Non-archimedean analysis on the extended hyperreal line ${}^*\mathbb{R}_{\mathbf{d}}$ and the solution of some very old transcendence conjectures over the field $\mathbb{Q}$.

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Abstract

In 1980 F. Wattenberg constructed the Dedekind completion ${}^*\mathbb{R}_{\mathbf{d}}$ of the Robinson non-archimedean field ${}^*\mathbb{R}$ and established basic algebraic properties of ${}^*\mathbb{R}_{\mathbf{d}}$ [6]. In 1985 H. Gonshor established further fundamental properties of ${}^*\mathbb{R}_{\mathbf{d}}$ [7].In [4] important construction of summation of countable sequence of Wattenberg numbers was proposed and corresponding basic properties of such summation were considered. In this paper the important applications of the Dedekind completion ${}^*\mathbb{R}_{\mathbf{d}}$ in transcendental number theory were considered. Given any analytic function of one complex variable $f \in \mathbb{Q}[[z]]$, we investigate the arithmetic nature of the values of $f(z)$ at transcendental points $e^n, n \in \mathbb{N}$. Main results are: (i) the both numbers $e+\pi$ and $e \times \pi$ are irrational, (ii) number e^e is transcendental. Nontrivial generalization of the Lindemann-Weierstrass theorem is obtained.

Keywords

Non-archimedean analysis,**Robinson transfer**, **Robinson non-archimedian field**,**Dedekind completion**, **Dedekind hyperreals**,**Wattenberg embeding**,**Gonshor idempotent theory**,**Gonshor transfer**.

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1.Introduction.

In 1873 French mathematician, Charles Hermite, proved that e is transcendental. Coming as it did 100 years after Euler had established the significance of e, this meant that the issue of transcendence was one mathematicians could not afford to ignore.Within 10 years of Hermite's breakthrough,his techniques had been extended by Lindemann and used to add π to the list of known transcendental numbers. Mathematician then tried to prove that other numbers such as $e+\pi$ and $e\times\pi$ are transcendental too,but these questions were too difficult and so no further examples emerged till today's time. The transcendence of e^{π} has been proved in1929 by A.O.Gel'fond.

Conjecture 1. Whether the both numbers $e+\pi$ and $e\times\pi$ are irrational.

Conjecture 2. Whether the numbers e and π are algebraically independent.

However, the same question with e^{π} and π has been answered:

Theorem.(Nesterenko, 1996 [1]) The numbers e^{π} and π are algebraically independent.

Throughout of 20-th century,a typical question: whether $f(\alpha)$ is a transcendental number for each algebraic number α has been investigated and answered many authors.Modern result in the case of entire functions satisfying a linear differential equation provides the strongest results, related with Siegel's E-functions [1],[2].Reference [1] contains references to the subject before 1998, including Siegel E and G functions.

Theorem.(Siegel C.L.) Suppose that $\lambda\in\mathbb{Q},\lambda\neq -1,-2,\ldots,\alpha\neq 0.$

$$\varphi_{\lambda}(z)=\sum_{n=0}^{\infty}\frac{z^n}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}. \tag{1.1}$$

Then $\varphi_{\lambda}(\alpha)$ is a transcendental number for each algebraic number $\alpha\neq 0$.

Let f be an analytic function of one complex variable $f\in\mathbb{Q}[[z]]$.

Conjecture 3.Whether $f(\alpha)$ is an irrational number for given transcendental number α.

Conjecture 4.Whether $f(\alpha)$ is a transcendental number for given transcendental number α.

In this paper we investigate the arithmetic nature of the values of $f(z)$ at transcendental points $e^n, n\in\mathbb{N}$.

Definition 1.1. Let $g(x):\mathbb{R}\to\mathbb{R}$ be any real analytic function such that

$$g_{\mathbb{Q}}(x)=\sum_{n=0}^{\infty}a_n x^n, |x|<r, \forall n[a_n\in\mathbb{Q}]. \tag{1.2}$$

We will call any function given by Eq.(1.2) $\mathbb{Q}$-analytic function and denoted by $g_{\mathbb{Q}}(x)$.

Definition 1.2.[3],[4]. A transcendental number $z\in\mathbb{R}$ is called #-transcendental number over field $\mathbb{Q}$, if there does not exist $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z)=0$,i.e. for every $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ the inequality $g_{\mathbb{Q}}(z)\neq 0$ is satisfied.

Definition 1.3.[3],[4].A transcendental number z is called w-transcendental number over field $\mathbb{Q}$,if z is not #-transcendental number over field $\mathbb{Q}$,i.e.there exists $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z)=0$.

Example. Number π is transcendental but number π is not #-transcendental number

over field $\mathbb{Q}$ as
(**1**) function $\sin x$ is a $\mathbb{Q}$-analytic and
(**2**) $\sin\left(\frac{\pi}{2}\right) = 1$,i.e.

$$-1 + \frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \ldots + \frac{(-1)^{2n+1}\pi^{2n+1}}{2^{2n+1}(2n+1)!} + \ldots = 0. \tag{1.3}$$

Main results are.
Theorem 1.1.[3],[4].Number e is #-transcendental over $\mathbb{Q}$.
From theorem 1.1 immediately follows.
Theorem 1.2.Number e^e is transcendental.
Theorem 1.3.[3],[4]. The both numbers $e + \pi$ and $e - \pi$ are irrational.
Theorem 1.4.For any $\xi \in \mathbb{Q}$ number e^{ξ} is #-transcendental over the field $\mathbb{Q}$.
Theorem 1.5.[3],[4]. The both numbers $e \times \pi$ and $e^{-1} \times \pi$ are irrational.
Theorem 1.6.[4] Let $f_l(z), l = 1,2,\ldots$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that for any $l \in \mathbb{N}$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l}$, $k_l \geq 1, l = 1,2,\ldots$ form a complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, \tag{1.4}$$

and $a_l \in \mathbb{Z}, l = 1,2,\ldots; a_0 \neq 0.$ Assume that

$$a_0 + \sum_{l=1}^{\infty} |a_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{1.5}$$

Then

$$a_0 + \sum_{l=1}^{\infty} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \tag{1.6}$$

2. Preliminaries.Short outline of Dedekind hyperreals and Gonshor idempotent theory

Let $\mathbb{R}$ be the set of real numbers and ${}^*\mathbb{R}$ a nonstandard model of $\mathbb{R}$ [5]. ${}^*\mathbb{R}$ is not Dedekind complete.For example, $\mu(0) = \{x \in {}^*\mathbb{R} \mid x \approx 0\}$ and $\mathbb{R}$ are bounded subsets of ${}^*\mathbb{R}$ which have no suprema or infima in ${}^*\mathbb{R}$.Possible completion of the field ${}^*\mathbb{R}$ can be constructed by Dedekind sections [6],[7]. In [6] Wattenberg constructed the Dedekind completion of a nonstandard model of the real numbers and applied the construction to obtain certain kinds of special measures on the set of integers. Thus was established that the Dedekind completion ${}^*\mathbb{R}_{\mathbf{d}}$ of the field ${}^*\mathbb{R}$ is a structure of interest not for its own sake only and we establish further important applications here. Important concept introduced by Gonshor [7] is that of the **absorption number** of an element $\mathbf{a} \in {}^*\mathbb{R}_{\mathbf{d}}$ which, roughly speaking, measures the degree to which the cancellation law $\mathbf{a}+b=\mathbf{a}+c \Rightarrow b=c$ fails for $\mathbf{a}$.

2.1 The Dedekind hyperreals ${}^*\mathbb{R}_{\mathbf{d}}$

Definition 2.1. Let ${}^*\mathbb{R}$ be a nonstandard model of $\mathbb{R}$ and $P({}^*\mathbb{R})$ the power set of ${}^*\mathbb{R}$.

A Dedekind hyperreal $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \alpha \notin {}^*\mathbb{R}$ is an ordered pair $\{U,V\} \in P({}^*\mathbb{R}) \times P({}^*\mathbb{R})$ that

satisfies the next conditions:

1.$\exists x \exists y(x \in U \wedge y \in V)$. 2. $U \cap V = \varnothing$. 3.$\forall x(x \in U \Leftrightarrow \exists y(y \in V \wedge x < y))$.

4. $\forall x(x \in V \Leftrightarrow \exists y(y \in V \wedge x < y))$. 5. $\forall x \forall y(x < y \Rightarrow x \in U \vee y \in V)$.

Compare the Definition 2.1 with original Wattenberg definition [6],(see [6] def.II.1).

Designation 2.1. Let $\{U,V\} \triangleq \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. We designate in this paper

$$U \triangleq \mathbf{cut}_{-}(\alpha), V \triangleq \mathbf{cut}_{+}(\alpha)$$

$$\alpha \triangleq \{\mathbf{cut}_{-}(\alpha), \mathbf{cut}_{+}(\alpha)\}$$

Designation 2.2. Let $\alpha \in {}^*\mathbb{R}$. We designate in this paper

$$\alpha^{\#} \triangleq \mathbf{cut}_{-}(\alpha), \alpha_{\#} \triangleq \mathbf{cut}_{+}(\alpha)$$

$$\alpha \triangleq \{\alpha^{\#}, \alpha_{\#}\}$$

Remark 2.1. The monad of $\alpha \in {}^*\mathbb{R}$ is the set: $\left\{x \in {}^*\mathbb{R} \mid x \approx \alpha\right\}$ is denoted by $\mu(\alpha)$.

Supremum of $\mu(0)$ is denoted by $\varepsilon_{\mathbf{d}}$. Supremum of $\mathbb{R}$ is denoted by $\Delta_{\mathbf{d}}$. Note that [6]

$$\varepsilon_{\mathbf{d}} = {}^*(-\infty,0] \cup \mu(0),$$

$$\Delta_{\mathbf{d}} = \bigcup_{n \in \mathbb{N}} [{}^*(-\infty,n)].$$

Let A be a subset of ${}^*\mathbb{R}$ bounded above. Then $\sup(A)$ exists in ${}^*\mathbb{R}_{\mathbf{d}}$ [6].

Example 2.1. (i) $\Delta_{\mathbf{d}} = \sup(\mathbb{R}_{+}) \in {}^*\mathbb{R}_{\mathbf{d}} \backslash {}^*\mathbb{R}$, (ii) $\varepsilon_{\mathbf{d}} = \sup\big(\mu(0)\big) \in {}^*\mathbb{R}_{\mathbf{d}} \backslash {}^*\mathbb{R}$.

Remark 2.2. Unfortunately the set ${}^*\mathbb{R}_{\mathbf{d}}$ inherits some but by **no means all** of the algebraic structure on ${}^*\mathbb{R}$. For example, ${}^*\mathbb{R}_{\mathbf{d}}$ is not a group with

respect to addition since if $x +_{{}^*\mathbb{R}_\mathbf{d}} y$ denotes the addition in ${}^*\mathbb{R}_\mathbf{d}$ then:
$\varepsilon_\mathbf{d} +_{{}^*\mathbb{R}_\mathbf{d}} \varepsilon_\mathbf{d} = \varepsilon_\mathbf{d} +_{{}^*\mathbb{R}_\mathbf{d}} 0_{{}^*\mathbb{R}_\mathbf{d}} = \varepsilon_\mathbf{d}$. Thus ${}^*\mathbb{R}_\mathbf{d}$ is not even a ring but pseudo-ring only.

Definition 2.2 We define:

1.The additive identity (zero cut) $0_{{}^*\mathbb{R}_\mathbf{d}}$, often denoted by $0^\#$ or simply 0 is
$0_{{}^*\mathbb{R}_\mathbf{d}} \triangleq \left\{x \in {}^*\mathbb{R} | \ x < 0_{{}^*\mathbb{R}}\right\}$.

2.The multiplicative identity $1_{{}^*\mathbb{R}_\mathbf{d}}$, often denoted by $1^\#$ or simply 1 is
$1_{{}^*\mathbb{R}_\mathbf{d}} \triangleq \left\{x \in {}^*\mathbb{R} | \ x <_{{}^*\mathbb{R}} 1_{{}^*\mathbb{R}}\right\}$.

Given two Dedekind hyperreal numbers $\alpha \in {}^*\mathbb{R}_\mathbf{d}$ and $\beta \in {}^*\mathbb{R}_\mathbf{d}$ we define:

3. Addition $\alpha +_{{}^*\mathbb{R}_\mathbf{d}} \beta$ of α and β often denoted by $\alpha + \beta$ is
$\alpha + \beta \triangleq \left\{x + y | \ x \in \alpha, y \in \beta\right\}$.

It is easy to see that $\alpha +_{{}^*\mathbb{R}_\mathbf{d}} 0_{{}^*\mathbb{R}_\mathbf{d}} = \alpha$ for all $\alpha \in {}^*\mathbb{R}_\mathbf{d}$.

It is easy to see that $\alpha +_{{}^*\mathbb{R}_\mathbf{d}} \beta$ is again a cut in ${}^*\mathbb{R}$ and $\alpha +_{{}^*\mathbb{R}_\mathbf{d}} \beta = \beta +_{{}^*\mathbb{R}_\mathbf{d}} \alpha$.

Another fundamental property of cut addition is associativity:
$(\alpha +_{{}^*\mathbb{R}_\mathbf{d}} \beta) +_{{}^*\mathbb{R}_\mathbf{d}} \gamma = \alpha +_{{}^*\mathbb{R}_\mathbf{d}} (\beta +_{{}^*\mathbb{R}_\mathbf{d}} \gamma)$.

This follows from the corresponding property of ${}^*\mathbb{R}$.

4.The opposite $-_{{}^*\mathbb{R}_\mathbf{d}} \alpha$ of α, often denoted by $(-\alpha)^\#$ or simply by $-\alpha$, is
$-\alpha \triangleq \left\{x \in {}^*\mathbb{R} | \ -x \notin \alpha, -x \text{ is not the least element of } {}^*\mathbb{R} \backslash \alpha\right\}$

5.We say that the cut α is positive if $0^\# < \alpha$ or negative if $\alpha < 0^\#$.

The absolute value of α, denoted $|\alpha|$, is $|\alpha| \triangleq \alpha$, if $\alpha \geq 0$ and $|\alpha| \triangleq -\alpha$, if $\alpha \leq 0$

6.If $\alpha, \beta > 0$ then multiplication $\alpha \times_{{}^*\mathbb{R}_\mathbf{d}} \beta$ of α and β often denoted $\alpha \times \beta$ is
$\alpha \times \beta \triangleq \left\{z \in {}^*\mathbb{R} | \ z = x \times y \text{ for some } x \in \alpha, y \in \beta \text{ with } x, y > 0\right\}$.

In general, $\alpha \times \beta = 0$ if $\alpha = 0$ or $\beta = 0$,
$\alpha \times \beta \triangleq |\alpha| \times |\beta|$ if $\alpha > 0, \beta > 0$ or $\alpha < 0, \beta < 0$,
$\alpha \times \beta \triangleq -(|\alpha| \cdot |\beta|)$ if $\alpha > 0, \beta < 0$, or $\alpha < 0, \beta > 0$.

7. The cut order enjoys on ${}^*\mathbb{R}_\mathbf{d}$ the standard additional properties of:

(**i**) transitivity: $\alpha \leq \beta \leq \gamma \Rightarrow \alpha \leq \gamma$.

(**ii**) trichotomy: eizer $\alpha < \beta, \beta < \alpha$ or $\alpha = \beta$ but only one of the three

(**iii**) translation: $\alpha \leq \beta \Rightarrow \alpha +_{{}^*\mathbb{R}_\mathbf{d}} \gamma \leq \beta +_{{}^*\mathbb{R}_\mathbf{d}} \gamma$.

2.2 The Wattenberg embeding ${}^*\mathbb{R}$ into ${}^*\mathbb{R}_\mathbf{d}$

Definition 2.3.[6]. Wattenberg hyperreal or #-hyperreal is a nonepty subset $\alpha \subsetneqq {}^*\mathbb{R}$ such that:

(**i**) For every $a \in \alpha$ and $b < a$, $b \in \alpha$.

(**ii**) $\alpha \neq \emptyset, \alpha \neq {}^*\mathbb{R}$.

(**iii**) α has no greatest element.

Definition 2.4.[6].In paper [6] Wattenberg embed ${}^*\mathbb{R}$ into ${}^*\mathbb{R}_\mathbf{d}$ by following way: if $\alpha \in {}^*\mathbb{R}$ the corresponding element, $\alpha^\#$, of ${}^*\mathbb{R}_\mathbf{d}$ is

$$\alpha^\# = \left\{x \in {}^*\mathbb{R} \middle| x < \alpha\right\} \tag{2.1}$$

Remark 2.3.[6]. In paper [6] Wattenberg pointed out that condition (iii) above is included only to avoid nonuniqueness. Without it $\alpha^\#$ would be represented by both

$\alpha^{\#}$and $\alpha^{\#} \cup \{\alpha\}$.

Remark 2.4.[7]. However in paper [7] H. Gonshor pointed out that the definition (2.1) in Wattenberg paper [6] is technically incorrect. Note that Wattenberg [6] defines $-\alpha$ in general by

$$-\alpha = \left\{a \in {}^{*}\mathbb{R} \,\middle|\, -a \notin \alpha\right\}. \tag{2.2}$$

If $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ i.e. ${}^{*}\mathbb{R}_{\mathbf{d}} \backslash \alpha$ has no mininum, then there is no any problem with definitions (2.1) and (2.2). However if $\alpha = a^{\#}$ for some $a \in {}^{*}\mathbb{R}$, i.e. $\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \,\middle|\, x < a\right\}$ then according to the latter definition (2.2)

$$-\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \,\middle|\, x \leq -a\right\} \tag{2.3}$$

whereas the definition of ${}^{*}\mathbb{R}_{\mathbf{d}}$ requires that:

$$-\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \,\middle|\, x < -a\right\}, \tag{2.4}$$

but this is a contradiction.

Remark 2.5.Note that in the usual treatment of Dedekind cuts for the ordinary real numbers both of the latter sets are regarded as equivalent so that no serious problem arises [7].

Remark 2.6.H.Gonshor [7] defines $-\alpha^{\#}$ by

$$-\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \,\middle|\, \exists b[b > a \wedge -b \notin a]\right\}, \tag{2.5}$$

Definition 2.5. (**Wattenberg embeding**) We embed ${}^{*}\mathbb{R}$ into ${}^{*}\mathbb{R}_{\mathbf{d}}$ of the following way: (i) if $\alpha \in {}^{*}\mathbb{R}$, the corresponding element $\alpha^{\#}$ of ${}^{*}\mathbb{R}_{\mathbf{d}}$ is

$$\alpha^{\#} \triangleq \left\{x \in {}^{*}\mathbb{R} | x \leq_{{}^{*}\mathbb{R}} \alpha\right\} \tag{2.6}$$

and

$$-\alpha^{\#} = \left\{a \in {}^{*}\mathbb{R} \,\middle|\, -a \notin \alpha\right\} \cup \{\alpha\}. \tag{2.7}$$

or in the equivalent way,i.e. if $\alpha \in {}^{*}\mathbb{R}$ the corresponding element $\alpha_{\#}$ of ${}^{*}\mathbb{R}_{\mathbf{d}}$ is

$$\alpha_{\#} \triangleq \left\{x \in {}^{*}\mathbb{R} | x \;{}_{{}^{*}\mathbb{R}}\geq \alpha\right\} \tag{2.8}$$

Thus if $\alpha \in {}^{*}\mathbb{R}$ then $\alpha^{\#} \triangleq A|B$ where

$$A = \left\{x \in {}^{*}\mathbb{R} | x \leq_{{}^{*}\mathbb{R}} \alpha\right\}, B = \left\{y \in {}^{*}\mathbb{R} | y \;{}_{{}^{*}\mathbb{R}}\geq \alpha\right\}. \tag{2.9}$$

Such embeding ${}^{*}\mathbb{R}$ into ${}^{*}\mathbb{R}_{\mathbf{d}}$ Such embeding we will name Wattenberg embeding and to designate by ${}^{*}\mathbb{R} \overset{\#}{\hookrightarrow} {}^{*}\mathbb{R}_{\mathbf{d}}$

Lemma 2.1.[6].

(**i**) Addition $(\circ +_{{}^*\mathbb{R}_{\mathbf{d}}} \circ)$ is commutative and associative in ${}^*\mathbb{R}_{\mathbf{d}}$.

(**ii**) $\forall \alpha \in {}^*\mathbb{R}_{\mathbf{d}} : \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} 0_{{}^*\mathbb{R}_{\mathbf{d}}} = \alpha$.

(**iii**) $\forall \alpha, \beta \in {}^*\mathbb{R} : \alpha^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta^{\#} = (\alpha +_{{}^*\mathbb{R}} \beta)^{\#}$.

Remark 2.7. Notice, here again something is lost going from ${}^*\mathbb{R}$ to ${}^*\mathbb{R}_{\mathbf{d}}$ since $a < \beta$ does not imply $\alpha + \alpha < \beta + \alpha$ since $0 < \varepsilon_{\mathbf{d}}$ but $0 + \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}} + \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}}$.

Lemma 2.2.[6].

(**i**) $\leq_{{}^*\mathbb{R}_{\mathbf{d}}}$ a linear ordering on ${}^*\mathbb{R}_{\mathbf{d}}$ often denoted $\leq$, which extends the usual ordering on ${}^*\mathbb{R}$.

(**ii**) $(\alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha') \wedge (\beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta') \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$.

(**iii**) $\left(\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha'\right) \wedge \left(\beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'\right) \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$.

(**iv**) ${}^*\mathbb{R}$ is dense in ${}^*\mathbb{R}_{\mathbf{d}}$. That is if $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$ in ${}^*\mathbb{R}_{\mathbf{d}}$ there is an $a \in {}^*\mathbb{R}$ then

$\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} a^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$.

(**v**) Suppose that $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ is bounded above then $\sup A = \sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \mathbf{cut}_{-}(\alpha)$ exist in ${}^*\mathbb{R}_{\mathbf{d}}$.

(**vi**) Suppose that $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ is bounded below then $\inf A = \inf_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \mathbf{cut}_{+}(\alpha)$ exist in ${}^*\mathbb{R}_{\mathbf{d}}$.

Remark 2.8.Note that in general case $\inf A = \inf_{\alpha \in A} \alpha \neq \bigcap_{\alpha \in A} \mathbf{cut}_{-}(\alpha)$. In particular the formula for $\inf A$ given in [6] on the top of page 229 is not quite correct [7], see Example 2.2. However by Lemma 2.2 (vi) this is no problem.

