Chain of a potential electric field

Radi I. Khrapko
Moscow Aviation Institute - Volokolamskoe shosse 4, 125993 Moscow, Russia
Email: khrapko_ri@hotmail.com

Abstract
Examples are presented that geometrical images of generated electromagnetic fields are emitted by the geometrical images of the electromagnetic fields, which are the sources

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1. Boundary of a potential

A potential electric field can be obtain as a gradient of an electric potential:

$$\mathbf{E} = -\operatorname{grad}\varphi, \quad E_i = -\partial_i \varphi . \tag{1}$$

Gradient is a covector, so this electric field (1) is a covector field. The geometric image of a covector is two parallel plane elements equipped with an outer orientation (see Fig. 1[1]).

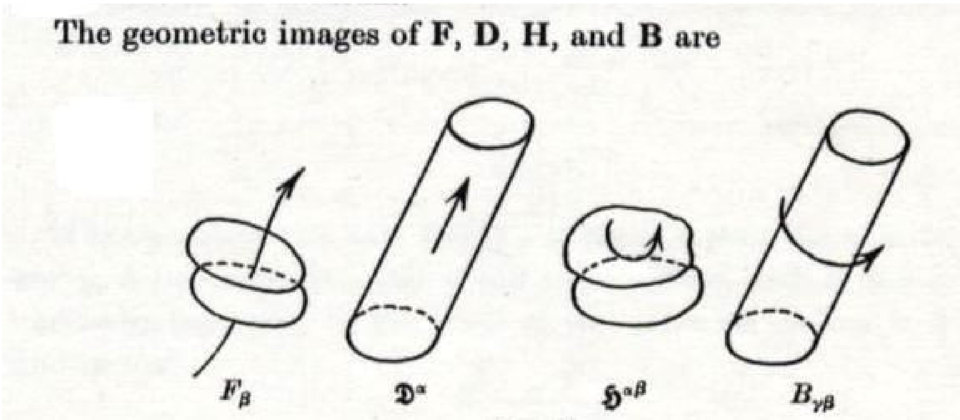


Fig. 1. Here $\mathbf{F} = \mathbf{E}$ is a covector E_i, $\mathbf{D}$ is a vector density $E^i_\wedge$, $\mathbf{H}$ is a bivector density $B^{ik}_\wedge$, $\mathbf{B}$ is a bicovector B_{ik}.

So, potential electric covector fields (1) are depicted by bisurfaces, not by field lines.

Meanwhile a scalar field, e.g. φ, may be depicted as a filling, which density is proportional to value of the scalar. Fig. 2c depicts roughly the potential of a charged sphere of radius R, $\varphi = 1/r$, $r > R$, and the corresponding covector field $\mathbf{E}$ (1). You see, the filling φ fills the closed bisurfaces of covector $\mathbf{E}$ (1), or the bisurfaces $\mathbf{E}$ bound the filling φ.

It may be said that the operation "gradient" creates a boundary of a scalar field and the field of gradient is a closed field, in correspondence with "boundary of a boundary is zero":

$$\operatorname{curl}\mathbf{E} = -\operatorname{curl}\operatorname{grad}\varphi = 0, \quad \partial_i E_k - \partial_k E_i = (-\partial_i\partial_k + \partial_k\partial_i)\varphi = 0 . \tag{2}$$

So, potential electric covector field (1) is a closed field and is depicted by closed bisurfaces, Fig. 2c.

It may be said that φ fills its boundary, which is the covector $\mathbf{E}$, and even that $\mathbf{E}$ generates φ, or $\mathbf{E}$ is a source of φ (in the sense that an area is generated by its boundary).

If we interested in the electric force, which exerts on a charge q and which is a vector, we must raise the lower index of the covector E_i by the metric tensor g^{ki}:

$$\mathbf{F} = q\mathbf{E}, \quad F^k = qE^k = qE_i g^{ik} . \tag{3}$$

[1] This is figure 23 from [1].

