# BEAL's Conjecture: A Complete Proof


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### Abstract

In 1997, Andrew Beal [1] announced the following conjecture : *Let $A, B, C, m, n$, and $l$ be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then $A, B$, and $C$ have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with $p, q$ integers depending of $A^m, B^n$ and $C^l$. We resolve $x^3 - px + q = 0$ and we obtain the three roots $x_1, x_2, x_3$ as functions of $p, q$ and a parameter $\theta$. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that $x_1, x_2, x_3$ are integers. Two numerical examples are given.



**Keywords:** Prime numbers, divisibility, roots of polynomials of third degree.


*O my Lord! Increase me further in knowledge.*

(Holy Quran, Surah Ta Ha, 20:114.)

***To my wife Wahida***

## Contents

# 1 Introduction

In 1997, Andrew Beal [1] announced the following conjecture :

***Conjecture*** 1.1. *Let $A, B, C, m, n,$ and $l$ be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1}$$

*then $A, B,$ and $C$ have a common factor.*

In this paper, we give a complete proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots $A^m, B^n$ and $-C^l$ with the condition (1.1). The paper is organized as follows. In Section 2 of preliminaries, we begin with the trivial case where $A^m = B^n$. Then we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. We express the three roots of $P(x) = x^3 - px + q = 0$ in function of two parameters $\rho, \theta$ that depend of $A^m, B^n, C^l$. The Section 3 is the main part of the paper. We write that $A^{2m} = \dfrac{4p}{3}cos^2\dfrac{\theta}{3}$. As $A^{2m}$ is an integer, it follows that $cos^2\dfrac{\theta}{3}$ must be written as $\dfrac{a}{b}$ where $a, b$ are two positive coprime integers. We discuss the conditions of divisibility of $p, a, b$ so that the expression of $A^{2m}$ is an integer. Depending on each individual case, we obtain that $A, B, C$ have or not a common factor. In the last Section, two numerical examples are presented.

# 2 Preliminaries

We begin with the trivial case when $A^m = B^n$. The equation (1.1) becomes:

$$2A^m = C^l \tag{2.1}$$

then $2|C^l \Longrightarrow 2|C \Longrightarrow \exists\, c \in \mathbb{N}^*/\ C = 2c$, it follows $2A^m = 2^l c^l \Longrightarrow A^m = 2^{l-1}c^l$. As $l > 2$, then $2|A^m \Longrightarrow 2|A \Longrightarrow 2|B^n \Longrightarrow 2|B$. The conjecture (1.1) is verified.

We suppose in the following that $A^m > B^n$.

## 2.1 General Case

Let $m, n, l \in \mathbb{N}^*\ > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \tag{2.2}$$

We call:

$$\begin{aligned} P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \\ +x[A^mB^n - C^l(A^m + B^n)] + C^lA^mB^n \end{aligned} \tag{2.3}$$

Using the equation (2.2), $P(x)$ can be written:

$$\boxed{P(x) = x^3 + x[A^mB^n - (A^m + B^n)^2] + A^mB^n(A^m + B^n)} \tag{2.4}$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^m B^n \tag{2.5}$$

$$q = A^m B^n (A^m + B^n) \tag{2.6}$$

As $A^m \neq B^n$, we have :

$$p > (A^m - B^n)^2 > 0 \tag{2.7}$$

Equation (2.4) becomes:

$$P(x) = x^3 - px + q \tag{2.8}$$

Using the equation (2.3), $P(x) = 0$ has three different real roots : $A^m, B^n$ and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \tag{2.9}$$

To resolve (2.9) let:

$$x = u + v \tag{2.10}$$

Then $P(x) = 0$ gives:

$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \Longrightarrow u^3 + v^3 + (u+v)(3uv - p) + q = 0 \tag{2.11}$$

To determine $u$ and $v$, we obtain the conditions:

$$u^3 + v^3 = -q \tag{2.12}$$

$$uv = p/3 > 0 \tag{2.13}$$

Then $u^3$ and $v^3$ are solutions of the second ordre equation:

$$X^2 + qX + p^3/27 = 0 \tag{2.14}$$

Its discriminant $\Delta$ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\overline{\Delta}}{27} \tag{2.15}$$

Let:

$$\begin{aligned} \overline{\Delta} = 27q^2 - 4p^3 &= 27(A^m B^n (A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \\ &= 27A^{2m} B^{2n} (A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \end{aligned} \tag{2.16}$$

Noting :

$$\alpha = A^m B^n > 0 \tag{2.17}$$

$$\beta = (A^m + B^n)^2 \tag{2.18}$$

we can write (2.16) as:

$$\overline{\Delta} = 27\alpha^2 \beta - 4(\beta - \alpha)^3 \tag{2.19}$$

As $\alpha \neq 0$, we can also rewrite (2.19) as :

$$\overline{\Delta} = \alpha^3 \left( 27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3 \right) \tag{2.20}$$

We call $t$ the parameter :

$$t = \frac{\beta}{\alpha} \tag{2.21}$$

$\overline{\Delta}$ becomes :

$$\overline{\Delta} = \alpha^3(27t - 4(t-1)^3) \tag{2.22}$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3 \tag{2.23}$$

Since $\alpha > 0$, the sign of $\overline{\Delta}$ is also the sign of $y(t)$. Let us study the sign of $y$. We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t) \tag{2.24}$$

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of $y$ is given below:

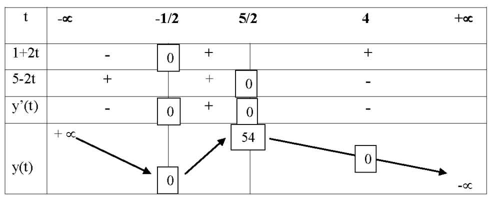


Fig. 1: The table of variation

The table of the variations of the function $y$ shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in ]0, 4[ \Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m + B^n)^2 > 4\alpha = 4A^mB^n \tag{2.25}$$

Then $y < 0 \Longrightarrow \overline{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (2.14) does not have real solutions $u^3$ and $v^3$. Let us find the solutions $u$ and $v$ with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

## 2.2 Demonstration

*Proof.* The solutions of (2.14) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{2.26}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \tag{2.27}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{2.28}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \tag{2.29}$$

Writing $X_1$ in the form:

$$X_1 = \rho e^{i\theta} \tag{2.30}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}} \tag{2.31}$$

$$\text{and } sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \tag{2.32}$$

$$cos\theta = -\frac{q}{2\rho} < 0 \tag{2.33}$$

Then $\theta\,[2\pi] \in ]+\frac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{2.34}$$

and:

$$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}} \tag{2.35}$$

hence the expression of $X_2$:

$$X_2 = \rho e^{-i\theta} \tag{2.36}$$

Let:

$$u = re^{i\psi} \tag{2.37}$$

$$\text{and } j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{2.38}$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j} \tag{2.39}$$

$j$ is a complex cubic root of the unity $\Longleftrightarrow j^3 = 1$. Then, the solutions $u$ and $v$ are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \tag{2.40}$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \tag{2.41}$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \tag{2.42}$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \tag{2.43}$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \tag{2.44}$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \tag{2.45}$$

We may now choose $u_k$ and $v_h$ so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \tag{2.46}$$

$$v_2 = \overline{u_2} \tag{2.47}$$

$$v_3 = \overline{u_3} \tag{2.48}$$

We obtain as real solutions of the equation (2.11):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\frac{\theta}{3} > 0 \tag{2.49}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\tfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) < 0 \tag{2.50}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\tfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) > 0 \tag{2.51}$$

We compare the expressions of $x_1$ and $x_3$, we obtain:

$$2\sqrt[3]{p}cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right)$$

$$3cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt{3}sin\tfrac{\theta}{3} \tag{2.52}$$

As $\frac{\theta}{3} \in ]+\frac{\pi}{6}, +\frac{\pi}{3}[$, then $sin\frac{\theta}{3}$ and $cos\frac{\theta}{3}$ are $> 0$. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{2.53}$$

which is true since $\frac{\theta}{3} \in ]+\frac{\pi}{6}, +\frac{\pi}{3}[$ then $x_1 > x_3$. As $A^m, B^n$ and $-C^l$ are the only real solutions of (2.9), we consider, as $A^m$ is supposed great than $B^n$, the expressions:

$$\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3} \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases} \tag{2.54}$$

□

## 3 Proof of the Main Theorem

**Main Theorem:** *Let $A, B, C, m, n,$ and $l$ be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{3.1}$$

*then $A, B,$ and $C$ have a common factor.*

*Proof.* $A^m = 2\sqrt[3]{\rho}cos\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3}$ is an integer. But:

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{3.2}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{3.3}$$

As $A^{2m}$ is an integer, and $p$ is an integer then $cos^2\dfrac{\theta}{3}$ must be written in the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{3.4}$$

with $b \in \mathbb{N}^*$, for the last condition $a \in \mathbb{N}^*$ and $a, b$ coprime.

