Quaternionic versus Maxwell based differential calculus

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Abstract

Two quite different forms of differential calculus exist that both have physical significance. The most simple version is quaternionic differential calculus. Maxwell based differential calculus is based on the equations that Maxwell and others have developed in order to describe electromagnetic phenomena. Both approaches can be represented by four-component “fields” and four-component differential operators. Both approaches result in a dedicated non-homogeneous wave equation. These wave equations differ and offer solutions that differ in details.

Maxwell based differential calculus uses coordinate time t, where quaternionic differential calculus uses proper time τ. The consequence is that also the interpretation of speed differs between the two approaches. A more intriguing fact is that these differences involve a different space-progression model and different charges and currents. The impact of these differences are not treated in this paper.

Physics formulated in Maxwell based differential calculus differs from physics formulated in quaternionic differential calculus. This choice influences the description of physical reality. It cannot not influence physical reality.

Quaternionic differential calculus fits better with the application of Hilbert spaces in quantum physics than Maxwell based differential calculus. However, Maxwell based differential calculus is the general trend in current physical theories.

Contents

1 Introduction

In this paper the quaternionic differential equations are compared to Maxwell based differential equations [1][2].

In order to ease the comparison of the two approaches, we apply four-component "fields" and four-component operators. The parameter space is represented in a similar way by a similar but flat four-component "field".

We start with a four-component "field" φ that is differentiable and we also define the corresponding four-component differential operator ∇. The four-component approach is sometimes implemented with the help of spinors and corresponding matrices. Here we could, but will not apply that methodology. The method confuses more than that it elucidates the situation. Instead, we consider the scalar part as a separate part and we apply base vectors $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ rather than the corresponding Pauli matrices [3].

Both approaches start with a basic "field" φ. Gravitation concerns applications where this "field" φ is always and everywhere present. This kind of field is suited as continuum for embedding discrete objects. It is also suited as long range transport medium for carriers of information and energy. Electromagnetic theory concerns applications where the existence of the "field" φ is determined by a set of charges in the form of local sources or local drains. These two kinds of basic fields are related, but that is subject of another paper [4].

Double differentiation results in a non-homogeneous wave equation that reveals how the basic "field" φ can be deformed or vibrated and how sources and drains control its existence.

2 Notation

Italic font face without subscript indicates four-component "fields" or four-component operators. Bold italic font face indicates 3D vectors or 3D operators.

The four-component "fields" consist of a combination of a scalar field and a field of 3D vectors.

$$\varphi = \{\varphi_{0,}\varphi_{1,}\varphi_{2,}\varphi_3\} = \{\varphi_{0,}\boldsymbol{\varphi}\} = \{\varphi_{0,}\ \boldsymbol{i}\,\varphi_1 + \boldsymbol{j}\,\varphi_2 + \boldsymbol{k}\,\varphi_3\} \tag{1}$$

Both approaches start with a basic "field" φ that is always and everywhere present or this "field" is generated by a set of local sources or drains. All other "fields" are supposed to be derived from the basic "field". For example the four-component "field" ϕ is defined as:

$$\phi = \nabla\varphi \tag{2}$$

Both approaches use the 3D nabla operator $\boldsymbol{\nabla}$**.**

$$\boldsymbol{\nabla} = \left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{3}$$

This vector operator can be used if the deformation of the subjected fields are not too violent.

In order to support the comparison, we will apply four-component "fields". We will use corresponding four-component differential operators and indicate these with symbol ∇.

$$\nabla = \{\nabla_{0,}\nabla_{1,}\nabla_{2,}\nabla_{3}\} = \{\nabla_{0,}\boldsymbol{\nabla}\} \tag{4}$$

The four-component differential operator differs between the two approaches. Quaternionic differential calculus uses proper time τ and partial derivative $\nabla_0 = \frac{\partial}{\partial\tau}$ and Maxwell based differential calculus uses coordinate time t and partial derivative $\nabla_t = \frac{\partial}{\partial t}$.

