Lemoine Circles Radius Calculus

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For the calculus of the first Lemoine Circle we will first prove:

1st Theorem (E. Lemoine – 1873).

The first Lemoine circle divides the sides of a triangle in segments proportional to the squares of the triangle's sides.

Each extreme segment is proportional to the corresponding adjacent side, and the chord-segment in the Lemoine circle is proportional to the square of the side that contains it.

Proof.

We will prove that $\frac{BC_2}{c^2} = \frac{C_2B_1}{a^2} = \frac{B_1C}{b^2}$.

In figure 1, K represents the symmedian center of the triangle ABC, and A_1A_2; B_1B_2; C_1C_2 represent Lemoine parallels.

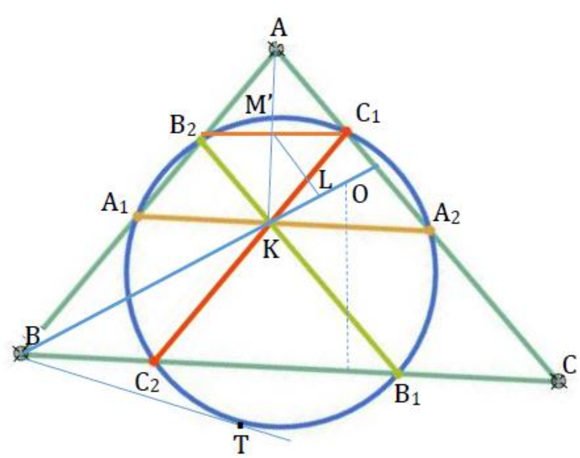

Figure 1

The triangles BC_2A_1; CB_1A_2 and KC_2A_1 have heights relative to the sides BC_2; B_1C and C_2B_1 equal ($A_1A_2 \parallel BC$).

Hence: $\frac{Area_\Delta\, BA_1C_2}{BC_2} = \frac{Area_\Delta\, KC_2A_1}{C_2B_1} = \frac{Area_\Delta\, CB_1A_2}{B_1C}$. (1)

On the other hand: A_1C_2 and B_1A_2 being antiparallels with respect to AC and AB, it follows that $\Delta BC_2A_1 \sim \Delta BAC$ and $\Delta CB_1A_2 \sim \Delta CAB$, likewise $KC_2 \parallel AC$ implies: $\Delta KC_2B_1 \sim \Delta ABC$.

We obtain:

$$\frac{Area_\Delta\, BC_2A_1}{Area_\Delta\, ABC} = \frac{BC_2^2}{c^2}\ ;\ \frac{Area_\Delta\, KC_2B_1}{Area_\Delta\, ABC} = \frac{C_2B_1^2}{a^2}\ ;\ \frac{Area_\Delta\, CB_1A_2}{Area_\Delta\, ABC} = \frac{CB_1^2}{b^2}. \quad (2)$$

If we note $Area_\Delta\, ABC = S$, we obtain from the relations (1) and (2):

$$\frac{BC_2}{c^2} = \frac{C_2B_1}{a^2} = \frac{B_1C}{b^2}.$$

Consequences.

C1. According to the 1st theorem we find that:

$$BC_2 = \frac{ac^2}{a^2+b^2+c^2};\ B_1C = \frac{ab^2}{a^2+b^2+c^2};\ B_1C_2 = \frac{a^3}{a^2+b^2+c^2}.$$

C2. We find that:

$\frac{B_1C_2}{a^3} = \frac{A_2C_1}{b^3} = \frac{A_1B_2}{c^3}$, meaning that:

"The chords determined by the first Lemoine circle on the triangle's sides are proportional to the cubes of the sides."

Due to this property, the first Lemoine circle is known in England by the name of triplicate ratio circle, Tucker.

1st Proposition.

The radius of the first Lemoine circle, R_{L_1} is given by the formula:

$$R_{L_1}^2 = \frac{1}{4} \cdot \frac{R^2(a^2+b^2+c^2) + a^2b^2c^2}{(a^2+b^2+c^2)^2}, \quad (3)$$

where R represents the radius of the circle inscribed in the triangle ABC.

Proof.

Let L be the center of the first Lemoine circle that is known to represent the middle of the segment (OK) – O being the center of the circle inscribed in the triangle ABC. Considering C1, we obtain $BB_1 = \frac{a\,(c^2+a^2)}{a^2+b^2+c^2}$.

Taking into account the power of point B in relation to the first Lemoine circle, we have:

$BC_2 \cdot BB_1 = BT^2 - LT^2$, ($BT$ is the tangent traced from B to the first Lemoine circle – see *Figure 1*)

Hence: $R_{L_1}^2 = BL^2 - BC_2 \cdot BB_1$. (4)

The median theorem in triangle BOK implies that:

$$BL^2 = \frac{2\cdot(BK^2+BO^2)-OK^2}{4}.$$

It is known that $K = \frac{(a^2+c^2)\cdot S_b}{a^2+b^2+c^2}$; $S_b = \frac{2ac\cdot m_b}{a^2+c^2}$, where S_b and m_b are the lengths of the symmedian and the median from B, and $OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)}$, see (3).

