Polignac conjecture is true for n=6: there are infinitely many pairs of "sexy" primes

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Consecutive primes whose gap is 6 are sometimes called sexy.

The first occurrence of such a pattern is the pair (23 29), belonging to the infinite set of pairs (30k-7 30k-1) which are good candidates to include sexy primes because both 30k-5 and 30k-3 need to be composite being 30 a multiple of both 3 and 5.

Not all the sexy primes belong to such set, nor all the pairs in such set are sexy, but it is sufficient to prove that infinitely many of them are sexy.

For that purpose, we need that both 30k-7 and 30k-1 are primes for infinitely many k.

In order not to be primes, they need to be sieved by at least one prime p greater than 5.

In fact, every prime p higher than 5 does sieve one of such candidate pairs, in at least one of its two positions, 2 times every p candidates, thus not sieving p-2 pairs every p candidates.

Then, in order to compute how many candidate pairs are not sieved by any of such infinitely many primes greater than 5, one needs to compute the product of the infinitely many fractions p-2/p over all the primes p greater than 5.

When p tends to the infinity, both the numerator and the denominator of such product tend to the infinity with the same strength (even if their fraction tends to be infinitesimal towards zero, quite slowly indeed, because p-2/p tends to increase toward 1 when p increases).

Thus, when the denominator tends to the infinity, also the numerator does, proving that there are infinitely many pairs of sexy primes not sieved by any lower prime, then proving the conjecture.