5g = (3, 2.5).

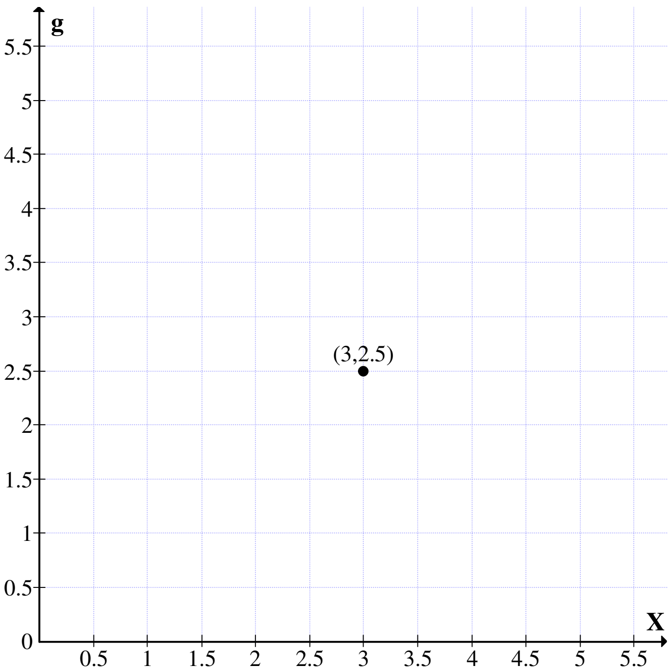

Figure 5.1

Clearly $R_n(m)(g)$ is only a MOD real dual number pseudo ring.

Consider

$$R_n(m)g[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(m)g, g^2 = 0 \right\}.$$

$R_n(m)g[x]$ is defined as the collection of MOD dual number coefficient polynomials $R_n(m)g[x]$ is only a pseudo ring.

Any $p(x) \in R_n(m)g[x]$ is a MOD polynomial function in the variable x.

One can differentiate and integrate these MOD functions only whenever they exist and are defined.

We will illustrate these situations by some examples.

Example 5.7: Let

$R_n(10)g[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(10)g = \{a + bg \mid a, b \in [0, 10), g^2 = 0\}\right\}$ be the MOD dual number polynomial pseudo ring.

Let $f(x) = (2.3 + 0.7g)x^3 + (3.2 + 5.1g)x^2 + (2.17 + 3.89g) \in C_n(10)g[x]$.

$$\frac{df(x)}{dx} = (3\,(2.3 + 0.7g)x^2 + 2(3.2 + 5.1g)x$$

$$= (6.9 + 2.1g)x^2 + (6.4 + 0.2g)x.$$

$$\frac{d^2f(x)}{dx^2} = 2\,(6.9 + 2.1g)x + (6.4 + 0.2g)$$

$$= (3.8 + 4.2g)x + (6.4 + 0.2g).$$

Thus the derivative of f(x) exists.

$$\int f(x)\,dx = \frac{(2.3+0.7g)x^4}{4} + \frac{(3.2+5.1g)x^3}{3} + (2.17 + 3.89g)x + C$$

Clearly $\int f(x)\,dx$ does not exist as $\frac{1}{4}$ is not defined in [0, 10).

Thus it is important to keep on record that the derivative may exist but the integral may not exist.

Similarly there may be MOD polynomials in which integrals may exist but the derivatives may not exist.

Study of these MOD functions is an interesting task.

Now we will proceed onto study the roots of these MOD functions.

Example 5.8: Let

$R_n(7)\,g[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(7)g = \{a + bg \mid a, b \in [0, 7), g^2 = 0\}\right\}$ be the MOD dual number pseudo polynomial function ring.

Let $p(x) = x + 3.25 + 6.08g \in R_n(7)g[x]$ only root of p(x) is $x = 3.75 + 0.92g$.

Now let $p(x) = (x + 6.5)(x + 2 + 4.5g) \in R_n(7)g[x]$.

The roots of p(x) are $x = 0.5$ and $x = 5 + 2.5g$

$p(0.5) = 0$ and $p(5 + 2.5g) = 0$.