We will use $\nabla_0 = \frac{\partial}{\partial\tau}$ rather than $\nabla_\tau = \frac{\partial}{\partial\tau}$. We suppose that ∇_0 and ∇_t commute with $\boldsymbol{\nabla}$.

3 Parameter spaces

The parameter space is represented by a four-component flat "field":

$$\{x_{0,}\ \boldsymbol{i}\, x_1 + \boldsymbol{j}\, x_2 + \boldsymbol{k}\, x_3\} = \{x_{0,}\ \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\};\ x_0 = \tau \text{ or } x_0 = t \tag{1}$$

Infinitesimal coordinate time steps and infinitesimal proper time steps are related by:

Coordinate time step vector $=$ proper time step vector $+$ spatial step vector (2)

Or in Pythagoras format:

$$(\Delta t)^2 = (\Delta\tau)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{3}$$

In the quaternionic model the formula indicates that the coordinate time step corresponds to the step of a full quaternion, which is a superposition of a proper time step and a pure spatial step.

An infinitesimal spacetime step Δs is usually presented as an infinitesimal proper time step $\Delta\tau$.

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{4}$$

The signs on the right side form the Minkowski signature $(+,-,-,-)$.

The quaternionic model offers a Euclidean signature $(+,+,+,+)$ as is shown in formula (3).

4 Definition of the differential

Locally, the deformation of the "field" φ is supposed to be sufficiently moderate, such that the nabla operator ∇ can be applied.

In the two approaches, the differentiations $\phi = \nabla\varphi$ have different definitions.

We do not go further than double differentiation. This double differentiation results in a non-homogeneous wave equation. In the two approaches, the non-homogeneous wave equations have a different format.

4.1 Quaternionic differential calculus

In the quaternionic differential calculus:

$$\phi = \nabla\varphi \equiv (\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla_0\varphi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \boldsymbol{\nabla}\varphi_0 + \nabla_0\boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{1}$$

The $\pm$ sign indicates the freedom of choice between a left handed and a right handed external vector product. Each term on the right side stands for a derived field.

Double quaternionic differentiation leads to a wave equation:

$$\zeta = \nabla^* \phi = \nabla^* \nabla \varphi = \{\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi = \frac{\partial^2 \varphi}{\partial \tau^2} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \tag{2}$$

4.2 Maxwell based differential calculus

In official Maxwell based differential calculus holds:

$$\phi = \nabla \varphi \equiv -\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \varphi_t + \nabla_t \boldsymbol{\varphi} - \boldsymbol{\nabla} \times \boldsymbol{\varphi}; \tag{3}$$

With gauge:

$$\alpha \nabla_t \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle = 0 \tag{4}$$

$$\alpha = \{-1,0,1\} \tag{5}$$

Using the proper gauge and double differentiation leads to the Maxwell based wave equation:

$$\rho = \{\nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi \tag{6}$$

The ***adapted Maxwell based calculus*** uses an extra scalar field $\varkappa$ that replaces the gauge:

$$\phi = \nabla \varphi \equiv -\varkappa - \boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{B}} = -\nabla_t \varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \boldsymbol{\nabla} \varphi_0 + \nabla_t \boldsymbol{\varphi} - \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{7}$$

$$\varkappa \equiv \nabla_t \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{8}$$

Double differentiation leads to the adapted Maxwell based wave equation:

$$\xi = \{\nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi \tag{9}$$

$\varkappa$, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are derived fields.

4.3 Wave equations

The quaternionic wave equations and the Maxwell based wave equations have different format and different solutions. They result in different speeds of the corresponding wave fronts.

Not only the format of the wave equations differ between the two approaches. The distributions ζ, ρ and ξ are density distributions of locations of charges, currents, sources, drains or embedding locations. The distributions ζ and ρ differ and ρ differs from ξ!

5 Mathematical facts

Here α is a scalar function. $\boldsymbol{a}$ is a vector function.

