On Maximal Acceleration, Born's Reciprocal Relativity and Strings in Tangent Bundle Backgrounds

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Abstract

Accelerated strings in tangent bundle backgrounds are studied in further detail than it has been done in the past. The worldsheet associated with the accelerated open string described in this work envisages a continuum family of worldlines of accelerated points. It is when one embeds the two-dim string worldsheet into the tangent bundle TM background (associated with a uniformly accelerated observer in spacetime) that the effects of the maximal acceleration are manifested. The induced worldsheet metric as a result of this embedding has a *null* horizon. It is the presence of this null horizon that limits the acceleration values of the points inside string. If the string crosses the null horizon some of its points will exceed the maximal acceleration value and that portion of the string will become causally disconnected from the rest of string outside the horizon. It is explained why our results differ from those in the literature pertaining the maximal acceleration modifications of the Rindler metric. We also find a modified Rindler metric which has a true curvature singularity at the location of the null horizon due to a finite maximal acceleration. One of the salient features of studying the geometry of the tangent bundle is that the underlying spacetime geometry becomes *observer dependent* as Gibbons and Hawking envisioned long ago. We conclude with some remarks about generalized QFT in accelerated frames and the black hole information paradox.

Keywords : Maximal Acceleration; Gravity; Rindler Spaces; Strings; Finsler Geometry; Born Reciprocity; Phase Space.

1 Born's Reciprocal Relativity in Phase Space and Maximal Acceleration

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality). The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where $X^i, T, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

The $U(1,3) = SU(1,3) \otimes U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dp_0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\sigma)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2}\left((dE)^2 - (dp_x)^2 - (dp_y)^2 - (dp_z)^2\right) \tag{1.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details.

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \otimes U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right) \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(1.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t \ = \ \sqrt{\frac{\hbar}{bc}}; \ \ \lambda_l \ = \ \sqrt{\frac{\hbar \ c}{b}}; \ \ \lambda_p \ = \ \sqrt{\frac{\hbar \ b}{c}}; \ \ \lambda_e \ = \ \sqrt{\hbar \ b \ c} \tag{1.3}$$

The gravitational constant can be written as $G = \alpha_G \ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales in eq-(1.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (1.2) invariant are [2]

$$T' \ = \ T \ cosh\xi \ + \ (\frac{\xi_v \ X}{c^2} \ + \ \frac{\xi_a \ P}{b^2}) \ \frac{sinh\xi}{\xi} \tag{1.4a}$$

$$E' \ = \ E \ cosh\xi \ + \ (-\xi_a \ X \ + \ \xi_v P) \ \frac{sinh\xi}{\xi} \tag{1.4b}$$

$$X' \ = \ X \ cosh\xi \ + \ (\xi_v \ T \ - \ \frac{\xi_a \ E}{b^2}) \ \frac{sinh\xi}{\xi} \tag{1.4c}$$

$$P' \ = \ P \ cosh\xi \ + \ (\frac{\xi_v \ E}{c^2} \ + \ \xi_a \ T) \ \frac{sinh\xi}{\xi} \tag{1.4d}$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) \ = \ \frac{F}{F_{max}} \tag{1.5}$$

It is straightforwad to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\sigma)^2 \ = \ (d\tau)^2 \ \left(1 \ - \ \frac{F^2}{F^2_{max}} \right) \tag{1.6}$$

is truly left invariant under force (acceleration) boosts (1.4). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = - \ dT \wedge E + dX \wedge dP$. One can verify that the transformations eqs-(1.4) are invariant under the discrete transformations

$$(T, X) \rightarrow (E, P); \ \ (E, P) \rightarrow (-T, -X), \ \ \mathbf{b} \rightarrow \frac{1}{\mathbf{b}} \tag{1.7}$$

we argued [6] that the latter transformation $\mathbf{b} \rightarrow \frac{1}{\mathbf{b}}$ is a manifestation of the large/small tension T-duality symmetry in string theory. In natural units of $\hbar = c = 1$, the maximal proper force $\mathbf{b}$ has the same dimensions as a string tension (energy per unit length) $(mass)^2$. Novel physical consequences of Born's Reciprocal Relativity can be found in [5].

We proceed next with the physics of maximal acceleration. A review of the arguments supporting the existence of a maximal acceleration for a massive particle and how different values for this upper limit can be predicted in different physical situations was provided by [10]. In particular, the maximal acceleration principle can be successfully used to prevent the occurrence of singularities in General Relativity, and of ultraviolet divergences in Quantum Field Theory, see [10] and references therein. From the historical point of view the maximal proper acceleration has been first derived starting from the principles of Quantum Mechanics and Relativity by Caianiello [4], [11].

Interesting results based on a simplified model (lacking covariance) were applied to the Schwarzschild metric by [12], the Reissner - Nordstrom metric by [13], the Kerr metric by [14] and the Robertson -Walker metric by [15]. A non-covariant $\frac{d^2x^\mu}{ds^2}$ acceleration should be replaced by the covariant expression $\frac{D^2x^\mu}{Ds^2} = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$. A fully covariant approach lead to a complete integrability of equations of motion in spacetimes of constant curvature [16].