Example 2.2.[7].The formula $\inf A = \inf_{\alpha \in A} = \bigcap_{\alpha \in A} \mathbf{cut}_{-}(\alpha)$ says

$$\inf_{\alpha \in A} = \left\{ a \,\middle|\, \exists d(d > 0) \left[a + d \in \bigcap_{\alpha \in A} \mathbf{cut}_{-}(\alpha) \right] \right\}$$

Let A be the set $A = \{a + d\}$ where d runs through the set of all positive numbers in ${}^*\mathbb{R}$, then $\inf A = a = \{x | x < a\}$. However $\bigcap_{\alpha \in A} \mathbf{cut}_{-}(\alpha) = \{x | x \leq a\}$.

Lemma 2.3.[6].

(**i**) If $\alpha \in {}^*\mathbb{R}$ then $-_{{}^*\mathbb{R}_{\mathbf{d}}} (\alpha^{\#}) = (-_{{}^*\mathbb{R}} \alpha)^{\#}$.

(**ii**) $-_{{}^*\mathbb{R}_{\mathbf{d}}}(-_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha) = \alpha$.

(**iii**) $\alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \Leftrightarrow -_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} -_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$.

(**iv**) $(-_{{}^*\mathbb{R}_{\mathbf{d}}}\alpha) +_{{}^*\mathbb{R}_{\mathbf{d}}} \left(-_{{}^*\mathbb{R}_{\mathbf{d}}} \beta\right) \leq_{{}^*\mathbb{R}_{\mathbf{d}}} -_{{}^*\mathbb{R}_{\mathbf{d}}}\left(\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta\right)$.

(**v**) $\forall a \in {}^*\mathbb{R} : (-_{{}^*\mathbb{R}} a)^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \left(-_{{}^*\mathbb{R}_{\mathbf{d}}} \beta\right) = -_{{}^*\mathbb{R}_{\mathbf{d}}} (a^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)$.

(**vi**) $\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} (-_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha) \leq_{{}^*\mathbb{R}_{\mathbf{d}}} 0_{{}^*\mathbb{R}_{\mathbf{d}}}$.

Proof.(**v**) By (iv): $(-a)^{\#} + (-\beta) \leq -(a^{\#} + \beta)$.

(1) Suppose now $c \in -(a^{\#} + \beta)$ this means

(2) $\exists c_1 [c < c_1 \in -(a^{\#} + \beta)]$ and therefore

(3) $-c_1 \notin (a^{\#}+\beta)$.

(4) Note that: $-c-a \notin \beta$ (since $-c-a \in \beta$ and $a-(c-c_1) \in a^{\#}$ imply $-c_1 = a-(c-c_1)+(-c-a) \in a^{\#}+\beta$ but this is a contradiction)

(5) Thus $-c-a \in \beta$ and therefore $c+a \in -\beta$.

(6) By similar reasoning one obtain: $c_1+a \in -\beta$.

(7) Note that: $-a-(c_1-c) \in a^{\#}$ and therefore $c = -a-(c_1-c)+(c_1+a) \in (-a)^{\#}+(-\beta)$.

Lemma 2.4.(**i**) $\forall a \in {}^*\mathbb{R}, \forall \beta \in {}^*\mathbb{R}_{\mathbf{d}}, \mu \in {}^*\mathbb{R}, \mu \geq 0 : (-\mu a)^{\#}+(-\mu^{\#}\beta) = -\mu^{\#}(a^{\#}+\beta)$,

(**ii**) $\forall a \in {}^*\mathbb{R}, \forall \beta \in {}^*\mathbb{R}_{\mathbf{d}}, \mu \in {}^*\mathbb{R}, \mu \geq 0 : (\mu a)^{\#}+\mu^{\#}\beta = \mu^{\#}(a^{\#}+\beta)$.

Proof.(**i**) For $\mu = 0$ the statement is clear. Suppose now without loss of generality $\mu > 0$. By Lemma 2.3.(iv): $(-\mu a)^{\#}+(-\mu^{\#}\beta) \leq -(\mu^{\#}a^{\#}+\mu^{\#}\beta)$.

(1) Suppose $c \in -\mu^{\#}(a^{\#}+\beta)$ and therefore $\frac{c}{\mu} \in -(a^{\#}+\beta)$, but this means

(2) $\exists c_1\left[\frac{c}{\mu} < \frac{c_1}{\mu} \in -(a^{\#}+\beta)\right]$ and therefore

(3) $-\frac{c_1}{\mu} \notin (a^{\#}+\beta)$.

(4) Note that: $-\frac{c}{\mu}-a \notin \beta$ (since $-\frac{c}{\mu}-a \in \beta$ and $a-\left(\frac{c}{\mu}-\frac{c_1}{\mu}\right) \in a^{\#}$ imply $-\frac{c_1}{\mu} = a-\left(\frac{c}{\mu}-\frac{c_1}{\mu}\right)+\left(-\frac{c}{\mu}-a\right) \in a^{\#}+\beta$ but this is a contradiction)

(5) Thus $-\frac{c}{\mu}-a \in \beta$ and therefore $c+\mu a \in -\mu^{\#}\beta$.

(6) By similar reasoning one obtain: $c_1+\mu a \in -\mu^{\#}\beta$.

(7) Note that: $-\mu a-(c_1-c) \in \mu^{\#}a^{\#}$ and therefore $c = -\mu a-(c_1-c)+(c_1+\mu a) \in (-\mu a)^{\#}+(-\mu^{\#}\beta)$.

(**ii**) Immediately follows from (**i**) by Lemma 2.3.

Definition 2.6. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. The absolute value of α written $|\alpha|$ is defined as follows:

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \;_{^*\mathbb{R}_{\mathbf{d}}}\geq 0_{^*\mathbb{R}_{\mathbf{d}}} \\ -_{^*\mathbb{R}_{\mathbf{d}}}\alpha & \text{if } \alpha \leq_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \end{cases}$$

Definition 2.7. Suppose $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{d}}$. The product $\alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta$, is defined as follows: **Case** (**1**) $\alpha, \beta \;_{^*\mathbb{R}_{\mathbf{d}}}> 0_{^*\mathbb{R}_{\mathbf{d}}}$:

$$\alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq \{a \times_{^*\mathbb{R}} b | (0_{^*\mathbb{R}_{\mathbf{d}}} <_{^*\mathbb{R}_{\mathbf{d}}} a^{\#} <_{^*\mathbb{R}_{\mathbf{d}}} \alpha) \wedge (0_{^*\mathbb{R}_{\mathbf{d}}} <_{^*\mathbb{R}_{\mathbf{d}}} b^{\#} <_{^*\mathbb{R}_{\mathbf{d}}} \beta)\} \cup ({}^*-\infty, {}^*0)^{\#}. \tag{2.10}$$

Case (**2**) $\alpha =_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \vee \beta =_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} : \alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq 0_{^*\mathbb{R}_{\mathbf{d}}}$.

Case (**3**) $(\alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}) \vee (\beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}) \vee (\alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \wedge \beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}})$

$$\begin{cases} \alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq |\alpha| \times_{^*\mathbb{R}_{\mathbf{d}}} |\beta| \text{ iff } \alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \wedge \beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}, \\ \alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq -_{^*\mathbb{R}_{\mathbf{d}}} (|\alpha| \times_{^*\mathbb{R}_{\mathbf{d}}} |\beta|) \text{ iff } (\alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}) \vee (\beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}). \end{cases} \tag{2.11}$$

Lemma 2.5.[6]. (**i**) $\forall a,b \in {}^*\mathbb{R} : (a \times_{{}^*\mathbb{R}} b)^{\#} = a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#}$.

(**ii**) Multiplication $(\bullet \times_{{}^*\mathbb{R}} \bullet)$ is associative and commutative:

$$(\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta) \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma = \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma),\ \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta = \beta \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha. \quad (2.12)$$

(**iii**) $1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha = \alpha;\ -1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha = -_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$, where $1_{{}^*\mathbb{R}_{\mathbf{d}}} = (1_{{}^*\mathbb{R}})^{\#}$.

(**iv**) $|\alpha| \times_{{}^*\mathbb{R}_{\mathbf{d}}} |\beta| = |\beta| \times_{{}^*\mathbb{R}_{\mathbf{d}}} |\alpha|$.

(**v**)

$$[(\alpha \geq 0) \wedge (\beta \geq 0) \wedge (\gamma \geq 0)] \Rightarrow \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma. \quad (2.13)$$

(**vi**)

$$0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha', 0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta' \Rightarrow \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'. \quad (2.14)$$

Lemma 2.6.Suppose $\mu \in {}^*\mathbb{R}$ and $\beta,\gamma \in {}^*\mathbb{R}_{\mathbf{d}}$. Then

$$[(\mu^{\#} \geq 0) \wedge (\beta \geq 0)] \Rightarrow \mu^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \mu^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma. \quad (2.15)$$

Proof. We choose now: (1) $a \in {}^*\mathbb{R}$ such that: $-\gamma + a^{\#} > 0$.
(2) Note that $\mu^{\#} \times (\beta - \gamma) = \mu^{\#} \times (\beta - \gamma) + \mu^{\#}a^{\#} - \mu^{\#}a^{\#}$.
Then from (2) by Lemma 2.4.(ii) one obtain
(3) $\mu^{\#} \times (\beta - \gamma) = \mu^{\#} \times [(\beta - \gamma) + a^{\#}] - \mu^{\#}a^{\#}$. Therefore
(4) $\mu^{\#} \times (\beta - \gamma) = \mu^{\#} \times [\beta + (a^{\#} - \gamma)] - \mu^{\#}a^{\#}$.
(5) Then from (4) by Lemma 2.5.(v) one obtain
(6) $\mu^{\#} \times (\beta - \gamma) = \mu^{\#} \times \beta + \mu^{\#} \times (a^{\#} - \gamma) - \mu^{\#}a^{\#}$.
Then from (6) by Lemma 2.4.(ii) one obtain
(7) $\mu^{\#} \times (\beta - \gamma) = \mu^{\#} \times \beta + \mu^{\#} \times a^{\#} - \mu^{\#}\gamma - \mu^{\#}a^{\#} = \mu^{\#} \times \beta - \mu^{\#}\gamma$.

Definition 2.8. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, 0 <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$ then $\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$ is defined as follows:
(**i**) $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha : \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \triangleq \inf\{a^{-1_{{}^*\mathbb{R}}} | a \in \alpha\}$,
(**ii**) $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} 0 : \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \triangleq -_{{}^*\mathbb{R}_{\mathbf{d}}} (-_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$.

Lemma 2.7.[6].

(**i**) $\forall a \in {}^*\mathbb{R} : (a^{\#})^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} =_w (a^{-1_{{}^*\mathbb{R}}})^{\#}$.
(**ii**) $(\alpha^{-1_{{}^*\mathbb{R}}})^{-1_{{}^*\mathbb{R}}} = \alpha$.
(**iii**) $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \Rightarrow \beta^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$.
(**iv**) $[(0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha) \wedge (0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)] \Rightarrow$
$\Rightarrow \left(\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right) \times_{{}^*\mathbb{R}_{\mathbf{d}}} \left(\beta^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right) \leq_{{}^*\mathbb{R}_{\mathbf{d}}} (\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$
(**v**) $\forall a \in {}^*\mathbb{R} : a \neq_{{}^*\mathbb{R}} 0_{{}^*\mathbb{R}} \Rightarrow (a^{\#})^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \left(\beta^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right) = (a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$.
(**vi**) $\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \leq_{{}^*\mathbb{R}_{\mathbf{d}}} 1_{{}^*\mathbb{R}_{\mathbf{d}}}$.

Lemma 2.8.[6]. Suppose that $a \in {}^*\mathbb{R}, a > 0, \beta,\gamma \in {}^*\mathbb{R}_{\mathbf{d}},\ \beta > 0, \gamma > 0$. Then $a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma$.

Theorem 2.1.Suppose that $\mathbf{S}$ is a non-empty subset of ${}^*\mathbb{R}_{\mathbf{d}}$ which is bounded from above, i.e. $\sup(\mathbf{S})$ exist and suppose that

$\xi \in {}^*\mathbb{R}, \xi > 0.$ Then

$$\sup_{x \in \mathbf{S}} \{\xi^{\#} \times x\} = \xi^{\#} \times \left(\sup_{x \in \mathbf{S}} \{x\} \right) = \xi^{\#} \times (\sup \mathbf{S}). \quad (2.16)$$

Proof.Let $B = \sup \mathbf{S}$. Then B is the smallest number such that, for any $x \in \mathbf{S}, x \le \mathbf{B}$. Let $\mathbf{T} = \{\xi^{\#} \times x | x \in \mathbf{S}\}$. Since $\xi^{\#} > 0, \xi^{\#} \times x \le \xi^{\#} \times B$ for any $x \in \mathbf{S}$. Hence $\mathbf{T}$ is bounded above by $\xi^{\#} \times B$. Hence $\mathbf{T}$ has a supremum $C_{\mathbf{T}} = \mathbf{s}\text{-}\sup \mathbf{T}$. Now we have to prove that $C_{\mathbf{T}} = \xi^{\#} \times B == \xi^{\#} \times (\sup \mathbf{S})$. Since $\xi^{\#} \times B = \xi^{\#} \times (\sup \mathbf{S})$ is an upper bound for $\mathbf{T}$and C is the smallest upper bound for $\mathbf{T}, C_{\mathbf{T}} \le \xi^{\#} \times B$. Now we repeat the argument above with the roles of $\mathbf{S}$ and $\mathbf{T}$ reversed. We know that $C_{\mathbf{T}}$ is the smallest number such that, for any $y \in \mathbf{T}, y \le C_{\mathbf{T}}$. Since $\xi^{\#} > 0$ it follows that $(\xi^{\#})^{-1} \times y \le (\xi^{\#})^{-1} \times C_{\mathbf{T}}$ for any $y \in \mathbf{T}$. But $\mathbf{S} = \left\{(\xi^{\#})^{-1} \times y | y \in \mathbf{T}\right\}$. Hence $(\xi^{\#})^{-1} \times C_{\mathbf{T}}$ is an upper bound for $\mathbf{S}$. But B is a supremum for $\mathbf{S}$. Hence $B \le (\xi^{\#})^{-1} \times C_{\mathbf{T}}$ and $\xi^{\#} \times B \le C_{\mathbf{T}}$. We have shown that $C_{\mathbf{T}} \le \xi^{\#} \times B$ and also that $\xi^{\#} \times B \le C_{\mathbf{T}}$. Thus $\xi^{\#} \times B = C_{\mathbf{T}}$.

2.3 Absorption numbers in ${}^*\mathbb{R}_{\mathbf{d}}$.

One of standard ways of defining the completion of ${}^*\mathbb{R}$ involves restricting oneself to subsets, which have the following property $\forall \varepsilon_{\varepsilon>0} \exists x_{x\in\alpha}\ \exists y_{y\in\alpha}\left[y - x < \varepsilon\right]$. It is well known that in this case we obtain a field. In fact the proof is essentially the same as the one used in the case of ordinary Dedekind cuts in the development of the standard real numbers, $\varepsilon_{\mathbf{d}}$,of course, does not have the above property because no infinitesimal works.This suggests the introduction of the concept of absorption part $\mathbf{ab.p.}(\alpha)$ of a number α for an element α of ${}^*\mathbb{R}_{\mathbf{d}}$ which, roughly speaking, measures how much α departs from having the above property [7].

Definition 2.9.[7]. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$, then

$$\mathbf{ab.p.}(\alpha) \triangleq \{d \ge 0 | \forall x_{x\in\alpha}[x + d \in \alpha]\}. \quad (2.17)$$

Example 2.5.

(**i**) $\forall \alpha \in {}^*\mathbb{R} : \mathbf{ab.p.}(\alpha^{\#}) = 0,$

(**ii**) $\mathbf{ab.p.}(\varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}},$

(**iii**) $\mathbf{ab.p.}(-\varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}},$

(**iv**) $\forall \alpha \in {}^*\mathbb{R} : \mathbf{ab.p.}(\alpha^{\#} + \varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}},$

(**v**) $\forall \alpha \in {}^*\mathbb{R} : \mathbf{ab.p.}(\alpha^{\#} - \varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}}.$

Lemma 2.9.[7].

(**i**) $c < \mathbf{ab.p.}(\alpha)$ and $0 \le d < c \Rightarrow d \in \mathbf{ab.p.}(\alpha)$

(**ii**) $c \in \mathbf{ab.p.}(\alpha)$ and $d \in \mathbf{ab.p.}(\alpha) \Rightarrow c + d \in \mathbf{ab.p.}(\alpha)$.

Remark 2.9. By Lemma 2.7 $\mathbf{ab.p.}(\alpha)$ may be regarded as an element of ${}^*\mathbb{R}_{\mathbf{d}}$ by adding on all negative elements of ${}^*\mathbb{R}_{\mathbf{d}}$ to $\mathbf{ab.p.}(\alpha)$. Of course if the condition $d \ge 0$ in the definition of $\mathbf{ab.p.}(\alpha)$ is deleted we automatically get all the negative elements to be in $\mathbf{ab.p.}(\alpha)$ since $x < y \in \alpha \Rightarrow x \in \alpha$. The reason for our definition is that the real interest lies in the non-negative numbers. A technicality occurs if $\mathbf{ab.p.}(\alpha) = \{0\}$. We then identify $\mathbf{ab.p.}(\alpha)$ with 0. [$\mathbf{ab.p.}(\alpha)$ becomes $\{x|x < 0\}$ which by our early convention is not in ${}^*\mathbb{R}_{\mathbf{d}}$].

Remark 2.10. By Lemma 2.7(ii), $\mathbf{ab.p.}(\alpha)$ is additive idempotent.

Lemma 2.10.[7].

(**i**) $\mathbf{ab.p.}(\alpha)$ is the maximum element $\beta \in {}^*\mathbb{R}_{\mathbf{d}}$ such that $\alpha + \beta = \alpha$.

(**ii**) $\mathbf{ab.p.}(\alpha) \leq \alpha$ for $\alpha > 0$.

(**iii**) If α is positive and idempotent then $\mathbf{ab.p.}(\alpha) = \alpha$.

Lemma 2.11.[7]. Let $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ satsify $\alpha > 0$. Then the following are equivalent. In what follows assume $a, b > 0$.

(**i**) α is idempotent,

(**ii**) $a, b \in \alpha \Rightarrow a + b \in \alpha$,

(**iii**) $a \in \alpha \Rightarrow 2a \in \alpha$,

(**iv**) $\forall n_{n \in \mathbb{N}}[a \in \alpha \Rightarrow n \cdot a \in \alpha]$,

(**v**) $a \in \alpha \Rightarrow r \cdot a \in \alpha$, for all finite $r \in {}^*\mathbb{R}$.

Theorem 2.2.[7]. $(-\alpha) + \alpha = -[\mathbf{ab.p.}(\alpha)]$.

Theorem 2.3.[7]. $\mathbf{ab.p.}(\alpha + \beta) \geq \mathbf{ab.p.}(\alpha)$.

Theorem 2.4.[7].

(**i**) $\alpha + \beta \leq \alpha + \gamma \Rightarrow -\mathbf{ab.p.}(\alpha) + \beta \leq \gamma$.

(**ii**) $\alpha + \beta = \alpha + \gamma \Rightarrow -[\mathbf{ab.p.}(\alpha)] + \beta = -[\mathbf{ab.p.}(\alpha)] + \gamma$.

Theorem 2.5.[7].Suppose $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{d}}$, then

(**i**) $\mathbf{ab.p.}(-\alpha) = \mathbf{ab.p.}(\alpha)$,

(**ii**) $\mathbf{ab.p.}(\alpha + \beta) = \max\{\mathbf{ab.p.}(\alpha), \mathbf{ab.p.}(\beta)\}$

Theorem 2.6.[7]. Assume $\beta > 0$. If α absorbs $-\beta$ then α absorbs β.

Theorem 2.7.[7]. Let $0 < \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. Then the following are equivalent

(**i**) α is an idempotent,

(**ii**) $(-\alpha) + (-\alpha) = -\alpha$,

(**iii**) $(-\alpha) + \alpha = -\alpha$.

(**iv**) Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$. Then $\Delta_2 + (-\Delta_1) = \Delta_2$.

2.4 Gonshor types of α with given $\mathbf{ab.p.}(\alpha)$.

Among elements of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ such that $\mathbf{ab.p.}(\alpha) = \Delta$ one can distinguish two many different types following [7].

Definition 2.10.[7].Assume $\Delta > 0$.

(**i**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 1 if $\exists x(x \in \alpha)\forall y[x + y \in \alpha \Rightarrow y \in \Delta]$,

(**ii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 2 if $\forall x(x \in \alpha)\exists y(y \notin \Delta)[x + y \in \alpha]$, i.e. $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 2 iff α does not have type 1.

(**iii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $1\mathbf{A}$ if $\exists x(x \notin \alpha)\forall y[x - y \notin \alpha \Rightarrow y \in \Delta]$,

(**iv**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $2\mathbf{A}$ if $\forall x(x \notin \alpha)\exists y(y \notin \alpha)[x - y \notin \alpha]$.

2.5 Robinson Part $\Re\mathbf{p}\{\alpha\}$ of absorption number $\alpha \in (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$

Theorem 2.8.[6].Suppose $\alpha \in (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$. Then there is a unique standard $x \in \mathbb{R}$, called Wattenberg standard part of α and denoted by $\mathbf{Wst}(\alpha)$, such that:

(**i**) $({}^*x)^{\#} \in [\alpha - \varepsilon_{\mathbf{d}}, \alpha + \varepsilon_{\mathbf{d}}]$.

(**ii**) $\alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$ implies $\mathbf{Wst}(\alpha) \leq \mathbf{Wst}(\beta)$.

(**iii**) The map $\mathbf{Wst}(\cdot) : {}^*\mathbb{R}_{\mathbf{d}} \to \mathbb{R}$ is continuous.

(**iv**) $\mathbf{Wst}(\alpha + \beta) = \mathbf{Wst}(\alpha) + \mathbf{Wst}(\beta)$.

(**v**) $\mathbf{Wst}(\alpha \times \beta) = \mathbf{Wst}(\alpha) \times \mathbf{Wst}(\beta)$.

(**vi**) $\mathbf{Wst}(-\alpha) = -\mathbf{Wst}(\alpha)$.

(**vii**) $\mathbf{Wst}(\alpha^{-1}) = [\mathbf{Wst}(\alpha)]^{-1}$ if $\alpha \notin [-\varepsilon_{\mathbf{d}}, \varepsilon_{\mathbf{d}}]$.

Theorem 2.9.[7].

(**i**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 1 iff $-\alpha$ has type $1\mathbf{A}$,

(**ii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ cannot have type 1 and type $1\mathbf{A}$ simultaneously.

(**iii**) Suppose $\mathbf{ab.p.}(\alpha) = \Delta > 0$. Then α has type 1 iff α has the form $a^{\#} + \Delta$ for some $a \in {}^*\mathbb{R}$.