## 3.1 Case $cos^2\dfrac{\theta}{3} = \dfrac{1}{b}$

We obtain :

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{3.5}$$

As $\dfrac{1}{4} < cos^2\dfrac{\theta}{3} < \dfrac{3}{4} \Rightarrow \dfrac{1}{4} < \dfrac{1}{b} < \dfrac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

### 3.1.1 Case $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

### 3.1.2 Case $b = 2$

$b = 2 \Rightarrow A^{2m} = p.\dfrac{4}{3}.\dfrac{1}{2} = \dfrac{2.p}{3} \Rightarrow 3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{2p}{3} = 2.p' \tag{3.6}$$

But :

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{1}{2}\right) = \frac{p}{3} = \frac{3p'}{3} = p' \tag{3.7}$$

On the one hand:

$$A^{2m} = (A^m)^2 = 2p' \Rightarrow 2|p' \Rightarrow p' = 2p"^2 \Rightarrow A^{2m} = 4p"^2$$
$$\Rightarrow A^m = 2p" \Rightarrow 2|A^m \Rightarrow 2|A$$

On the other hand:
$B^nC^l = p' = 2p"^2 \Rightarrow 2|B^n$ or $2|C^l$. If $2|B^n \Rightarrow 2|B$. As $C^l = A^m + B^n$ and $2|A$ and $2|B$, it follows $2|A^m$ and $2|B^n$ then $2|(A^m + B^n) \Rightarrow 2|C^l \Leftrightarrow 2|C$.

Then, we have : $A$,$B$ and $C$ solutions of (2.2) have a common factor. Also if $2|C^l$, we obtain the same result : $A$,$B$ and $C$ solutions of (2.2) have a common factor.

### 3.1.3 Case $b = 3$

$b = 3 \Rightarrow A^{2m} = p.\dfrac{4}{3}.\dfrac{1}{3} = \dfrac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p'$ with $p' \neq 1$ since $9 \ll p$ then $A^{2m} = 4p' \Longrightarrow p'$ is not a prime. Let $\mu$ a prime with $\mu|p' \Rightarrow \mu|A^{2m} \Rightarrow \mu|A$.

On the other hand:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 5p'$$

Then $\mu|B^n$ or $\mu|C^l$. If $\mu|B^n \Rightarrow \mu|B$. As $C^l = A^m + B^n$ and $\mu|A$ and $\mu|B$, it follows $\mu|A^m$ and $\mu|B^n$ then $\mu|(A^m + B^n) \Rightarrow \mu|C^l \Longrightarrow \mu|C$.

Then, we have : $A$,$B$ and $C$ solutions of (2.2) have a common factor. Also if $\mu|C^l$, we obtain the same result : $A$,$B$ and $C$ solutions of (2.2) have a common factor.

## 3.2 Case $a > 1$, $cos^2\frac{\theta}{3} = \frac{a}{b}$

That is to say:

$$cos^2\frac{\theta}{3} = \frac{a}{b} \tag{3.8}$$

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{3.9}$$

and $a, b$ verify one of the two conditions:

$$\boxed{\{3|p \quad and \quad b|4p\}} \quad \text{or} \quad \boxed{\{3|a \quad and \quad b|4p\}} \tag{3.10}$$

and using the equation (2.35), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{3.11}$$

In these conditions, respectively, $A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (3.10).

### 3.2.1 Hypothesis: $\{3|p \quad and \quad b|4p\}$

**3.2.1.1. Case $b = 2$ and $3|p$ :** $3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a \tag{3.12}$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow a < 2 \Rightarrow a = 1 \tag{3.13}$$

But $a > 1$ then the case $b = 2$ and $3|p$ is impossible.

**3.2.1.2. Case $b = 4$ and $3|p$ :** We have $3|p \Longrightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a \tag{3.14}$$

and:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2 \tag{3.15}$$

But $a, b$ are coprime. Then the case $b = 4$ and $3|p$ is impossible.

**3.2.1.3. Case:** $b \neq 2, b \neq 4$**,** $b|p$ **and** $3|p$ **:** As $3|p$ then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4\times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b} \tag{3.16}$$

We consider the case: $b|p' \Longrightarrow p' = bp"$ and $p" \neq 1$ (if $p" = 1$, then $p = 3b$, see sub-paragraph **II. Case k'=1** of paragraph **3.2.1.8**). Hence :

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{3.17}$$

Let us calculate $B^nC^l$:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = b.p".\frac{3b-4a}{b} = p".(3b-4a) \tag{3.18}$$

Finally, we have the two equations:

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{3.19}$$

$$B^nC^l = p".(3b-4a) \tag{3.20}$$

**I. Case** $p"$ **is prime:**
From (3.19), $p"|A^{2m} \Rightarrow p"|A^m \Rightarrow p"|A$. From (3.20), $p"|B^n$ or $p"|C^l$. If $p"|B^n \Rightarrow p"|B$, as $C^l = A^m + B^n \Rightarrow p"|C^l \Rightarrow p"|C$. If $p"|C^l \Rightarrow p"|C$, as $B^n = C^l - A^m \Rightarrow p"|B^n \Rightarrow p"|B$.

Then $A$,$B$ and $C$ solutions of (2.2) have a common factor.

**II. Case** $p"$ **is not prime:**
Let $\lambda$ one prime divisor of $p"$. From (3.19), we have :

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{3.21}$$

From (3.20), as $\lambda|p"$ we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or}\ \lambda|C^l \tag{3.22}$$

If $\lambda|B^n$, $\lambda$ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also :

$$\lambda|C^l \quad \text{as}\,\lambda\,\text{is prime, then} \quad \lambda|C \tag{3.23}$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$.

Then: $A, B$ and $C$ solutions of (2.2) have a common factor.

Let us verify the condition (3.11) given by:

$$b < 4a < 3b$$

In our case, the last equation becomes:

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^mB^n \tag{3.24}$$

The condition $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ is verified.
If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^mB^n - B^{2n} > 0$$

We put $Q(Y) = 2Y^2 - B^nY - B^{2n}$, the roots of $Q(Y) = 0$ are $Y_1 = -\frac{B^n}{2}$ and $Y_2 = B^n$. $Q(Y) > 0$ for $Y < Y_1$ and $Y > Y_2 = B^n$. In our case, we take $Y = A^m$. As $A^m > B^n$ then $p < 3A^{2m}$ is verified. Then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify $A^m > B^n$ which is true.