To study the geometry behind a maximal proper force and/or maximal acceleration, we shall follow next the description by the authors [7], [8] where one may study in detail the geometry of Lagrange-Finsler and Hamilton-Cartan spaces and their higher order (jet bundles) generalizations The metric associated with the tangent space TM_d can be written in the following block diagonal form

$$(ds)^2 \ = \ g_{ij}(x^k, y^a)\, dx^i d\, x^j + h_{ab}(x^i, y^a)\, \delta y^a \ \delta y^b, \ \ i,j,k = 1,2,3, ...d; \ a,b,c = 1,2,3, ...d \tag{1.9}$$

if instead of the standard coordinate-basis one introduces the anholonomic frames (non-coordinate basis) defined as

$$\delta_i \ = \ \partial_i - N^b_i(x,y)\ \partial_b \ = \ \partial/\partial x^i - N^b_i(x,y)\ \partial_b; \quad \partial_a \ = \ \frac{\partial}{dy^a} \tag{1.10}$$

and its dual basis is

$$\delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \quad \delta^a = dy^a + N^a_k(x,y)\ dx^k) \tag{1.11}$$

where the N–coefficients define a nonlinear connection, N–connection structure, see details in [7], [8]. As a very particular case one recovers the ordinary linear connections if $N^a_i(x,y) = \Gamma^a_{bi}(x)\, y^b$.

The $8D$ tangent bundle infinitesimal interval is given by

$$(d\sigma)^2 \ = \ g_{ij}(x,y)\, dx^i\, dx^j + h_{ab}(x,y)\, (dy^a - N^a_c(x,y)\, dx^c)\, (dy^b - N^b_d(x,y)\, dx^d) \tag{1.12}$$

In the flat tangent bundle case, when there is a maximal proper acceleration M, one can find a coordinate system so that one has $N^a_c = 0$, $g_{ij} = \eta_{ij}, h_{ab} = \frac{\eta_{ab}}{M^2}$.

The $8D$ cotangent space (phase-space) infinitesimal interval is

$$(d\omega)^2 \ = \ g_{ij}(x,p)\, dx^i\, dx^j + h^{ab}(x,p)\,(dp_a - N_{ac}(x,p)\, dx^c)\,(dp_b - N_{bd}(x,p)\, dx^d) \quad (1.13)$$

In the flat cotangent bundle case, when there is a maximal proper force $\mathbf{b}$, one can find a coordinate system so that one has $N_{ac} = 0, g_{ij} = \eta_{ij}, h^{ab} = \frac{\eta^{ab}(x,p)}{\mathbf{b}^2}$.

In another particular case, when

$$g_{ij}(x,y) \ = \ g_{ij}(x), \quad h_{ab}(x,y) = \frac{g_{ab}(x)}{M^2}, \quad N^b_d(x,y) = \Gamma^b_{dc}(x) y^c \quad (1.14)$$

the $8D$ tangent bundle infinitesimal interval (1.12) becomes

$$(d\sigma)^2 \ = \ g_{ij}(x)\, dx^i\, dx^j + \frac{g_{ab}(x)}{M^2}\,(dy^a \ - \ \Gamma^a_{ce}(x)\, y^e\, dx^c)\,(dy^b \ - \ \Gamma^b_{de}(x)\, y^e\, dx^d) \quad (1.15)$$

by factoring out the terms $(ds)^2 = g_{ij}(x)\, dx^i\, dx^j$, after writing $y^e = \frac{dx^e}{ds}$, gives in the right hand side of eq-(1.15)

$$(ds)^2 \left(1 \ + \frac{g_{ab}}{M^2}\,(\frac{dy^a}{ds} \ - \ \Gamma^a_{ce}(x)\, \frac{dx^e}{ds}\, \frac{dx^c}{ds})\,(\frac{dy^b}{ds} \ - \ \Gamma^b_{de}(x)\, \frac{dx^e}{ds}\, \frac{dx^d}{ds}) \right) \quad (1.16a)$$

therefore, one ends up by having

$$(d\sigma)^2 \ = \ (ds)^2 \left(1 \ - \ \frac{g^2(s)}{M^2} \right) \quad (1.16b)$$

where the (spacelike) *covariant* proper acceleration squared is

$$g^2(s) \ = \ - \ g_{ab}(x)\,(\frac{d^2x^a}{ds^2} - \Gamma^a_{ce}(x)\, \frac{dx^e}{ds}\, \frac{dx^c}{ds})\,(\frac{d^2x^b}{ds^2} - \Gamma^b_{de}(x)\, \frac{dx^e}{ds}\, \frac{dx^d}{ds}) \quad (1.17)$$

after writing $\frac{dy^a}{ds} = \frac{d^2x^a}{ds^2}$. To sum up, in the special case described by eq-(1.14), the $8D$ tangent bundle infinitesimal interval (1.12) can be written in terms of the *covariant* proper acceleration $\frac{D^2x^\mu}{Ds^2}$ of a particle in the underlying spacetime as displayed in eqs-(1.16).