(**iv**) Suppose $\mathbf{ab.p.}(\alpha) = -\Delta, \Delta > 0. \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $1\mathbf{A}$ iff α has the form $a^{\#} + (-\Delta)$ for some $a \in {}^*\mathbb{R}$.

(**v**) If $\mathbf{ab.p.}(\alpha) > \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 1 iff α has type 1.

(**vi**) If $\mathbf{ab.p.}(\alpha) = \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 2 iff either α or β has type 2.

Proof (**iii**) Let $\alpha = a + \Delta$. Then $\mathbf{ab.p.}(\alpha) = \Delta$.Since $\Delta > 0, a \in a + \Delta$ (we chose $d \in \Delta$ such that $0 < d$ and write a as $(a - d) + d$).
It is clear that a works to show that α has type 1.
Conversely, suppose α has type 1 and choose $a \in \alpha$ such that: $\forall y[a + y \in \alpha \Rightarrow y \in \Delta]$. Then we claim that: $\alpha = a + \Delta$.
By definition of $\mathbf{ab.p.}(\alpha)$ certainly $a + \Delta \leq \alpha$. On the other hand by choice of a, every element of α has the form $a + d$ with $d \in \Delta$.
Choose $d' \in \Delta$ such that $d' > d$, then $a + d = \left[a - \left(d' - d\right)\right] + d' \in a + \Delta$.
Hence $\alpha \leq a + \Delta$. Therefore $\alpha = a + \Delta$.

Examples. (**i**) $\varepsilon_{\mathbf{d}}$ has type 1 and therefore $-\varepsilon_{\mathbf{d}}$ has type $1\mathbf{A}$. Note that also $-\varepsilon_{\mathbf{d}}$ has type 2. (**ii**) Suppose $\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R}$. Then $\varepsilon^{\#} \times \varepsilon_{\mathbf{d}}$ has type 1 and therefore $-\varepsilon^{\#} \times \varepsilon_{\mathbf{d}}$ has type $1\mathbf{A}$.

(**ii**) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = \varepsilon_{\mathbf{d}} > 0$, i.e. α has type 1 and therefore by Theorem 2.9 α has the form $({}^*a)^{\#} + \varepsilon_{\mathbf{d}}$ for some unique $a \in \mathbb{R}, a = \mathbf{Wst}(\alpha)$. Then, we define unique Robinson part $\Re\mathbf{p}[\alpha]$ of absorption number α by formula

$$\begin{cases} \Re\mathbf{p}\{\alpha\} \triangleq ({}^*a)^{\#}, \\ \Re\mathbf{p}\{\alpha\} = ({}^*\mathbf{Wst}(\alpha))^{\#}. \end{cases} \tag{2.18}$$

(**ii**) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = -\varepsilon_{\mathbf{d}}$, i.e. α has type $1\mathbf{A}$ and therefore by Theorem 2.9 α has the form $({}^*a)^{\#} - \varepsilon_{\mathbf{d}}$ for some unique $a \in \mathbb{R}, a = \mathbf{Wst}(\alpha)$. Then we define unique

Robinson part $\Re\mathbf{p}[\alpha]$ of absorption number α by formula

$$\begin{cases} \Re\mathbf{p}\{\alpha\} \triangleq ({}^*a)^{\#}, \\ \Re\mathbf{p}\{\alpha\} = ({}^*\mathbf{Wst}(\alpha))^{\#}. \end{cases} \tag{2.19}$$

(**iii**) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = \Delta, \Delta > 0$ and α has type $1\mathbf{A}$, i.e. α has the form $a^{\#} + \Delta$ for

some $a \in {}^*\mathbb{R}$. Then, we define Robinson part $\Re\mathbf{p}\{\alpha\}$ of absorption number α by formula

$$\Re\mathbf{p}\{\alpha\} \triangleq a^{\#}. \tag{2.20}$$

(**iv**) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = -\Delta, \Delta > 0$ and α has type $1\mathbf{A}$, i.e. α has the

form $a^{\#} + (-\Delta)$ for some $a \in {}^*\mathbb{R}$. Then, we define Robinson part $\Re\mathbf{p}\{\alpha\}$ of absorption

number α by formula

$$\Re\mathbf{p}\{\alpha\} \triangleq a^{\#}. \tag{2.21}$$

Remark 2.11. Note that in general case, i.e. if $\alpha \notin (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$ Robinson part $\Re\mathbf{p}\{\alpha\}$ of

absorption number α is not unique.

Remark 2.12. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ and $\alpha \in (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$ has type 1or type $1\mathbf{A}$. Then by definitions

above one obtain the representation

$$\alpha = \Re\mathbf{p}\{\alpha\} + \mathbf{ab.p.}(\alpha).$$

2.6 The pseudo-ring of Wattenberg hyperintegers ${}^*\mathbb{Z}_{\mathbf{d}}$

Lemma 2.12. [6].Suppose that $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. Then the following two conditions on α are equivalent:

(i) $\alpha = \sup\{\nu^{\#} | (\nu \in {}^*\mathbb{Z}) \wedge (\nu^{\#} \leq \alpha)\}$,

(ii) $\alpha = \inf\{\nu^{\#} | (\nu \in {}^*\mathbb{Z}) \wedge (\alpha \leq \nu^{\#})\}$.

Definition 2.11.[6].If α satisfies the conditions mentioned above α is said to be the Wattenberg hyperinteger. The set of all Wattenberg hyperintegers is denoted by ${}^*\mathbb{Z}_{\mathbf{d}}$.

Lemma 2.13. [6]. Suppose $\alpha, \beta \in {}^*\mathbb{Z}_{\mathbf{d}}$. Then

(i) $\alpha + \beta \in {}^*\mathbb{Z}_{\mathbf{d}}$.

(ii) $-\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$.

(iii) $\alpha \times \beta \in {}^*\mathbb{Z}_{\mathbf{d}}$.

The set of all positive Wattenberg hyperintegers is called the Wattenberg hypernaturals and

is denoted by ${}^*\mathbb{N}_{\mathbf{d}}$.

Definition 2.12.Suppose that (i) $\lambda \in {}^*\mathbb{N}, \nu \in {}^*\mathbb{Z}_{\mathbf{d}}$, (ii) $\widehat{\lambda} = \lambda^{\#}, \widehat{\nu} = \nu^{\#}$ and (iii) $\lambda | \nu$.

If $\widehat{\lambda} \in {}^*\mathbb{N}_{\mathbf{d}}$ and $\widehat{\nu} \in {}^*\mathbb{Z}_{\mathbf{d}}$ satisfies these conditions then we say that $\widehat{\nu}$ is divisible by $\widehat{\lambda}$ and we denote this by $\lambda^{\#}|\nu^{\#}$.

Definition 2.13.Suppose that (i) $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ and (ii) there exists $\lambda^{\#} \in {}^*\mathbb{N}_{\mathbf{d}}$ such that

(1) $\alpha = \sup\{\nu^{\#}|(\nu \in {}^*\mathbb{Z}) \wedge (\lambda|\nu) \wedge (\nu^{\#} \leq \alpha)\}$ or

(2) $\alpha = \inf\{\nu^{\#}|(\nu \in {}^*\mathbb{Z}) \wedge (\lambda|\nu) \wedge (\alpha \leq \nu^{\#})\}$.

If α satisfies the conditions mentioned above is said α is divisible by $\lambda^{\#}$ and we denote this by$\lambda^{\#}|\alpha$.

Theorem 2.10. (**i**) Let $\mathbf{p} \in {}^*\mathbb{N}$, $M(\mathbf{p}) \in {}^*\mathbb{N}$, be a prime hypernaturals such that (i) $\mathbf{p} \nmid M(\mathbf{p})$.

Let $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ be a Wattenberg hypernatural such that (i) $p|\alpha$. Then

$$\left|(M(\mathbf{p}))^{\#} + \alpha\right| > 1.$$

(**ii**) $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ has type 1 iff $-\alpha$ has type $1\mathbf{A}$,

(**iii**) $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ cannot have type 1 and type $1\mathbf{A}$ simultaneously.

(**iv**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = \Delta > 0$. Then α has type 1 iff α has the form $a^{\#} + \Delta$ for some $a \in \alpha, a \in {}^*\mathbb{Z}$.

(**v**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$, $\mathbf{ab.p.}(\alpha) = -\Delta, \Delta > 0. \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $1\mathbf{A}$ iff α has the form

$a^{\#} + (-\Delta)$ for some $a \in \alpha, a \in {}^*\mathbb{Z}$.

(**vi**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$. If $\mathbf{ab.p.}(\alpha) > \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 1 iff α has type 1.

(**vii**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$.If $\mathbf{ab.p.}(\alpha) = \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 2 iff either α or β has type 2.

Proof. (**i**) Immediately follows from definitions (2.12)-(2.13).

(**iv**) Let $\alpha = a + \Delta$. Then $\mathbf{ab.p.}(\alpha) = \Delta$.Since $\Delta > 0, a \in a + \Delta$ (we chose $d \in \Delta$ such that $0 < d$ and write a as $(a - d) + d$).

It is clear that a works to show that α has type 1.

Conversely, suppose α has type 1 and choose $a \in \alpha$ such that:

$\forall y[a + y \in \alpha \Rightarrow y \in \Delta]$.Then we claim that: $\alpha = a + \Delta$.

By definition of $\mathbf{ab.p.}(\alpha)$ certainly $a + \Delta \leq \alpha$. On the other hand by choice of a,every element of α has the form $a + d$ with $d \in \Delta$.

Choose $d' \in \Delta$ such that $d' > d$, then $a + d = [a - (d' - d)] + d' \in a + \Delta$.

Hence $\alpha \leq a + \Delta$.Therefore $\alpha = a + \Delta$.

2.7 The integer part $\mathbf{Int.p}(\alpha)$ of Wattenberg hyperreals $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$

Definition 2.14. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \alpha \geq 0$.Then, we define $\mathbf{Int.p}(\alpha) = [\alpha] \in {}^*\mathbb{N}_{\mathbf{d}}$ by formula

$$[\alpha] \triangleq \sup\{\nu^{\#}|(\nu \in {}^*\mathbb{N}) \wedge (\nu^{\#} \leq \alpha)\}.$$

Obviously there are two possibilities:

1. A set $\{\nu^{\#}|(\nu \in {}^*\mathbb{N}) \wedge (\nu^{\#} \leq \alpha)\}$ has no greatest element. In this case valid only the

Property I: $[\alpha] = \alpha$

since $[\alpha] < \alpha$ implies $\exists a \in {}^*\mathbb{R}$ such that $[\alpha] < a^\# < \alpha$. But then $[a^\#] < \alpha$ which implies $[a^\#] + 1 < \alpha$ contradicting $[\alpha] < a^\# < [a^\#] + 1$.

2. A set $\left\{v^\# | \left(v \in {}^*\mathbb{N}\right) \wedge (v^\# \leq \alpha)\right\}$ has a greatest element, $\upsilon \in {}^*\mathbb{N}$. In this case valid the

Property II: $[\alpha] = \upsilon$

and obviously $\upsilon = [\alpha] \leq \alpha < [\alpha] + 1 = \upsilon + 1$.

Definition 2.15. Suppose $\alpha \in {}^*\mathbb{R}_\mathbf{d}$. Then, we define $\mathbf{Int.p}(\alpha) \in {}^*\mathbb{Z}_\mathbf{d}$ by formula

$$\mathbf{Int.p}(\alpha) = \begin{cases} [\alpha] \text{ for } \alpha \geq 0 \\ -[\alpha] \text{ for } \alpha < 0. \end{cases}$$

Note that obviously: $\mathbf{Int.p}(-\alpha) = -\mathbf{Int.p}(\alpha)$.

2.8 External sum of the countable infinite series in ${}^*\mathbb{R}_\mathbf{d}$

This subsection contains key definitions and properties of summ of countable sequence of Wattenberg hyperreals.

Definition 2.16.[4]. Let $\{\mathbf{s}_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$. such that

(i) $\forall n(s_n \geq 0)$ or (ii) $\forall n(s_n < 0)$ or

(iii) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1 \in \mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2 \in \mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0]$,

$\forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$.

Then external sum (#-sum)

$$\#Ext\text{-}\sum_{n \in \mathbb{N}} s_n^\#$$

of the corresponding countable sequence ${}^*s_n : \mathbb{N} \to {}^*\mathbb{R}$ is defined by

$$\begin{cases} \textbf{(i)} \quad \forall n(s_n \geq 0): \\ \#Ext\text{-}\sum_{n \in \mathbb{N}} s_n^\# \triangleq \sup_{k \in \mathbb{N}} \left\{\sum_{n \leq k} ({}^*s_n)^\#\right\}, \\ \textbf{(ii)} \quad \forall n(s_n < 0): \\ \#Ext\text{-}\sum_{n \in \mathbb{N}} s_n^\# \triangleq \inf_{k \in \mathbb{N}} \left\{\sum_{n \leq k} s_n^\#\right\} = -\sup_{k \in \mathbb{N}} \left\{\sum_{n \leq k} (|{}^*s_n|)^\#\right\}, \\ \textbf{(iii)} \quad \forall n_1(n_1 \in \mathbb{N}_1)[s_{n_1} \geq 0], \\ \forall n_2(n_2 \in \mathbb{N}_2)[s_{n_2} < 0], \mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2: \\ \#Ext\text{-}\sum_{n \in \mathbb{N}} s_n^\# \triangleq \#Ext\text{-}\sum_{n_1 \in \widehat{\mathbb{N}}_1} s_{n_1}^\# + \#Ext\text{-}\sum_{n_2 \in \widehat{\mathbb{N}}_2} s_{n_2}^\#. \end{cases} \tag{2.22}$$

Theorem 2.11.(**i**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_{n+1} > s_n]$ and $\lim_{n \to \infty} s_n = \eta$. Then

$$\sup_{n \in \mathbb{N}} \left\{({}^*s_n)^\#\right\} = ({}^*\eta)^\# - \varepsilon_\mathbf{d}.$$

(**ii**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_{n+1} < s_{n+1}]$ and $\lim_{n\to\infty} s_n = \eta$. Then

$$\inf_{n\in\mathbb{N}} \left\{({}^*s_n)^{\#}\right\} = ({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}}.$$

(**iii**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_n \geq 0]$, $\sum_{n=1}^{\infty} s_n = \eta < \infty$ and infinite series $\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k} ({}^*s_n)^{\#}\right\} = ({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \qquad (2.23)$$

(**iv**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_n < 0]$, $\sum_{n=1}^{\infty} s_n = \eta > -\infty$ and infinite series$\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{\sum_{n\leq k} ({}^*s_n)^{\#}\right\} = ({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \qquad (2.24)$$

(**v**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that (1) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0], \forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0]$, $\mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$ and (2) $\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1} = \eta_1 < \infty, \sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2} = \eta_2 > -\infty.$ Then

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1}^{\#} + \#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2}^{\#} = ({}^*\eta_1)^{\#} + ({}^*\eta_2)^{\#} - \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \quad (2.25)$$

Proof. (**i**) Let $\forall n(n \in \mathbb{N})[s_{n+1} > s_n]$ and $\lim_{n\to\infty} s_n = \eta$. Then obviously: $\forall n(n \in \mathbb{N})[s_n < \eta]$.

Thus $\forall \varepsilon \in \mathbb{R}$ there exists $M \in \mathbb{N}$ such that (1)

$$(1) \quad \forall k \in \mathbb{N} : \eta - \varepsilon < s_{M+k} < \eta.$$

Therefore from (1) by Robinson transfer one obtains (2)

$$(2) \;\; \forall \varepsilon \in \mathbb{R}, \forall k \in \mathbb{N} : ({}^*\eta) - ({}^*\varepsilon) < ({}^*s_{M+k}) < ({}^*\eta).$$

Using now Wattenberg embedding from (2) we obtain (3)

$$(3) \;\; \forall \varepsilon \in \mathbb{R}, \forall k \in \mathbb{N} : ({}^*\eta^{\#}) - ({}^*\varepsilon^{\#}) < ({}^*s_{M+k}^{\#}) < ({}^*\eta^{\#}).$$

From (3) one obtains (4)

$$(4) \;\; \forall \varepsilon \in \mathbb{R} : ({}^*\eta^{\#}) - ({}^*\varepsilon^{\#}) < \sup_{k\in\mathbb{N}} ({}^*s_{M+k}^{\#}) < ({}^*\eta^{\#}).$$

Note that $\forall \delta[(\delta \in \mathbb{R}) \wedge (\delta \approx 0)]$ obviously

$$(5) \quad \sup_{n\in\mathbb{N}} ({}^*s_n^{\#}) < ({}^*\eta^{\#}) - \delta^{\#}.$$

From (4) and (5) one obtains (6)

$$(6)\ \ \forall\varepsilon(\varepsilon\in\mathbb{R})\forall\delta[(\delta\in\mathbb{R})\wedge(\delta\approx 0)]\left\{({}^*\eta^\#)-({}^*\varepsilon^\#)<\sup_{n\in\mathbb{N}}({}^*s_n^\#)<({}^*\eta^\#)-\delta^\#\right\}.$$

Thus (**i**) immediately from (6) and from definition of the idempotent $-\varepsilon_{\mathbf{d}}$.

Proof.(**ii**) Immediately from (**i**) by Lemma 2.3 (**v**).

Proof.(**iii**) Let $\eta_m=\sum_{n=1}^m s_n$. Then obviously: $\eta_m<\eta$ and $\lim_{m\to\infty}\eta_m=\eta$. Thus $\forall\varepsilon\in\mathbb{R}$ there exists $M\in\mathbb{N}$ such that (1)

$$(1)\quad \forall k\in\mathbb{N}:\eta-\varepsilon<\eta_{M+k}<\eta.$$

Therefore from (1) by Robinson transfer one obtains (2)

$$(2)\ \ \forall\varepsilon\in\mathbb{R},\forall k\in\mathbb{N}:({}^*\eta)-({}^*\varepsilon)<({}^*\eta_{M+k})<({}^*\eta).$$

Using now Wattenberg embedding from (2) we obtain (3)

$$(3)\ \ \forall\varepsilon\in\mathbb{R},\forall k\in\mathbb{N}:({}^*\eta^\#)-({}^*\varepsilon^\#)<({}^*\eta_{M+k}^\#)<({}^*\eta^\#).$$

From (3) one obtains (4)

$$(4)\ \ \forall\varepsilon\in\mathbb{R}:({}^*\eta^\#)-({}^*\varepsilon^\#)<\sup_{k\in\mathbb{N}}({}^*\eta_{M+k}^\#)<({}^*\eta^\#).$$

From (4) by Definition 2.16 (i) one obtains

$$(5)\ \ \forall\varepsilon\in\mathbb{R}:({}^*\eta^\#)-({}^*\varepsilon^\#)<\#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*s_n^\#)<({}^*\eta^\#).$$

Note that $\forall\delta[(\delta\in\mathbb{R})\wedge(\delta\approx 0)]$ obviously

$$(6)\quad \#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*s_n^\#)<({}^*\eta^\#)-\delta^\#.$$

From (5)-(6) follows (7)

$$(7)\ \ \forall\varepsilon(\varepsilon\in\mathbb{R})\forall\delta[(\delta\in\mathbb{R})\wedge(\delta\approx 0)]\left\{({}^*\eta^\#)-({}^*\varepsilon^\#)<\#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*s_n^\#)<({}^*\eta^\#)-\delta^\#\right\}.$$

Thus Eq.(2.23) immediately from (7) and from definition of the idempotent $-\varepsilon_{\mathbf{d}}$.

Proof.(**iv**) Immediately from (**iii**) by Lemma 2.3 (**v**).

Proof.(**v**) From Definition 2.16.(iii) and Eq.(2.23)-Eq.(2.24) by Theorem 2.7.(iii) one obtain

$$\begin{cases}\#Ext\text{-}\sum_{n\in\mathbb{N}}s_n^\#\triangleq\#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1}s_{n_1}^\#+\#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2}s_{n_2}^\#=({}^*\eta_1)^\#-\varepsilon_{\mathbf{d}}+\left(({}^*\eta_2)^\#+\varepsilon_{\mathbf{d}}\right)=\\ \qquad=({}^*\eta_1)^\#+({}^*\eta_2)^\#-\varepsilon_{\mathbf{d}}+\varepsilon_{\mathbf{d}}=({}^*\eta_1)^\#+({}^*\eta_2)^\#-\varepsilon_{\mathbf{d}}\in{}^*\mathbb{R}_{\mathbf{d}}.\end{cases}$$

Theorem 2.12.Let $\{a_n\}_{n=1}^\infty$ be a countable sequence $a_n:\mathbb{N}\to\mathbb{R}$ such that $\forall n(a_n\geq 0)$ and infinite series $\sum_{n=1}^\infty a_n$ absolutely converges in $\mathbb{R}$. Let $s=\#Ext\text{-}\sum_{n\in\mathbb{N}}a_n^\#$ be external sum of the corresponding countable sequence $\{{}^*a_n\}_{n=1}^\infty$. Let $\{b_n\}_{n=1}^\infty$ be a countable sequence where $b_n=a_{m(n)}$ is any rearrangement of terms of the sequence $\{a_n\}_{n=1}^\infty$. Then external sum $\sigma=\#Ext\text{-}\sum_{n\in\mathbb{N}}b_n^\#$ of the corresponding countable sequence $\{{}^*b_n\}_{m=1}^\infty$ has the same value s as external sum of the countable sequence $\{{}^*a_n\}$, i.e. $\sigma = s - \varepsilon_{\mathbf{d}}$.