**3.2.1.4. Case $b = 3$ and $3|p$ :** As $3|p \Longrightarrow p = 3p'$ and we write :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4\times 3p'}{3}\frac{a}{3} = \frac{4p'a}{3} \tag{3.25}$$

As $A^{2m}$ is an integer and that $a$ and $b$ are coprime and $cos^2\dfrac{\theta}{3}$ can not be one in reference to the equation (2.34), then we have necessary $3|p' \Longrightarrow p' = 3p"$ with $p" \neq 1$, if not $p = 3p' = 3\times 3p" = 9$ but $p = A^{2m} + B^{2n} + A^mB^n > 9$, the hypothesis $p" = 1$ is impossible, then $p" > 1$. hence:

$$A^{2m} = \frac{4p'a}{3} = \frac{4\times 3p"a}{3} = 4p"a \tag{3.26}$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p"(9-4a)}{3} = p".(9-4a) \tag{3.27}$$

As $\dfrac{1}{4} < cos^2\dfrac{\theta}{3} = \dfrac{a}{b} = \dfrac{a}{3} < \dfrac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow a = 2$ as $a > 1$. $a = 2$, we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4\times 3p"a}{3} = 4p"a = 8p" \tag{3.28}$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p"(9-4a)}{3} = p" \tag{3.29}$$

The two last equations give that $p"$ is not prime. Then we use the same methodology described above for the case **3.2.1.3.**, and we have : $A$,$B$ and $C$ solutions of (2.2) have a common factor.

**3.2.1.5. Case $3|p$ and $b = p$ :** We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$$

and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{p} = \frac{4a}{3} \tag{3.30}$$

As $A^{2m}$ is an integer, this implies that $3|a$, but $3|p \Longrightarrow 3|b$. As $a$ and $b$ are coprime, hence the contradiction. Then the case $3|p$ and $b = p$ is impossible.

**3.2.1.6. Case $3|p$ and $b = 4p$ :** $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{a}{3} \Longrightarrow 3|a \tag{3.31}$$

because $A^{2m}$ is an integer. But $3|p \Longrightarrow 3|\left[(4p) = b\right]$, that is in contradiction with the hypothesis $a, b$ are coprime. Then the case $b = 4p$ is impossible.

**3.2.1.7. Case $3|p$ and $b = 2p$ :** $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{2a}{3} \Longrightarrow 3|a \tag{3.32}$$

because $A^{2m}$ is an integer. But $3|p \Longrightarrow 3|(2p) \Longrightarrow 3|b$, that is in contradiction with the hypothesis $a, b$ are coprime. Then the case $b = 2p$ is impossible.

**3.2.1.8. Case $3|p$ and $b \neq 3$ is a divisor of $p$ :** We have $b = p' \neq 3$, and $p$ is written as:

$$p = kp' \quad with \quad 3|k \Longrightarrow k = 3k' \tag{3.33}$$

and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4 \times 3.k'p'}{3}\frac{a}{p'} = 4ak' \tag{3.34}$$

We calculate $B^nC^l$:

$$B^nC^l = \frac{p}{3}.\left(3 - 4cos^2\frac{\theta}{3}\right) = k'(3p' - 4a) \tag{3.35}$$

**I. Case $k' \neq 1$:**
We suppose $k' \neq 1$, we use the same methodology described for the case **3.1.2.3.**, and we obtain: $A, B$ and $C$ solutions of (2.2) have a common factor.

**II. Case $k' = 1$:**
We have $k' = 1 \Longrightarrow p = 3b$, then we have:

$$A^{2m} = 4a \Longrightarrow a \quad \text{is even} \tag{3.36}$$

and :

$$A^mB^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a \tag{3.37}$$

let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2b\sqrt{3}sin\frac{2\theta}{3} \tag{3.38}$$

The left member of (3.38) is an integer and $b$ also, then $2\sqrt{3}sin\dfrac{2\theta}{3}$ can be written in the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{3.39}$$

where $k_1, k_2$ are two coprime integers and $k_2|b \Longrightarrow b = k_2.k_3$.

**II.1. Case $k_3 \neq 1$:**

We suppose $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^m B^n = k_3.k_1 \tag{3.40}$$

Let $\mu$ is an prime integer such that $\mu|k_3$. If $\mu = 2 \Rightarrow 2|b$, but $2|a$ that is contradiction with $a, b$ coprime. We suppose $\mu \neq 2$ and $\mu|k_3$, then:

$$\boxed{\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m \text{ or } \mu|(A^m + 2B^n)} \tag{3.41}$$

**II.1.1. Case $\mu|A^m$:**
If $\mu|A^m \Longrightarrow \mu|A^{2m} \Longrightarrow \mu|4a \Longrightarrow \mu|a$. As $\mu|k_3 \Longrightarrow \mu|b$ and that $a, b$ are coprime hence the contradiction.

**II.1.2. Case $\mu|(A^m + 2B^n)$:**
If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \tag{3.42}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of $p$, we obtain:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{3.43}$$

As $p = 3b = 3k_2.k_3$ and $\mu|k_3$ hence $\mu|p \Longrightarrow \; p = \mu\mu'$, so we have :

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{3.44}$$

then:

$$\boxed{\mu|B^n(B^n - A^m) \Longrightarrow \mu|B^n \text{ or } \mu|(B^n - A^m)} \tag{3.45}$$

**II.1.2.1. Case $\mu|B^n$:**
If $\mu|B^n \Longrightarrow \mu|B$ which is in contradiction with case **II.1.2.** above.

**II.1.2.2. Case $\mu|(B^n - A^m)$:**
If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\mu|3B^n \tag{3.46}$$

**II.1.2.2.1. Case $\mu|B^n$:**
If $\mu|B^n$, using the result above of **II.1.2.1.** of this paragraph, it is impossible.

**II.1.2.2.2. Case $\mu = 3$:**
If $\mu = 3 \Longrightarrow 3|k_3 \Longrightarrow k_3 = 3k'_3$, and we have $b = k_2k_3 = 3k_2k'_3$, it follows $p = 3b = 9k_2k'_3$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then :

$$9k_2k'_3 - 3A^mB^n = (A^m - B^n)^2$$

we write it as :

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \tag{3.47}$$

hence :

$$\boxed{3|(3k_2k_3' - A^mB^n) \Longrightarrow\ 3|A^mB^n \Longrightarrow 3|A^m \text{ or } 3|B^n} \tag{3.48}$$

**II.1.2.2.2.1. Case $3|A^m$:**
If $3|A^m \Longrightarrow 3|A$ and we have also $3|A^{2m}$, but $A^{2m} = 4a \Longrightarrow 3|4a \Longrightarrow 3|a$. As $b = 3k_2k_3'$ then $3|b$, but $a, b$ are coprime hence the contradiction. Then $3 \nmid A$.

**II.1.2.2.2.2. Case $3|B^n$:**
If $3|B^n \Longrightarrow 3|B$, but the (3.47) gives $3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n) \Longrightarrow 3|A^m \Longrightarrow 3|(A^{2m} = 4a) \Rightarrow 3|a$. As $3|b$ then the contradiction with $a, b$ coprime.

Then the hypothesis $k_3 \neq 1$ is impossible.

**III. Case $k_3 = 1$:**
Now we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$. We have then:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{3.49}$$

with $k_1, b$ coprime. We write (3.49) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $cos^2\dfrac{\theta}{3}$ by $\dfrac{a}{b}$, we obtain:

$$3 \times 4^2.a(b-a) = k_1^2 \tag{3.50}$$

which implies that :

$$\boxed{3|a \quad or \quad 3|(b-a)} \tag{3.51}$$

**III.1. Case $3|a$:**
If $3|a$, as $A^{2m} = 4a \Longrightarrow 3|A^{2m} \Longrightarrow 3|A$ and $3|a$. But $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \Longrightarrow 3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n)$. But $3|A$ hence $3|B^n \Longrightarrow 3|B$, as $m \geq 3 \Longrightarrow 3^2|p$, it follows $3|b$ then the contradiction with $a, b$ coprime.

**III.2. Case $3|(b-a)$:**
Considering now that $3|(b-a)$. As $k_1 = A^m(A^m + 2B^n)$ by the equation (3.40) and that $3|k_1 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow \boxed{3|A^m \ or \ 3|(A^m + 2B^n)}$.