Next we shall discuss the maximal acceleration modifications to the Rindler metric and explain why our results differ from those in [9]. The hyperbolic wordline in Minkowski spacetime of a uniformly accelerated observer whose constant proper acceleration is g is given by

$$x^0 \ = \ \frac{1}{g}\ sinh(gs), \quad x^1 \ = \ \frac{1}{g}\ cosh(gs) \quad (1.18)$$

from which one infers that

$$(\frac{d^2x^0}{ds^2})^2 \ - \ (\frac{d^2x^1}{ds^2})^2 \ = \ - \, g^2 \quad (1.19)$$

when the signature is chosen to be $(+,-)$. The Rindler (curvilinear) coordinates ξ, η are defined by the equations

$$x^0 \;=\; \xi \; sinh(\eta), \quad x^1 \;=\; \xi \; cosh(\eta), \tag{1.20}$$

Caution must be taken *not* to confuse the hyperbolic worldlines of Rindler spacetime with the hyperbolic worldline of an accelerated observer whose constant (uniform) proper acceleration is g. Only at the very specific values $\xi = \frac{1}{g}$, and $\eta = gs$, *one* of the hyperbolic worldlines of the Rindler spacetime *coincides* with the hyperbolic worldline of an accelerated observer whose constant proper acceleration is g. The hyperbolic worldlines of Rindler spacetime consist of a continuous family of uniformly accelerated observers $\mathcal{O}, \mathcal{O}', \mathcal{O}'', \ldots$ in a two-dim spacetime M_2 with proper accelerations $g, g', g'', \ldots$ and whose affine parameters associated with their hyperbolic worldlines are $s, s', s'', \ldots$ such that

$$\eta \;=\; gs \;=\; g's' \;=\; g''s'' \;=\; \ldots \; ; \;\; g \;=\; \frac{1}{\xi}, \;\; g' \;=\; \frac{1}{\xi'}, \;\; g'' \;=\; \frac{1}{\xi''}, \;\; \ldots \tag{1.21a}$$

Inverting the coordinate transformations of a Rindler spacetime one has

$$(x^1)^2 \;-\; (x^0)^2 \;=\; (\xi)^2, \;\; \frac{x^0}{x^1} \;=\; tanh(\eta) \;\Rightarrow\; arctanh(\frac{x^0}{x^1}) \;=\; \eta \tag{1.21b}$$

and the Rindler metric is given by

$$(ds)^2_{Rindler} \;=\; \xi^2 \; (d\eta)^2 \;-\; (d\xi)^2 \;=\; (dx^0)^2 \;-\; (dx^1)^2 \tag{1.22}$$

the constant ξ lines are hyperbolas, and the constant η lines are straight lines in the x^0, x^1 axes.

If ones chooses for conformal factor in eq-(1.16) the one associated with the continuous family of uniformly accelerated observers $g = \frac{1}{\xi}$ in Rindler spacetimes

$$\Omega^2 \;=\; 1 \;-\; \frac{g^2}{M^2} \;=\; 1 \;-\; \frac{1}{\xi^2 \; M^2} \tag{1.23}$$

the maximal acceleration corrections to the Rindler spacetime interval are then given by

$$(d\tilde{s})^2 \;=\; (1 - \frac{1}{\xi^2 M^2}) \; \left(\; \xi^2 (d\eta)^2 \;-\; (d\xi)^2 \; \right) \tag{1.24}$$

The metric (1.24) is conformally flat, when $M^2 = \infty$ the conformal factor is unity as expected. One can find a particle motion $x^\mu(s)$ whose non-uniform proper acceleration is such that eq-(1.19) becomes now

$$(\frac{d^2x^0}{ds^2})^2 \;-\; (\frac{d^2x^1}{ds^2})^2 \;=\; -\,g^2(x^0, x^1) \;=\; -\;\frac{1}{\xi^2} \;=\; \frac{1}{(x^0)^2 - (x^1)^2} \tag{1.25a}$$

subject to the condition

$$(\frac{dx^0}{ds})^2 \; - \; (\frac{dx^1}{ds})^2 \; = \; 1 \qquad (1.25b)$$

stemming from the normalization condition of the velocity $g_{\mu\nu}\frac{dx^\mu}{ds}\;\frac{dx^\nu}{ds} = 1$. We may notice that our modified Rindler metric in eq-(1.24) $differs$ from the one in [9]

$$(d\tilde{s})^2 \; = \; (1 - \frac{1}{\xi^2 M^2}) \; \xi^2 (d\eta)^2 \; - \; (d\xi)^2 \; = \; (\xi^2 - \frac{1}{M^2}) \; (d\eta)^2 \; - \; (d\xi)^2 \quad (1.26)$$

which is *not* conformally flat as it should be according to eqs-(1.16). The reason the authors [9] obtained eq-(1.26) is because after the $first$ differentiation of eq-(1.20), giving $\dot{x}^0 = g\xi cosh(\eta), \dot{x}^1 = g\xi sinh(\eta)$, they went ahead and substituted g for $\frac{1}{\xi}$, *prior* to performing the *second* differentiation $d\dot{x}^0, d\dot{x}^1$ in eq-(1.15) (when the connection $\Gamma = 0$). The correct way to proceed is to use eqs-(1.16) when the non-uniform proper acceleration is determined by eqs-(1.25).

The scalar curvature corresponding to the conformally flat metric (1.24) is

$$\tilde{R} \; = \; \Omega^{-2} \; [\; R \; - 2 \; (d-1) \; \nabla_a \nabla^a ln \; \Omega \; - \; (d-2) \; (d-1) \; (\nabla_a ln \; \Omega) \; (\nabla^a ln \; \Omega) \;] \qquad (1.27a)$$

in $d = 2$, due to the fact that $R = 0$, one arrives after plugging in the expression (1.23) for the conformal factor, at

$$\tilde{R} \; = \; - \; \frac{4}{\xi^4 M^2} \left(1 \; - \; \frac{1}{\xi^2 M^2} \right)^{-3} \qquad (1.27b)$$

after using

$$\nabla_a \nabla^a ln \; \Omega \; = \; \frac{1}{\sqrt{|g|}} \; \partial_a (\sqrt{|g|} \; g^{ab} \partial_b \; ln \; \Omega), \quad \sqrt{|g|} \; = \; \xi, \quad g^{00} = \xi^{-2}, \quad g^{11} = -1 \qquad (1.28)$$