Theorem 2.13.(**i**) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}_{\mathbf{d}}$, such that (1) $\forall n(a_n \geq 0)$, (2) infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges to $\eta \neq +\infty$ in $\mathbb{R}$ and let $\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#}$ be external sum of the corresponding sequence $\{{}^*a_n\}_{n=1}^{\infty}$. Then for any $c \in {}^*\mathbb{R}_+$ the equality is satisfied

$$\begin{cases} c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#}\right) = \#Ext\text{-}\sum_{n\in\mathbb{N}} c^{\#} \times a_n^{\#} = \\ = c^{\#} \times ({}^*\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (2.26)$$

(**ii**) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that (1) $\forall n(a_n < 0)$, (2) infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges to $\eta \neq -\infty$ in $\mathbb{R}$ and let $\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#}$ be external sum of the corresponding sequence $\{{}^*a_n\}_{n=1}^{\infty}$. Then for any $c \in {}^*\mathbb{R}_+$ the equality is satisfied:

$$\begin{cases} c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#}\right) = \#Ext\text{-}\sum_{n\in\mathbb{N}} c^{\#} \times a_n^{\#} = \\ = c^{\#} \times ({}^*\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (2.27)$$

(**iii**) Let $\{s_n\}_{n=1}^{\infty}$ be a countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that

(1) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1(n_1 \in \mathbb{N}_1)[s_{n_1} \geq 0], \forall n_2(n_2 \in \mathbb{N}_2)[s_{n_2} < 0]$, $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$,
(2) infinite series $\sum_{n=1}^{\infty} s_{n_1}$ absolutely converges to $\eta_1 \neq +\infty$ in $\mathbb{R}$,
(3) infinite series $\sum_{n=1}^{\infty} s_{n_2}$ absolutely converges to $\eta_2 \neq -\infty$ in $\mathbb{R}$.
Then the equality is satisfied:

$$\begin{cases} c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#}\right) = \\ = \#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} c^{\#} \times s_{n_1}^{\#} + \#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} c^{\#} \times s_{n_2}^{\#} = \\ = c^{\#} \times \left(({}^*\eta_1)^{\#} + ({}^*\eta_2)^{\#}\right) - c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (2.28)$$

Proof.(i) From Definition 2.16.(i) by Theorem 2.1,Theorem 2.11.(i) and Lemma (2.4) (ii) one obtain

$$\begin{cases} \#Ext\text{-}\sum_{n\in\mathbb{N}} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#}\right) = \\ = c^{\#} \times \left(({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}}\right) = c^{\#} \times ({}^*\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases}$$

(ii) Straightforward from Definition 2.16.(i) and Theorem 2.1,Theorem 2.11.(iii) and Lemma (2.4) (ii) one obtain

$$\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} c^{\#}\times a_n^{\#}\right) = c^{\#}\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#}\right) =$$
$$= c^{\#}\times\left(({}^*\eta)^{\#}+\varepsilon_{\mathbf{d}}\right) = c^{\#}\times({}^*\eta)^{\#}+c^{\#}\times\varepsilon_{\mathbf{d}}.$$

(iii) By Theorem 2.11.(iv) and Lemma (2.4).(ii) one obtain

$$c^{\#}\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#}\right) = c^{\#}\times\left(({}^*\eta_1)^{\#}+\ ({}^*\eta_2)^{\#}-\varepsilon_{\mathbf{d}}\right) =$$
$$= c^{\#}\times\left(({}^*\eta_1)^{\#}+\ ({}^*\eta_2)^{\#}\right)-c^{\#}\times\varepsilon_{\mathbf{d}}.$$

But other side from (i) and (ii) follows

$$\#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} c^{\#}\times s_{n_1}^{\#}+\#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} c^{\#}\times s_{n_2}^{\#} =$$
$$= c^{\#}\times({}^*\eta_1)^{\#}-c^{\#}\times\varepsilon_{\mathbf{d}})+c^{\#}\times({}^*\eta)^{\#}+c^{\#}\times\varepsilon_{\mathbf{d}} =$$
$$c^{\#}\times\left(({}^*\eta_1)^{\#}+\ ({}^*\eta_2)^{\#}\right)-c^{\#}\times\varepsilon_{\mathbf{d}}.$$

Definition 2.17. Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n:\mathbb{N}\to\mathbb{R}$, such that infinite

series $\sum_{n=1}^{\infty}a_n$ absolutely converges in $\mathbb{R}$ to $\eta\neq\pm\infty$. We assume now that:

(i) there exists $m>1$ such that $\forall k\geq m:\sum_{n=1}^{k}a_n>\eta$, or

(ii) there exists $m>1$ such that $\forall k\geq m:\sum_{n=1}^{k}a_n<\eta$, or

(iii) there exists infinite sequence $n_i, i=1,2,\ldots$ such that

(a) $\forall i,m:\sum_{i=1}^{m}a_{n_i}>\eta$ and infinite series $\sum_{i=1}^{\infty}a_{n_i}$ absolutely converges in $\mathbb{R}$ to η and

(b) there exists infinite sequence $n_j, j=1,2,\ldots$ such that $\forall j,m:\sum_{j=1}^{m}a_{n_j}<\eta$ and infinite

series $\sum_{j=1}^{\infty}a_{n_j}$ absolutely converges in $\mathbb{R}$ to η.

Then: (i) external upper sum (#-upper sum) of the corresponding countable sequence

${}^*a_n:\mathbb{N}\to\mathbb{R}$ is defined by

$$\begin{cases} \quad\quad\quad\quad (\mathbf{i}) \\ \#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^{*}a_n)^{\#} \right\}, \\ \quad\quad\quad\quad (\mathbf{ii}) \\ \#Ext\text{-}\overset{\vee}{\sum_{i\in\mathbb{N}}} a_{n_i}^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{ \sum_{i\leq k} ({}^{*}a_{n_i})^{\#} \right\}, \end{cases} \tag{2.29}$$

(ii) external lower sum (#-lower sum) of the corresponding countable sequence ${}^{*}a_n : \mathbb{N} \to \mathbb{R}$
is defined by

$$\begin{cases} \quad\quad\quad\quad (\mathbf{i}) \\ \#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^{*}a_n)^{\#} \right\}, \\ \quad\quad\quad\quad (\mathbf{ii}) \\ \#Ext\text{-}\overset{\wedge}{\sum_{j\in\mathbb{N}}} a_{n_j}^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{ \sum_{j\leq k} ({}^{*}a_{n_j})^{\#} \right\}. \end{cases} \tag{2.30}$$

Theorem 2.14.Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that infinite
series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$ to $\eta \neq \pm\infty$. We assume now that:
(i) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n > \eta$, or
(ii) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n < \eta$, or
(iii) there exists infinite sequence $n_i, i = 1,2,\ldots$ such that
(a) $\forall i,m : \sum_{i=1}^{m} a_{n_i} > \eta$ and infinite series $\sum_{i=1}^{\infty} a_{n_i}$ absolutely converges in $\mathbb{R}$ to η and
(b) there exists infinite sequence $n_j, j = 1,2,\ldots$ such that $\forall j,m : \sum_{j=1}^{m} a_{n_j} < \eta$ and infinite
series $\sum_{j=1}^{\infty} a_{n_j}$ absolutely converges in $\mathbb{R}$ to η. Then

$$\begin{cases} \#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^{*}a_n)^{\#} \right\} = ({}^{*}\eta)^{\#} + \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^{*}a_n)^{\#} \right\} = ({}^{*}\eta)^{\#} - \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \tag{2.31}$$

and

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#} \triangleq \inf\limits_{k\in\mathbb{N}} \left\{\sum_{i\leq k}({}^*a_{n_i})^{\#}\right\} = ({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} a_{n_j}^{\#} \triangleq \sup\limits_{k\in\mathbb{N}} \left\{\sum_{j\leq k}({}^*a_{n_j})^{\#}\right\} = ({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.32)$$

Proof. straightforward from definitions and by Theorem 2.11 (i)-(ii).

Theorem 2.15. (1) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$ to $\eta \neq \pm\infty$. We assume now that:

(i) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n > \eta$, or

(ii) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n < \eta$, or

(iii) there exists infinite sequence $n_i, i = 1,2,\ldots$ such that

(a) $\forall i,m : \sum_{i=1}^{m} a_{n_i} > \eta$ and infinite series $\sum_{i=1}^{\infty} a_{n_i}$ absolutely converges in $\mathbb{R}$ to η and

(b) there exists infinite sequence $n_j, j = 1,2,\ldots$ such that $\forall j,m : \sum_{j=1}^{m} a_{n_j} < \eta$ and infinite series $\sum_{j=1}^{\infty} a_{n_j}$ absolutely converges in $\mathbb{R}$ to η.

Then for any $c \in {}^*\mathbb{R}_+$ the equalities are satisfied

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#}\right) = c^{\#} \times ({}^*\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} a_n^{\#}\right) = c^{\#} \times ({}^*\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.33)$$

and

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} c^{\#} \times a_{n_i}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#}\right) = c^{\#} \times ({}^*\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} c^{\#} \times a_{n_j}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} a_{n_j}^{\#}\right) = c^{\#} \times ({}^*\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.34)$$

Proof. Copy the proof of the Theorem 2.13.

Theorem 2.16. (1) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$ to $\eta = 0$. We assume now that:

(i) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n > 0$, or

(ii) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n < 0$, or

(iii) there exists infinite sequence $n_i, i = 1,2,\ldots$ such that

(a) $\forall i,m : \sum_{i=1}^{m} a_{n_i} > 0$ and infinite series $\sum_{i=1}^{\infty} a_{n_i}$ absolutely converges in $\mathbb{R}$ to $\eta = 0$

and

(b) there exists infinite sequence $n_j, j = 1,2,\ldots$ such that $\forall j,m : \sum_{j=1}^{m} a_{n_j} < 0$ and infinite

series $\sum_{j=1}^{\infty} a_{n_j}$ absolutely converges in $\mathbb{R}$ to $\eta = 0$.

Then for any $c \in {}^*\mathbb{R}_+$ the equalities are satisfied

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#} \right) = c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} a_n^{\#} \right) = -c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.35)$$

and

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} c^{\#} \times a_{n_i}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#} \right) = c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} c^{\#} \times a_{n_j}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\wedge} a_{n_j}^{\#} \right) = -c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.36)$$

Proof. (1) From Eq.(2.31) we obtain

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#} = +\varepsilon_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} a_n^{\#} = -\varepsilon_{\mathbf{d}}. \end{cases} \quad (2.37)$$

From Eq.(2.37) by Theorem 2.1 we obtain directly

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#} \right) = c^{\#} \times \varepsilon_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} a_n^{\#} \right) = -c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (2.38)$$

(2) From Eq.(2.32) we obtain

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#} = +\varepsilon_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} a_{n_j}^{\#} = -\varepsilon_{\mathbf{d}}. \end{cases} \quad (2.39)$$

From Eq.(2.39) by Theorem 2.1 we obtain directly

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} c^{\#}\times a^{\#}_{n_i} = c^{\#}\times\left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a^{\#}_{n_i}\right) = c^{\#}\times\varepsilon_{\mathbf{d}}\in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} c^{\#}\times a^{\#}_{n_j} = c^{\#}\times\left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\wedge} a^{\#}_{n_i}\right) = -c^{\#}\times\varepsilon_{\mathbf{d}}\in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.40)$$

Remark 2.13. Note that we have proved Eq.(2.35) and Eq.(2.36) without any reference to the Lemma 2.4.

Definition 2.18. (**i**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a countable sequence $\alpha_n : \mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}$, such that

$$\forall n(n \geq m > 0)[\alpha_n > 0] \text{ and } \forall n(n \leq m-1)\left[(\alpha_n = a^{\#}_n) \wedge \left(\alpha_n \in {}^{*}\mathbb{R}\right)\right] \quad (2.41)$$

Then external countable lower sum (#-lower sum) of the countable sequence $\alpha_n : \mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}$ is defined by

$$\begin{gathered} \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \alpha_n = \sum_{n=0}^{m-1} \alpha_n + \#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \\ \#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \triangleq \sup_{k\in\mathbb{N}} \sum_{n=m}^{k} \alpha_n. \end{gathered} \quad (2.42)$$

In particular if $\{\alpha_n\}_{n=1}^{\infty} = \{a^{\#}_n\}_{n=1}^{\infty}$, where $\forall n \in \mathbb{N}\left[a_n \in {}^{*}\mathbb{R}\right]$ the external countable lower sum (#-lower sum) of the countable sequence $\alpha_n : \mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}$ is defined by

$$\begin{gathered} \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \alpha_n = \sum_{n=0}^{m-1} \alpha^{\#}_n + \#Ext\text{-}\sum_{n=m}^{\infty} \alpha^{\#}_n, \\ \#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \triangleq \sup_{k\in\mathbb{N}} \sum_{n=m}^{k} \alpha^{\#}_n. \end{gathered} \quad (2.43)$$

(**ii**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a countable sequence $\alpha_n : \mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}$, such that

$$\forall n(n \geq m > 0)[\alpha_n < 0] \text{ and } \forall n(n \leq m-1)\left[(\alpha_n = a^{\#}_n) \wedge \left(a_n \in {}^{*}\mathbb{R}\right)\right] \quad (2.44)$$

Then external countable upper sum (#-upper sum) of the countable sequence $a_n : \mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}$ is defined by

$$\begin{gathered} \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee} \alpha_n = \sum_{n=0}^{m-1} \alpha_n + \#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \\ \#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \triangleq \inf_{k\in\mathbb{N}} \sum_{n=m}^{k} \alpha_n. \end{gathered} \quad (2.45)$$

In particular if $\{\alpha_n\}_{n=1}^{\infty} = \{a^{\#}_n\}_{n=1}^{\infty}$, where $\forall n \in \mathbb{N}\left[a_n \in {}^{*}\mathbb{R}\right]$ the external countable upper sum (#-upper sum) of the countable sequence $\alpha_n : \mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}$ is defined by

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}\alpha_n = \sum_{n=0}^{m-1}a_n^{\#} + \#Ext\text{-}\sum_{n=m}^{\infty}a_n^{\#},$$
$$\#Ext\text{-}\sum_{n=m}^{\infty}\alpha_n \triangleq \inf_{k\in\mathbb{N}}\sum_{n=m}^{k}a_n^{\#}. \tag{2.46}$$

Theorem 2.17. (**i**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$, such that valid the property (2.41). Then for any $c \in {}^*\mathbb{R}_+$ the equality is satisfied

$$c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}\alpha_n\right) = \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}c^{\#} \times \alpha_n =$$
$$= \sum_{n=0}^{m-1}c^{\#} \times a_n^{\#} + \#Ext\text{-}\sum_{n=m}^{\infty}c^{\#} \times a_n^{\#}. \tag{2.47}$$

(**ii**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be an countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$, such that valid the property (2.44).

Then for any $c \in {}^*\mathbb{R}_+$ the equality is satisfied

$$c^{\#} \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}\alpha_n\right) = \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}c^{\#} \times \alpha_n =$$
$$= \sum_{n=0}^{m-1}c^{\#} \times a_n^{\#} + \#Ext\text{-}\sum_{n=m}^{\infty}c^{\#} \times a_n^{\#}. \tag{2.48}$$

Proof. Immediately from Definition 2.18 by Theorem 2.1.

Definition 2.19. Let $\{z_n\}_{n=1}^{\infty} = \{a_n + ib_n\}_{n=1}^{\infty}$ be a countable sequence $z_n = a_n + ib_n : \mathbb{N} \to \mathbb{C}$ such that infinite series $\sum_{n=1}^{\infty} z_n$ absolutely converges in $\mathbb{C}$ to $z, |z| \neq \infty$. Then: (i) external complex sum (complex #-sum), (ii) external upper complex sum (upper complex #-sum) and (iii) external lover complex sum (lover complex #-sum) of the corresponding countable sequence ${}^*z_n : \mathbb{N} \to {}^*\mathbb{C}$ is defined by

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}z_n^{\#} = \#Ext\text{-}\sum_{n\in\mathbb{N}}a_n^{\#} + i \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}}b_n^{\#}\right),$$
$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}z_n^{\#} = \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}a_n^{\#} + i \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}b_n^{\#}\right)$$
$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}z_n^{\#} = \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}a_n^{\#} + i \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}b_n^{\#}\right). \tag{2.49}$$

correspondingly.

Note that any properties of this sum immediately follow from the properties of the real external sum.

Definition 2.20. (i) We define now Wattenberg complex plane ${}^*\mathbb{C}_{\mathbf{d}}$ by ${}^*\mathbb{C}_{\mathbf{d}} = {}^*\mathbb{R}_{\mathbf{d}} \oplus i \times {}^*\mathbb{R}_{\mathbf{d}}$ with $i^2 = 1$. Thus for any $z \in {}^*\mathbb{C}_{\mathbf{d}}$ we obtain $z = x + iy$, where $x,y \in {}^*\mathbb{R}_{\mathbf{d}}$, (ii) for any $z \in {}^*\mathbb{C}_{\mathbf{d}}$ such that $z = x + iy$ we define $|z|^2$ by $|z|^2 = x^2 + y^2 \in {}^*\mathbb{R}_{\mathbf{d}}$.

Theorem 2.18. Let $\{z_n\}_{n=1}^{\infty} = \{a_n + ib_n\}_{n=1}^{\infty}$ be a countable sequence $z_n = a_n + ib_n : \mathbb{N} \to \mathbb{C}$ such that infinite series $\sum_{n=1}^{\infty} z_n$ absolutely converges in $\mathbb{C}$ to $z = \zeta_1 + i\zeta_2$ and $|z| \neq \infty$. Then

(i)

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} z_n^{\#} = \#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#} + i \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} b_n^{\#}\right) =$$

$$\left[({}^*\zeta_1)^{\#} - \varepsilon_{\mathbf{d}}\right] + i\left[({}^*\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}\right] = ({}^*\zeta_1)^{\#} + i({}^*\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}(1+i)$$

$$\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} z_n^{\#} =$$

$$\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i \times \left(\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right) = ({}^*\zeta_1)^{\#} + i({}^*\zeta_2)^{\#} + \varepsilon_{\mathbf{d}}(1+i)$$

$$\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} z_n^{\#} =$$

$$\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i \times \left(\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right) = ({}^*\zeta_1)^{\#} + i({}^*\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}(1+i)$$

(ii)

$$\left|\#Ext\text{-}\sum_{n\in\mathbb{N}} z_n^{\#}\right|^2 =$$

$$= \left|\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#} + i \times \left(\#Ext\text{-}\sum_{n\in\mathbb{N}} b_n^{\#}\right)\right|^2 = \left|({}^*\zeta_1)^{\#} + i({}^*\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}(1+i)\right|^2,$$

$$\left|\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} z_n^{\#}\right|^2 =$$

$$\left|\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i \times \left(\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right)\right|^2 = \left|({}^*\zeta_1)^{\#} + i({}^*\zeta_2)^{\#} + \varepsilon_{\mathbf{d}}(1+i)\right|^2,$$

$$\left|\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} z_n^{\#}\right|^2 =$$

$$\left|\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i \times \left(\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right)\right|^2 = \left|({}^*\zeta_1)^{\#} + i({}^*\zeta_2)^{\#} + \varepsilon_{\mathbf{d}}(1+i)\right|^2.$$

2.9 Gonshor transfer

Definition 2.21.[7]. Let $[S]_{\mathbf{d}} = \{x|\exists y(y \in S)[x \leq y]\}$.

Note that $[S]_{\mathbf{d}}$ satisfies the usual axioms for a closure operator,i.e. if (i) $S \neq \varnothing, S' \neq \varnothing$ and

(ii) S has no maximum, then $[S]_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}$.

Let f be a continuous strictly increasing function in each variable from a subset of $\mathbb{R}^n$ into $\mathbb{R}$. Specifically, we want the domain to be the cartesian product $\prod_{i=1}^{n} A_i$, where $A_i = \{x|x > a_i\}$ for some $a_i \in \mathbb{R}$. By Robinson transfer f extends to a function ${}^{*}f : {}^{*}\mathbb{R}^n \to {}^{*}\mathbb{R}$ from the corresponding subset of ${}^{*}\mathbb{R}^n$ into ${}^{*}\mathbb{R}$ which is also strictly increasing in each variable and continuous in the Q topology (i.e. ε and δ range over arbitrary positive elements in ${}^{*}\mathbb{R}$).We now extend ${}^{*}f$ to $[{}^{*}f]_{\mathbf{d}}$

$$[{}^{*}f]_{\mathbf{d}} : {}^{*}\mathbb{R}_{\mathbf{d}}^{n} \to {}^{*}\mathbb{R}_{\mathbf{d}}. \tag{2.50}$$

Definition 2.22.[7]. Let $\alpha_i \in {}^{*}\mathbb{R}_{\mathbf{d}}$, $\alpha_i > a_i$, $b_i \in {}^{*}\mathbb{R}$, then

$$[{}^{*}f]_{\mathbf{d}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left[\left\{ {}^{*}f(b_1, b_2, \ldots, b_n) | \, a_i < b_i \in \alpha_i \right\} \right]_{\mathbf{d}}. \tag{2.51}$$

Theorem 2.20.[7]. If f and g are functions of one variable then

$$[{}^{*}(f \cdot g)]_{\mathbf{d}}(\alpha) = ([{}^{*}f]_{\mathbf{d}}(\alpha)) \cdot ([{}^{*}g]_{\mathbf{d}}(\alpha)). \tag{2.52}$$

Theorem 2.21.[7].Let f be a function of two variables. Then for any $\alpha \in {}^{*}\mathbb{R}$ and $a \in {}^{*}\mathbb{R}$

$$[{}^{*}f]_{\mathbf{d}}(\alpha, a) = [{}^{*}f(b, c) | b \in \alpha, c < a]. \tag{2.53}$$

Theorem 2.22.[7].Let f and g be any two terms obtained by compositions of strictly

increasing continuous functions possibly containing parameters in ${}^{*}\mathbb{R}$. Then any relation ${}^{*}f = {}^{*}g$ or ${}^{*}f < {}^{*}g$ valid in ${}^{*}\mathbb{R}$ extends to ${}^{*}\mathbb{R}_{\mathbf{d}}$, i.e.

$$[{}^{*}f]_{\mathbf{d}}(\alpha) = [{}^{*}g]_{\mathbf{d}}(\alpha) \text{ or } [{}^{*}f]_{\mathbf{d}}(\alpha) < [{}^{*}g]_{\mathbf{d}}(\alpha). \tag{2.54}$$

Remark 2.14. For any function ${}^{*}f : {}^{*}\mathbb{R}^n \to {}^{*}\mathbb{R}$ we often write for short $f^{\#}$ instead of $[{}^{*}f]_{\mathbf{d}}$.

Theorem 2.23.[7].(1) For any $a, b \in {}^{*}\mathbb{R}_{+}$

$$\begin{cases} \exp^{\#}(a^{\#} + b^{\#}) = \exp^{\#}(a^{\#}) \exp^{\#}(b^{\#}), \\ (\exp^{\#}(a^{\#}))^{b^{\#}} = \exp^{\#}(b^{\#} a^{\#}). \end{cases} \tag{2.55}$$

For any $\alpha, \beta \in {}^{*}\mathbb{R}_{\mathbf{d}}, \alpha, \beta > 0$

$$\begin{cases} \exp^{\#}(\alpha + \beta) = \exp^{\#}(\alpha) \exp^{\#}(\beta), \\ (\exp^{\#}(\alpha))^{\beta} = \exp^{\#}(\beta \alpha). \end{cases} \tag{2.56}$$

(2) For any $a, b \in {}^{*}\mathbb{R}$

$$(a^{b})^{\#} = (a^{\#})^{b^{\#}}. \tag{2.57}$$

(3) For any $\alpha,\beta,\gamma \in {}^*\mathbb{R}_{\mathbf{d}}, \alpha,\beta,\gamma > 0$

$$(\alpha^{\beta})^{\gamma} = \alpha^{\gamma\beta} \tag{2.58}$$

(4) For any $a \in {}^*\mathbb{R}$

$$\begin{aligned} \ln^{\#}(\exp^{\#}(a^{\#})) &= a^{\#}, \\ \exp^{\#}(\ln^{\#}(a^{\#})) &= a^{\#}. \end{aligned} \tag{2.59}$$

Note that we must always beware of the restriction in the domain when it comes to multiplication

Theorem 2.24.[7].The map $\alpha \mapsto [\exp]_{\mathbf{d}}(\alpha)$ maps the set of additive idempotents onto the set of all multiplicative idempotents other than 0.