**III.2.1. Case $3|A^m$:**
If $3|A^m \Longrightarrow 3|A \Longrightarrow 3|A^{2m}$ then $3|4a \Longrightarrow 3|a$. But $3|(b-a) \Longrightarrow 3|b$ hence the contradiction with $a, b$ are coprime.

**III.2.2. Case $3|(A^m + 2B^n)$:**
If:

$$3|(A^m + 2B^n) \Longrightarrow 3|(A^m - B^n) \tag{3.52}$$

But $p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n$ then $p - 3A^mB^n = (A^m - B^n)^2 \Longrightarrow 9|(p - 3A^mB^n)$ or $9|(3b - 3A^mB^n)$, then $3|(b - A^mB^n)$ but $3|(b-a) \Longrightarrow$

$3|(a - A^m B^n)$. As $A^{2m} = 4a = (A^m)^2 \Longrightarrow \exists a' \in \mathbb{N}^*$ and $a = a'^2 \Longrightarrow A^m = 2a'$. We arrive to:

$$\boxed{3|(a'^2 - 2a'B^n) \Rightarrow 3|a'(a' - 2B^n) \Rightarrow 3|a' \quad or \quad 3|(a' - 2B^n)} \tag{3.53}$$

**III.2.2.1. Case $3|a'$:**
If $3|a' \Rightarrow 3|a'^2 \Rightarrow 3|a$, but $3|(b-a) \Rightarrow 3|b$, then the contradiction with $a, b$ coprime.

**III.2.2.2. Case $3|(a' - 2B^n)$:**
Now if $3|(a' - 2B^n) \Rightarrow 3|(2a' - 4B^n) \Rightarrow 3|(A^m - 4B^n) \Rightarrow 3|(A^m - B^n)$, we refind the case **III.2.2.**, equation (3.52), that has a solution given by the case **2.2.1.** above.

Then, the study of the case **3.2.1.8.** is finished.

**3.2.1.9 Case $3|p$ and $b|4p$:** As $3|p \Rightarrow p = 3p'$ and $b|4p \Rightarrow \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1 b$.

**I. Case $k_1 = 1$:**
If $k_1 = 1$, then $b = 12p'$, ($p' \neq 1$ if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} = \dfrac{12p'}{3}\dfrac{a}{b} = \dfrac{4p'.a}{12p'} = \dfrac{a}{3} \Rightarrow 3|a$ because $A^{2m}$ is an integer, then the contradiction with $a, b$ coprime.

**II. Case $k_1 = 3$:**
If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} = \dfrac{k_1.a}{3} = a$.
Let us calculate $A^m B^n$:

$$A^m B^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - \frac{a}{2} \tag{3.54}$$

Let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2p'\sqrt{3}sin\frac{2\theta}{3} \tag{3.55}$$

The left member of the equation (3.55) is an integer and also $p'$, then $2\sqrt{3}sin\dfrac{2\theta}{3}$ can be written as :

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{k_3} \tag{3.56}$$

where $k_2, k_3$ are two coprime integers and:

$$k_3|p' \Longrightarrow \exists k_4 \in \mathbb{N}^* \quad and \quad p' = k_3.k_4 \tag{3.57}$$

**II.1. Case $k_4 \neq 1$:**
We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2.k_4 \tag{3.58}$$

Let $\mu$ one prime integer with:

$$\mu|k_4 \tag{3.59}$$

Then :

$$\boxed{\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m \quad \text{or} \quad \mu|(A^m + 2B^n)} \tag{3.60}$$

**II.1.1. Case $\mu|A^m$:**
If $\mu|A^m \Longrightarrow \mu|A^{2m} \Longrightarrow \mu|a$. As $\mu|k_4 \Longrightarrow \mu|p' \Rightarrow \mu|(4p'=b)$. But $a, b$ are coprime then the contradiction.

**II.1.2. Case $\mu|(A^m+2B^n)$:**
If $\mu|(A^m+2B^n) \Longrightarrow \mu\nmid A^m$ and $\mu\nmid 2B^n$ then $\mu\neq 2$ and $\mu\nmid B^n$. $\mu|(A^m+2B^n)$, we can write:

$$A^m+2B^n=\mu.t' \quad t'\in\mathbb{N}^* \tag{3.61}$$

It follows:

$$A^m+B^n=\mu t'-B^n \Longrightarrow A^{2m}+B^{2n}+2A^mB^n=\mu^2t'^2-2t'\mu B^n+B^{2n}$$

Using the expression of $p$, we obtain:

$$p=t'^2\mu^2-2t'B^n\mu+B^n(B^n-A^m) \tag{3.62}$$

As $p=3p'$ and $\mu|p' \Rightarrow \mu|(3p') \Rightarrow \mu|p$, we can write $:\exists\mu'\in\mathbb{N}^*$ and $p=\mu\mu'$, then we obtain :

$$\mu'\mu=\mu(\mu t'^2-2t'B^n)+B^n(B^n-A^m) \tag{3.63}$$

and:

$$\boxed{\mu|B^n(B^n-A^m) \Longrightarrow \mu|B^n \quad \text{or} \quad \mu|(B^n-A^m)} \tag{3.64}$$

**II.1.2.1. Case $\mu|B^n$:**
If $\mu|B^n \Longrightarrow \mu|B$ which is in contradiction with the case **II.1.2.** above.

**II.1.2.2. Case $\mu|(B^n-A^m)$:**
If $\mu|(B^n-A^m)$ and using $\mu|(A^m+2B^n)$, we obtain:

$$\boxed{\mu|3B^n} \tag{3.65}$$

**II.1.2.2.1. Case $\mu|B^n$:**
If $\mu|B^n$ it is impossible, see the case **II.1.2.1.** above.

**II.1.2.2.2 Case $\mu=3$:**
If $\mu=3 \Longrightarrow 3|k_4 \Longrightarrow k_4=3k'_4$, and we obtain $p'=k_3k_4=3k_3k'_4$, it follows $p=3p'=9k_3k'_4$ then $9|p$, but $p=(A^m-B^n)^2+3A^mB^n$, then:

$$9k_4k'_5-3A^mB^n=(A^m-B^n)^2$$

that we write :

$$3(3k_4k'_5-A^mB^n)=(A^m-B^n)^2 \tag{3.66}$$

then $3|(3k_4k'_5-A^mB^n) \Longrightarrow 3|A^mB^n \Longrightarrow \boxed{3|A^m \quad or \quad 3|B^n}$.

**II.1.2.2.2.1. Case $3|A^m$:**
If $3|A^m \Longrightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with $a, b$ coprime. Then $3\nmid A$.

**II.1.2.2.2.2. Case $3|B^n$:**

If $3|B^n$ and using (3.61), we have $A^m = \mu t' - 2B^n = 3t' - 2B^n \Longrightarrow 3|A^m \Rightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with $a, b$ coprime.

Then the hypothesis $k_4 \neq 1$ is impossible.

**II.2. Case $k_4 = 1$:**
We suppose that $\boxed{k_4 = 1} \Longrightarrow p' = k_3 k_4 = k_3$. Then we obtain:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{p'} \tag{3.67}$$

with $k_2, p'$ coprime, we write (3.67) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b-a) = k_2^2 \tag{3.68}$$

that implies:

$$\boxed{3|a \quad or \quad 3|(b-a)} \tag{3.69}$$

**II.2.1. Case $3|a$:**
If $3|a \Rightarrow 3|A^{2m} \Rightarrow 3|A$, as $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \Longrightarrow 3|(A^m - B^n)^2 \Longrightarrow 9|(A^m - B^n)^2$. But $(A^m - B^n)^2 = p - 3A^mB^n = 3b - 3A^mB^n \Longrightarrow 3|(b - A^mB^n)$. As $3|A^m \Longrightarrow 3|b \Longrightarrow$ the contradiction with $a, b$ coprime.