The most salient features of the maximal acceleration modifications of the Rindler metric (1.24) is that there is a null horizon $(d\tilde{s})^2 = 0$ and a true singularity $\tilde{R} \; = \; \infty$ at the location of the maximal acceleration hyperbola given by $\xi = \frac{1}{g_{max}} = \frac{1}{M}$. $\mathcal{R} = 0$ at $\xi = 0$ and $\xi = \infty$. One has a $shifted$ horizon associated with the modified Rindler metric (1.24). The null horizon corresponds to the hyperbola $(x^1)^2 - (x^0)^2 = \frac{1}{M^2}$ rather than the straight lines (asymptotes) $x^0 = \pm x^1 \Rightarrow \xi = 0, \eta = \pm\infty$ in Rindler (flat) spacetimes.

Another example is given by the worldline of an observer (particle) in $d = 2$ dimensions defined by the equations

$$\frac{d^2 x^0}{ds^2} \; = \; F(x^0), \quad \frac{d^2 x^1}{ds^2} \; = \; H(x^0) \qquad (1.29)$$

one can integrate the above differential equations by quadratures giving

$$ s \ = \ \frac{1}{\sqrt{2}} \int \ \frac{dx^0}{\sqrt{\int \ F(x^0) \ dx^0}} \tag{1.30a} $$

$$ s \ = \ \frac{1}{\sqrt{2}} \int \ \frac{dx^1}{\sqrt{\int \ H(x^1) \ dx^1}} \tag{1.30b} $$

the functions $F(x^0), H(x^1)$ must obey the constraints

$$ \int \ F(x^0) \ dx^0 \ - \ \int \ H(x^1) \ dx^1 \ = \ \frac{1}{2} \tag{1.31} $$

resulting from the condition $g_{\mu\nu} \frac{dx^\mu}{ds} \ \frac{dx^\nu}{ds} = 1$. The proper acceleration squared is now

$$ (\frac{d^2x^0}{ds^2})^2 \ - \ (\frac{d^2x^1}{ds^2})^2 \ = \ F(x^0)^2 \ - \ H(x^1)^2 \ = \ - \ g^2(x^0, x^1) \tag{1.32} $$

furnishing the spacetime dependent conformal factor

$$ \Omega^2 \ = \ 1 \ - \ \frac{g^2(x^0, x^1)}{M^2} \tag{1.33} $$

and leading to a more complicated expression for the scalar curvature. To conclude, when the worldline of an observer (particle) is *not* one of uniform acceleration $g = constant$, we have $\frac{d^2x^\mu}{ds^2} \neq g^2 x^\mu$ and the conformal factor Ω^2 is a more complicated function leading to different expressions for the scalar curvature $\tilde{R}$. Thus, the spacetime geometry becomes *observer dependent* as the authors [18] envisioned long ago.

Liouville's equation is the nonlinear partial differential equation satisfied by the conformal factor Ω with respect to a flat Euclidean metric in two dimensions such that the scaled metric $\Omega^2(dx^2 + dy^2)$ leads to a surface of *constant* Gaussian curvature K. One can also obtain the Minkowski version of Liouville's equation directly from eq-(1.26) in $d = 2$, when $R = 0$, giving

$$ \nabla_a \nabla^a ln \ \Omega \ = \ - \ \frac{K}{2} \ \Omega^2 \tag{1.34} $$

in Euclidean signature one can use complex coordinates $z = x^1 + ix^0, \bar{z} = x^1 - ix^0$ so that the most general solution for Ω^2 is given in terms of two arbitrary meromorphic functions $\rho(z), \bar{\rho}(\bar{z})$, and its derivatives, as follows

$$ \Omega^2 \ = \ \left(\ 4 \ \frac{\rho'(z) \ \bar{\rho}'(\bar{z})}{(\ 1 \ + \ K \ \rho(z) \ \bar{\rho}(\bar{z}) \ )^2} \ \right)^{\frac{1}{2}} \tag{1.35} $$

thus an *analytic* continuation (via a Wick rotation) of the solutions (1.35) will give the expression for the non-uniform proper acceleration squared $g^2(x^0, x^1) = M^2(1 - \Omega^2)$ associated with the surfaces of constant Gaussian curvature.

A more rigorous procedure requires to study the geometry in curved (co) tangent spaces following the Lagrange-Finsler and Hamilton-Cartan spaces [7], [8]. More recently a study of gravity in curved phase spaces was performed by [19]. The geometry of the (co) tangent bundle of spacetime was studied via the introduction of nonlinear connections associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. The curvature tensors in the (co) tangent bundle of spacetime were explicitly constructed leading to the analog of the Einstein vacuum field equations. The geometry of Hamilton Spaces associated with curved phase spaces followed. An explicit construction of a gauge theory of gravity in the $8D$ co-tangent bundle T^*M of spacetime was provided, and based on the gauge group $SO(6,2) \times_s R^8$ which acts on the tangent space to the cotangent bundle $T_{(x,p)}T^*M$ at each point $(\mathbf{x}, \mathbf{p})$. Several gravitational actions associated with the geometry of curved phase spaces were presented.