3. The proof of the #-transcendence of the numbers $e^k, k \in \mathbb{N}$.

In this section we will prove the #-transcendence of the numbers $e^k, k \in \mathbb{N}$. Key idea of this proof reduction of the statement of e is #-transcendental number to equivalent statement in ${}^*\mathbb{Z}_{\mathbf{d}}$ by using pseudoring of Wattenberg hyperreals ${}^*\mathbb{R}_{\mathbf{d}} \supset {}^*\mathbb{Z}_{\mathbf{d}}$ [6] and Gonshor idempotent theory [7]. We obtain this reduction by three steps, see subsections 3.2.1-3.2.3.

3.1. The basic definitions of the Shidlovsky quantities

In this section we remind the basic definitions of the Shidlovsky quantities [8].Let $M_0(n,p), M_k(n,p)$ and $\varepsilon_k(n,p)$ be the Shidlovsky quantities:

$$M_0(n,p) = \int_0^{+\infty} \left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!} \right] dx \neq 0, \tag{3.1}$$

$$M_k(n,p) = e^k \int_k^{+\infty} \left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!} \right] dx, k = 1,2,\ldots \tag{3.2}$$

$$\varepsilon_k(n,p) = e^k \int_0^{k} \left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!} \right] dx, k = 1,2,\ldots \tag{3.3}$$

where $p \in \mathbb{N}$ this is any prime number.Using Eqs.(3.1)-(3.3.) by simple calculation one obtains:

$$M_k(n,p) + \varepsilon_k(n,p) = e^k M_0(n,p) \neq 0, k = 1,2,\ldots. \tag{3.4}$$

and consequently

$$\begin{cases} e^k = \dfrac{M_k(n,p) + \varepsilon_k(n,p)}{M_0(n,p)} \\ k = 1,2,\ldots \end{cases} \tag{3.5}$$

Lemma 3.1.[8]. Let p be a prime number. Then $M_0(n,p) = (-1)^n(n!)^p + p\Theta_1, \Theta_1 \in \mathbb{Z}$.

Proof. ([8], p.128) By simple calculation one obtains the equality

$$\begin{cases} x^{p-1}[(x-1)\ldots(x-n)]^p = (-1)^n(n!)^p x^{p-1} + \sum\limits_{\mu=p+1}^{(n+1)\times p} c_{\mu-1}x^{\mu-1}, \\ c_\mu \in \mathbb{Z}, \mu = p, p+1, \ldots, [(n+1)\times p] - 1, n > 0, \end{cases} \tag{3.6}$$

where p is a prime. By using equality $\Gamma(\mu) = \int_0^\infty x^{\mu-1}e^{-x}dx = (\mu-1)!$, where $\mu \in \mathbb{N}$, from Eq.(3.1) and (3.6) one obtains

$$\begin{cases} M_0(n,p) = (-1)^n(n!)^p \dfrac{\Gamma(p)}{(p-1)!} + \sum\limits_{\mu=p+1}^{(n+1)\times p} c_{\mu-1}\dfrac{\Gamma(\mu)}{(p-1)!} = \\ = (-1)^n(n!)^p + c_p p + c_{p+1}p(p+1) + \ldots = \\ = (-1)^n(n!)^p + p \times \Theta_1, \Theta_1 \in \mathbb{Z}. \end{cases} \tag{3.7}$$

Thus

$$M_0(n,p) = (-1)^n(n!)^p + p \bullet \Theta_1(n,p), \Theta_1(n,p) \in \mathbb{Z}. \tag{3.8}$$

Lemma 3.2.[8]. Let p be a prime number. Then $M_k(n,p) = p \bullet \Theta_2(n,p)$, $\Theta_2(n,p) \in \mathbb{Z}$, $k = 1,2,\ldots,n$.

Proof.([8], p.128) By subsitution $x = k + u \Rightarrow dx = du$ from Eq.(3.3) one obtains

$$\begin{cases} M_k(n,p) = \int\limits_0^{+\infty} \left[\dfrac{(u+k)^{p-1}[(u+k-1)\times\ldots\times u\times\ldots\times(u+k-n)]^p e^{-u}}{(p-1)!} \right] du \\ k = 1,2,\ldots \end{cases} \tag{3.9}$$

By using equality

$$\begin{cases} (u+k)^{p-1}[(u+k-1)\times\ldots\times u\times\ldots\times(u+k-n)]^p = \sum\limits_{\mu=p+1}^{(n+1)\times p} d_{\mu-1}u^{\mu-1}, \\ d_\mu \in \mathbb{Z}, \mu = p, p+1, \ldots, [(n+1)\times p] - 1, \end{cases} \tag{3.10}$$

and by subsitution Eq.(3.10) into RHS of the Eq.(3.9) one obtains

$$M_k(n,p) = \frac{1}{(p-1)!}\int_0^{+\infty}\sum_{\mu=p+1}^{(n+1)\times p} d_{\mu-1}u^{\mu-1}du = p\bullet\Theta_2(n,p), \tag{3.11}$$

$$\Theta_2(n,p)\in\mathbb{Z}, k = 1,2,\ldots .$$

Lemma 3.3.[8]. (i) There exists sequences $a(n), n\in\mathbb{N}$ and $g(n), n\in\mathbb{N}$ such that

$$|\varepsilon_k(n,p)| \leq \frac{n\bullet g(n)\bullet[a(n)]^{p-1}}{(p-1)!}, \tag{3.12}$$

where sequences $a(n), n\in\mathbb{N}$ and $g(n), n\in\mathbb{N}$ does not depend on number p. (ii) For any $n\in\mathbb{N} : \varepsilon_k(n,p)\to 0$ if $p\to\infty$.

Proof.([8], p.129) Obviously there exists sequences $a(n), n\in\mathbb{N}$ and $g(n), k\in\mathbb{N}, n\in\mathbb{N}$ such that $a(n), n\in\mathbb{N}$ and $g(n), n\in\mathbb{N}$ does not depend on number p

$$|x(x-1)\ldots(x-n)| < a(n), 0\leq x\leq n \tag{3.13}$$

and

$$|(x-1)\ldots(x-n)e^{-x+k}| < g(n), 0\leq x\leq n, k = 1,2,\ldots,n. \tag{3.14}$$

Substitution inequalities (3.13)-(3.14) into RHS of the Eq.(3.3) by simple calculation gives

$$\varepsilon_k(n,p) \leq g(n)\frac{[a(n)]^{p-1}}{(p-1)!}\int_0^k dx \leq \frac{n\bullet g(n)\bullet[a(n)]^{p-1}}{(p-1)!}. \tag{3.15}$$

Statement (i) follows from (3.15). Statement (ii) immediately follows from a statement (ii).

Lemma 3.4.[8]. For any $k\leq n$ and for any δ such that $0<\delta<1$ there exists $p\in\mathbb{N}$ such that

$$\left|e^k - \frac{M_k(n,p)}{M_0(n,p)}\right| < \delta. \tag{3.16}$$

Proof.From Eq.(3.5) one obtains

$$\left|e^k - \frac{M_k(n,p)}{M_0(n,p)}\right| = \frac{|\varepsilon_k(n,p)|}{M_0(n,p)}. \tag{3.17}$$

From Eq.(3.17) by using Lemma 3.3.(ii) one obtains (3.17).

Remark 3.1.We remind now the proof of the transcendence of e following Shidlovsky proof is given in his book [8].

Theorem 3.1. The number e is transcendental.

Proof.([8], pp.126-129) Suppose now that e is an algebraic number; then it satisfies some relation of the form

$$a_0 + \sum_{k=1}^{n} a_k e^k = 0, \tag{3.18}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ integers and where $a_0 > 0$. Having substituted RHS of the Eq.(3.5) into Eq.(3.18) one obtains

$$a_0 + \sum_{k=1}^{n} a_k \frac{M_k(n,p) + \varepsilon_k(n,p)}{M_0(n,p)} = a_0 + \sum_{k=1}^{n} a_k \frac{M_k(n,p)}{M_0(n,p)} + \sum_{k=1}^{n} a_k \frac{\varepsilon_k(n,p)}{M_0(n,p)} = 0. \tag{3.19}$$

From Eq.(3.19) one obtains

$$a_0 M_0(n,p) + \sum_{k=1}^{n} a_k M_k(n,p) + \sum_{k=1}^{n} a_k \varepsilon_k(n,p) = 0. \tag{3.20}$$

We rewrite the Eq.(3.20) for short in the form

$$\begin{cases} a_0 M_0(n,p) + \sum_{k=1}^{n} a_k M_k(n,p) + \sum_{k=1}^{n} a_k \varepsilon_k(n,p) = \\ = a_0 M_0(n,p) + \Xi(n,p) + \sum_{k=1}^{n} a_k \varepsilon_k(n,p) = 0, \\ \Xi(n,p) = \sum_{k=1}^{n} a_k M_k(n,p). \end{cases} \tag{3.21}$$

We choose now the integers $M_1(n,p), M_2(n,p), \ldots, M_n(n,p)$ such that:

$$\begin{cases} p | M_1(n,p), p | M_2(n,p), \ldots, p | M_n(n,p) \\ \text{where } p > |a_0| \end{cases} \tag{3.22}$$

and $p \not| \; M_0(n,p)$. Note that $p | \; \Xi(n,p)$. Thus one obtains

$$p \not| \;\; a_0 M_0(n,p) + \Xi(n,p) \tag{3.23}$$

and therefore

$$\begin{cases} a_0 M_0(n,p) + \Xi(n,p) \in \mathbb{Z}, \\ \text{where} \\ a_0 M_0(n,p) + \Xi(n,p) \neq 0. \end{cases} \tag{3.24}$$

By using Lemma 3.4 for any δ such that $0 < \delta < 1$ we can choose a prime number $p = p(\delta)$ such that:

$$\left| \sum_{k=1}^{n} a_k \varepsilon_k(n,p) \right| < \delta \sum_{k=1}^{n} |a_k| = \epsilon < 1. \tag{3.25}$$

From (3.25) and Eq.(3.21) we obtain

$$a_0 M_0(n,p) + \Xi(n,p) + \epsilon = 0. \tag{3.26}$$

From (3.26) and Eq.(3.24) one obtains the contradiction.This contradiction finalized the proof.

3.2 The proof of the #-transcendence of the numbers $e^k, k \in \mathbb{N}$. We will divide the proof into four parts

3.2.1. Part I.The Robinson transfer of the Shidlovsky quantities $M_0(n,p), M_k(n,p), \varepsilon_k(n,p)$

In this subsection we will replace using Robinson transfer the Shidlovsky quantities $M_0(n,p), M_k(n,p), \varepsilon_k(n,p)$ by corresponding nonstandard quantities ${}^*M_0(\mathbf{n},\mathbf{p}), {}^*M_k(\mathbf{n},\mathbf{p}),\ {}^*\varepsilon_k(\mathbf{n},\mathbf{p})$. The properties of the nonstandard quantities ${}^*M_0(\mathbf{n},\mathbf{p}), {}^*M_k(\mathbf{n},\mathbf{p}),\ {}^*\varepsilon_k(\mathbf{n},\mathbf{p})$ one obtains directly from the properties of the standard quantities $M_0(n,p), M_k(n,p), \varepsilon_k(n,p)$ using Robinson transfer principle [4],[5].

1.Using Robinson transfer principle [4],[5] from Eq.(3.8) one obtains directly

$$\begin{cases} {}^*M_0(\mathbf{n},\mathbf{p}) = (-1)^{\mathbf{n}}(\mathbf{n}!)^{\mathbf{p}} + \mathbf{p} \times\ {}^*\Theta_1(\mathbf{n},\mathbf{p}), \\ {}^*\Theta_1(\mathbf{n},\mathbf{p}) \in\ {}^*\mathbb{Z}_\infty, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \\ \mathbb{N}_\infty \triangleq\ {}^*\mathbb{N} \backslash \mathbb{N}. \end{cases} \quad (3.27)$$

From Eq.(3.11) using Robinson transfer principle one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{cases} {}^*M_k(\mathbf{n},\mathbf{p}) == \mathbf{p} \times \left({}^*\Theta_2(\mathbf{n},\mathbf{p}) \right), \\ {}^*\Theta_2(\mathbf{n},\mathbf{p}) \in\ {}^*\mathbb{Z}_\infty, k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.28)$$

Using Robinson transfer principle from inequality (3.15) one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{cases} {}^*\varepsilon_k(\mathbf{n},\mathbf{p}) \leq \dfrac{\mathbf{n} \cdot ({}^*g(\mathbf{n})) \cdot \left([{}^*a(\mathbf{n})]^{\mathbf{p}-1}\right)}{(\mathbf{p}-1)!}, \\ k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.29)$$

Using Robinson transfer principle, from Eq.(3.5) one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{cases} {}^*(e^k) = ({}^*e)^k = \dfrac{{}^*M_k(\mathbf{n},\mathbf{p}) +\ ({}^*\varepsilon_k(\mathbf{n},\mathbf{p}))}{{}^*M_0(\mathbf{n},\mathbf{p})}, \\ k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.30)$$

Lemma 3.5. Let $\mathbf{n} \in {}^*\mathbb{N}_\infty$, then for any $k \in \mathbb{N}$ and for any $\delta \approx 0, \delta \in\ {}^*\mathbb{R}$ there exists

$\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$\left| {}^*e^k - \frac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})} \right| < \delta. \quad (3.31)$$

Proof. From Eq.(3.30) we obtain $\forall k(k \in \mathbb{N})$:

$$\begin{cases} \left| {}^*e^k - \dfrac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})} \right| = \dfrac{|{}^*\varepsilon_k(\mathbf{n},\mathbf{p})|}{|{}^*M_0(\mathbf{n},\mathbf{p})|}, \\ k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.32)$$

From Eq.(3.32) and (3.29) we obtain (3.31).

3.2.2. Part II.The Wattenberg imbedding ${}^{*}(e^{k})$ into ${}^{*}\mathbb{R}_{\mathbf{d}}$

In this subsection we will replace by using Wattenberg imbedding [6] and Gonshor transfer the nonstandard quantities ${}^{*}(e^{k})$ and the nonstandard Shidlovsky quantities ${}^{*}M_{0}(\mathbf{n},\mathbf{p}), {}^{*}M_{k}(\mathbf{n},\mathbf{p}), {}^{*}\varepsilon_{k}(\mathbf{n},\mathbf{p})$ by corresponding Wattenberg quantities ${}^{*}(e^{k})^{\#}, ({}^{*}M_{0}(\mathbf{n},\mathbf{p}))^{\#}, ({}^{*}M_{k}(\mathbf{n},\mathbf{p}))^{\#}, ({}^{*}\varepsilon_{k}(\mathbf{n},\mathbf{p}))^{\#}$. The properties of the Wattenberg quantities ${}^{*}(e^{k})^{\#}, ({}^{*}M_{0}(\mathbf{n},\mathbf{p}))^{\#}, ({}^{*}M_{k}(\mathbf{n},\mathbf{p}))^{\#}, ({}^{*}\varepsilon_{k}(\mathbf{n},\mathbf{p}))^{\#}$ one obtains directly from the properties of the corresponding nonstandard quantities ${}^{*}(e^{k}), {}^{*}M_{0}(\mathbf{n},\mathbf{p}), {}^{*}M_{k}(\mathbf{n},\mathbf{p}), {}^{*}\varepsilon_{k}(\mathbf{n},\mathbf{p})$ using Gonshor transfer principle [4],[7].

1.By using Wattenberg imbedding ${}^{*}\mathbb{R} \overset{\#}{\hookrightarrow} {}^{*}\mathbb{R}_{\mathbf{d}}$, from Eq.(3.30) one obtains

$$\begin{cases} [{}^{*}(e^{k})]^{\#} = \left[({}^{*}e)^{\#}\right]^{k} = \dfrac{[{}^{*}M_{k}(\mathbf{n},\mathbf{p})]^{\#} + [{}^{*}\varepsilon_{k}(\mathbf{n},\mathbf{p})]^{\#}}{[{}^{*}M_{0}(\mathbf{n},\mathbf{p})]^{\#}}, \\ k = 1,2,\ldots; k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}. \end{cases} \quad (3.33)$$

2.By using Wattenberg imbedding ${}^{*}\mathbb{R} \overset{\#}{\hookrightarrow} {}^{*}\mathbb{R}_{\mathbf{d}}$, and Gonshor transfer (see subsection 2.9 Theorem 2.19) from Eq.(3.27) one obtains

$$\begin{cases} [{}^{*}M_{0}(\mathbf{n},\mathbf{p})]^{\#} = [(-1)^{\mathbf{n}}]^{\#} \times [(\mathbf{n}!)^{\mathbf{p}}]^{\#} + \mathbf{p}^{\#} \times [{}^{*}\Theta_{1}(\mathbf{n},\mathbf{p})]^{\#} = \\ = \left[(-1^{\#})^{\mathbf{n}^{\#}}\right] \times \left[\left((\mathbf{n}!)^{\#}\right)^{\mathbf{p}^{\#}}\right] + \mathbf{p}^{\#} \times [{}^{*}\Theta_{1}(\mathbf{n},\mathbf{p})]^{\#}, \\ {}^{*}\Theta_{1}(\mathbf{n},\mathbf{p}) \in {}^{*}\mathbb{Z}_{\infty,\mathbf{d}}, \mathbf{n},\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}. \end{cases} \quad (3.34)$$

3.By using Wattenberg imbedding ${}^{*}\mathbb{R} \overset{\#}{\hookrightarrow} ({}^{\#}\mathbb{R})$, from Eq.(3.28) one obtains

$$\begin{cases} [{}^{*}M_{k}(\mathbf{n},\mathbf{p})]^{\#} == \mathbf{p}^{\#} \times \left[{}^{*}\Theta_{2}(\mathbf{n},\mathbf{p})\right]^{\#}, \\ [{}^{*}\Theta_{2}(\mathbf{n},\mathbf{p})]^{\#} \in {}^{*}\mathbb{Z}_{\infty,\mathbf{d}}, \\ k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}. \end{cases} \quad (3.35)$$

Lemma 3.6. Let $\mathbf{n} \in {}^{*}\mathbb{N}_{\infty}$, then for any $k \in \mathbb{N}$ and for any $\delta \approx 0, \delta \in {}^{*}\mathbb{R}$ there exists
$\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}$ such that

$$\left|({}^{*}e^{k})^{\#} - \frac{[{}^{*}M_{k}(\mathbf{n},\mathbf{p})]^{\#}}{[{}^{*}M_{0}(\mathbf{n},\mathbf{p})]^{\#}}\right| < \delta^{\#}. \quad (3.36)$$

Proof. Inequality (3.36) immediately follows from inequality (3.31) by using Wattenberg imbedding ${}^{*}\mathbb{R} \overset{\#}{\hookrightarrow} {}^{*}\mathbb{R}_{\mathbf{d}}$ and Gonshor transfer.

3.2.3.Part III.Reduction of the statement of e is #-transcendental number to equivalent statement in ${}^{*}\mathbb{Z}_{\mathbf{d}}$

using Gonshor idempotent theory

To prove that e is #-transcendental number we must show that e is not w-transcendental, i.e., there does not exist real $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} a_n x^n$ with rational coefficients $a_0, a_1, \ldots, a_n, \ldots \in \mathbb{Q}$ such that

$$\begin{cases} \sum_{n=0}^{\infty} a_k e^n = 0, \\ \sum_{n=0}^{\infty} |a_k| e^n \neq \infty. \end{cases} \tag{3.37}$$

Suppose that e is w-transcendental, i.e., there exists an $\mathbb{Q}$-analytic function $\breve{g}_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} \breve{a}_n x^n$, with rational coefficients:

$$\begin{cases} \breve{a}_0 = \frac{k_0}{m_0}, \breve{a}_1 = \frac{k_1}{m_1}, \ldots, \breve{a}_n = \frac{k_n}{m_n}, \ldots \in \mathbb{Q}, \\ \left| \breve{a}_0 \right| > 0, \end{cases} \tag{3.38}$$

such that the equality is satisfied:

$$\begin{cases} \sum_{n=0}^{\infty} \breve{a}_n e^n = 0. \\ \sum_{n=0}^{\infty} |a_k| e^n \neq \infty. \end{cases} \tag{3.39}$$

In this subsection we obtain an reduction of the equality given by Eq.(3.39) to equivalent equality given by Eq.(3.). The main tool of such reduction that external countable sum defined in subsection 2.8.

Lemma 3.7.Let $\Delta_{\leq}(k)$ and $\Delta_{>}(k)$ be the sum correspondingly

$$\begin{cases} \Delta_{\leq}(k) = \breve{a}_0 + \sum_{n=1}^{k \geq 1} \breve{a}_n e^n, \\ \Delta_{>}(k) = \sum_{n=k+1}^{\infty} \breve{a}_n e^n. \end{cases} \tag{3.40}$$

Then $\Delta_{>}(k) \neq 0, k = 1, 2, \ldots$

Proof. Suppose there exists k such that $\Delta_{>}(k) = 0$. Then from Eq.(3.39) follows $\Delta_{\leq}(k) = 0$. Therefore by Theorem 3.1 one obtains the contradiction.

Remark 3.2.Note that from Eq.(3.39) follows that in generel case there is a sequence $\{m_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} m_i = \infty,$$
$$\forall(i\in\mathbb{N})\left[\sum_{n=1}^{m_i}\breve{a}_n e^n < 0\right], \tag{3.41}$$
$$\breve{a}_0 + \lim_{i\to\infty}\left(\sum_{n=1}^{m_i}\breve{a}_n e^n\right) = 0,$$

or there is a sequence $\{m_j\}_{j=1}^{\infty}$ such that

$$\begin{cases} \lim\limits_{i\to\infty} m_j = \infty, \\ \forall(j\in\mathbb{N})\left[\sum\limits_{n=1}^{m_j}\breve{a}_n e^n > 0\right], \\ \breve{a}_0 + \lim\limits_{j\to\infty}\left(\sum\limits_{n=1}^{m_j}\breve{a}_n e^n\right) = 0, \end{cases} \tag{3.42}$$

or both sequences $\{m_i\}_{i=0}^{\infty}$ and $\{m_j\}_{j=0}^{\infty}$ with a property that is specified above exist.