**II.2.2. Case $3|(b-a)$:**
We consider that $3|(b-a)$. As $k_2 = A^m(A^m + 2B^n)$ given by the equation (3.58) and that $3|k_2 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow \boxed{3|A^m \quad or \quad 3|(A^m + 2B^n)}$.

**II.2.2.1. Case $3|A^m$:**
If $3|A^m \Longrightarrow 3|A^{2m} \Longrightarrow 3|a$, but $3|(b-a) \Longrightarrow 3|b$ then the contradiction with $a, b$ coprime.

**II.2.2.2. Case $3|(A^m + 2B^n)$:**
If:

$$3|(A^m + 2B^n) \Longrightarrow 3|(A^m - B^n) \tag{3.70}$$

but $p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n$ then $p - 3A^mB^n = (A^m - B^n)^2 \Longrightarrow 9|(p - 3A^mB^n)$ or $9|(3p' - 3A^mB^n)$, then $3|(p' - A^mB^n) \Rightarrow 3|4(p' - 4A^mB^n) \Rightarrow 3|(b - 4A^mB^n)$ but $3|(b-a) \Longrightarrow 3|(a - A^mB^n)$. As $3|(A^{2m} - 4A^mB^n) \Rightarrow \boxed{3|A^m(A^m - 4B^n)}$.

**II.2.2.2.1. Case $3|A^m$:**
If $3|A^m \Longrightarrow 3|A^{2m} \Longrightarrow 3|a$, but $3|(b-a) \Longrightarrow 3|b$ then the contradiction with $a, b$ coprime.

**II.2.2.2.2. Case $3|(A^m - 4B^n)$:**

Now if $3|(A^m-4B^n) \Longrightarrow 3|(A^m-B^n)$, we refind the hypothesis of the beginning (3.70) above, that has a solution **II.2.2.2.1.**.

**III. Case $k_1 \neq 3$ and $3|k_1$:**
We suppose $k_1 \neq 3$ and $3|k_1 \Rightarrow k_1 = 3k'1$ with $k'_1 \neq 1$. We have $4p = 12p' = k_1 b = 3k'_1 b \Rightarrow 4p' = k'_1 b$. $A^{2m}$ can be written as :

$$A^{2m} = \frac{4p}{3} cos^2\frac{\theta}{3} = \frac{3k'_1 b}{3}\frac{a}{b} = k'_1 a \tag{3.71}$$

and $B^nC^l$:

$$B^nC^l = \frac{p}{3}\left(3-4cos^2\frac{\theta}{3}\right) = \frac{k'_1}{4}(3b-4a) \tag{3.72}$$

As $B^nC^l$ is an integer, we must have $\boxed{4|(3b-4a) \quad \text{or} \quad 4|k'_1}$.

**III.1. Case $4|(3b-4a)$:**
We suppose that $4|(3b-4a) \Rightarrow \dfrac{3b-4a}{4} = c \in \mathbb{N}^*$, and we obtain:

$$A^{2m} = k'_1 a$$
$$B^nC^l = k'_1 c$$

**III.1.1. Case $k'_1$ is prime:**
If $k'_1$ is prime, then $k'_1|A^{2m} \Rightarrow k'_1|A$ and $k'_1|B^nC^l \Rightarrow k'_1|B^n$ or $k'_1|C^l$. If $k'_1|B^n \Rightarrow k'_1|B$, then $k'_1|C^l \Rightarrow k'_1|C$. With the same method if $k'_1|C^l$, we arrive to $k'_1|B$.

We obtain: $A$,$B$ and $C$ solutions of (2.2) have a common factor.

**III.1.2. Case $k'_1$ not a prime:**
We suppose $k'_1$ not a prime. Let $\mu$ a prime divisor of $k'_1$, as described in **III.1.1.** above, we obtain : $A$,$B$ and $C$ solutions of (2.2) have a common factor.

**III.2. Case $4|k'_1$:**
Now, we suppose that $4|k'_1$.

**III.2.1. Case $k'_1 = 4$:**
We suppose $k'_1 = 4$, then $A^{2m} = 4a$ and $B^nC^l = 4c$, It is easy to verify that 2 is a common factor of $A, B, C$.

We obtain: $A$,$B$ and $C$ solutions of (2.2) have a common factor.

**III.2.2. Case $k'_1 = 4k"_1$:**
If $k'_1 = 4k"_1$ with $k"_1 > 1$. Then, we have:

$$A^{2m} = 4k"_1 a \tag{3.73}$$
$$B^nC^l = k"_1(3b-4a) \tag{3.74}$$

**III.2.2.1. Case $k"_1$ prime:**
If $k"_1$ is prime, then $k"_1|A^{2m} \Rightarrow k"_1|A$ and $k"_1|B^nC^l \Rightarrow k"_1|B^n$ or $k"_1|C^l$. If $k"_1|B^n \Rightarrow k"_1|B$, then $k"_1|C^l \Rightarrow k"_1|C$. With the same method if $k"_1|C^l$, we arrive to $k"_1|B$.

We obtain: $A$,$B$ and $C$ solutions of (2.2) have a common factor.

**III.2.2.2. Case $k"_1$ not a prime:**
If $k"_1$ not a prime. Let $\mu$ a prime divisor of $k"_1$, as described in case **III.2.2.1.** above, we obtain : $A$,$B$ and $C$ solutions of (2.2) have a common factor.

### 3.2.2 Hypothesis : $\{3|a \quad and \quad b|4p\}$

We have :

$$3|a \Longrightarrow \exists a' \in \mathbb{N}^* \ / \ a = 3a' \tag{3.75}$$

**3.2.2.1. Case $b = 2$ and $3|a$ :** $A^{2m}$ is written as :

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{3.76}$$

Using the equation (3.75), $A^{2m}$ becomes:

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{3.77}$$

But $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

**3.2.2.2. Case $b = 4$ and $3|a$ :** $A^{2m}$ is written as :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{3.78}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{3.79}$$

which is impossible.

Then the case $b = 4$ is impossible.

**3.2.2.3. Case $b = p$ and $3|a$ :** Then:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{3.80}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{3.81}$$

$$\exists a" \in \mathbb{N}^* \ / \ a' = a"^2 \tag{3.82}$$

We calculate $A^mB^n$, hence:

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$

$$or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{3.83}$$

The left member of (3.83) is an integer and $p$ is also, then $2\dfrac{\sqrt{3}}{3}sin\dfrac{2\theta}{3}$ will be written as :

$$2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{3.84}$$

where $k_1, k_2$ are two coprime integers and $k_2|p \Longrightarrow p = b = k_2.k_3,\ k_3 \in \mathbb{N}^*$.

**I. Case $k_3 \neq 1$:**
We suppose that $k_3 \neq 1$. We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \tag{3.85}$$

Let us $\mu$ a prime integer with $\mu|k_3$, then $\mu|b$ and $\mu|A^m(A^m + 2B^n)$. Hence:

$$\boxed{\mu|A^m \quad or \quad \mu|(A^m + 2B^n)} \tag{3.86}$$

**I.1. Case $\mu|A^m$:**
If $\mu|A^m \Longrightarrow \mu|A$ and $\mu|A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu|4a' \Longrightarrow (\mu = 2$ but $2|a')$ or $\mu|a'$. Then $\mu|a$ hence the contradiction with $a, b$ coprime.

**I.2. Case $\mu|(A^m + 2B^n)$:**
If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu|(A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \tag{3.87}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of $p$:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{3.88}$$

Since $p = b = k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists \mu' \in \mathbb{N}^*$ and $b = \mu\mu'$, so we can write:

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{3.89}$$

From the last equation, we get $\mu|B^n(B^n - A^m) \Longrightarrow \boxed{\mu|B^n \quad or \quad \mu|(B^n - A^m)}$.