2 Maximal Acceleration and Strings in Tangent Bundle Backgrounds

The Polyakov bosonic string action is

$$S \ = \ T \int d\sigma \ d\tau \ \sqrt{|det \ h_{\alpha\beta}|} \ h^{\alpha\beta} \ \partial_\alpha x^\mu \ \partial_\beta x^\nu \ g_{\mu\nu} \tag{2.1}$$

T is the string tension. $h_{\alpha\beta}(\sigma,\tau)$ is an auxiliary two-dim worldsheet metric $\alpha,\beta = 1,2$. $x^\mu(\sigma,\tau), \mu,\nu = 1,2,\ldots,D$ are the D embedding functions of the two-dim string worldsheet into the D-dim target spacetime background whose metric is $g_{\mu\nu}(x^\rho)$. In the conformal gauge $h_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$ the Polyakov action becomes

$$S \ = \ T \int d\sigma \ d\tau \ g_{\mu\nu} \ \left(\frac{\partial x^\mu}{\partial\tau} \ \frac{\partial x^\nu}{\partial\tau} \ - \ \frac{\partial x^\mu}{\partial\sigma} \ \frac{\partial x^\nu}{\partial\sigma} \right) \tag{2.2}$$

In flat target spacetime backgrounds $g_{\mu\nu} = \eta_{\mu\nu}$, the equations of motion associated with the Polyakov action in the conformal gauge are

$$\frac{\delta S}{\delta x^\mu} \ = \ 0 \ \Rightarrow \ \frac{\partial^2 x^\mu}{\partial\tau^2} \ - \ \frac{\partial^2 x^\mu}{\partial\sigma^2} \ = \ 0, \quad \mu = 1,2,....,D \tag{2.3}$$

in curved target spacetime backgrounds the equations of motion are more complicated and involve the Christoffel connection $\Gamma^\mu_{\nu\rho}$ of the D-dim target spacetime background [1]

$$\frac{\partial^2 x^\mu}{\partial\tau^2} \ - \ \frac{\partial^2 x^\mu}{\partial\sigma^2} \ + \ \Gamma^\mu_{\nu\rho} \ (\frac{\partial x^\mu}{\partial\tau} \ + \ \frac{\partial x^\mu}{\partial\sigma}) \ (\frac{\partial x^\rho}{\partial\tau} \ + \ \frac{\partial x^\rho}{\partial\sigma}) \ = \ 0 \tag{2.4}$$

[1]In flat spaces the connection coefficients are not necessarily zero if one uses curvilinear coordinates

In addition to the equations of motion one must include the constraint equations resulting from a variation of the Polyakov action with respect to the auxiliary world sheet metric metric $h_{\alpha\beta}$, prior to fixing the conformal gauge,

$$g_{\mu\nu} \left(\frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau} + \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \sigma} \right) = 0 \tag{2.5}$$

$$g_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \sigma} = 0 \tag{2.6}$$

In flat target spacetime backgrounds $g_{\mu\nu} = \eta_{\mu\nu}$, the most general solutions to eqs-(2.3, 2.5, 2.6) are of the form

$$x^\mu(\sigma, \tau) = f^{(\mu)}(\sigma + \tau) + h^{(\mu)}(\sigma - \tau) \tag{2.7}$$

where $f^{(\mu)}(\sigma + \tau), h^{(\mu)}(\sigma - \tau)$ are D arbitrary functions of $\sigma \pm \tau$, respectively.

Of particular physical interest are the solutions in the $D = 2$ case given by

$$f^{(0)}(\sigma + \tau) = \frac{L}{2} e^{\kappa(\sigma+\tau)}, \quad h^{(0)}(\sigma - \tau) = - \frac{L}{2} e^{\kappa(\sigma-\tau)} \tag{2.8a}$$

$$f^{(1)}(\sigma + \tau) = \frac{L}{2} e^{\kappa(\sigma+\tau)}, \quad h^{(1)}(\sigma - \tau) = \frac{L}{2} e^{\kappa(\sigma-\tau)} \tag{2.8b}$$

and from which we infer that

$$x^0 = L e^{\kappa\sigma} sinh(\kappa\tau), \quad x^1 = L e^{\kappa\sigma} cosh(\kappa\tau), \quad \kappa > 0, \quad L > 0 \tag{2.9}$$

the spatial variable σ ranges from $-\frac{\lambda}{2} \leq \sigma \leq \frac{\lambda}{2}$ where the proper string length is λ.

The physical relevance of the above solutions is that they have the same form as a continuous family of hyperbolic worldlines

$$x^0 = \xi(\sigma) \; sinh(\eta(\tau)), \quad x^1 = \xi(\sigma) \; cosh(\eta(\tau)) \tag{2.10}$$

like those comprising the Rindler spacetime coordinates. In a Rindler frame of reference (accelerated frame) the solutions (2.9) to the string equations of motion are simply

$$\xi(\sigma, \tau) = L e^{\kappa\sigma}, \quad \eta(\sigma, \tau) = \kappa \, \tau \tag{2.11}$$

In the inertial frame of reference involving the coordinates x^0, x^1 one has an open string extended along the x^1 axis whose extreme ends (and all the intermediate points) have $different$ accelerations along the x^1 axis. There is an acceleration $gradient$ along the points of the string. One may envision two heavy quarks with the same mass at the end points of the open string lying along the x^1-axis and subjected to $different$ accelerations (forces). The left end point has a greater acceleration than the right end point. As time evolves the string length gets *larger*. This may seem paradoxical, and we shall explain *why* it is so at the end of this section, contrary to the naive intuition.