Consider

$$\begin{aligned} q(x) &= (x + 6.5)(x + 2 + 4.5g) \\ &= x^2 + 6.5x + 2x + 4.5gx + 6 + 1.25g \\ &= x^2 + (1.5 + 4.5g)x + 0.25g. \end{aligned}$$

$$\begin{aligned} q(0.5) &= (0.5)^2 + (1.5 + 4.5g)\,0.5 + 0.25g \\ &= 0.25 + 0.75 + 2.25g + 0.25g \\ &= 1 + 2.5g \neq 0. \end{aligned}$$

So 0.5 is not a root of q(x).

$$\begin{aligned} q(5 + 2.5g) &= (5 + 2.5g)^2 + (1.5 = 4.5g)(5 + 2.5g) + 0.25g \\ &= 4 + 4g + 0.5 + 1.5g + 3.75g + 0.25g \end{aligned}$$

$= 4.5 + 2.5g \neq 0.$

Thus $5 + 2.5g$ is not a root of $q(x)$.

Hence the problem of finding roots for these MOD polynomial functions happens to be a challenging work.

Next we proceed onto define the notion of MOD special dual like number functions built using the MOD special dual like number plane $R_n(m)g$; $g^2 = g$.

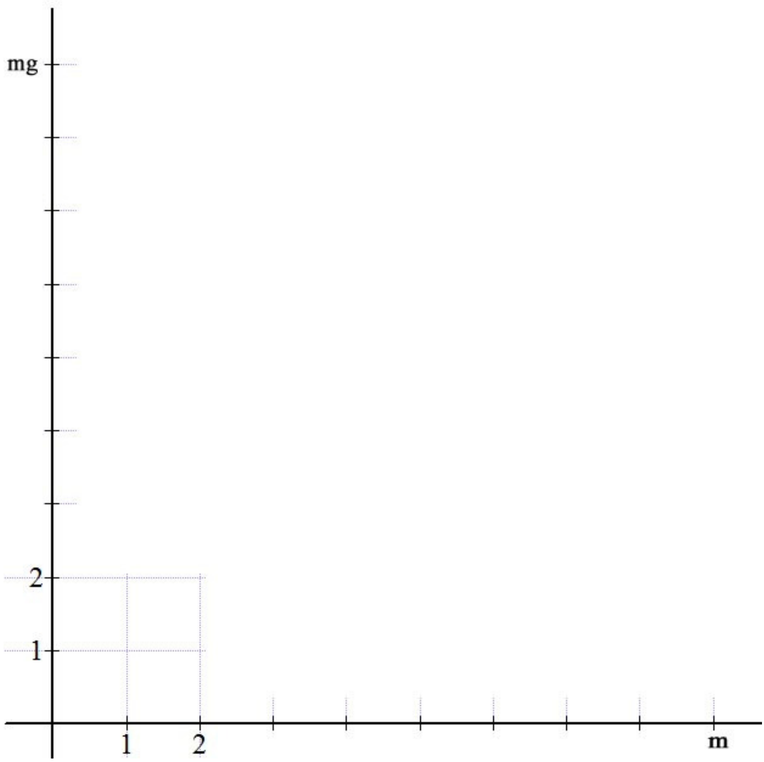

Figure 5.2

Define

$$R_n(m)g[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(m)g;\ g^2 = g \right.$$

where $R_n(m)g = \{a + bg \mid a, b \in [0, m)\}\}$ to be the collection of all MOD special dual like number functions.

Clearly $R_n(m)g[x]$ is a pseudo ring.

Here through examples we show how these MOD special dual like functions behave.

Example 5.9: Let

$R_n(12)g[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(12)g; = \{a + bg \mid g^2 = g,\ a, b \in [0, 12)\}\right\}$ be the collection of all MOD functions.

Solving equations are difficult as

$$p(x) = (x + \alpha_1)(x + \alpha_2)(x + \alpha_3)(x + \alpha_4)$$

$$\neq x^4 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)x^3 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_3\alpha_2 + \alpha_2\alpha_4 + \alpha_4\alpha_3)x^2 - (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4)x + \alpha_1\alpha_2\alpha_3\alpha_4 = q(x).$$

So roots of p(x) are not roots of q(x). It is difficult to find roots; for the situation in case of MOD polynomials happens to be a challenging one.

Next integrating or differentiating these functions is also a difficult task.