Remark 3.3. We assume now for short but without loss of generelity that (3.41) is satisfied. Then from (3.41) by using Definition 2.17 and Theorem 2.14 (see subsection 2.8) one obtains the equality [4]

$$({}^*\breve{a}_0)^\# + \left[\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}({}^*\breve{a}_n)^\# \times ({}^*e^n)^\#\right] = -\varepsilon_{\mathbf{d}}. \tag{3.43}$$

Remark 3.4.Let $\Delta^\#_\leq(k)$ and $\Delta^\#_>(k)$ be the upper external sum defined by

$$\begin{cases} \Delta^\#_\leq(k) = \breve{a}_0 + \sum\limits_{n=1}^{k\geq 1}({}^*\breve{a}_n)^\# \times ({}^*e^n)^\#, \\ \Delta^\#_>(k) = \#Ext\text{-}\sum\limits_{\substack{n\in\mathbb{N} \\ n=k+1}}^{\wedge}\breve{a}_n e^n. \end{cases} \tag{3.44}$$

Note that from Eq.(3.43)-Eq.(3.44) follows that

$$\Delta^\#_\leq(k) + \Delta^\#_>(k) = -\varepsilon_{\mathbf{d}}. \tag{3.45}$$

Remark 3.5. Assume that $\alpha,\beta \in {}^*\mathbb{R}_{\mathbf{d}}$ and $\beta \notin {}^*\mathbb{R}$. In this subsection we will write for a short $\mathbf{ab}[\alpha|\beta]$ iff β absorbs α, i.e. $\beta+\alpha=\beta$.

Lemma 3.8. $\neg\mathbf{ab}[\Delta^\#_\leq(k)|\Delta^\#_>(k)], k = 1,2,\ldots$

Proof.Suppose there exists $k\in\mathbb{N}$ such that $\mathbf{ab}[\Delta^\#_\leq(k)|\Delta^\#_>(k)]$. Then from Eq.(3.45) one obtains

$$\Delta^\#_>(k) = -\varepsilon_{\mathbf{d}}. \tag{3.46}$$

From Eq.(3.46) by Theorem 2.11 follows that $\Delta_>(k) = 0$ and therefore by Lemma 3.7 one obtains the contradiction.

Theorem 3.2.[4] The equality (3.43) is inconsistent.

Proof.Let us consider hypernatural number $\Im \in {}^*\mathbb{N}_\infty$ defined by countable sequence

$$\Im = (m_0, m_0 \times m_1, \ldots, m_0 \times m_1 \times \ldots \times m_n, \ldots) \tag{3.47}$$

From Eq.(3.43) and Eq.(3.47) one obtains

$$\Im^\# \times ({}^*\breve{a}_0)^\# + \Im^\# \times \left[\#Ext\text{-}\widehat{\sum_{n\in\mathbb{N}}} ({}^*\breve{a}_n)^\# \times ({}^*e^n)^\# \right] = -\Im^\# \times \varepsilon_{\mathbf{d}}. \tag{3.48}$$

Remark 3.6.Note that from inequality (3.27) by Wattenberg transfer one obtains

$$[{}^*\varepsilon_n(\mathbf{n},\mathbf{p})]^\# \leq \frac{\mathbf{n}^\# \cdot [g_n(\mathbf{n})]^\# \cdot \left[[a(\mathbf{n})]^{\mathbf{p}-1}\right]^\#}{[(\mathbf{p}-1)!]^\#}, \tag{3.49}$$
$$n \in \mathbb{N}, \mathbf{n}, \mathbf{p} \in {}^*\mathbb{N}_\infty.$$

Substitution Eq.(3.30) into Eq.(3.48) gives

$$\begin{cases} \Im_0^\# + \left[\#Ext\text{-} \widehat{\sum_{n\in\mathbb{N}\setminus\{0\}}} (\Im_n)^\# \times ({}^*e^n)^\# \right] = \\ \Im_0^\# + \left[\#Ext\text{-}\widehat{\sum_{n\in\mathbb{N}}} (\Im_n)^\# \times \dfrac{[{}^*M_n(\mathbf{n},\mathbf{p})]^\# + [{}^*\varepsilon_n(\mathbf{n},\mathbf{p})]^\#}{[{}^*M_0(\mathbf{n},\mathbf{p})]^\#} \right] = -\Im^\# \times \varepsilon_{\mathbf{d}}, \\ \Im_n^\# \triangleq \Im^\# \times ({}^*\breve{a}_n)^\#, n \in \mathbb{N}, \Im_0^\# = \Im^\# \times ({}^*\breve{a}_0)^\#. \end{cases} \tag{3.50}$$

Multiplying Eq.(3.50) by Wattenberg hyperinteger $[{}^*M_0(\mathbf{n},\mathbf{p})]^\# \in {}^*\mathbb{Z}_{\mathbf{d}}$ by Theorem 2.13 (see subsection 2.8) one obtains

$$\begin{cases} \Im_0^\# \times [{}^*M_0(\mathbf{n},\mathbf{p})]^\# + \#Ext\text{-}\widehat{\sum_{n\in\mathbb{N}}} \left\{ (\Im_n)^\# \times [{}^*M_n(\mathbf{n},\mathbf{p})]^\# + \Im_k^\# \times [{}^*\varepsilon_n(\mathbf{n},\mathbf{p})]^\# \right\} = \\ = -\Im^\# \times [{}^*M_0(\mathbf{n},\mathbf{p})]^\# \times \varepsilon_{\mathbf{d}}. \end{cases} \tag{3.51}$$

By using inequality (3.49) for a given $\delta \in {}^*\mathbb{R}$, $\delta \approx 0$ we will choose infinite prime integer $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that:

$$\#Ext\text{-}\widehat{\sum_{k\in\mathbb{N}}} (\Im_k)^\# \times [{}^*\varepsilon_k(\mathbf{n},\mathbf{p})]^\# \subseteq \Im^\# \times [{}^*M_0(\mathbf{n},\mathbf{p})]^\# \times \delta^\# \times \varepsilon_{\mathbf{d}} \tag{3.52}$$

Now using the inequality (3.49) we are free to choose a prime hyperinteger $\mathbf{p} \in {}^*\mathbb{N}_\infty$ and

$\delta^\# \in {}^*\mathbb{R}_{\mathbf{d}}$, $\delta^\# = \delta^\#(\mathbf{p}) \approx 0$ in the Eq.(3.51) for a given $\epsilon \in {}^*\mathbb{R}, \epsilon \approx 0$ such that:

$$\Im^\# \times [{}^*M_0(\mathbf{n},\mathbf{p})]^\# \times \delta^\#(\mathbf{p}) = \epsilon^\#. \tag{3.53}$$

Hence from Eq.(3.52) and Eq.(3.53) we obtain

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} (\Im_n)^{\#} \times [{}^*\varepsilon_n(\mathbf{n},\mathbf{p})]^{\#} \subseteq -\epsilon^{\#} \times \varepsilon_{\mathbf{d}}. \tag{3.54}$$

Therefore from Eq.(3.51) and (3.54) by using definition (2.15) of the function $\mathbf{Int.\,p}(\alpha)$ given by Eq.(2.20)-Eq.(2.21) and corresponding basic property **I** (see subsection 2.7) of the function $\mathbf{Int.\,p}(\alpha)$ we obtain

$$\begin{cases} \mathbf{Int.\,p}\left(\Im_0^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} + \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \left\{\Im_n^{\#} \times [{}^*M_n(\mathbf{n},\mathbf{p})]^{\#} + \Im_n^{\#} \times [{}^*\varepsilon_n(\mathbf{n},\mathbf{p})]^{\#}\right\}\right) = \\ \Im_0^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} + \#Ext\text{-}\sum_{k\in\mathbb{N}}^{\wedge} \left\{\Im_n^{\#} \times [{}^*M_n(\mathbf{n},\mathbf{p})]^{\#}\right\} = \\ = -\mathbf{Int.\,p}\left(\Im^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}}\right) = -\Im^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \tag{3.55}$$

From Eq.(3.55) using basic property **I** of the function $\mathbf{Int.\,p}(\alpha)$ finally we obtain the main equality

$$\Im_0^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} + \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \left\{(\Im_k)^{\#} \times [{}^*M_n(\mathbf{n},\mathbf{p})]^{\#}\right\} = \Im^{\#} \times [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}}. \tag{3.56}$$

We will choose now infinite prime integer $\mathbf{p}$ in Eq.(3.56) $\mathbf{p} = \widehat{\mathbf{p}} \in {}^*\mathbb{N}_\infty$ such that

$$\widehat{\mathbf{p}}^{\#} > \mathbf{max}(|\Im_0^{\#}|, \mathbf{n}^{\#}.) \tag{3.57}$$

Hence from Eq.(3.34) follows

$$\widehat{\mathbf{p}}^{\#} \nmid [{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#}. \tag{3.58}$$

Note that $[{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#} \neq 0^{\#}$. Using (3.57) and (3.58) one obtains:

$$\widehat{\mathbf{p}}^{\#} \nmid [{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#} \times (\Im_0)^{\#}. \tag{3.59}$$

Using Eq.(3.35) one obtains

$$\widehat{\mathbf{p}}^{\#} \mid [{}^*M_n(\mathbf{n},\widehat{\mathbf{p}})]^{\#}, n = 1,2,\ldots. \tag{3.60}$$

3.2.4.Part IV.The proof of the inconsistency of the main equality (3.56)

In this subsection we wil prove that main equality (3.56) is inconsistent. This prooff is based on the Theorem 2.10 (v), see subsection 2.6.

Lemma 3.9.The equality (3.56) under conditions (3.59)-(3.60) is inconsistent.

Proof. (**I**) Let us rewrite Eq.(3.56) in the short form

$$\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}, \tag{3.61}$$

where

$$\begin{cases} \Sigma^\wedge(\mathbf{n},\widehat{\mathbf{p}}) = \#Ext\text{-}\sum\limits_{\substack{n\in\mathbb{N}\\ n\geq 1}}^{\wedge} \left\{(\Im_n)^\# \times [{}^*M_n(\mathbf{n},\widehat{\mathbf{p}})]^\#\right\}, \\ \Gamma(\mathbf{n},\widehat{\mathbf{p}}) = \Im_0^\# \times [{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]^\#, \\ \Lambda^\#(\widehat{\mathbf{p}}) = \Im^\# \times [{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]^\#. \end{cases} \quad (3.62)$$

From (3.59)-(3.60) follows that

$$\begin{cases} \widehat{\mathbf{p}}^\# \nmid \Gamma(\mathbf{n},\widehat{\mathbf{p}}), \\ \widehat{\mathbf{p}}^\# \,|\, \Sigma^\wedge(\mathbf{n},\widehat{\mathbf{p}}). \end{cases} \quad (3.63)$$

Remark 3.7.Note that $\Sigma^\wedge(\mathbf{n},\widehat{\mathbf{p}}) \notin {}^*\mathbb{R}$. Otherwise we obtain that $\mathbf{ab}.\mathbf{p}(\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^\wedge(\mathbf{n},\widehat{\mathbf{p}})) = \{\varnothing\}$. But the other hand from Eq.(3.61) follows that $\mathbf{ab}.\mathbf{p}(\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^\wedge(\mathbf{n},\widehat{\mathbf{p}})) = -\Lambda^\#(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}$. But this is a contradiction. This contradiction completed the proof of the statement (**I**)

(**II**) Let $\widetilde{\Delta}^\#_\leq(k,\mathbf{n},\widehat{\mathbf{p}}), \widetilde{\Delta}^\#_>(k,\mathbf{n},\widehat{\mathbf{p}}), \widetilde{\Delta}^\#_\leq(k_1,k_2,\mathbf{n},\widehat{\mathbf{p}})$ and $\widetilde{\Delta}^\#_\leq(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^\#), \widetilde{\Delta}^\#_>(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^\#)$, be the external sum correspondingly

$$\begin{cases} \widetilde{\Delta}^\#_\leq(k,\mathbf{n},\widehat{\mathbf{p}}) = \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \sum\limits_{n=1}^{k\geq 1} \left\{\Im_n^\# \times [{}^*M_n(\mathbf{n},\widehat{\mathbf{p}})]^\#\right\}, \\ \widetilde{\Delta}^\#_>(k,\mathbf{n},\widehat{\mathbf{p}}) = \#Ext\text{-}\sum\limits_{\substack{n\in\mathbb{N}\\ n\geq k+1}}^{\wedge} \left\{\Im_n^\# \times [{}^*M_n(\mathbf{n},\widehat{\mathbf{p}})]^\#\right\}, \\ \widetilde{\Delta}^\#_\leq(k_1,k_2,\mathbf{n},\widehat{\mathbf{p}}) = \sum\limits_{n=k_1}^{k_2} \left\{\Im_n^\# \times [{}^*M_n(\mathbf{n},\widehat{\mathbf{p}})]^\#\right\}, \\ \widetilde{\Delta}^\#_\leq(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^\#) = \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \sum\limits_{n=1}^{k\geq 1} \left\{\Im_n^\# \times [{}^*M_n(\mathbf{n},\widehat{\mathbf{p}})]^\# + \Im_n^\# \times [{}^*\varepsilon_n(\mathbf{n},\mathbf{p})]^\#\right\}, \\ \widetilde{\Delta}^\#_>(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^\#) = \#Ext\text{-}\sum\limits_{\substack{n\in\mathbb{N}\\ n\geq k+1}}^{\wedge} \left\{\Im_n^\# \times [{}^*M_n(\mathbf{n},\widehat{\mathbf{p}})]^\# + \Im_n^\# \times [{}^*\varepsilon_n(\mathbf{n},\mathbf{p})]^\#\right\}, \end{cases} \quad (3.64)$$

Note that from Eq.(3.61) and Eq.(3.64) follows that

$$\widetilde{\Delta}^\#_\leq(k,\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^\#_>(k,\mathbf{n},\widehat{\mathbf{p}}) = -\Lambda^\#(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \quad (3.65)$$

Lemma 3.10. (**i**) Under conditions (3.59)-(3.60)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^\#_\leq(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^\#) \,\middle|\, \widetilde{\Delta}^\#_>(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^\#)\right], k = 1,2,\ldots \quad (3.66)$$

And (**ii**) Under conditions (3.59)-(3.60)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^\#_\leq(k,\mathbf{n},\widehat{\mathbf{p}}) \,\middle|\, \widetilde{\Delta}^\#_>(k,\mathbf{n},\widehat{\mathbf{p}})\right], k = 1,2,\ldots \quad (3.67)$$

Proof. (**i**) First note that under conditions (3.59)-(3.60) one obtains

$$\forall k\left[\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) \neq 0\right] \tag{3.68}$$

Suppose that there exists a $k \geq 0$ such that $\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) \middle| \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n})\right]$. Then from Eq.(3.65) one obtains

$$\widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{3.69}$$

From Eq.(3.69) by Theorem 2.17 one obtains

$$\begin{gathered}-\varepsilon_{\mathbf{d}} = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1} \times \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1} \times \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = \\ = \Delta^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}}).\end{gathered} \tag{3.70}$$

Thus

$$-\varepsilon_{\mathbf{d}} = \Delta^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}}). \tag{3.71}$$

From Eq.(3.71) by Theorem 2.11 follows that $\Delta_{>}(k) = 0$ and therefore by Lemma 3.7 one obtains the contradiction. This contradiction finalized the proof of the Lemma 3.10 (i).

Proof. (**ii**) This is immediate from the Definition 2.14 (**Property I**), see subsection 2.7.

Part (**III**)

Remark 3.8.(**i**) Note that from Eq.(3.62) by Theorem 2.10 (v) follws that $\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}})$ has the form

$$\begin{gathered}\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = \mathbf{q}^{\#} + \mathbf{ab}.\mathbf{p}(\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}})) = \\ = \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}),\end{gathered} \tag{3.72}$$

where

$$\begin{gathered}\mathbf{q}^{\#} \in \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = \widetilde{\Delta}^{\#}_{>}(1,\mathbf{n},\widehat{\mathbf{p}}), \\ \mathbf{q} \in {}^{*}\mathbb{Z}_{\infty} \text{ and} \widehat{\mathbf{p}}|\mathbf{q}.\end{gathered} \tag{3.73}$$

(**ii**) Substitution by Eq.(3.72) into Eq.(3.61) gives

$$\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{3.74}$$

Remark 3.9. Note that from (3.74) by definitions follows that

$$\mathbf{ab}[(\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#})|(-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}})]. \tag{3.75}$$

Remark 3.10. Note that from (3.73) by construction of the Wattenberg integer $\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}})$ obviously follows that there exist some $k,d \in \mathbb{N}$ such that

$$\begin{gathered}\widetilde{\Delta}^{\#}_{\leq}(1,k,\mathbf{n},\widehat{\mathbf{p}}) < \mathbf{q}^{\#} \leq \widetilde{\Delta}^{\#}_{\leq}(1,d,\mathbf{n},\widehat{\mathbf{p}}), \\ k < d.\end{gathered} \tag{3.76}$$

Therefore

$$\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,k,\mathbf{n},\widehat{\mathbf{p}}) < \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} \leq \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,d,\mathbf{n},\widehat{\mathbf{p}}). \tag{3.77}$$

Note that under conditions (3.59)-(3.60) and (3.73) obviously one obtains

$$\begin{gathered} 0 \neq \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,k,\mathbf{n},\widehat{\mathbf{p}}) < \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} \leq \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,d,\mathbf{n},\widehat{\mathbf{p}}) \lneqq 0, \\ \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} \neq 0. \end{gathered} \tag{3.78}$$

From Eq.(3.74) follows that

$$\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{3.79}$$

Therefore

$$(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] + (-\varepsilon_{\mathbf{d}}) = -\varepsilon_{\mathbf{d}}. \tag{3.80}$$

From (3.78) follows that

$$\begin{gathered} 0 \neq (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\left[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,k,\mathbf{n},\widehat{\mathbf{p}})\right] < (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] \leq \\ \leq (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\left[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,d,\mathbf{n},\widehat{\mathbf{p}})\right] \lneqq 0, \\ (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] \lneqq 0. \end{gathered} \tag{3.81}$$

Note that by Theorem 2.8 (see subsection 2.5) and formula (3.44) one otains

$$\begin{gathered} 0 \neq \mathbf{Wst}\left\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\left[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,k,\mathbf{n},\widehat{\mathbf{p}})\right]\right\} = \mathbf{Wst}\left[({}^{*}a_0)^{\#} + \Delta^{\#}_{\leq}(1,k,\mathbf{n},\widehat{\mathbf{p}})\right], \\ \mathbf{Wst}\left\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\left[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,d,\mathbf{n},\widehat{\mathbf{p}})\right]\right\} = \mathbf{Wst}\left[({}^{*}a_0)^{\#} + \Delta^{\#}_{\leq}(1,d,\mathbf{n},\widehat{\mathbf{p}})\right] \lneqq 0, \\ \mathbf{Wst}\left\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\right\} \neq 0. \end{gathered} \tag{3.82}$$

From Eq.(3.81)-Eq.(3.82) follows that

$$\begin{gathered} 0 \neq \mathbf{Wst}\left[({}^{*}\breve{a}_0)^{\#} + \Delta^{\#}_{\leq}(1,k,\mathbf{n},\widehat{\mathbf{p}})\right] < \mathbf{Wst}\left\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\right\} \leq \\ \leq \mathbf{Wst}\left[({}^{*}\breve{a}_0)^{\#} + \Delta^{\#}_{\leq}(1,d,\mathbf{n},\widehat{\mathbf{p}})\right] \lneqq 0, \\ \mathbf{Wst}\left\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\right\} \lneqq 0. \end{gathered} \tag{3.83}$$

Thus

$$\neg\mathbf{ab}\left[(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\,\middle|\,(-\varepsilon_{\mathbf{d}})\right] \tag{3.84}$$

and therefore

$$(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] + (-\varepsilon_{\mathbf{d}}) \neq -\varepsilon_{\mathbf{d}}. \tag{3.85}$$

But this is a contradiction. This contradiction completed the proof of the Lemma 3.9.

4.Generalized Lindemann-Weierstrass theorem

In this section we remind the basic definitions of the Shidlovsky quantities,see [8] p.132- 134.

Theorem 4.1.[8] Let $f_l(z), l = 1,2,\ldots,r$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that

for any $l = 1,2,\ldots,r$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l}$, $k_l \geq 1, l = 1,2,\ldots,r$

form a complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots,r \tag{4.1}$$

and $a_l \in \mathbb{Z}, l = 1,2,\ldots,r, a_0 \neq 0$. Then

$$a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \tag{4.2}$$

Let $f_r(z)$ be a polynomial such that

$$\begin{cases} f_r(z) = \prod_{l=1}^{r} f_l(z) = b_0 + b_1 z + \ldots + b_{N_r} z^{N_r} = \\ = b_{N_r} \prod_{l=1}^{r} \prod_{k=1}^{k_l} (z - \beta_{k,l}), b_0 \neq 0, b_N > 0, N_r = \sum_{l=1}^{r} k_l. \end{cases} \tag{4.3}$$

Let $M_0(N_r,p), M_{k,l}(N_r,p)$ and $\varepsilon_{k,l}(N_r,p)$ be the quantities [8]:

$$M_0(N_r,p) = \int_0^{+\infty} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \tag{4.4}$$

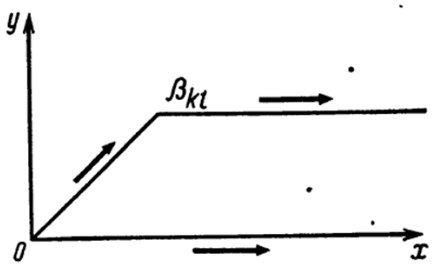
where in (4.4) we integrate in complex plane $\mathbb{C}$ along line $[0,+\infty]$, see Pic.1.

$$M_{k,l}(N_r,p) = e^{\beta_{k,l}} \int_{\beta_{k,l}}^{+\infty} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \tag{4.5}$$

where $k = 1,\ldots,k_l$ and where in (4.5) we integrate in complex plane $\mathbb{C}$ along line with initial point $\beta_{k,l} \in \mathbb{C}$ and which are parallel to real axis of the complex plane $\mathbb{C}$, see Pic.1.

$$\varepsilon_{k,l}(N_r,p) = e^{\beta_{k,l}} \int_0^{\beta_{k,l}} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \tag{4.6}$$

where $k = 1,\ldots,k_l$ and where in (4.6) we integrate in complex plane $\mathbb{C}$ along contour $[0,\beta_{k,l}]$, see Pic.1.



Pic.1.Contour $[0,\beta_{k,l}]$ in complex plane $\mathbb{C}$.