The picture is different in the Rindler accelerated frame displayed by eqs-(2.11). The string just *sits* idly between the fixed end points $\xi_1 = L\, e^{-\frac{\kappa\lambda}{2}}; \xi_2 \ = \ L\, e^{\frac{\kappa\lambda}{2}}$ and the (dimensionless) temporal coordinate $\eta = \kappa\tau$ simply keeps ticking away linearly with the string's clock time τ. This is the analogy of a particle sitting at rest with a clock ticking away.

These results (2.9) are very *different* from the findings of [17] where the line joining the two heavy masses (at the end points of the string) is always parallel to the y-axis (x^2-axis). These masses were both subjected to the *same* acceleration (forces) along the horizontal x^1 direction. The spatial shape of the string at any given time is a catenary (hyperbola) of finite length. The string dynamics can be envisioned as an accelerated catenary with an acceleration gradient from the center point to the end points (the center of the string had the greatest acceleration).

One can now embed the two-dim string's worldsheet into a four-dim background corresponding to the four-dim tangent bundle TM_2 of a 2-dim spacetime M_2. If one chooses for the tangent bundle metric the one corresponding to the maximal acceleration corrections, the embedding condition reads

$$h_{\alpha\beta}\ d\sigma^\alpha\ d\sigma^\beta \ = \ (1 - \frac{g^2}{M^2})\ g_{\mu\nu}\ dx^\mu\ dx^\nu \tag{2.12}$$

from which one obtains the induced worldsheet metric as a result of the embedding

$$h_{\alpha\beta} \ = \ (1 - \frac{g^2}{M^2})\ g_{\mu\nu}\ \frac{\partial x^\mu}{\partial\sigma^\alpha}\ \frac{\partial x^\mu}{\partial\sigma^\beta} \tag{2.13}$$

let us now use for embedding functions $x^\mu(\sigma,\tau)$ those resulting from the solutions to the string equations of motion in flat target backgrounds $g_{\mu\nu} = \eta_{\mu\nu}$ and given by eqs-(2.9). Eqs-(2.9, 2.13) will then yield

$$h_{00} \ = \ (1 - \frac{g^2}{M^2})\ \left((\frac{\partial x^0}{\partial\tau})^2 \ - \ (\frac{\partial x^1}{\partial\tau})^2 \right) \ = \ (1 - \frac{g^2}{M^2})\ L^2\ \kappa^2\ e^{2\kappa\sigma} \tag{2.14a}$$

$$h_{11} \ = \ (1 - \frac{g^2}{M^2})\ \left((\frac{\partial x^0}{\partial\sigma})^2 \ - \ (\frac{\partial x^1}{\partial\sigma})^2 \right) \ = \ - \ (1 - \frac{g^2}{M^2})\ L^2\ \kappa^2\ e^{2\kappa\sigma} \tag{2.14b}$$

hence, the induced worldsheet interval as a result of the embedding is

$$(ds)^2 \ = \ (1 - \frac{g^2}{M^2})\ L^2\ \kappa^2\ e^{2\kappa\sigma}\ (\ (d\tau)^2 \ - \ (d\sigma)^2\ ) \tag{2.15}$$

one still has a conformally flat metric, whose conformal factor is [2]

[2]It was a conformally flat worldsheet metric $h_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$ in the Polyakov that led to the string equations of motion and its solutions.

$$(1 - \frac{g^2}{M^2})\ L^2\ \kappa^2\ e^{2\kappa\sigma} \tag{2.16}$$

The metric has a *null* horizon at $g = M$.

Before continuing we must include the role of the remaining two coordinates $\dot{x}^0 = \frac{dx^0}{ds}; \dot{x}^1 = \frac{dx^1}{ds}$ of the four-dim tangent bundle TM_2 corresponding to the motion of a uniformly accelerated observer $\mathcal{O}$ in a two-dim spacetime M_2 with proper constant acceleration g and whose affine parameter associated with its hyperbolic worldline is s. A careful analysis reveals that the condition $\eta = gs = \kappa\tau$, deduced from eqs-(1.21a), will allow us to *replace* $cosh(\kappa\tau), sinh(\kappa\tau)$ for $cosh(gs), sinh(gs)$, respectively, in the string solutions (2.9) and perform the derivatives

$$\dot{x}^0 \ = \ \frac{dx^0}{ds} \ = \ g\ x^1 \ = \ g\ L\ e^{\kappa\sigma}\ cosh(\kappa\tau),$$

$$\dot{x}^1 \ = \ \frac{dx^1}{ds} \ = \ g\ x^0 \ = \ g\ L\ e^{\kappa\sigma}\ sinh(\kappa\tau) \tag{2.17}$$

Therefore, eqs-(2.9) and eqs-(2.17) are the relevant solutions we are looking for the accelerated open string in the tangent bundle TM_2. They describe the four embedding coordinates $x^0(\sigma,\tau), x^1(\sigma,\tau), \dot{x}^0(\sigma,\tau), \dot{x}^1(\sigma,\tau)$ corresponding to an accelerated open string moving in a $flat$ target tangent bundle TM_2 background with metric $g_{\mu\nu} = \eta_{\mu\nu}; g_{\dot{\mu}\dot{\nu}} = \frac{\eta_{\mu\nu}}{M^2}, \mu, \nu = 0, 1$.