For let

$$f(x) = (3.4 + 4.5g)x^4 + (4 + 8g)x^3 + (2.5 + 3.5g)x^2 + (7.1 + 10.3g) \in R_n(12)\, g[x].$$

$$f'(x) = 4(3.4 + 4.5g)x^3 + 3(4 + 8g)x^2 + 2\,(2.5 + 3.5g)x$$
$$= (1.6 + 6g)x^3 + (5 + 7g)x$$

$$f'\,(x) = 3\,(1.6 + 6g)x^2 + 5 + 7g$$
$$= (4.8 + 6g)x^2 + 5 + 7g$$

$$f'''\,(x) = 2(4.8 + 6g)x$$
$$= 9.6x.$$

$$f^4(x) = 9.6.$$

Thus the fifth derivative is zero.

Now

$$\int f(x)\,dx = \frac{(3.4+4.5g)x^5}{5} + \frac{(4+8g)x^4}{4} + \frac{(2.5+3.5g)x^3}{3}$$

$$+ (7.1 + 10.3)gx + C.$$

Clearly $\frac{1}{4}$ or $\frac{1}{3}$ is not defined in [0, 12) so the integral does not exist.

Consider $s(x) = 5.2x^{12} + (6g + 4)x^6 + (4.5 + 2.3g)x^3 + 5g$

$$s'(x) = 12 \times 5.2x^{11} + 6(6g + 4)x^5 + 3(4.5 + 2.3g)x^2 + 0$$

$$= 2.4x^{11} + (1.5 + 6.9g)x^2\,.$$

$$s''(x) = 2.4 \times 11x^{10} + 2(1.5 + 6.9g)x$$

$$= 2.4x^{10} + (3 + 1.8g)x.$$

We can find the 3rd derivative

$$\frac{d^3s(x)}{dx^3} = 10 \times 2.4x^9 + (3 + 1.8g)$$

$= \quad 3 + 1.8g.$

The forth derivative is zero.

However this MOD function is a polynomial of degree 12 and the forth derivative zero is not in keeping with the usual form of differentiation.

Next we find the integral of s(x).

$$\int s(x)\, dx = \frac{5.12x^{13}}{13} + \frac{(6g+4)x^7}{7} + \frac{(4.5+2.3g)x^4}{4} + 5gx + C$$

Clearly this integral is also not defined as $\frac{1}{4}$ is not defined in the MOD interval [0, 12).

Example 5.10: Let $R_n(10)g[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(10)g = \right.$ $\{a + bg \mid a, b \in [0, 10), g^2 = g\}\}$ be the collection of MOD dual like number polynomial functions.

Let $p(x) = (6.2 + 4.3g)x^6 + 0.2x^5 + (5 + 2g)x^2 + (7 + 3g) \in R_n(10)g[x]$.

$$p'(x) = 6(6.2 + 4.3g)x^5 + x^4 + 2(5 + 2g)x$$

$$= (7.2 + 5.8g)x^5 + x^4 + 4gx.$$

If the integration and differentiation are the inverse process of each other.

We find $\int p'(x)\, dx$

$$\frac{(7.2+5.8g)x^6}{6} + \frac{x^5}{5} + \frac{4gx^2}{2}.$$

Clearly $\int p'(x)\, dx$ is not defined.

However we find

$$\int p(x)\, dx = \frac{(6.2+4.3g)x^7}{7} + \frac{0.2x^6}{6} + \frac{(5+2g)x^3}{3} +$$

$$(7 + 3g)x + C.$$

Clearly $\int p(x)\, dx$ is also not defined.

$$\text{Thus } \frac{d(\int p(x)dx)}{dx} \neq \int \frac{dp(x).dx.}{dx}$$

in general for any MOD polynomial function in $R_n(10)g[x]$.

Example 5.11: Let $R_n(7)g[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(7)g = \{a + bg \mid a, b \in [0,7), g^2 = g\}\right\}$ be the MOD special dual like number functions.

$$\text{Let } p(x) = (6.3 + 2g)x^5 + (3 + 6.5g)x^3 + (4 + 2g) \in R_n(7)g[x].$$

$$p'(x) = 5(6.3 + 2g)x^4 + 3(3 + 6.5g)x^2$$

$$= (3.5 + 3g)x^4 + (2 + 5.5g)x^2 .$$

$$\int p'(x)\, dx = \frac{(3.5+3g)x^5}{5} + \frac{(2+5.5g)x^3}{3} + C$$

$$= (3.5 + 2g)x^5 + (3 + 6.5g)x^3 + C$$

$$\neq p(x).$$

$$\int p(x)\,dx = \frac{(6.3+2g)x^6}{6} + \frac{(3+6.5g)x^4}{4} + (4 + 2g)x + C$$