From Eq.(4.3) one obtains

$$b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) = b_{N_r}^{(N_r-1)p-1} b_0^p z^{p-1} + \sum_{s=p+1}^{(N_r+1)p} c_{s-1} z^{s-1}, \tag{4.7}$$

where $b_{N_r} b_0 \neq 0, c_s \in \mathbb{Z}, s = p,\ldots,(N_r-1)p-1$. Now from Eq.(4.4) and Eq.(4.7) using formula

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx = (s-1)!, s \in \mathbb{N}$$

one obtains

$$\begin{cases} M_0(N_r,p) = \dfrac{b_{N_r}^{(N_r-1)p-1} b_0^p}{(p-1)!} \displaystyle\int_0^{+\infty} z^{p-1} e^{-z} dz + \sum_{s=p+1}^{(N_r+1)p} \dfrac{c_{s-1}}{(p-1)!} \int_0^{+\infty} z^{s-1} e^{-z} dz = \\ b_{N_r}^{(N_r-1)p-1} b_0^p + \displaystyle\sum_{s=p+1}^{(N_r-1)p} \dfrac{(s-1)!}{(p-1)!} c_{s-1} = b_{N_r}^{(N_r-1)p-1} b_0^p + pC, \end{cases} \tag{4.8}$$

where $b_{N_r} b_0 \neq 0, C \in \mathbb{Z}$. We choose now a prime p such that $p > \max(|a_0|, b_{N_r}, |b_0|)$. Then from Eq.(4.8) follows that

$$p \nmid a_0 M_0(N_r,p). \tag{4.9}$$

From Eq.(4.3) and Eq.(4.5) one obtains

$$M_{k,l}(N_r,p) = \frac{e^{\beta_{k,l}}}{(p-1)!} \int_{\beta_{k,l}}^{+\infty} \left\{ b_{N_r}^{N_r p-1} z^{p-1} z^{p-1} \left[\prod_{j=1}^{r} \prod_{i=1}^{k_j} (z-\beta_{i,j})^p \right] \right\} e^{-z+\beta_{k,l}} dz, \tag{4.10}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. By change of the variable integration $z = u + \beta_{k,l}$ in RHS of the Eq.(4.10) we obtain

$$M_{k,l}(N_r,p) = \frac{1}{(p-1)!} \int_0^{+\infty} \left\{ b_{N_r}^{N_r p-1} (u+\beta_{k,l})^{p-1} u^p e^{-u} \left[\prod_{\substack{j=1 \\ j\neq l}}^{r} \prod_{\substack{i=1 \\ i\neq k}}^{k_j} (z+\beta_{k,l}-\beta_{i,j})^p \right] \right\} du, \tag{4.11}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. Let us rewrite now Eq.(4.11) in the following form

$$\left\{ \begin{array}{c} M_{k,l}(N_r,p) = \\ \dfrac{1}{(p-1)!}\int\limits_0^{+\infty}\left\{ (b_{N_r}u + b_{N_r}\beta_{k,l})^{p-1}u^p e^{-u}\left[\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{r_i=1\\ i\neq k}}^{k_j}(b_{N_r}u + b_{N_r}\beta_{k,l} - b_{N_r}\beta_{i,j})^p \right]\right\} du \end{array} \right. \quad (4.12)$$

Let $\mathbb{Z}_{\mathbf{A}}$ be a ring of the all algebraic integers. Note that [8]

$$\alpha_{i,j} = b_{N_r}\beta_{i,j} \in \mathbb{Z}_{\mathbf{A}}, i = 1,\ldots,k_j, j = 1,\ldots,r. \quad (4.13)$$

Let us rewrite now Eq.(4.12) in the following form

$$M_{k,l}(N_r,p) = \frac{1}{(p-1)!}\int\limits_0^{+\infty}(b_{N_r}u + \alpha_{k,l})^{p-1}u^p e^{-u}\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{i=1\\ i\neq k}}^{k_j}(b_{N_r}u + \alpha_{k,l} - \alpha_{i,j})^p du \quad (4.14)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ From Eq.(4.14) one obtains

$$\left\{ \begin{array}{c} \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) = \int\limits_0^{\infty}\dfrac{u^p e^{-u}\Phi_r(u)}{(p-1)!}du, \\ \Phi_r(u) = \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l}(b_{N_r}u + \alpha_{k,l})^{p-1}u^p e^{-u}\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{i=1\\ i\neq k}}^{k_j}(b_{N_r}u + \alpha_{k,l} - \alpha_{i,j})^p \end{array} \right. \quad (4.15)$$

The polynomial $\Phi_r(u)$ is a symmetric polynomial on any system Δ_l of variables $\alpha_{1,l}, \alpha_{2,l}, \ldots, \alpha_{k_l,l}$, where

$$\begin{array}{c} \Delta_l = \{\alpha_{1,l}, \alpha_{2,l}, \ldots, \alpha_{k_l,l}\}, l = 1,\ldots,r. \\ \alpha_{1,l}, \alpha_{2,l}, \ldots, \alpha_{k_l,l} \in \mathbb{Z}_{\mathbf{A}}, l = 1,\ldots,r. \end{array} \quad (4.16)$$

It well known that $\Phi_r(u) \in \mathbb{Z}[u]$ (see [8] p.134) and therefore

$$u^p\Phi_r(u) = \sum_{s=p+1}^{(N_r+1)p} c_{s-1}u^{s-1}, c_s \in \mathbb{Z}. \quad (4.17)$$

From Eq.(4.15) and Eq.(4.17) one obtains

$$\left\{ \begin{array}{c} \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) = \int\limits_0^{\infty}\dfrac{u^p e^{-u}\Phi_r(u)}{(p-1)!}du = \\ \sum_{s=p+1}^{(N_r+1)p}\dfrac{c_{s-1}}{(p-1)!}\int\limits_0^{\infty}u^{s-1}e^{-u}du = \sum_{s=p+1}^{(N_r+1)p} c_{s-1}\dfrac{(s-1)!}{(p-1)!} = pC, C \in \mathbb{Z}. \end{array} \right. \quad (4.18)$$

Therefore

$$\Xi(N_r,p) = \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) \in \mathbb{Z}, \qquad (4.19)$$
$$p|\Xi(N_r,p).$$

Let $O_R \subset \mathbb{C}$ be a circle wth the centre at point $(0,0)$. We assume now that $\forall k \forall l(\beta_{k,l} \in O_R)$. We will designate now

$$g_{k,l}(r) = \max_{|z| \leq R} |b_{N_r}^{-1} f_r(z) e^{-z+\beta_{k,l}}|,$$
$$g_0(r) = \max_{1 \leq k \leq k_l, 1 \leq l \leq r} g_{k,l}(r), g(r) = \max_{|z| \leq R} |b_{N_r}^{-1} z f_r(z)|. \qquad (4.20)$$

From Eq.(4.6) and Eq.(4.20) one obtains

$$|\varepsilon_{k,l}(N_r,p)| = \left| \int_0^{\beta_{k,l}} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z+\beta_{k,l}} dz}{(p-1)!} \right| \leq$$
$$\frac{1}{(p-1)!} \int_0^{\beta_{k,l}} |b_{N_r}^{-1} f(z) e^{-z+\beta_{k,l}}| [|b_{N_r}^{-1} z f_r(z)|]^{p-1} dz \leq \frac{g_0(r) g^{p-1}(r) |\beta_{k,l}|}{(p-1)!} \leq \frac{g_0(r) g^{p-1}(r) R}{(p-1)!}, \qquad (4.21)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. Note that

$$\frac{g_0(r) g^{p-1}(r) R}{(p-1)!} \to 0 \text{ if } p \to \infty. \qquad (4.22)$$

From (4.22) follows that for any $\epsilon \in [0,\delta]$ there exists a prime number p such that

$$\sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \varepsilon_{k,l}(N_r,p) = \epsilon(p) < 1. \qquad (4.23)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. From Eq.(4.4)-Eq.(4.6) follows

$$e^{\beta_{k,l}} = \frac{M_{k,l}(N_r,p) + \varepsilon_{k,l}(N_r,p)}{M_0(N_r,p)} \qquad (4.24)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. Assume now that

$$a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} = 0. \qquad (4.25)$$

Having substituted RHS of the Eq.(4.24) into Eq.(4.25) one obtains

$$\begin{cases} a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{M_{k,l}(N_r,p) + \varepsilon_{k,l}(N_r,p)}{M_0(N,p)} = \\ a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{M_{k,l}(N_r,p)}{M_0(N_r,p)} + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{\varepsilon_{k,l}(N_r,p)}{M_0(N_r,p)} = 0. \end{cases} \qquad (4.26)$$

From Eq.(4.26) by using Eq.(4.19) one obtains

$$a_0 + \Xi(N_r,p) + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \varepsilon_{k,l}(N_r,p) = 0. \tag{4.27}$$

We choose now a prime $p \in \mathbb{N}$ such that $p > \max(|a_0|,|b_0|,|b_{N_r}|)$ and $\epsilon(p) < 1$. Note that $p|\Xi(N_r,p)$ and therefore from Eq.(4.19) and Eq.(4.27) one obtains the contradiction. This contradiction completed the proof.

5.Generalized Lindemann-Weierstrass theorem

Theorem 5.1.[4] Let $f_l(z), l = 1,2,\ldots,$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that

for any $l \in \mathbb{N}$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l},\ k_l \geq 1, l = 1,2,\ldots$ form a

complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots \tag{5.1}$$

and $a_l \in \mathbb{Q}, a_0 \neq 0, l = 1,2,\ldots,$. We assume now that

$$\sum_{l=1}^{\infty} |a_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{5.2}$$

Then

$$a_0 + \sum_{l=1}^{\infty} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \tag{5.3}$$

We will divide the proof into three parts

Part I.The Robinson transfer

Let $f(z) = f_{\mathbf{r}}(z) \in {}^*\mathbb{Z}[z], z \in {}^*\mathbb{C}, l = 1,2,\ldots,\mathbf{r}, \mathbf{r} \in {}^*\mathbb{N}_\infty$ be a nonstandard polynomial such that

$$\begin{cases} f(z) = f_{\mathbf{r}}(z) = \prod_{l=1}^{\mathbf{r}} f_l(z) = \mathbf{b}_0 + \mathbf{b}_1 z + \ldots + \mathbf{b}_{\mathbf{N}} z^{\mathbf{N}} = \\ = \mathbf{b}_{\mathbf{N}} \prod_{l=1}^{\mathbf{r}} \prod_{k=1}^{{}^*k_l} (z - ({}^*\beta_{k,l})), \mathbf{b}_0 \neq 0, \mathbf{b}_{\mathbf{N}} > 0, \\ \mathbf{N} = \mathbf{N}_{\mathbf{r}} = \sum_{l=1}^{\mathbf{r}} ({}^*k_l) \in {}^*\mathbb{N}_\infty. \end{cases} \tag{5.4}$$

Let ${}^*M_0(\mathbf{N},\mathbf{p}), {}^*M_{k,l}(\mathbf{N},\mathbf{p})$ and ${}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})$ be the quantities:

$$ {}^*M_0(\mathbf{N},\mathbf{p}) = \int_0^{{}^*(+\infty)} \frac{b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!}, \tag{5.5}$$

$$\mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty,$$

where in (5.5) we integrate in nonstandard complex plaine ${}^*\mathbb{C}$ along line

${}^*[0,+\infty]$, see Pic.1.

$$ {}^*M_{k,l}(\mathbf{N},\mathbf{p}) = ({}^*e^{{}^*\beta_{k,l}}) \int\limits_{{}^*\beta_{k,l}}^{{}^*(+\infty)} \frac{\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!}, \quad (5.6) $$
$$ \mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty, $$

where $k = 1,\dots,{}^*k_l$ and where in (5.6) we integrate in nonstandard complex plain ${}^*\mathbb{C}$ along line with initial point ${}^*\beta_{k,l} \in {}^*\mathbb{C}$ and which are parallel to real axis of the complex plane ${}^*\mathbb{C}$, see Pic.1.

$$ {}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}) = ({}^*e^{{}^*\beta_{k,l}}) \int\limits_{0}^{{}^*\beta_{k,l}} \frac{b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!}, \quad (5.7) $$
$$ \mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty, $$

where $k = 1,\dots,{}^*k_l$ and where in (5.7) we integrate in nonstandard complex plain ${}^*\mathbb{C}$ along contour ${}^*[0,{}^*\beta_{k,l}]$, see Pic.1.

1.Using Robinson transfer principle [4],[5],[6] from Eq.(5.5) and Eq.(4.8) one obtains directly

$$ {}^*M_0(\mathbf{N},\mathbf{p}) = b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1}\mathbf{b}_0^{\mathbf{p}} + \mathbf{p}\mathbf{C}, \quad (5.8) $$

where $\mathbf{b}_{\mathbf{N}}\mathbf{b}_0 \neq 0, \mathbf{C} \in {}^*\mathbb{Z}_\infty$. We choose now infinite prime $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$ \{ \mathbf{p} > \max(|\mathbf{a}_0|, \mathbf{b}_{\mathbf{N}}, |\mathbf{b}_0|). \quad (5.9) $$

2.Using Robinson transfer principle from Eq.(5.6) and Eq.(4.19) one obtains directly

$$ \forall r(r \in \mathbb{N}) : $$
$$ {}^*\Xi(\mathbf{N},\mathbf{p},r) = \sum_{l=1}^{r}({}^*a_l)\sum_{k=1}^{k_l}({}^*M_{k,l}(\mathbf{N},\mathbf{p})) = \mathbf{p}\mathbf{C}_r \in {}^*\mathbb{Z}_\infty. \quad (5.10) $$

and therefore

$$ \forall r(r \in \mathbb{N}) : $$
$$ \mathbf{p} | {}^*\Xi(\mathbf{N},\mathbf{p},r). \quad (5.11) $$

3.Using Robinson transfer principle from Eq.(5.7) and Eq.(4.21) one obtains directly

$$\begin{cases} |{}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})| = \left| ({}^*e^{{}^*\beta_{k,l}}) \int\limits_0^{{}^*\beta_{k,l}} \frac{\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!} \right| \leq \\ \frac{1}{(\mathbf{p}-1)!} \int\limits_0^{{}^*\beta_{k,l}} \left| b_{\mathbf{N}}^{-1} f(z)({}^*e^{-z+({}^*\beta_{k,l})}) \right| [|b_{N_r}^{-1} z f(z)|]^{\mathbf{p}-1} dz \leq \frac{[{}^*g_0(\mathbf{r})][{}^*g^{\mathbf{p}-1}(\mathbf{r})]|{}^*\beta_{k,l}|}{(\mathbf{p}-1)!} \\ \leq \frac{[{}^*g_0(\mathbf{r})][{}^*g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!}, \end{cases} \quad (5.12)$$

where $k = 1,\ldots,{}^*k_l, l = 1,\ldots,\mathbf{r}$. Note that $\forall\epsilon(\epsilon \in {}^*\mathbb{R})[\epsilon \approx 0]$, there exists $\mathbf{p} = \mathbf{p}(\epsilon)$

$$\frac{[{}^*g_0(\mathbf{r})][{}^*g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!} \leq \epsilon. \quad (5.13)$$

4. From (5.13) follows that for any $\epsilon \in [0,\delta]$ there exists an infinite prime $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$\forall r(r \in \mathbb{N}) : \quad \sum_{l=1}^{r}({}^*a_l)\sum_{k=1}^{k_l}({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})) = \epsilon(\mathbf{p}) < 1 \quad (5.14)$$

where $k = 1,\ldots,{}^*k_l, l = 1,\ldots,\mathbf{r}.$.

5. From Eq.(5.5)-Eq.(5.7) we obtain

$$\left[{}^*e^{{}^*\beta_{k,l}} = \frac{{}^*M_{k,l}(\mathbf{N},\mathbf{p}) + ({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}))}{{}^*M_0(\mathbf{N},\mathbf{p})} \right], \quad (5.15)$$

where $k = 1,\ldots,{}^*k_l, l = 1,\ldots,\mathbf{r}$.

Part II.The Wattenberg imbedding ${}^*e^{{}^*\beta_{k,l}}$ into ${}^*\mathbb{R}_{\mathbf{d}}$

1.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{d}}$, and Gonshor transfer (see subsection 2.8 Theorem 2.17) from Eq.(5.8) one obtains

$$\begin{cases} ({}^*M_0(\mathbf{N},\mathbf{p}))^\# = \left(\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1}\mathbf{b}_0^{\mathbf{p}}\right)^\# + \mathbf{p}^\#\mathbf{C}^\# = \\ = (\mathbf{b}_{\mathbf{N}}^\#)^{(\mathbf{N}^\#-1)\mathbf{p}^\#-1}(\mathbf{b}_0^\#)^{\mathbf{p}^\#} + \mathbf{p}^\#\mathbf{C}^\# \end{cases} \quad (5.16)$$

where $\mathbf{b}_{\mathbf{N}}^\#\mathbf{b}_0^\# \neq 0^\#, \mathbf{C}^\# \in {}^*\mathbb{Z}_{\mathbf{d}}$. We choose now an infinite prime $\mathbf{p} \in {}^*\mathbb{N}$ such that

$$\left\{ \mathbf{p}^\# > \max(|\mathbf{a}_0^\#|, \mathbf{b}_{\mathbf{N}}^\#, \left|\mathbf{b}_0^\#\right|). \right. \quad (5.17)$$

2.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{d}}$, and Gonshor transfer from Eq.(5.10) one obtains directly

$$\begin{cases} \forall r(r \in \mathbb{N}) : \\ ({}^*\Xi(\mathbf{N},\mathbf{p},r))^\# = \sum_{l=1}^{r}\left(({}^*a_l)^\#\right)\sum_{k=1}^{k_l}({}^*M_{k,l}(\mathbf{N},\mathbf{p}))^\# = \mathbf{p}^\#\mathbf{C}_r^\# \in {}^*\mathbb{Z}_\mathbf{d} \end{cases} \quad (5.18)$$

and therefore

$$\forall r(r \in \mathbb{N})\left[\mathbf{p}^\# | ({}^*\Xi(\mathbf{N},\mathbf{p},r))^\#\right]. \quad (5.19)$$

3.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_\mathbf{d}$, and Gonshor transfer from Eq.(5.14) one obtains directly

$$\begin{cases} \forall r(r \in \mathbb{N}) : \\ \sum_{l=1}^{r}\left(({}^*a_l)^\#\right)\sum_{k=1}^{k_l}({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}))^\# = \epsilon^\#(\mathbf{p}^\#) < 1. \end{cases} \quad (5.20)$$

4.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_\mathbf{d}$, and Gonshor transfer from Eq.(5.15) one obtains directly

$$\left\{ e^{\beta_{k,l}^\#} \triangleq ({}^*e^{{}^*\beta_{k,l}})^\# = \frac{({}^*M_{k,l}(\mathbf{N},\mathbf{p}))^\# + ({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}))^\#}{({}^*M_0(\mathbf{N},\mathbf{p}))^\#}, \right. \quad (5.21)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,\mathbf{r} \in {}^*\mathbb{N}$.

Part III.Main equality

Remark 5.1 Note that in this subsection we often write for a short $a^\#$ instead $({}^*a)^\#, a \in \mathbb{R}$. For example we write

$$\begin{cases} \forall r(r \in \mathbb{N}) : \\ e^{\beta_{k,l}^\#} = \dfrac{M_{k,l}^\#(\mathbf{N},\mathbf{p})^\# + \varepsilon_{k,l}^\#(\mathbf{N},\mathbf{p})}{M_0^\#(\mathbf{N},\mathbf{p})} \end{cases}$$

instead Eq.(5.21).

Assumption 5.1. Let $f_l(z), l = 1,2,\ldots,$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that

for any $l \in \mathbb{N}$ algebraic numbers over the field $\mathbb{Q} : \beta_{1,l},\ldots,\beta_{k_l,l}$,
$k_l \geq 1, l = 1,2,\ldots,r$ form a

complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots \quad (5.22)$$

$l = 1,2,\ldots,a_0 \in \mathbb{Q}, a_0 \neq 0, r = 1,2,\ldots$.

Note that from Assumption 5.1 by Robinson transfer follows that algebraic numbers over

${}^*\mathbb{Q} : {}^*\beta_{1,l},\ldots,{}^*\beta_{k_l,l},\ k_l \geq 1, l = 1,2,\ldots,$ for any $l = 1,2,\ldots,$ form a complete set of the roots

of ${}^*f_l(z)$ such that

$${}^*f_l(z) \in {}^*\mathbb{Z}[z], \deg({}^*f_l(z)) = k_l, l = 1,2,\ldots . \quad (5.23)$$

Assumption 5.2. We assume now that there exists a sequence

$$\breve{a}_l = \frac{q_l}{m_l} \in \mathbb{Q}, l = 1,2,\ldots; r = 1,2,\ldots \tag{5.24}$$

and rational number

$$\breve{a}_0 = \frac{q_0}{m_0} \in \mathbb{Q}, \tag{5.25}$$

such that

$$\sum_{l=1}^{\infty} |\breve{a}_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{5.26}$$

and

$$\breve{a}_0 + \sum_{l=1}^{\infty} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} = 0. \tag{5.27}$$

Assumption 5.3. We assume now for a short that the all roots ${}^*\beta_{1,l},\ldots,{}^*\beta_{k_l,l}$, $k_l \geq 1, l = 1,2,\ldots$ of ${}^*f_l(z)$ are real.

In this subsection we obtain an reduction of the equality given by Eq.(5.27) in $\mathbb{R}$ to some equivalent equality given by Eq.(3.) in ${}^*\mathbb{R}_{\mathbf{d}}$. The main tool of such reduction that external countable sum defined in subsection 2.8.

Lemma 5.1.Let $\Delta_{\leq}(r)$ and $\Delta_{>}(r)$ be the sum correspondingly

$$\begin{cases} \Delta_{\leq}(r) = \breve{a}_0 + \sum_{l=1}^{r\geq 1} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}}, \\ \Delta_{>}(r) = \sum_{l=r+1}^{\infty} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}}. \end{cases} \tag{5.28}$$

Then $\Delta_{>}(r) \neq 0, r = 1,2,\ldots$

Proof. Suppose there exist r such that $\Delta_{>}(r) = 0$. Then from Eq.(5.27) follows $\Delta_{\leq}(r) = 0$. Therefore by Theorem 4.1 one obtains the contradiction.