The following formulas are just mathematical facts that generally hold for vector differential calculus:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \alpha = \boldsymbol{0} \tag{1}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = 0 \tag{2}$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{\nabla}, \boldsymbol{a} \rangle = \mathbf{0} \tag{3}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{a}; \tag{4}$$

6 Quaternionic differential calculus

Quaternions are a combination of a real scalar a_0 and a 3D vector $\boldsymbol{a}$, which forms the imaginary part. Quaternionic number systems are division rings. This means that every non-zero element has an inverse. Hilbert spaces can only cope with number systems that are division rings. Quaternionic number systems form the most elaborate division ring.

Continuous quaternionic functions represent skew fields. Quaternionic differential calculus uses proper time τ as progression parameter. For that reason all quaternionic differential equations are inherently Lorentz invariant.

Due to their four dimensions quaternionic number systems exist in 16 symmetry flavors that only differ in their discrete symmetry sets [1][4].

$$a \equiv a_0 + \boldsymbol{a} \tag{1}$$

The quaternionic conjugate is defined as:

$$a^* \equiv a_0 - \boldsymbol{a} \tag{2}$$

The norm is defined as:

$$\|a\| \equiv \sqrt{a^* a} \tag{3}$$

The quaternionic product is defined as:

$$c = ab = a_0 b_0 + a_0 \boldsymbol{b} + b_0 \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{4}$$

The $\pm$ sign indicates the freedom of choice between a left handed and a right handed external vector product.

The quaternionic nabla is defined by:

$$\nabla \equiv \left\{\frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \equiv \frac{\partial}{\partial \tau} + \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} = \nabla_0 + \boldsymbol{\nabla} \tag{5}$$

$$\nabla^* = \nabla_0 - \boldsymbol{\nabla} \tag{6}$$

In quaternionic differential calculus ***the differential can be defined as a product***.

$$\phi = \nabla \varphi \equiv (\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla_0 \varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \boldsymbol{\nabla} \varphi_0 + \nabla_0 \boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{7}$$

$$\phi_0 = \nabla_0 \varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{8}$$

$$\boldsymbol{\phi} = \boldsymbol{\nabla} \varphi_0 + \nabla_0 \boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{9}$$

The second derivative delivers an non-homogeneous wave equation:

$$\zeta = \nabla^*\phi = \nabla^*\nabla\varphi \equiv \{\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\}\varphi \tag{10}$$

$$\zeta_0 = \nabla_0\phi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\phi}\rangle \tag{11}$$

$$\boldsymbol{\zeta} = \boldsymbol{\nabla}\phi_0 + \nabla_0\boldsymbol{\phi} \pm \boldsymbol{\nabla} \times \boldsymbol{\phi} \tag{12}$$

Notice that:

$$\nabla_0\phi = \nabla_0\nabla_0\varphi_0 - \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \nabla_0\boldsymbol{\nabla}\varphi_0 + \nabla_0\nabla_0\boldsymbol{\varphi} \pm \nabla_0\boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{13}$$

$$-\boldsymbol{\nabla}\phi = -\boldsymbol{\nabla}\nabla_0\varphi_0 + \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 + \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \mp \nabla_0\boldsymbol{\nabla} \times \boldsymbol{\varphi} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \tag{14}$$

$$= -\boldsymbol{\nabla}\nabla_0\varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 + \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \mp \nabla_0\boldsymbol{\nabla} \times \boldsymbol{\varphi} + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{\varphi}$$

6.1 Solutions of the quaternionic wave equation

Solutions of the wave equation depend on start and boundary conditions.

$$\nabla^*\nabla\varphi \equiv \{\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\}\varphi = \frac{\partial^2\varphi}{\partial\tau^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \zeta \tag{1}$$

Apart from more detailed conditions the equation can be reduced to special forms.

6.1.1 Poisson Equations

In the screened Poisson equation the first term is reduced to multiplication with a real constant λ:

$$\nabla_0\nabla_0\varphi = -\lambda^2\,\varphi \tag{1}$$

$$\{-\lambda^2 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\}\varphi = \zeta \tag{2}$$

The corresponding solution is superposition of screened Green's functions.