$$= (2.8 + 5g)x^6 + (6 + 6g)x^4 + (4 + 2g)x + C.$$

$$\frac{d(\int p(x)dx)}{dx} = 6\,(2.8 + 5g)x^5 + 4(6 + 6g)x^3 + (4 + 2g)$$

$$= (2.8 + 2g)x^5 + (3 + 3g)x^3 + 4 + 2g$$

$$\neq p(x).$$

Hence

$$\frac{d(\int p(x)dx)}{dx} \neq p(x)$$

$$\int p'(x)\,dx \neq p(x).$$

Thus in no way one can relate the MOD functions derivatives and integrals.

Next the concept of special quasi dual number functions is described by some examples.

Let $R_n(m)g[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n(m)g = \{a + bg \mid a, b \in [0,m) \text{ and } g^2 = (m-1)g\}\right\}$ be the collection of all MOD special quasi dual number polynomial functions.

Clearly $\int p'(x)\,dx \neq p(x)$.

$$\frac{d}{dx}\int p(x)dx \neq p(x) \text{ in general.}$$

Further the roots of these MOD polynomial with special quasi dual number coefficients behave in a chaotic way.

These situations are described by an example or two.

Example 5.12: Let

$$R_n(9)g[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in R_n(9)g = \{a + bg \mid a, b \in [0,9) \text{ and } g^2 = 8g\}\right\}$$

be the collection of all MOD polynomials functions with coefficients from the special quasi dual number $R_n(9)g$.

Let

$$p(x) = x + (3.7 + 2.5g) \in R_n(9)g[x].$$

The root of p(x) is

$$x = 5.3 + 6.5g.$$

Consider

$$q(x) = x^2 + 7.2 + 4.3g \in R_n(9)g[x]$$

find the roots of q(x) is a difficult task.

Let

$$p(x) = (x + 3 + 6.5g)(x + 2.1 + 3g) \in R_n(9)g[x].$$

The roots of p(x) are x = 6 + 2.5g and x = 6.9 + 6g

Consider

$$(x + 3 + 6.5g)(x + 2.1 + 3g)$$

$$= x^2 + 3x + 6.5gx + 2.1x + 6.3 + 4.65 + 3gx + 0 + 1.5x \times 8g$$

$$= x^2 + (5.1 + 0.5g)x + 4.95$$

$$= q(x).$$

is x = 6 + 2.5g and x = 6.9 + 6g roots of q(x).

$$q(6 + 2.5g) = (6 + 2.5g)^2 + (5.1 + 0.5g)(6 + 2.5g) + 4.95$$

$$= 6.25 \times 4g + 7.5g + 3.6 + 3g + 3.75g + g + 4.95$$

$= \quad 4.25 + 8.55 \neq 0.$

So 6 + 2.5g is not a root.

Let us consider

$$
\begin{aligned}
q(6.9 + 6g) &= (6.9 + 6g)^2 + (5.2 + 0.5g) \times (6.9 + 6g) + 4.95 \\
&= 2.61 + 1.8g + 8.19 + 3.6g + 6g + 7.5g + 4.95 \\
&= 7.75 + 0.9g \neq 0.
\end{aligned}
$$

So 6.9 + 6g is also not a root of q(x).

Let us consider the MOD polynomial function.

$$
\begin{aligned}
p(x) &= (3.6 + 4.2g)x^6 + (2.1 + 0.6g)x^4 + (4.5 + 2.5g)x^2 \\
&\quad + (5 + 7g) \in R_n(9)[x].
\end{aligned}
$$

$$
\begin{aligned}
p'(x) &= 6(3.6 + 4.2g)x^5 + 4(2.1 + 0.6g)x^3 + 2(4.5 + 2.5g)x \\
&= (3.6 + 7.2g)x^5 + (8.4 + 2.4g)x^3 + 5gx.
\end{aligned}
$$

$$\int p'(x)\,dx = \frac{(3.6+7.2g)x^6}{6} + \frac{(8.4+2.4g)x^4}{4} + \frac{5gx^2}{2} + C$$

$$\neq p(x).$$

$\int p'(x)\,dx$ is not defined as $\frac{1}{6}$ is not defined in [0, 9).

$$\int p(x)\,dx = \frac{(3.6+4.2g)x^7}{7} + \frac{(2.1+0.6g)x^5}{5} + \frac{(4.5+2.5g)x^3}{3}$$

$$+ (5 + 7g)x + C.$$

This is not defined as $\frac{1}{3}$ is not defined in [0, 9).