**I.2.1. Case $\mu|B^n$:**
If $\mu|B^n$ which is contradiction with $\mu \nmid B^n$.

**I.2.2. Case $\mu|(B^n - A^m)$:**
If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we arrive to:

$$\mu|3B^n \Longrightarrow \begin{cases} \boxed{\mu|B^n} \\ or \\ \boxed{\mu = 3} \end{cases} \tag{3.90}$$

**I.2.2.1. Case $\mu|B^n$:**
If $\mu|B^n$ which is contradiction with $\mu \nmid B$ from **I.2. Case** $\mu|(A^m + 2B^n)$.

**I.2.2.2. Case $\mu = 3$:**
If $\mu = 3$, then $b = 3\mu'$, but $3|a$ then the contradiction with $a, b$ coprime.

**II. Case $k_3 = 1$:**
We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^m B^n = k_1 \tag{3.91}$$
$$b = k_2 \tag{3.92}$$
$$\frac{2\sqrt{3}}{3} sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{3.93}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2\frac{2\theta}{3} = \frac{k_1^2}{b^2}$$
$$\frac{16}{3} sin^2\frac{\theta}{3} cos^2\frac{\theta}{3} = \frac{k_1^2}{b^2}$$
$$\frac{16}{3} sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2 a'(p - a) = k_1^2 \tag{3.94}$$

but $a' = a"^2$ then $p - a$ is a square. Let us:

$$\lambda^2 = p - a \tag{3.95}$$

The equation (3.94) becomes:

$$4^2 a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = 4a"\lambda \tag{3.96}$$

taking the positive square root. Using (3.91), we get :

$$k_1 = 4a"\lambda \tag{3.97}$$

But $k_1 = A^m(A^m + 2B^n) = 2a"(A^m + 2B^n)$, it follows:

$$A^m + 2B^n = 2\lambda \tag{3.98}$$

Let $\lambda_1$ prime $\neq 2$, a divisor of $\lambda$ (if not, $\lambda_1 = 2|\lambda \Longrightarrow 2|\lambda^2 \Longrightarrow 2|(p - a)$ but $a$ is even, then $2|p \Longrightarrow 2|b$ which is contradiction with $a, b$ coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|\lambda^2 \quad and \quad \lambda_1|(A^m + 2B^n) \tag{3.99}$$
$$\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{3.100}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(p = b) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a" \Longrightarrow \lambda_1|a \tag{3.101}$$

hence the contradiction with $a, b$ coprime.

**II.1. Case $\lambda_1 \nmid A^m$ and $\lambda_1|(A^m+2B^n)$:**
We assume now $\lambda_1 \nmid A^m$. $\lambda_1|(A^m+2B^n) \Longrightarrow \lambda_1|(A^m+2B^n)^2$ that is $\lambda_1|(A^{2m}+4A^mB^n+4B^{2n})$, we write it as $\lambda_1|(p+3A^mB^n+3B^{2n}) \Longrightarrow \lambda_1|(p+3B^n(A^m+2B^n)-3B^{2n})$. But $\lambda_1|(A^m+2B^n) \Longrightarrow \lambda_1|(p-3B^{2n})$, as $\lambda_1|(p-a)$ hence by difference, we obtain $\lambda_1|(a-3B^{2n})$ or $\lambda_1|(3a'-3B^{2n}) \Longrightarrow \lambda_1|3(a'-B^{2n})$, Then:

$$\boxed{\lambda_1 = 3 \quad or \quad \lambda_1|(a'-B^{2n})} \tag{3.102}$$

**II.1.1. Case $\lambda_1 = 3$:**
If $\lambda_1 = 3$ but $3|a$, as $\lambda_1|(p-a) \Longrightarrow 3|(p=b)$ hence the contradiction with $a, b$ coprime.

**II.1.2. Case $\lambda_1|(a'-B^{2n})$:**
If $\lambda_1|(a'-B^{2n}) \Longrightarrow \lambda_1|(a"^2-B^{2n}) \Longrightarrow \boxed{\lambda_1|(a"-B^n)(a"+B^n)} \Longrightarrow \lambda_1|(a"+B^n)$ or $\lambda_1|(a"-B^n)$, because $(a"-B^n) \neq 1$, if not, we obtain $a"^2-B^{2n} = a"+B^n \Longrightarrow a"^2-a" = B^n-B^{2n}$. The left member is positive and the right member is negative, then the contradiction.

**II.1.2.1. Case $\lambda_1|(a"-B^n)$:**
If $\lambda_1|(a"-B^n) \Longrightarrow \lambda_1|2(a"-B^n) \Longrightarrow \lambda_1|(A^m-2B^n)$ but $\lambda_1|(A^m+2B^n)$ hence $\lambda_1|2A^m \Longrightarrow \lambda_1|A^m$ as $\lambda_1 \neq 2$, it follows $\lambda_1|A^m$ hence the contradiction with (3.100).

**II.1.2.2. Case $\lambda_1|(a"+B^n)$:**
If $\lambda_1|(a"+B^n) \Longrightarrow \lambda_1|2(a"+B^n) \Longrightarrow \lambda_1|(2a"+2B^n) \Rightarrow \lambda_1|(A^m+2B^n)$. We find the case **II.1.** that has solutions.

Then the case $k_3 = 1$ is impossible.

**3.2.2.4. Case $b|p \Rightarrow\ p = b.p', p' > 1$, $b \neq 2$, $b \neq 4$ and $3|a$ :**

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{3.103}$$

We calculate $B^nC^l$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{3.104}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, hence using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b-4a') \tag{3.105}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^nC^l = p'(b-4a') \tag{3.106}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{3.107}$$

**I. Case $\lambda$ a prime divisor of $p'$:**

Let $\lambda$ a prime divisor of $p'$ (we suppose $p'$ not prime ). From (3.107), we have:

$$\lambda | A^{2m} \Rightarrow \lambda | A^m \quad \text{as}\, \lambda\, \text{is a prime, then} \;\; \lambda | A \tag{3.108}$$

From (3.106), as $\lambda | p'$ we have:

$$\lambda | B^n C^l \Rightarrow \lambda | B^n \quad \text{or } \lambda | C^l \tag{3.109}$$

If $\lambda | B^n$, $\lambda$ is a prime $\lambda | B$, but $C^l = A^m + B^n$, then we have also :

$$\lambda | C^l \quad \text{as}\, \lambda\, \text{is a prime, then } \lambda | C \tag{3.110}$$

By the same way, if $\lambda | C^l$, we obtain $\lambda | B$. then : $A, B$ and $C$ solutions of (2.2) have a common factor.

**II. Case $p'$ is a prime number:**
We suppose now that $p'$ is prime, from the equations (3.106) and (3.107), we obtain that:

$$p' | A^{2m} \Rightarrow p' | A^m \Rightarrow p' | A \tag{3.111}$$

and:

$$p' | B^n C^l \Rightarrow p' | B^n \quad \text{or } p' | C^l \tag{3.112}$$

$$\text{If} \quad p' | B^n \Rightarrow p' | B \tag{3.113}$$

$$\text{As} \quad C^l = A^m + B^n \quad \text{and that } \; p'|A, p'|B \Rightarrow p'|A^m, p'|B^n \Rightarrow p'|C^l$$
$$\Rightarrow p' | C \tag{3.114}$$

By the same way, if $p' | C^l$, we arrive to $p' | B$.

Hence: $A$ ,$B$ and $C$ solutions of (2.2) have a common factor.

**3.2.2.5. Case $b = 2p$ and $3|a$ :** We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' \Longrightarrow 2|A^m \Longrightarrow 2|a \Longrightarrow 2|a'$$

Then $2|a$ and $2|b$ which is contradiction with $a, b$ coprime.

**3.2.2.6. Case $b = 4p$ and $3|a$ :** We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a'$$

Calculate $A^m B^n$, we obtain:

$$A^m B^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$
$$A^m B^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} \tag{3.115}$$

let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{3.116}$$

The left member of (3.116) is an integer and $p$ is an integer, then $\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3}$ will be written:

$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{k_2} \tag{3.117}$$

where $k_1, k_2$ are two coprime integers and $k_2|p \Longrightarrow p = k_2.k_3$.