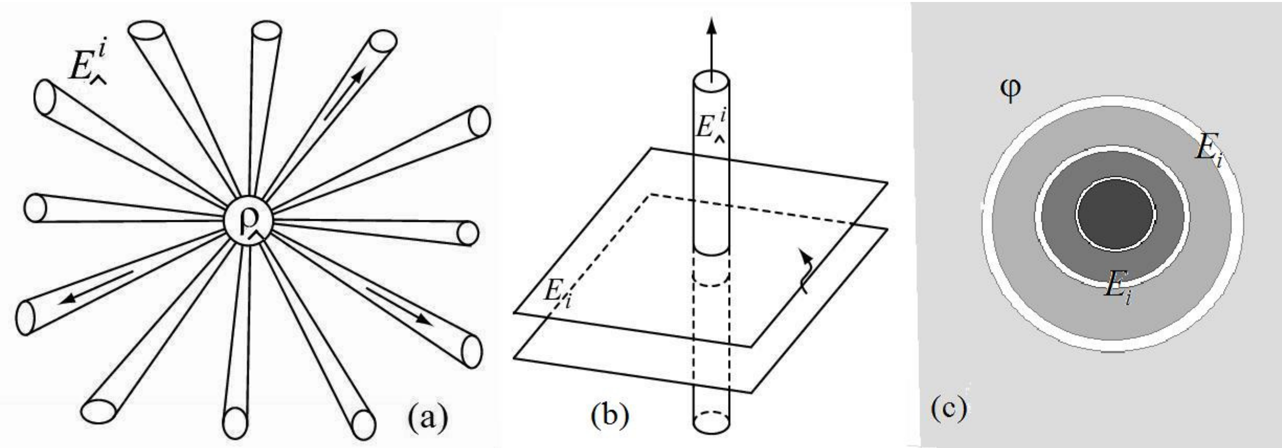


Fig. 2. Links of the potential electric field chain (6)
(a) A charge density $\rho_\wedge$ emits the tubes of electric vector density $E^i_\wedge$. **(b)** The conjugation. The vector density tube $E^i_\wedge$ changes into two parallel plane elements (bielement) of the covector E_i.
(c) The scalar field φ fills the bispheres of the covector field E_i.

2. Source of a potential electric field

The boundary of the covector potential field $\mathbf{E}$ (1) is zero, according to (2). But a potential electric field has a source. A charge density $\rho_\wedge$ is a source of the potential electric field, i.e. a charge density $\rho_\wedge$ generates the potential electric field, according to

$$\rho_\wedge = \operatorname{div}\mathbf{E}, \quad \rho_\wedge = \partial_i E^i_\wedge . \tag{4}$$

Therefore, the electric field has no boundary, but it has a source. How can this be?

Here we must recognize that the electromagnetism involves geometrical quantities of two types [1]. These are: covariant (antisymmetric) tensors, e.g. $\mathbf{E} = E_i$, $\mathbf{B} = B_{ik}$, which are named exterior differential forms or simply forms, and contravariant (antisymmetric) tensor *densities*, e.g. $\rho_\wedge$, $\mathbf{E} = E^i_\wedge$, $\mathbf{B} = B^{ik}_\wedge$ (the geometric images of E_i, $E^i_\wedge$, $B^{ik}_\wedge$, B_{ik}, see in Fig. 1). Mathematics and physicists often use Gothic fonts while writing densities. We do not use a gothic font; instead, we mark densities with the symbol "wedge" $\wedge$. For example, we name Schouten's displacement vector density $\mathfrak{D}^{\alpha}$ $E^i_\wedge$. This notation was used by Kunin in his Russian translation [2] of the monograph [1]. The square root of the metric tensor determinant, which is a scalar density of the weight $+1$, is denoted by $\sqrt{g}_\wedge$.

As you see, the potential electric field $E^i_\wedge$, which is generated by a charge density $\rho_\wedge$ according to (4), is a contravariant vector density. The geometric image of a vector density is a cylinder with an inner orientation. So this electric field is depicted by tubes emitted by the charge density $\rho_\wedge$ (Fig. 1a). Thus there are two different forms of the potential electric field. Covector potential electric field E_i (1) has no boundary, according to (2), but vector density potential electric field $E^i_\wedge$, according to (4), has charge density $\rho_\wedge$ as its source and its boundary.

3. The conjugation

The transition between covector E_i and vector density $E^i_\wedge$ is performed by the metric tensor density $g^{ik}_\wedge = g^{ik}\sqrt{g}_\wedge$, or $g^\wedge_{ik} = g_{ik}/\sqrt{g}_\wedge$. The transition is referred to as *the conjugation* [3,4] and is designated by the five-pointed asterisk $\star$ (in contrast to the Hodge star operation $*$), namely

$$\star E_i = g^{ik}_\wedge E_i = E^k_\wedge, \quad \star E^k_\wedge = g^\wedge_{ik} E^k_\wedge = E_i \tag{5}$$

The conjugation changes the geometric image of an electric field as it is shown in Fig. 1b.

4. Conclusion

So, we have the chain of the fields:

$$\rho_{\wedge} \quad \partial \quad E^{i}_{\wedge} \quad \star \quad E_i \quad \partial \quad \varphi \tag{6}$$

Our symbol ∂ designates differential operations: grad, or div, or curl. These operators create boundaries. In particular, grad creates a boundary of a scalar, div creates a boundary of a tensor density, curl creates a boundary of a differential form.

In chain (6), charge density $\rho_{\wedge}$ generates the vector density $E^{i}_{\wedge}$. The conjugation $\star$ transforms the vector density $E^{i}_{\wedge}$ into the closed covector E_i, which, in turn, generates potential φ.