Thus the natural laws of derivative and integral are not in general true in case of MOD function.

Now we proceed onto define some new types of MOD functions.

Let f be a MOD function defined on the MOD real plane or MOD neutrosophic plane or MOD complex modulo integer plane or MOD neutrosophic complex modulo integer plane or MOD dual number plane or MOD special dual like number plane or MOD special quasi dual number plane.

We say f is a special dual MOD function if f o f = 0.

We call f a special dual like MOD function if f o f = f.

We call f to be a special quasi dual function if

$$f \circ f = (m - 1)f$$

where f is defined over the MOD plane relative to the interval [0, m).

It is left as an exercise for the reader to construct examples of these new type of MOD functions.

Chapter Six

SUGGESTED PROBLEMS

In this chapter we suggest a few problems for the reader. Some of them are difficult and some of them are simple and some are open problems.

1. Obtain some interesting properties enjoyed by decimal polynomial rings in the MOD plane.

2. Why the MOD differentiation is different from that of usual differentiation in case of functions in the MOD plane?

3. Prove by an example a function which is continuous in R is not continuous in the MOD plane $R_n(m)$.

4. What are the special and interesting features enjoyed by functions in the MOD plane?

5. Study the function $y = 5x + 3 \in R[x]$ in the MOD plane $R_n(7)$ and $R_n(4)$.

   (i) Are they continuous in these planes?
   (ii) What are the zeros in these MOD functions?
   (iii) When is the function differentiable?

(iv) Does there exist a MOD function f(x) whose derivative $f'(x)$ exists?
(v) Can this function have infinite number of zero divisors in any MOD plane?
(vi) Does there exists a MOD plane in which the function $y = 5x + 3$ has only 5 zero divisors?

6. Let $y = 7x^2 + 3x + 1 \in R[x]$ be the function in the real plane.

   Study questions (i) to (vi) of problem 5 for y in the MOD planes $R_n(7)$, $R_n(3)$, $R_n(5)$ and $R_n(10)$.

7. Let $y = 3x^3 - 7 \in R[x]$ be the function in the real plane.

   Study questions (i) to (vi) of problem 5 in the MOD planes $R_n(7)$, $R_n(2)$ and $R_n(6)$.

8. Let $y = 3x^3 + 2x^2 - 5x + 8 \in R[x]$ be the function.

   Study questions (i) to (vi) of problem 5 in the MOD planes $R_n(6)$, $R_n(8)$, $R_n(5)$, $R_n(10)$ and $R_n(4)$.

9. Let $y = x^2 + 4 \in R[x]$ be the function.

   Study questions (i) to (vi) of problem 5 in the MOD planes $R_n(3)$, $R_n(7)$ and $R_n(4)$.

10. Let $y = 9x^5 - 3x^2 + 7 \in R[x]$ be the real function.

    Study questions (i) to (vi) of problem 5 in the MOD planes $R_n(7)$, $R_n(3)$, $R_n(2)$ and $R_n(10)$.

11. What is the difference in any function in the MOD planes $R_n(p)$ and $R_n(m)$; p a prime and m is a non prime?

12. Study question (11) for the function $y = 7x^3 - 5x + 1$ in $R_n(11)$ and $R_n(12)$.

13. Can all functions in the MOD plane $R_n(m)[x]$ be integrable?

14. How can we adopt integration and differentiation in case of the functions in MOD plane $R_n(m)[x]$?

15. List out the difficulties in adopting integration to functions in $R_n(m)[x]$.

16. Prove derivatives of functions in $R_n(m)[x]$ do not satisfy the basic properties of derivatives.

17. Let $f(x) = 5x^7 + 5x^3 + 1.7x + 3 \in R_n(7)[x]$.

    (i) Find $f'(x)$.
    (ii) What is $\frac{d^7 f(x)}{dx^7}$?
    (iii) Find $\int f(x)\, dx$.
    (iv) Is f(x) continuous in $R_n(7)[x]$?
    (v) Can f(x) be continuous on $R_n(6)[x]$?