**I. Case $k_3 \neq 1$:**
Firstly, we suppose that $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^m B^n = k_3.k_1 \tag{3.118}$$

Let $\mu$ a prime integer and $\mu|k_3$, then $\mu|A^m(A^m+2B^n) \Longrightarrow \boxed{\mu|A^m \quad or \quad \mu|(A^m + 2B^n)}$.

**I.1. Case $\mu|A^m$:**
If $\mu|A^m \Longrightarrow \mu|(A^{2m} = a') \Rightarrow \mu|(3a' = a)$. As $\mu|k_3 \Longrightarrow \mu|p \Rightarrow \mu|(4p = b)$. Then the contradiction with $a, b$ coprime.

**I.2. Case $\mu|(A^m + 2B^n)$:**
If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{3.119}$$

$\mu|(A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \tag{3.120}$$

Then :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

$$\Longrightarrow p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{3.121}$$

As $b = 4p = 4k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists \mu' \in \mathbb{N}^*$ that $b = \mu\mu'$, we obtain:

$$\mu'\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m) \tag{3.122}$$

The last equation implies $\mu|4B^n(B^n-A^m)$, but $\mu \neq 2$ then $\boxed{\mu|B^n \quad or \quad \mu|(B^n - A^m)}$.

**I.2.1. Case $\mu|B^n$:**
If $\mu|B^n$ then the contradiction with (3.119).

**I.2.2. Case $\mu|(B^n - A^m)$:**
If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\boxed{\mu|3B^n \Longrightarrow \mu|B^n \quad or \quad \mu = 3} \tag{3.123}$$

**I.2.2.1. Case $\mu|B^n$:**
If $\mu|B^n$ it is contradiction with (3.119).

**I.2.2.2. Case $\mu = 3$:**

If $\mu = 3$, then $b = 3\mu'$, but $3|a$ which is contradiction with $a, b$ coprime.

**II. Case $k_3 = 1$:**
We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^m B^n = k_1 \tag{3.124}$$

$$p = k_2 \tag{3.125}$$

$$\frac{2\sqrt{3}}{3} sin\frac{2\theta}{3} = \frac{k_1}{p} \tag{3.126}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3}sin^2\frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3}sin^2\frac{\theta}{3}cos^2\frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3}sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{3.127}$$

but $a' = a"^2$ then $4p - 3a'$ is a square. Let us:

$$\lambda^2 = 4p - 3a' = 4p - a = b - a \tag{3.128}$$

The equation (3.127) becomes:

$$a"^2\lambda^2 = k_1^2 \Longrightarrow k_1 = a"\lambda \tag{3.129}$$

taking the positive square root. Using (3.124), we get :

$$k_1 = a"\lambda \tag{3.130}$$

But $k_1 = A^m(A^m + 2B^n) = a"(A^m + 2B^n)$, it follows:

$$(A^m + 2B^n) = \lambda \tag{3.131}$$

Let $\lambda_1$ prime $\neq 2$, a divisor of $\lambda$ (if not $\lambda_1 = 2|\lambda \Longrightarrow 2|\lambda^2$. As $2|(b = 4p) \Longrightarrow 2|(a = 3a')$ which is contradiction with $a, b$ coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|(A^m + 2B^n) \tag{3.132}$$

$$\Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{3.133}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(b = 4p) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a" \Longrightarrow \lambda_1|a \tag{3.134}$$

hence the contradiction with $a, b$ coprime.

**II.1. Case $\lambda_1 \nmid A^m$, $\lambda_1 \nmid B^n$ and $\lambda_1|(A^m + 2B^n)$:**
We assume now $\lambda_1 \nmid A^m$, $\lambda_1 \nmid B^n$. $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(A^m + 2B^n)^2$ that is $\lambda_1|(A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1|(p + 3A^mB^n + 3B^{2n}) \Longrightarrow \lambda_1|(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(p - 3B^{2n})$, as $\lambda_1|(4p - a)$ hence by difference, we obtain $\lambda_1|(a - 3(B^{2n} + p))$ or $\lambda_1|(3a' - 3(B^{2n} + p)) \Longrightarrow \lambda_1|3(a' - B^{2n} - p) \Longrightarrow \boxed{\lambda_1 = 3 \text{ or } \lambda_1|(a' - (B^{2n} + p))}$.

**II.1.1. Case $\lambda_1 = 3$:**
If $\lambda_1 = 3|\lambda \Rightarrow 3|\lambda^2 \Rightarrow 3|b - a$ but $3|a \Longrightarrow 3|(p = b)$ hence the contradiction with $a, b$ coprime.

**II.1.2. Case $\lambda_1|(a' - (B^{2n} + p))$:**
If $\lambda_1 \neq 3$ and $\lambda_1|(a' - B^{2n} - p) \Longrightarrow \lambda_1|(A^mB^n + B^{2n}) \Longrightarrow \lambda_1|B^n(A^m + 2B^n) \Longrightarrow \boxed{\lambda_1|B^n \quad or \quad \lambda_1|(A^m + 2B^n)}$.

**II.1.2.1. Case $\lambda_1|B^n$:**
If $\lambda_1|B^n$ that is in contradiction with the hypothesis $\lambda_1 \nmid B$ cited above case II.1.

**II.1.2.2. Case $\lambda_1|(A^n + 2B^n)$:**
If $\lambda_1|(A^n + 2B^n)$. We refind this condition in the case II.1.

Then the case $k_3 = 1$ is impossible.

**3.2.2.7. Case $3|a$ and $b = 2p'$ $b \neq 2$ with $p'|p$ :** $3|a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, hence:

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a' \tag{3.135}$$

Calculate $B^nC^l$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{3.136}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$ hence en using $cos^2\frac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a') \tag{3.137}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^nC^l = k(p' - 2a') \tag{3.138}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{3.139}$$

**I. Case $\lambda$ is a prime divisor of $k$:**
We suppose that $\lambda$ is a prime divisor of $k$ (we suppose $k$ not a prime ). From (3.139), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{3.140}$$

From (3.138), as $\lambda | k$, we have:

$$\lambda | B^n C^l \Rightarrow \lambda | B^n \quad \text{or} \quad \lambda | C^l \tag{3.141}$$

If $\lambda | B^n$, $\lambda$ is prime $\lambda | B$, and as $C^l = A^m + B^n$ then we have also:

$$\lambda | C^l \quad \text{as}\, \lambda\, \text{is prime then} \quad \lambda | C \tag{3.142}$$

By the same way, if $\lambda | C^l$, we obtain $\lambda | B$. Then : $A, B$ and $C$ solutions of (2.2) have a common factor.

**II. Case $k$ is prime:**
Now, we suppose now that $k$ is prime, from the equations (3.138) and (3.139), we obtain:

$$k | A^{2m} \Rightarrow k | A^m \Rightarrow k | A \tag{3.143}$$

and:

$$k | B^n C^l \Rightarrow k | B^n \quad \text{or } k | C^l \tag{3.144}$$

$$\text{if} \quad k | B^n \Rightarrow k | B \tag{3.145}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m,\, k|B^n \Rightarrow k|C^l$$
$$\Rightarrow k | C \tag{3.146}$$

By the same way, if $k | C^l$, we arrive to $k | B$.

Hence: $A$ ,$B$ and $C$ solutions of (2.2) have a common factor.