Remark 5.2. Note that from Eq.(5.27) follows that in generel case there is a sequence $\{m_i\}_{i=1}^{\infty}$ such that

$$\begin{gathered} \lim_{i\to\infty} m_i = \infty, \\ \forall (i \in \mathbb{N}) \left[\breve{a}_0 + \sum_{l=1}^{m_i} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} < 0 \right], \\ \breve{a}_0 + \lim_{i\to\infty} \left(\sum_{l=1}^{m_i} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \right) = 0, \end{gathered} \tag{5.29}$$

or there is a sequence $\{m_j\}_{j=1}^{\infty}$ such that

$$\begin{cases} \lim\limits_{i\to\infty} m_j = \infty, \\ \forall (j \in \mathbb{N}) \left[\breve{a}_0 + \sum\limits_{l=1}^{m_j} \breve{a}_l \sum\limits_{k=1}^{k_l} e^{\beta_{k,l}} > 0 \right], \\ \breve{a}_0 + \lim\limits_{j\to\infty} \left(\sum\limits_{l=1}^{m_j} \breve{a}_l \sum\limits_{k=1}^{k_l} e^{\beta_{k,l}} \right) = 0, \end{cases} \quad (5.30)$$

or both sequences $\{m_i\}_{i=0}^{\infty}$ and $\{m_j\}_{j=0}^{\infty}$ with a property that is specified above exist.

Remark 5.3. We assume now for short but without loss of generelity that (5.29) is satisfied. Then from (5.29) by using Definition 2.17 and Theorem 2.14 (see subsection 2.8) one obtains the equality [4]

$$({}^*\breve{a}_0)^{\#} + \left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge} ({}^*\breve{a}_l)^{\#} \sum_{k=1}^{k_l} ({}^*e^{{}^*\beta_{k,l}})^{\#} \right] = -\varepsilon_{\mathbf{d}}. \quad (5.31)$$

Remark 5.4.Let $\Delta^{\#}_{\leq}(r)$ and $\Delta^{\#}_{>}(r)$ be the upper external sum defined by

$$\begin{cases} \Delta^{\#}_{\leq}(r) = \breve{a}_0 + \sum\limits_{l=1}^{r\geq 1} ({}^*\breve{a}_l)^{\#} \sum\limits_{k=1}^{k_l} ({}^*e^{{}^*\beta_{k,l}})^{\#}, \\ \Delta^{\#}_{>}(r) = \#Ext\text{-}\sum\limits_{\substack{n\in\mathbb{N} \\ l=r+1}}^{\wedge} ({}^*\breve{a}_l)^{\#} \sum\limits_{k=1}^{k_l} ({}^*e^{{}^*\beta_{k,l}})^{\#}. \end{cases} \quad (5.32)$$

Note that from Eq.(5.31)-Eq.(5.32) follows that

$$\Delta^{\#}_{\leq}(r) + \Delta^{\#}_{>}(r) = -\varepsilon_{\mathbf{d}}. \quad (5.33)$$

Remark 5.5. Assume that $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{d}}$ and $\beta \notin {}^*\mathbb{R}$. In this subsection we will write for a short $\mathbf{ab}[\alpha|\beta]$ iff β absorbs α,i.e. $\beta + \alpha = \beta$.

Lemma 5.2. $\neg\mathbf{ab}[\Delta^{\#}_{\leq}(r)|\Delta^{\#}_{>}(r)], k = 1,2,\ldots$

Proof.Suppose there exists $r \in \mathbb{N}$ such that $\mathbf{ab}[\Delta^{\#}_{\leq}(r)|\Delta^{\#}_{>}(r)]$. Then from Eq.(5.33) one obtains

$$\Delta^{\#}_{>}(r) = -\varepsilon_{\mathbf{d}}. \quad (5.34)$$

From Eq.(5.34) by Theorem 2.11 follows that $\Delta_{>}(r) = 0$ and therefore by Lemma 5.1 one obtains the contradiction.

Theorem 5.2.[4] The equality (5.31) is inconsistent.

Proof.Let us considered hypernatural number $\mathfrak{I} \in {}^*\mathbb{N}_{\infty}$ defined by countable

sequence

$$\mathfrak{I} = (m_0, m_0 \times m_1, \ldots, m_0 \times m_1 \times \ldots \times m_n, \ldots) \quad (5.35)$$

From Eq.(5.31) and Eq.(5.35) one obtains

$$\mathfrak{I}^{\#}\times({}^{*}\breve{a}_{0})^{\#}+\mathfrak{I}^{\#}\times\left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}({}^{*}\breve{a}_{l})^{\#}\sum_{k=1}^{k_l}({}^{*}e^{{}^{*}\beta_{k,l}})^{\#}\right]=$$
$$=\mathfrak{I}_{0}^{\#}+\left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\mathfrak{I}_{l}^{\#}\times\sum_{k=1}^{k_l}({}^{*}e^{{}^{*}\beta_{k,l}})^{\#}\right]=-\mathfrak{I}^{\#}\times\varepsilon_{\mathbf{d}} \quad (5.36)$$

where

$$\mathfrak{I}_{0}^{\#}=\mathfrak{I}^{\#}\breve{a}_{0}=\frac{\mathfrak{I}^{\#}q_{0}^{\#}}{m_{0}^{\#}},$$
$$\mathfrak{I}_{l}^{\#}=\mathfrak{I}^{\#}\breve{a}_{l}^{\#}=\frac{\mathfrak{I}_{0}^{\#}q_{l}^{\#}}{m_{l}^{\#}}. \quad (5.37)$$

Remark 5.6.Note that from inequality (5.12) by Gonshor transfer one obtains

$$\left|{}^{*}\varepsilon_{k,l}(\mathbf{N},\mathbf{p})^{\#}\right|\leq\frac{[{}^{*}g_{0}(\mathbf{r})]^{\#}[{}^{*}g^{\mathbf{p}-1}(\mathbf{r})]^{\#}|{}^{*}\beta_{k,l}^{\#}|}{(\mathbf{p}^{\#}-1)!^{\#}}$$
$$\mathbf{N},\mathbf{p}\in{}^{*}\mathbb{N}_{\infty}. \quad (5.38)$$

Substitution Eq.(5.21) into Eq.(5.36) gives

$$\mathfrak{I}_{0}^{\#}+\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\mathfrak{I}_{l}^{\#}\times\sum_{k=1}^{k_l}\frac{M_{k,l}^{\#}(\mathbf{N},\mathbf{p})^{\#}+\ \varepsilon_{k,l}^{\#}(\mathbf{N},\mathbf{p})}{M_{0}^{\#}(\mathbf{N},\mathbf{p})}=-\mathfrak{I}^{\#}\times\varepsilon_{\mathbf{d}}. \quad (5.39)$$

Multiplying Eq.(5.39) by Wattenberg hyperinteger $[{}^{*}M_{0}(\mathbf{N},\mathbf{p})]^{\#}\in{}^{*}\mathbb{Z}_{\mathbf{d}}$ by Theorem 2.13 (see subsection 2.8) we obtain

$$\mathfrak{I}_{0}^{\#}\times M_{0}^{\#}(N_{r},\mathbf{p})+\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\mathfrak{I}_{l}^{\#}\times\sum_{k=1}^{k_l}\left[M_{k,l}^{\#}(\mathbf{N},\mathbf{p})+\ \varepsilon_{k,l}^{\#}(\mathbf{N},\mathbf{p})\right]=$$
$$=-\mathfrak{I}^{\#}\times[{}^{*}M_{0}(\mathbf{N},\mathbf{p})]^{\#}\times\varepsilon_{\mathbf{d}}. \quad (5.40)$$

By using inequality (5.38) for a given $\delta\in{}^{*}\mathbb{R},\ \delta\approx0$ we will choose infinite prime integer $\mathbf{p}\in{}^{*}\mathbb{N}_{\infty},\mathbf{p}=\mathbf{p}(\delta)$ such that:

$$\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\mathfrak{I}_{l}^{\#}\sum_{k=1}^{k_l}\ \varepsilon_{k,l}^{\#}(\mathbf{N},\mathbf{p})\subseteq-\delta^{\#}\times\varepsilon_{\mathbf{d}}. \quad (5.41)$$

Therefore from Eq.(5.40) and (5.41) by using definition (2.15) of the function $\mathbf{Int}.\mathbf{p}(\alpha)$ given by Eq.(2.20)-Eq.(2.21) and corresponding basic property **I** (see subsection 2.7) of the function $\mathbf{Int}.\mathbf{p}(\alpha)$ we obtain

$$\mathbf{Int.\,p}\left(\Im_0^\# \times M_0^\#(\mathbf{N},\mathbf{p}) + \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l}\left[M_{k,l}^\#(\mathbf{N},\mathbf{p}) + \varepsilon_{k,l}^\#(\mathbf{N},\mathbf{p})\right]\right) =$$
$$\Im_0^\# \times M_0^\#(\mathbf{N},\mathbf{p}) + \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l} M_{k,l}^\#(\mathbf{N},\mathbf{p}) = \tag{5.42}$$
$$-\mathbf{Int.\,p}\left(\Im^\# \times [{}^*M_0(\mathbf{N},\mathbf{p})]^\# \times \varepsilon_{\mathbf{d}}\right) = -\Im^\# \times [{}^*M_0(\mathbf{N},\mathbf{p})]^\# \times \varepsilon_{\mathbf{d}}.$$

From Eq.(5.42) finally we obtain the main equality

$$\Im_0^\# \times M_0^\#(\mathbf{N},\mathbf{p}) + \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l} M_{k,l}^\#(\mathbf{N},\mathbf{p}) = -\Im^\# \times [{}^*M_0(\mathbf{N},\mathbf{p})]^\# \times \varepsilon_{\mathbf{d}}. \tag{5.43}$$

We will choose now infinite prime integer $\mathbf{p}$ in Eq.(3.56) $\mathbf{p} = \widehat{\mathbf{p}} \in {}^*\mathbb{N}_\infty$ such that

$$\left\{ \widehat{\mathbf{p}}^\# > \max(|\mathbf{a}_0^\#|, \mathbf{b}_{\mathbf{N}}^\#, |\mathbf{b}_0^\#|, \Im_0^\#). \right. \tag{5.44}$$

Hence from Eq.(5.16) follows

$$\widehat{\mathbf{p}}^\# \nmid M_0^\#(\mathbf{N},\widehat{\mathbf{p}}). \tag{5.45}$$

Note that $[{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]^\# \neq 0^\#$. Using (5.44) and (5.45) one obtains:

$$\widehat{\mathbf{p}}^\# \nmid M_{k,l}^\#(\mathbf{N},\widehat{\mathbf{p}},r) \times \Im_0^\#. \tag{5.46}$$

Using Eq.(5.11) one obtains

$$\widehat{\mathbf{p}}^\# \mid M_{k,l}^\#(\mathbf{N},\widehat{\mathbf{p}}), k,l = 1,2,\ldots. \tag{5.47}$$

Part IV.The proof of the inconsistency of the main equality (5.43)

In this subsection we wil prove that main equality (5.43) is inconsistent. This prooff is based on the Theorem 2.10 (**v**), see subsection 2.6.

Lemma 5.3.The equality (5.43) under conditions (5.46)-(5.47) is inconsistent.

Proof. (**I**) Let us rewrite Eq.(5.43) in the short form

$$\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \Sigma^\wedge(\mathbf{N},\widehat{\mathbf{p}}) = -\Lambda^\#(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}, \tag{5.48}$$

where

$$\left\{\begin{array}{c} \Sigma^\wedge(\mathbf{N},\widehat{\mathbf{p}}) = \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l} M_{k,l}^\#(\mathbf{N},\mathbf{p}), \\ \Gamma(\mathbf{n},\widehat{\mathbf{p}}) = \Im_0^\# \times [{}^*M_0(\mathbf{N},\widehat{\mathbf{p}})]^\#, \Lambda^\#(\widehat{\mathbf{p}}) = \Im^\# \times [{}^*M_0(\mathbf{N},\widehat{\mathbf{p}})]^\#. \end{array}\right. \tag{5.49}$$

From (5.46)-(5.47) follows that

$$\begin{cases} \widehat{\mathbf{p}}^{\#} \nmid \Gamma(\mathbf{N}, \widehat{\mathbf{p}}), \\ \widehat{\mathbf{p}}^{\#} \left| \Sigma^{\wedge}(\mathbf{N}, \widehat{\mathbf{p}}). \right. \end{cases} \tag{5.50}$$

Remark 5.7.Note that $\Sigma^{\wedge}(\mathbf{N}, \widehat{\mathbf{p}}) \not\in {}^{*}\mathbb{R}$. Otherwise we obtain that

$$\mathbf{ab}.\,\mathbf{p}(\Gamma(\mathbf{N}, \widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{N}, \widehat{\mathbf{p}})) = \{\varnothing\}. \tag{5.51}$$

But the other hand from Eq.(5.48) follows that

$$\mathbf{ab}.\,\mathbf{p}(\Gamma(\mathbf{N}, \widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{N}, \widehat{\mathbf{p}})) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{5.52}$$

But this is a contradiction. This contradiction completed the proof of the statement (**I**).

(**II**) Let $\widetilde{\Delta}^{\#}_{\leq}(k, \mathbf{N}, \widehat{\mathbf{p}}), \widetilde{\Delta}^{\#}_{>}(k, \mathbf{N}, \widehat{\mathbf{p}}), \widetilde{\Delta}^{\#}_{\leq}(k_1, k_2, \mathbf{N}, \widehat{\mathbf{p}})$ and $\widetilde{\Delta}^{\#}_{\leq}(k, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{n}), \widetilde{\Delta}^{\#}_{>}(k, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{n})$, be the external sum correspondingly

$$\begin{cases} \widetilde{\Delta}^{\#}_{\leq}(r, \mathbf{N}, \widehat{\mathbf{p}}) = \Gamma(\mathbf{N}, \widehat{\mathbf{p}}) + \sum_{l=1}^{r\geq 1} \sum_{k=1}^{k_l} \Im^{\#}_{l} \times \sum_{k=1}^{k_l} M^{\#}_{k,l}(\mathbf{N}, \mathbf{p}), \\ \widetilde{\Delta}^{\#}_{>}(r, \mathbf{N}, \widehat{\mathbf{p}}) = \sum_{l\geq r+1}^{\wedge} \sum_{k=1}^{k_l} \Im^{\#}_{l} \times \sum_{k=1}^{k_l} M^{\#}_{k,l}(\mathbf{N}, \mathbf{p}), \\ \widetilde{\Delta}^{\#}_{\leq}(r_1, r_2, \mathbf{N}, \widehat{\mathbf{p}}) = \sum_{l=r_1}^{r_2} \sum_{k=1}^{k_l} \Im^{\#}_{l} \times \sum_{k=1}^{k_l} M^{\#}_{k,l}(\mathbf{N}, \mathbf{p}), \\ \widetilde{\Delta}^{\#}_{\leq}(r, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{k,l}) = \Gamma(\mathbf{n}, \widehat{\mathbf{p}}) + \sum_{l=1}^{r\geq 1} \sum_{k=1}^{k_l} \Im^{\#}_{l} \times \sum_{k=1}^{k_l} \{M^{\#}_{k,l}(\mathbf{N}, \mathbf{p}) + \varepsilon^{\#}_{k,l}(\mathbf{N}, \mathbf{p})\}, \\ \widetilde{\Delta}^{\#}_{>}(r, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{k,l}) = \#Ext\text{-}\sum_{l\geq r+1}^{\wedge} \sum_{k=1}^{k_l} \Im^{\#}_{l} \times \sum_{k=1}^{k_l} \{M^{\#}_{k,l}(\mathbf{N}, \mathbf{p}) + \varepsilon^{\#}_{k,l}(\mathbf{N}, \mathbf{p})\}. \end{cases} \tag{5.53}$$

Note that from Eq.(5.43) and Eq.(5.53) follows that

$$\widetilde{\Delta}^{\#}_{\leq}(r, \mathbf{N}, \widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{>}(r, \mathbf{N}, \widehat{\mathbf{p}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}, \quad r = 1, 2, \ldots \tag{5.54}$$

Lemma 5.4. (**i**) Under conditions (5.46)-(5.47)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(r, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{k,l}) \left| \widetilde{\Delta}^{\#}_{>}(r, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{k,l}) \right.\right], r = 1, 2, \ldots\ . \tag{5.55}$$

And (**ii**) Under conditions (5.46)-(5.47)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(r, \mathbf{N}, \widehat{\mathbf{p}}) \left| \widetilde{\Delta}^{\#}_{>}(r, \mathbf{N}, \widehat{\mathbf{p}}) \right.\right], r = 1, 2, \ldots\ . \tag{5.56}$$

Proof. (**i**) First note that under conditions (5.46)-(5.47) one obtains

$$\left[\widetilde{\Delta}^{\#}_{\leq}(r, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{k,l}) \neq 0\right], r = 1, 2, \ldots \tag{5.57}$$

Suppose that there exists $r \geq 0$ such that $\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(r, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{k,l}) \left| \widetilde{\Delta}^{\#}_{>}(r, \mathbf{N}, \widehat{\mathbf{p}}, \varepsilon^{\#}_{k,l}) \right.\right]$. Then from Eq.(5.54) one obtains

$$\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{5.58}$$

From Eq.(5.58) by Theorem 2.17 one obtains

$$-\varepsilon_{\mathbf{d}} = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1} \times \widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1} \times \widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = \Delta^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}). \tag{5.59}$$

Thus

$$-\varepsilon_{\mathbf{d}} = \Delta^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}). \tag{5.60}$$

From Eq.(5.60) by Theorem 2.11 follows that $\Delta_{>}(r) = 0$ and therefore by Lemma 5.2 one obtains the contradiction. This contradiction finalized the proof of the Lemma 5.4 (**i**)

Proof. (**ii**) This is immediate from the Definition 2.14 (**Property I**), see subsection 2.7.

(**III**)

Remark 5.8.(**i**) Note that from Eq.(5.49) by Theorem 2.10 (**v**) follws that $\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}})$ has the form

$$\begin{aligned}\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}}) &= \mathbf{q}^{\#} + \mathbf{ab}.\mathbf{p}(\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}})) = \\ &= \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}})\end{aligned} \tag{5.61}$$

where

$$\begin{gathered}\mathbf{q}^{\#} \in \Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}}) = \widetilde{\Delta}^{\#}_{>}(1,\mathbf{N},\widehat{\mathbf{p}}), \\ \mathbf{q} \in {}^{*}\mathbb{Z}_{\infty} \text{ and} \widehat{\mathbf{p}}|\mathbf{q}.\end{gathered} \tag{5.62}$$

(**ii**) Substitution by Eq.(5.61) into Eq.(5.48) gives

$$\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}}) = \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{5.63}$$

Remark 5.9. Note that from (5.63) by definitions follows that

$$\mathbf{ab}[(\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#})|(-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}})]. \tag{5.64}$$

Remark 5.10. Note that from (5.62) by construction of the Wattenberg integer $\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}})$ obviously follows that there exists some $r_1, r_2 \in \mathbb{N}$ such that

$$\begin{gathered}\widetilde{\Delta}^{\#}_{\leq}(1,r_1,\mathbf{N},\widehat{\mathbf{p}}) < \mathbf{q}^{\#} \leq \widetilde{\Delta}^{\#}_{\leq}(1,r_2,\mathbf{N},\widehat{\mathbf{p}}), \\ r_1 < r_2.\end{gathered} \tag{5.65}$$

Therefore

$$\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_1,\mathbf{N},\widehat{\mathbf{p}}) < \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} \leq \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_2,\mathbf{N},\widehat{\mathbf{p}}). \tag{5.66}$$

Note that under conditions (5.46)-(5.47) and (5.66) obviously one obtains

$$\begin{gathered}0 \neq \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_1,\mathbf{N},\widehat{\mathbf{p}}) < \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} \leq \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_2,\mathbf{N},\widehat{\mathbf{p}}) \lneqq 0, \\ \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} \neq 0.\end{gathered} \tag{5.67}$$

From Eq.(5.63) follows that

$$\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{5.68}$$

Therefore

$$(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] + (-\varepsilon_{\mathbf{d}}) = -\varepsilon_{\mathbf{d}}. \tag{5.69}$$

From (5.69) follows that

$$\begin{gathered}0 \neq (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\Big[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_1,\mathbf{N},\widehat{\mathbf{p}})\Big] < (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] \leq \\ \leq (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\Big[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_2,\mathbf{N},\widehat{\mathbf{p}})\Big] \lneqq 0, \\ (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] \lneqq 0.\end{gathered} \tag{5.70}$$

Note that from (5.70) by Theorem 2.8 (see subsection 2.5) and formula (5.32) one otains

$$\begin{gathered}0 \neq \mathbf{Wst}\Big\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\Big[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_1,\mathbf{N},\widehat{\mathbf{p}})\Big]\Big\} = \mathbf{Wst}\big[({}^*a_0)^{\#} + \Delta^{\#}_{\leq}(1,r_1,\mathbf{n},\widehat{\mathbf{p}})\big], \\ \mathbf{Wst}\Big\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}\Big[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{\leq}(1,r_2,\mathbf{n},\widehat{\mathbf{p}})\Big]\Big\} = \mathbf{Wst}\big[({}^*a_0)^{\#} + \Delta^{\#}_{\leq}(1,r_2,\mathbf{N},\widehat{\mathbf{p}})\big] \lneqq 0, \\ \mathbf{Wst}\big\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\big\} \neq 0.\end{gathered} \tag{5.71}$$

From Eq.(5.70)-Eq.(5.71) follows that

$$\begin{gathered}0 \neq \mathbf{Wst}\big[({}^*\breve{a}_0)^{\#} + \Delta^{\#}_{\leq}(1,r_1,\mathbf{N},\widehat{\mathbf{p}})\big] < \mathbf{Wst}\big\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\big\} \leq \\ \leq \mathbf{Wst}\big[({}^*\breve{a}_0)^{\#} + \Delta^{\#}_{\leq}(1,r_2,\mathbf{N},\widehat{\mathbf{p}})\big] \lneqq 0, \\ \mathbf{Wst}\big\{(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\big\} \lneqq 0.\end{gathered} \tag{5.72}$$

Thus

$$\neg\mathbf{ab}\big[(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}]\,\big|\,(-\varepsilon_{\mathbf{d}})\big] \tag{5.73}$$

and therefore

$$(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] + (-\varepsilon_{\mathbf{d}}) \neq -\varepsilon_{\mathbf{d}}. \tag{5.74}$$

But this is a contradiction. This contradiction completed the proof of the Lemma 5.3.

Remark 5.11. Note that by Definitions 2.19-2.20 and Theorem 2.18 from Assumption 5.1 and Assumption 5.2 follows

$$\left|({}^*\breve{a}_0)^{\#} + \left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}({}^*\breve{a}_l)^{\#}\sum_{k=1}^{k_l}({}^*e^{{}^*\beta_{k,l}})^{\#}\right]\right|^2 = |-\varepsilon_{\mathbf{d}}|^2 = \varepsilon_{\mathbf{d}}. \tag{5.75}$$

Theorem 5.3.The equality (5.75) is inconsistent.

Proof. The proof of the Theorem 5.3 obviously copies in main details the proof of the

Theorem 5.2.

Theorem 5.3 completed the proof of the main Theorem 1.6.

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