Consequently: $BK^2 = \frac{2a^2c^2(a^2+c^2)-a^2b^2c^2}{(a^2+b^2+c^2)^2}$, and

$$4BL^2 = R^2 + \frac{4a^2c^2(a^2+c^2)+a^2b^2c^2}{(a^2+b^2+c^2)^2}.$$

As: $BC_2 \cdot BB_1 = \frac{a^2c^2(a^2+c^2)}{(a^2+b^2+c^2)^2}$, by replacing in (4) we obtain formula (3).

2nd Proposition.

The radius of the second Lemoine circle, R_{L_2}, is given by the formula:

$$R_{L_2} = \frac{abc}{a^2+b^2+c^2} \quad (5)$$

Proof.

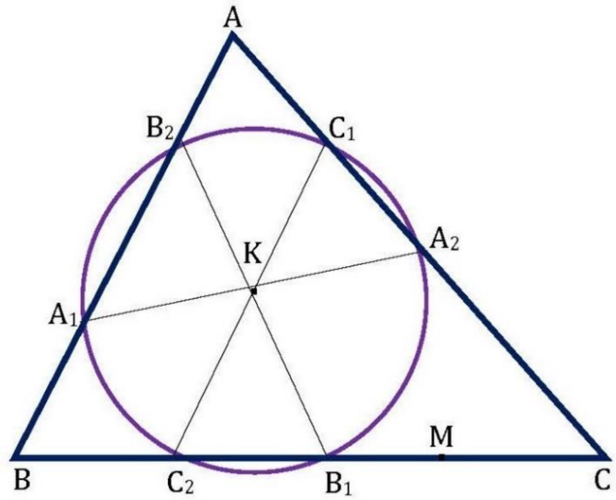

Figure 2

In *Figure 2*, A_1A_2; B_1B_2; C_1C_2 are Lemoine antiparallels traced through symmedian center K that is the center of the second Lemoine circle, thence:

$R_{L_2} = KA_1 = KA_2$.

If we note with S and M the feet of the symmedian and the median from A, it is known that:

$\frac{AK}{KS} = \frac{b^2+c^2}{a^2}$.

From the similarity of triangles AA_2A_1 and ABC, we have: $\frac{A_1A_2}{BC} = \frac{AK}{AM}$.

But: $\frac{AK}{AS} = \frac{b^2+c^2}{a^2+b^2+c^2}$ and $AS = \frac{2bc}{b^2+c^2} \cdot m_a$.

$A_1A_2 = 2R_{L_2}$, $BC = a$, consequently:

$R_{L_2} = \frac{AK \cdot a}{2m_a}$, and as $AK = \frac{2bc \cdot m_a}{a^2+b^2+c^2}$ formula (5) is the consequence.

Observations.

1. If we use $tg\omega = \frac{4S}{a^2+b^2+c^2}$, ω being the Brocard angle (see [2]), we obtain: $R_{L_2} = R \cdot tg\omega$.

2. If, in *Figure 1*, we denote with M_1 the middle of the antiparallel B_2C_1, which is equal to R_{L_2} (due to their similarity), we thus find from the rectangular triangle LM_1C_1 that:

$LC_1^2 = LM_1^2 + M_1C_1^2$, but $LM_1^2 = \frac{1}{4}a^2$ and $M_1C_2 = \frac{1}{2}R_{L_2}$; it follows that:

$$R_{L_1}^2 = \frac{1}{4}\left(R^2 + R_{L_2}^2\right) = \frac{R^2}{4}(1 + tg^2\omega).$$

We obtain:

$$R_{L_1} = \frac{R}{2} \cdot \sqrt{1 + tg^2\omega}\,.$$

3rd Proposition.

The chords determined by the sides of the triangle in the second Lemoine circle are respectively proportional to the opposing angles cosines.

Proof.

KC_2B_1 is an isosceles triangle, $\sphericalangle KC_2B_1 = \sphericalangle KB_1C_2 = \sphericalangle A$; as $KC_2 = R_{L_2}$ we have that $\cos A = \frac{C_2B_1}{2R_{L_2}}$, deci $\frac{C_2B_1}{\cos A} = 2R_{L_2}$, similary:

$$\frac{A_2C_1}{\cos B} = \frac{B_2A_1}{\cos C} = 2R_{L_2}.$$

Remark.

Due to this property of the Lemoine's second circle, in England this circle is known as the cosine circle.

References.

[1] D. Efremov, *The new geometry of the triangle*, translated in Romanian from Russian by Mihai Micutite, GIL Publiching, Zalau, 2010.

[2] F. Smarandache and I. Patrascu, *The Geometry of Homological Triangles*, The Education Publisher, Ohio, USA, 2012.

[3] I. Patrascu and F. Smarandache, *Variance on Topics of Plane Geometry*, Educational Publisher, Ohio, USA, 2013.