We shall examine carefully the solutions (2.9, 2.17) to the string equations of motion in the tangent bundle TM_2. g is the uniform proper acceleration of a particle (let us say an "observer" $\mathcal{O}$). Its hyperbolic world line $x^\mu(g,s)$ will *coincide* with the hyperbolic world line $x^\mu(\sigma,\tau)$ of a certain point inside the interior of the string if $\eta = \kappa\tau = gs$, and there is a value of $\sigma = \sigma_*$ such that $\xi(\sigma_*) = Le^{\kappa\sigma_*} = \frac{1}{g}$. In particular, there is a null horizon in the string's worldsheet metric (2.15) when there exists a value of $\sigma = \sigma_h$ such

$$\xi_h \ = \ Le^{\kappa\sigma_h} \ = \ \frac{1}{g} \ = \ \frac{1}{M} \ \Rightarrow \ \sigma_h \ = \ - \ \frac{1}{\kappa}\ ln(LM), \quad \kappa > 0, \quad L > 0 \tag{2.18}$$

The σ spatial variable of an open string of proper length λ ranges from $-\frac{\lambda}{2} \leq \sigma \leq \frac{\lambda}{2}$. If σ_h lies within this interval : $-\frac{\lambda}{2} < -\frac{1}{\kappa} ln(LM) < \frac{\lambda}{2}$, there are two causally *disconnected* regions in the open string worldsheet. Namely, there is a portion of the open string worldsheet that lies to the left (inside) of the horizon and another portion which lies to the right (outside) of the horizon.

Whereas the string worldsheet lies outside the horizon if $-\frac{1}{\kappa} ln(LM) < -\frac{\lambda}{2}$, and now there are *no* causally disconnected regions in the open string worldsheet. In this case the open string worldsheet lies entirely to the right (outside) of the horizon. In the limiting case when the horizon lies at the very left-end of the open string one has

$$\frac{1}{\kappa}\ ln(LM) \ = \ \frac{\lambda}{2}, \quad \kappa > 0, \quad L > 0 \tag{2.19}$$

from which one learns that a *finite* open string proper length λ is *not* compatible with an *infinite* maximal acceleration value $M \rightarrow \infty$. Since the proper string length $\lambda \geq 0$ cannot be negative one infers from eq-(2.19) that the maximal acceleration must obey $M \geq \frac{1}{L} \Rightarrow L \geq \frac{1}{M}$. In the point particle limit case $\lambda = 0$, eq-(2.19) furnishes a lower bound for the maximal acceleration $M = \frac{1}{L}$.

The spatial separation between the extreme ends of the open string at $\tau = 0$ is

$$\Delta x(\tau = 0) \ = \ x(\frac{\lambda}{2}, \tau = 0) \ - \ x(-\frac{\lambda}{2}, \tau = 0) \ = \ 2L \ sinh(\frac{\kappa\lambda}{2}) \qquad (2.20)$$

Because the string at $\tau = x^0 = 0$ is at *rest*, [3] one may equate $\Delta x(\tau = 0)$ to the string proper length λ, and which in turn, it allows us to solve for κ as follows

$$sinh(\frac{\kappa\lambda}{2}) \ = \ \frac{\lambda}{2L} \ \Rightarrow \ \frac{\kappa\lambda}{2} \ = \ sinh^{-1}(\frac{\lambda}{2L}) \ =$$

$$ln \left(\frac{\lambda}{2L} \ + \ \sqrt{1 + (\frac{\lambda}{2L})^2} \right) \ \Rightarrow$$

$$\kappa \ = \ \frac{2}{\lambda} \ ln \left(\frac{\lambda}{2L} \ + \ \sqrt{1 + (\frac{\lambda}{2L})^2} \right) \qquad (2.21)$$

In this limiting case, when the string barely lies outside the horizon, upon inserting eq-(2.21) into eq-(2.19) yields

$$ln(LM) \ = \ \frac{\kappa\lambda}{2} \ = \ ln \left(\frac{\lambda}{2L} \ + \ \sqrt{1 + (\frac{\lambda}{2L})^2} \right) \qquad (2.22$$

leading finally to

$$LM \ = \ \frac{\lambda}{2L} \ + \ \sqrt{1 + (\frac{\lambda}{2L})^2} \ \geq \ 1 \ \Rightarrow \ M \ \geq \ \frac{1}{L} \qquad (2.23)$$

if one chooses the plus sign under the square root and due to $L > 0$.

One should notice that it is *not* required to set $L = \lambda$. The upshot of having $L \neq \lambda$ is that after setting $\lambda = 0$ in eq-(2.23) it gives $LM = 1 \Rightarrow M = \frac{1}{L}$, so it is still possible to have a *finite* maximal acceleration $M \neq \infty$ by simply choosing $L \neq 0$, despite having a zero proper string length $\lambda = 0$. Therefore, when the string collapses to a point, the point particle can still move with a *finite* maximal acceleration. If $\lambda \neq 0$, and $L \neq \lambda$, the analysis of the above equations reveals that the finite maximal acceleration M is greater than the acceleration of all the points inside the string (except the left-end point). In particular, one has $M > \frac{1}{L}$ where $\frac{1}{L}$ is the acceleration of the center point ($\sigma = 0$) of the string. Similar findings were obtained by [17].