18. Does this MOD function $f(x) = 6x^3 + 7x^2 + 1 \in R_n(8)[x]$ continuous in the MOD plane $R_n(8)$.

19. Is $f(x) \in R_n(8)[x]$ in problem 18 differentiable?

20. Is $f(x) \in R_n(8)[x]$ in problem 18 integrable?

21. Give a function $f(x) \in R_n(12)[x]$ which is both differentiable and integrable.

22. Give a function $f(x) \in R_n(10)[x]$ which is neither integrable nor follow any properties of derivation?

23. Finding roots of a MOD polynomials in $R_n(m)[x]$ is a open problem.

24. Prove MOD polynomials do not satisfy the fundamental theorem of algebra;

    "A nth degree polynomial has n and only n roots"

25. Give an example of a MOD polynomial of degree n which has less than n roots.

26. Give an example of a MOD polynomial of degree n which has more than n-roots.

27. Let $p(x) = 8x^8 + 5x^4 + 2 \in R_n(10)[x]$ be the MOD polynomial.

    Find all the roots of p(x).

28. Let $p(x) = 5x^3 + 3x + 1 \in R_n(7)[x]$ be the MOD polynomial.

    Find all the roots of p(x).

29. Is it possible to find all the roots of $q(x) = x^4 + 2 \in R_n(3)[x]$?

    (i) Can q(x) have more than 4 roots?
    (ii) Does q(x) have less than four roots?
    (iii) Find $\int q(x)\,dx$.
    (iv) Find $\frac{dq(x)}{dx}$.

30. Let $p(x) \in R_n(6)[x]$ where $p(x) = x^6 + 1$.

    Study questions (i) to (iv) of problem 29 for this p(x).

31. Find a polynomial in $R_n(5)[x]$ of degree 3 which has only three roots.

32. Find $p(x) \in R_n(4)[x]$ be a polynomial of degree four which has more than four roots.

33. Find a fifth degree polynomial in $R_n(5)[x]$ which has less than five roots.

34. Does there exist a MOD plane $R_n(m)$ such that every polynomial $p(x) \in R_n(m)[x]$ follows the classical properties of polynomials?

35. Let $f(x) = 3x^3 + 5x^2 + 1 \in R_n(10)[x]$.

    Is f(x) a continuous function in the MOD plane $R_n(10)$?

36. Give an example of a function f(x) which is continuous in R[x] but f(x) is not continuous in $R_n(7)[x]$.

37. Let $f(x) = x^3 + 1 \in R_n(6)[x]$ be a MOD function.

    (i) Is f(x) a continuous function?
    (ii) Draw the graph of f(x) in the MOD plane $R_n(6)$?
    (iii) Can $f(x) = x^3 + 1$ be continuous in any other MOD plane?
    (iv) Can all the MOD planes in which the function $f(x) = x^3 + 1$ is continuous be characterized?

38. Let $f(x) = 3x^2 + 1 \in R_n(5)[x]$ be the MOD function.

    Study questions (i) to (iv) of problem 37 for this f(x).

39. Obtain some special features enjoyed by MOD complex plane $C_n(m)$.

40. Prove every function in the complex plane C can be transformed into a function in the MOD complex plane $C_n(m)$.

41. Let $\lim_{z \to i} f(z)$ where $f(z) = \frac{z^4 - 1}{z - i} \in C$.

Find the corresponding MOD function of f(z) in $C_n(10)$, $C_n(7)$, $C_n(16)$, $C_n(11)$ and $C_n(15)$.

42. Let $f(z) = \frac{z^2 - 8i}{z^2 - 8z + 8} \in C$; limit $z \to 2 + 2i$

    (i) Find all the complex MOD planes $C_n(m)$ in which f(z) has a limit.
    (ii) Find all these complex MOD planes $C_n(m)$ in which f(z) has no limit.

43. Does there exist a function in the complex plane which has limit points for all values of $z \to z'$ such that the f(z) has limit for all $z \in C_n(m)$?
    Justify your claim.

44. Let $f(z) = \frac{z^2 + z - 2 + i}{z^2 - 2z + 1} \in C$, let $z \to 1 + i$ be the function in the complex plane.

    Transform f(z) to the MOD planes (i) $C_n(20)$, (ii) $C_n(5)$, (iii) $C_n(7)$, (iv) $C_n(4t)$ (v) $C_n(16)$, (vi) $C_n(29)$, (vii) $C_n(15)$ and (viii) $C_n(21)$
    In which of these planes the limit exists?