**3.2.2.8. Case $3|a$ and $b = 4p'$ $b \neq 2$ with $p'|p$ :** $3|a \Longrightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$, if not, $b = 4p$ a case that has been studied (paragraph **3.2.2.6**), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a' \tag{3.147}$$

Writing $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{3.148}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, hence en using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - a') \tag{3.149}$$

As $p = b.p'$, and $p' > 1$, we have:

$$B^n C^l = k(p' - 2a') \tag{3.150}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{3.151}$$

**I. Case $\lambda$ a prime divisor of $k$:**
Let $\lambda$ a prime divisor of $k$ (we suppose $k$ not a prime). From (3.151), we have:

$$\lambda | A^{2m} \Rightarrow \lambda | A^m \quad \text{as}\, \lambda\, \text{is prime then} \quad \lambda | A \tag{3.152}$$

From (3.150), as $\lambda|k$ we obtain:

$$\lambda|B^nC^l \Rightarrow \boxed{\lambda|B^n \quad or \quad \lambda|C^l} \tag{3.153}$$

**I.1 Case $\lambda|B^n$ or $\lambda|C^n$:**
If $\lambda|B^n$, $\lambda$ is a prime, then $\lambda|B$, and as $\lambda|A \Rightarrow \lambda|(A^m + B^n = C^l) \Rightarrow \lambda|C$. By the same way if $\lambda|C^l$, we obtain $\lambda|B$. Then : $A, B$ and $C$ solutions of (2.2) have a common factor.

**II. Case $k$ is prime:**
We suppose now that $k$ is prime, from the equations (3.150) and (3.151), we have:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \tag{3.154}$$

and:

$$k|B^nC^l \Rightarrow k|B^n \quad \text{or } k|C^l \tag{3.155}$$

$$\text{if} \quad k|B^n \Rightarrow k|B \tag{3.156}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m,\, k|B^n \Rightarrow k|C^l$$

$$\Rightarrow k|C \tag{3.157}$$

By the same way if $k|C^l$, we arrive to $k|B$.

Hence: $A$ ,$B$ and $C$ solutions of (2.2) have a common factor.

**3.2.2.9. Case $3|a$ and $b|4p$ :** $a = 3a'$ and $4p = k_1b$ with $k_1 \in \mathbb{N}^*$. As $A^{2m} = \dfrac{4p}{3}cos^2\dfrac{\theta}{3} = \dfrac{4p}{3}\dfrac{3a'}{b} = k_1a'$ and $B^nC^l$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3a'}{b}\right) = \frac{k_1}{4}(b-4a') \tag{3.158}$$

As $B^nC^l$ is an integer, we must have $\boxed{4|k_1 \quad or \quad 4|(b-4a')}$.
**I. Case $k_1 = 1$:**
If $k_1 = 1 \Rightarrow b = 4p$ : it is the case (**3.2.2.6**) above.

**II. Case $k_1 = 4$:**
If $k_1 = 4 \Rightarrow p = b$ : it is the case (**3.2.2.3**) above.

**III. Case $4|k_1$:**
We suppose that $4|k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k_1'$, then we have:

$$A^{2m} = 4k_1'a'$$
$$B^nC^l = k_1'(b - 4a')$$

By discussing $k_1'$ is a prime integer or not, we arrive easily to: $A$ ,$B$ and $C$ solutions of (2.2) have a common factor.

**III.1. Case $4 \nmid (b - 4a')$ and $4 \nmid k_1'$:**

If $4 \nmid (b-4a')$ and $4 \nmid k'_1$ it is impossible.

**III.2. Case** $4|(b-4a')$**:**
If $4|(b-4a') \Rightarrow (b-4a') = 4c$, with $c \in \mathbb{N}^*$, then we obtain:

$$A^{2m} = k_1 a'$$
$$B^n C^l = k_1 c$$

By discussing $k_1$ is a prime integer or not, we arrive easily to: $A$ ,$B$ and $C$ solutions of (2.2) have a common factor.

□

**The main theorem is proved**.

# 4 Numerical Examples

## 4.1 Example 1:

We consider the example:

$$6^3 + 3^3 = 3^5 \tag{4.1}$$

with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$p = 3^6 \times 73, \tag{4.2}$$
$$q = 8 \times 3^{11}, \tag{4.3}$$
$$\overline{\Delta} = 4 \times 3^{11}(3^6 \times 4^2 - 73^3) < 0, \tag{4.4}$$
$$\rho = \frac{p\sqrt{p}}{3\sqrt{3}} = \frac{3^8 \times 73\sqrt{73}}{3}, \tag{4.5}$$
$$cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{4.6}$$

As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{3 \times 2^4}{73} = \dfrac{a}{b} \Longrightarrow a = 3 \times 2^4,\ b = 73;$ then:

$$cos\frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}} \tag{4.7}$$
$$p = 3^6 b \tag{4.8}$$

Let us verify the equation (4.6) using the equation (4.7):

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{4\sqrt{3}}{\sqrt{73}}\right)^3 - 3\frac{4\sqrt{3}}{\sqrt{73}} = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{4.9}$$

That's OK. For this example, we can use the two conditions of (3.10) as $3|p$ ,$b|4p$ and $3|a$. The cases **3.2.1.3** and **3.2.2.4** are respectively used. We find for both cases that $A^m, B^n$ and $C^l$ of the equation (4.1) have a common prime factor which is true.

## 4.2 Example 2:

Let the second example:

$$7^4 + 7^3 = 14^3 \Rightarrow 2401 + 343 = 2744 \tag{4.10}$$

With the notations of the paper, we take:

$$A^m = 7^4 \tag{4.11}$$

$$B^n = 7^3 \tag{4.12}$$

$$C^l = 14^3 \tag{4.13}$$

We obtain:

$$p = 57 \times 7^6 = 3 \times 19 \times 7^6 \tag{4.14}$$

$$q = 8 \times 7^{10} \tag{4.15}$$

$$\begin{aligned} \overline{\Delta} = 27q^2 - 4p^3 &= 27 \times 4 \times 7^{18}(16 \times 49 - 19^3) \\ &= -27 \times 4 \times 7^{18} \times 6075 < 0 \end{aligned} \tag{4.16}$$

$$\rho = \tfrac{p\sqrt{p}}{3\sqrt{3}} = 19 \times 7^9 \times \sqrt{19} \tag{4.17}$$

$$cos\theta = \frac{-q}{2\rho} = -\frac{4 \times 7}{19\sqrt{19}} \tag{4.18}$$

As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{7^2}{4 \times 19} = \dfrac{a}{b} \Longrightarrow a = 7^2,\ b = 4 \times 19;$ then:

$$cos\frac{\theta}{3} = \frac{7}{2\sqrt{19}} \tag{4.19}$$

$$3|p \quad and \quad b|(4p) \tag{4.20}$$

Let us verify the equation (4.18) using the equation (4.19):

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{7}{2\sqrt{19}}\right)^3 - 3\frac{7}{2\sqrt{19}} = -\frac{4 \times 7}{19\sqrt{19}} \tag{4.21}$$

It is the same value of (4.18)!

Now, from (4.20), we have $3|p \Rightarrow p = 3p'$, $b|(4p)$ with $b \neq 2, 4$ then $12p' = k_1 b = 3 \times 7^6 b$. It concerns the paragraph **3.2.1.9.** of the first hypothesis. As $k_1 = 3 \times 7^6 = 3k'_1$ with $k'_1 = 7^6 \neq 1$. It is the case **III.**, with the two conditions: $4|(3b - 4a)$ or $4|k'_1$. We take $4|(3b - 4a)$. Let us calculate $3b - 4a$:

$$3b - 4a = 3 \times 4 \times 19 - 4 \times 7^2 = 32 \Longrightarrow 4|(3b - 4a) \tag{4.22}$$

Then it is the sous-case **III.1.** with $A^{2m} = 7^8 = 7^6 \times 7^2 = k'_1.a$ with $k'_1$ not a prime, we find the sous-case **III.1.2** with the result that $A, B$ and $C$ have a common factor namely the prime number 7 a divisor of $k'_1 = 7^6$!.

## References

[1] R. Daniel Mauldin. *A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem.* Notice of AMS, Vol 44, $n°11$, 1997, pp 1436-1437.