[3]The velocity of all the points inside the string is zero at $\tau = x^0 = 0$, despite that there is acceleration

If one were to choose $L = \lambda$, we have a $finite$ maximal acceleration $M \neq \infty$ associated to a *nonzero* proper string length $\lambda \neq 0$. However, in this case, when $\lambda = 0$ (point particle) eq-(2.23) yields $M > \frac{1}{L} = \frac{1}{\lambda}$ leading to an infinite maximal acceleration $M = \infty$.

Based on these findings, it is preferable to have $L \neq \lambda$ so that one can still have a $finite$ maximal acceleration. The Planck scale L_P has been postulated to be the minimal length scale in nature [20] yielding a finite maximal acceleration of $M = \frac{1}{L_P}$ (in natural $c = 1$ units) . Eq-(2.23) implies in this case that $M = \frac{1}{L_P} \geq \frac{1}{L} \Rightarrow L_P \leq L$. [4]

The string spatial length at a time τ is

$$\mathcal{L}(\tau) \ = \ 2L \ sinh(\frac{\kappa\lambda}{2}) \ cosh(\kappa\tau) \tag{2.24}$$

when $\lambda = 0 \Rightarrow \mathcal{L}(\tau) = 0$ as expected when the string collapses to a point particle. The results for the string length in eq-(2.24) are exact and simpler than those obtained by [17] involving elliptic functions.

We discussed earlier that in the Rindler accelerated frame of reference the open string just *sits* idly between the fixed points $\xi_1 = L\, e^{-\frac{\kappa\lambda}{2}}$ and $\xi_2 \ = \ L\, e^{\frac{\kappa\lambda}{2}}$. In that frame of reference the string length is *constant* and given by eq-(2.20). Since the string energy and length remain constant, the string tension (given by the energy per unit length) is constant in the Rindler accelerated frame of reference, as it should.

Matters are quite different in the inertial frame of reference (x^0, x^1 axes). The spatial string length *increases* with time as displayed in eqs-(2.24). Since the string is being accelerated (due to the external forces acting on the two masses at the end points), with an acceleration gradient along its points, the string's energy also keeps increasing with time such that the energy per unit length (tension) still remains constant.

Eq-(1.21a) will help us find the reason why the string stretches with time in the inertial frame, and the left end point will never catch up with the right end point despite $g_1 > g_2$. This is because the proper times s_1, s_2 associated with the respective hyperbolic lines of the two end points obey $s_1 < s_2$, such that $g_1 s_1 = g_2 s_2$. One "must give time to time" in oder for one end point to catch up with the other, which is not the case here since $s_1 < s_2$.

In essence, to conclude, the worldsheet associated with the accelerated open string described in this work envisages a continuum family of worldlines of accelerated points, whose proper accelerations are $g(\sigma) = \frac{1}{\xi(\sigma)} = (Le^{\kappa\sigma})^{-1}$. $\eta = \kappa\tau = gs$, where g, s *vary* along the hyperbolic worldlines of the points of the string, but κ is *constant*. It is when one embeds the two-dim string worldsheet into the four-dim tangent bundle TM_2 background (associated with a uniformly accelerated point particle/observer) that the effects of the maximal acceleration M are manifested The induced worldsheet interval as a result of the embedding in TM_2

[4] If the minimal Planck scale postulate holds one cannot have $\lambda = 0$. Nevertheless one can still view a "point" particle as the center of mass of the string.

$$(ds)^2 \ = \ (1 - \frac{g^2}{M^2}) \ L^2 \ \kappa^2 \ e^{2\kappa\sigma} \ (\ (d\tau)^2 \ - \ (d\sigma)^2 \ ) \qquad (2.25)$$

has a *null* horizon at $g = M$. It is the presence of this null horizon that limits the acceleration values of the points inside string. If the string crosses the null horizon, some of its points will exceed the maximal acceleration value and that portion of the string will become causally disconnected from the rest of string outside the horizon.

The study of accelerated strings in *curved* backgrounds, like those reflected by the metric of eq-(1.24), is more difficult. The scalar curvature (1.27) has a true singularity at the horizon location. One could generalize the results of this work to the case of strings propagating in the cotangent bundle T^*M (phase spaces). Born Reciprocal Relativity postulates a maximal proper force which one could fix to be $m_P c^2/L_P$, and given in terms of the Planck length, mass and speed of light. A photon has zero mass, and since the proper time of a photon path in spacetime is null, one needs another affine parameter to describe the motion of a photon. Hence it is plausible to have an infinite maximal proper acceleration for a photon such that $m_{photon} a_{photon} = m_P c^2/L_P = finite$. Consequently one could have an infinite maximal acceleration and finite maximal speed for the photon, while still having a finite maximal proper force.

To finalize, it is worth mentioning that, recently, Dasgupta [21] re-investigated the Bogoliubov transformations which relate the Minkowski inertial vacuum to the vacuum of an accelerated observer. He implemented the transformation using a non-unitary operator used in formulations of irreversible systems by Prigogine [22]. An attempt was discussed to generalize Quantum Field Theory (QFT) for accelerated frames using this new connection to Prigogine transformations. It is warranted to build a generalized QFT in accelerated frames which is compatible with the Quaplectic group transformations in Born's Reciprocal Relativity [1]. This may shed some light into the resolution of the black hole information paradox by recurring to *novel* physical principles and which are beyond the many current proposals based on standard QFT in curved Riemannian spacetimes. This generalized QFT in accelerated frames would allow for pure states to evolve into mixed states so that the Hawking black hole emission process would no longer be paradoxical.

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