45. Let $f(z) = z^2 + 1$ be a MOD complex function in $C_n(7)$.

    (i) Is f(z) continuous in $C_n(7)$?
    (ii) Is f(z) differential?
    (iii) Find zeros of f(z).

46. Does there exist a function f(z) in $C_n(5)$ which is continuous in $C_n(5)$?

47. Obtain more properties related with MOD complex functions.

48. When will a complex MOD function be continuous at all points? Give one example of it.

49. $f(x) = 3.75x^3 + 0.4x + 3.28 \in R_n(4)[x]$ be a MOD polynomial in $R_n(4)[x]$.

    (i) Find all roots of f(x).
    (ii) Is f(x) a continuous function?
    (iii) Does f(x) contain more than 3 roots?
    (iv) Find all roots of f(x).
    (v) Find a method by which these MOD equations can be solved.

50. Solve $p(x) = x^3 + 1 \in R_n(5)[x]$.

51. Find roots of $p(x) = x^3 + 1 \in R_n(4)[x]$.

52. Find roots of p(x) in problems 50 and 52; which has more zeros?

53. Let $f(x) = x^2 + 3 \in R_n(m)$, m = 4, 5, 6, 7, 8, 9, 10 and 11 be the MOD function.

    (i) Find the plane $R_n(m)$ in which f(x) has least number of zeros; $4 \leq m \leq 11$.
    (ii) Find the plane $R_n(m)$ in which f(x) has greatest number of zeros; $4 \leq m \leq 11$.
    (iii) Is f(x) a continuous function?
    (iv) Is f(x) an integrable function in $R_n(m)$; $4 \leq m \leq 11$?
    (v) Is f(x) differential in $R_n(m)$; $4 \leq m \leq 11$?
    (vi) Trace the function f(x) in all the MOD planes $R_n(m)$; $4 \leq m \leq 11$.

54. Let $f(x) = x^3 + 0.8 \in R_n(m)$; $2 \leq m \leq 15$ be the MOD function.

    Study questions (i) to (vi) of problem 53 for this f(x).

55. Let $p(x) = 0.7x^{5.2} + 0.9x^{0.2} + 3 \in R_n(m)$; $4 \leq m \leq 10$ be the MOD function.

    Study questions (i) to (vi) of problem 53 for this f(x).

56. Let $f(z) = z^2 + 0.3z + 1 \in C_{n(}m)$; $2 \leq m \leq 5$ be the complex MOD function.

    (i) Is f(z) continuous in all planes?
    (ii) Is f(z) differentiable?
    (iii) Find lim f(z); $z \in C_n(m)$, $z \to 1 + i_F$; $2 \leq m \leq 5$.

57. Let $f(z) = 3z^2 + z + 1 \in C_n(m)$; $4 \leq m \leq 8$ be the complex MOD function.

    Study questions (i) to (iii) of problem 56.

58. Obtain any special and interesting feature enjoyed by complex MOD planes $C_n(m)$, $2 \leq m < \infty$.

59. Derive the special identities related with MOD trigonometric functions.

60. Obtain all the special features related with MOD trigonometric functions.

61. What are the advantages of using MOD trigonometric functions?

62. Draw the MOD graph of the MOD trigonometric function $y_n = n \tan x$ in the MOD plane $R_n(10)$.

63. Draw the MOD graph of the MOD trigonometric function $y_n = n \tan 2x$ in the MOD plane $R_n(10)$.

    Compare the graphs in problems 62 and 63.

64. Draw the MOD trigonometric graph of the function $y_n = n \sin 3x$ in the MOD plane $R_n(9)$.

65. Let y_n = n cos 3x be the MOD trigonometric function. Find the MOD graph of y_n = n cos 3x in the MOD plane $R_n(19)$.

    Compare y_n = n cos 3x with y_n = n sin 3x.

66. Prove or disprove $\text{nsin}^{-1}x + \text{ncos}^{-1}x = \pi / 2$ if $x \in [0, 1]$.

67. Prove or disprove $\text{ntan}^{-1}x + \text{ncot}^{-1}x = \pi / 2$ if $x \in [0, 1]$.

68. Prove or disprove $\text{nsec}^{-1}x + \text{ncos}^{-1}x = \pi / 2$ if $x \in [0, 1]$.

69. Prove or disprove $\text{nsin}^{-1}x + \text{n sin}^{-1}x =$ $\text{n sin}^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2})$.