References

[1] Schouten J A 1951 *Tensor Analysis for Physicists* (Oxford: Clarendon).

[2] Schouten J A 1965 *Tensor Analysis for Physicists. Тензорный анализ для физиков* (Nauka, Moscow).

[3] Khrapko R I 2011 V*isible representation of exterior differential forms and pseudo forms. Electromagnetism in terms of sources and generation of fields. Наглядное представление дифференциальных форм и псевдоформ. Электромагнетизм в терминах источников и порождений полей.* (Saarbrucken: Lambert). http://khrapkori.wmsite.ru/ftpgetfile.php?id=105&module=files

[4] Khrapko R I 2001 *Violation of the gauge equivalence* arXiv:physics/0105031

EJP quality

EJP Board does not know the difference between vector and covector and does not want to know. They rejected a paper **"Chain of a potential electric field"** EJP-101418 and ignore author's objection. Please see

Sent: Tuesday, August 25, 2015
BOARD MEMBER REPORT:
This short paper deals with geometry of the electromagnetic field. Most of the textbooks on electromagnetism ignore the geometrical subtleties; there is no need to discuss them if the analysis is restricted to the Cartesian coordinates.
The geometry of the electromagnetic field is of interest and I will have nothing against a paper on the geometry of the electromagnetic fields.
The present analysis is misleading. First, there is an important difference between the E field (electric field) and the D field(induction field). The E field gives the force acting on a charge, the D field is generated by charge density. In the units used in the paper (and in many textbooks) both vectors are equal (E=D), and the difference between them is not transparent. (Geometrical differences between E and D are more transparent in the SI units). One of the Maxwell equations (div D= \rho) relates D (rather than E) to \rho. The other equation(in the static case), curl E =0 is for the E field. The author avoids to specify the relation between E and D, instead he uses vectors with upper and lower indices (vectors and convectors). This is correct in principle, but requires introduction of the metric tensor. In the Cartesian coordinates this tensor is equal to 1, and there is no difference between vectors and convectors. The difference becomes clear in curvilinear coordinates. However the role of the metric tensor is not explained in the paper.
The pictorial presentation of vectors, vector densities etc., as presented, is not helpful at all in understanding the underlying geometrical structure, in my opinion at least. The pictures presented do not build intuition regarding the electromagnetic field.
I do not think that the paper is acceptable for publication.

Author's objection.

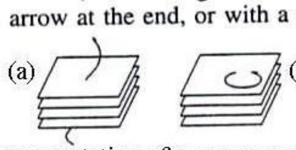
Dear Editors, thank you for the report, but

Geometric images of geometrical quantities are independent of units used and of coordinates in use. Referee believes that this is not the case. It is inadmissible.

In reality, a vector density is depicted by a tube element, and a covector is depicted by two parallel plane elements no matter what coordinates are used, curvilinear or Cartesian. In particular, the vector density **D**, which satisfies $\mathrm{div}\mathbf{D} = \rho$, is depicted by a tube element, and covector **E**, which satisfies $\mathbf{E} = -\mathrm{grad}\,\varphi$, is depicted by two parallel plane elements. **D** is not equal to covector **E** no matter what coordinates and units are used. Referee believes that this is not the case, and it is inadmissible.

The objective difference between vector density and covector, i.e. between **D** and **E**, is indicated not only by Schouten. J. Napolitano and R. Lichtenstein explained the difference in “Answer to Question #55, Are the pictorial examples that distinguish covariant and contravariant vectors?” Am. J. Phys. **65**,1037-1038 (1997)

Now consider a covector. This should be familiar to most students in terms of a gradient. We can picture a gradient best in terms of the equipotential surfaces to which it refers, and this is the basis of the pictorial representation. That is, draw the surfaces themselves, along with some sense of direction, which might be indicated by a wavy line with an arrow at the end, or with a whorl on one of the sheets:



The authors refer to this pictorial representation of a covector as a "lasagna vector". Covector (a) has an outer orientation. (b) represents a pseudo covector; it has an inner orientation.

A force, which satisfies **F** = -grad U, is a covector, not a vector no matter what coordinates and units are used. If Referee wants force as a vector, he must use the metric tensor $F^i = g^{ik} F_k$. The metric tensor equals the Kronecker symbol in the Cartesian coordinates, but the tensor has an important geometrical sense: it implements the transition between a covector and a vector, as I wrote in my papers.

In whole, the Referee's comment is not adequate to my paper "Chain of a potential electric field". I do not use **D** in this paper at all, and I use **E** as a gradient. Referee does not see this. Referee does not see the paper at all. It is unacceptable.

The delusions of Referee prove urgency of my papers "Chain of a potential electric field": EJP-101418 and "Depicting of electric fields": EJP-101291. If these papers seem to be too short, I am ready to present a more detailed content.