70. Find ntan^{-1} (n tan $3\pi/4$).

71. Find n cos (n $\tan^{-1}$ ¾).

72. Find ncos^{-1} (n cos $7\pi/6$) $\in R_n(m)$, $m \geq 9$.

73. Find nsin (n $\cot^{-1}x$).

74. Find nsin 2A.

75. Find n cos2A.

76. Is $\text{nsin}^2 A = \dfrac{1 - \text{n}\cos 2A}{2}$?

77. Obtain any interesting result about logarithmic MOD functions.

78. Obtain the special features enjoyed by exponential MOD functions.

79. Give examples of MOD neutrosophic real functions.

80. Let $f(x) \in R_n^I(12)[x] = S$ a MOD neutrosophic polynomial of degree t.

    (i) When will the derivatives of f(x) exist?
    (ii) Does there exist a $p(x) \in R_n^I(12)[x]$ which satisfies both the laws of differentiation and integration?
    (iii) Give an example of a MOD neutrosophic polynomial functions in S which does not satisfy both the laws of differentiation and integration.
    (iv) Give an example of a MOD neutrosophic function in S which satisfies the properties of derivatives but not the properties of integrals.
    (v) Give an example of a MOD neutrosophic function in S which satisfies the properties of integration but does not satisfy the properties of differentiation.
    (vi) Does there exist a polynomial function p(x) of degree 5 in S which has only 5 roots?
    (vii) If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_8$ are roots of $p(x) = (x + \alpha_1) \ldots (x + \alpha_8)$ in S will $q(x) = x^8 - (\alpha_1 + \ldots + \alpha_8)x^7 + \ldots \pm \alpha_1 \ldots \alpha_8$ have $\alpha_1, \alpha_2, \ldots, \alpha_8$ to be its root?

81. Let $M = R_n^I(13)[x]$ be the collection of MOD neutrosophic polynomial functions.

    Study questions (i) to (vii) of problem 80 for this M.

82. Let $P = C_n^I(15)[x]$ be the MOD neutrosophic complex modulo integer functions.

    Study questions (i) to (vii) of problem 80 for this P.

83. Let $W = R_n(20)g[x]$ be the collection of all MOD dual number functions.

    Study questions (i) to (vii) of problem 80 for this W.

84. Let S = $R_n(m)h[x]$ be the collection of special dual like number MOD functions.

    Study questions (i) to (vii) of problem 80 for this S.

85. Let V = $R_n(24)k[x]$ be the special quasi dual number MOD functions.

    Study questions (i) to (vii) of problem 80 for this V.

86. Let B = $R_n(13)g[x]$ ($g^2 = 0$) be the MOD dual number functions.

    Study questions (i) to (vii) of problem 80 for this B.

87. Give examples of dual MOD functions f such that f o f = 0, f defined over any of the MOD planes described in chapter III.

88. Define a dual MOD function of $R_n^I(27)$.

89. Can dual MOD functions in general behave like usual functions or behave like MOD functions? Justify.

90. Give examples of MOD special dual like functions g so that g o g = g.

91. Can MOD transformations over MOD planes lead to such g's in problem 90?

92. Obtain any special or unique property associated with MOD dual functions in general.

93. Describe and develop some interesting properties associated with MOD special dual like functions g; g o g = 0.

94. Can the collection of MOD dual functions related with $C_n(20)$ have any nice algebraic structure?

95. Give examples of special quasi dual MOD functions f associated with $R_n(19)$; f o f = 18f.

96. Develop special features enjoyed by special quasi dual MOD functions associated with $R_n^I(18)$.

97. Prove or disprove dual MOD functions on $R_n(18)$ has no relation with MOD dual number polynomial functions $R_n(18)g[x]$; $g^2 = 0$.

98. Prove or disprove that special dual like functions f on $R_n(17)$, f o f = f has no relation with MOD special dual like number functions $R_n(17)$ g[x], $g^2 = g$.

99. Prove there exist no relation between special quasi dual MOD function f defined on $R_n(9)$ where f o f = 8f and the MOD special quasi dual number polynomial functions in $R_n(9)g[x]$; $g^2 = 8g$.

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INDEX

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MOD functions are those functions which are defined on the MOD planes. These MOD functions behave in a very distinct way. Functions which are continuous in real plane are not so in MOD planes.

Further, the exponential and trigonometric functions like sin x and cos x happen to be discontinuous in the MOD plane. Several open conjectures are proposed in this book.

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