Green functions represent solutions for point sources.

$$\{-\lambda^2 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\}G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{3}$$

$$\varphi = \iiint G(\boldsymbol{r} - \boldsymbol{r}')\,\zeta(\boldsymbol{r}')\,d^3\boldsymbol{r}' \tag{4}$$

In this case the Green's function for spherical symmetric conditions ($\boldsymbol{r} = \boldsymbol{0}$) is:

$$G(r) = \frac{\exp(-\lambda\, r)}{r} \tag{5}$$

A zero value of λ offers the normal Poisson equation.

6.1.2 Coherent swarm of charges

A coherent swarm of charges that can be described by a continuous quaternionic location density function represents a blurred Green's function. For example, in case of an isotropic Gaussian distribution ζ_0 the N contributions of the swarm elements to the integral φ will on average equal $\mathfrak{G}(r) = \mathrm{ERF}(r)/r$. $N\, \mathfrak{G}(r)$ represents the local potential.

6.1.3 The homogeneous quaternionic wave equation

Locally, the wave function is considered to act in a rather flat continuum φ.

$$\nabla^*\nabla\varphi = \nabla_0\nabla_0\varphi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi = 0 \tag{1}$$

First we look at:

$$\nabla^*\nabla\varphi_0 = 0 \tag{2}$$

χ_0 is a scalar function. For isotropic conditions in three participating dimensions equation (2) has three dimensional spherical wave fronts as one group of its solutions.

By changing to polar coordinates it can be deduced that a general solution is given by:

$$\varphi_0(r, \tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} \tag{3}$$

where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

Next we consider the vector function $\boldsymbol{\chi}$

$$\nabla^*\nabla\boldsymbol{\varphi} = 0 \tag{4}$$

Equation (4) has one dimensional wave fronts as one group of its solutions:

$$\boldsymbol{\varphi}(z, \tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{5}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function.

The imaginary $\boldsymbol{i}$ represents the base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the one dimensional wave front.

7 Maxwell based equations

The formulas that are used here are taken from Bo Thidé; "Electromagnetic field theory"; second edition.

We use these formulas without units. Thus $c = 1;\ \varepsilon_0 = 1; \mu = 1$.

The Maxwell equations use coordinate time t.

∇_t stands for $\frac{\partial}{\partial t}$.

We use symbol $\boldsymbol{\mathcal{E}}$ for the electrical field and symbol $\boldsymbol{\mathcal{B}}$ for the magnetic field.

We introduce $\varkappa$ as a new scalar field.

$$\varkappa \equiv \alpha\ \nabla_t \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{1}$$

In EMFT this scalar field is taken as a zero valued gauge with

$\alpha = 1$; Lorentz gauge

$\alpha = 0$; Coulomb gauge

$\alpha = -1$; Kirchhoff gauge.

In this paper we also consider $\varkappa$ with $\alpha = 1$ as a separate field.

$$\phi = \nabla\varphi = -\varkappa - \boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{B}} \stackrel{?}{=} -\nabla_t\varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \boldsymbol{\nabla}\varphi_0 + \nabla_t \boldsymbol{\varphi} - \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{2}$$

Thus here

$$\phi = \nabla\varphi \neq (\nabla_t + \boldsymbol{\nabla})(\varphi_0 + \ \boldsymbol{\varphi}) = \nabla_t\varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \boldsymbol{\nabla}\varphi_0 + \nabla_t \boldsymbol{\varphi} - \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{3}$$

$$\phi_0 = \nabla_t\varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{4}$$

$$\boldsymbol{\phi} = \boldsymbol{\nabla}\varphi_0 + \nabla_t \boldsymbol{\varphi} - \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{5}$$

$$\boldsymbol{\mathcal{E}}\ \equiv\ -\boldsymbol{\nabla}\varphi_0 - \nabla_t \boldsymbol{\varphi} \tag{6}$$

$$\boldsymbol{\mathcal{B}}\ \equiv\ \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{7}$$

Charge and current distributions are defined differently from the way that it is done in quaternionic differential calculus.

$$\rho_0 \equiv \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle \tag{8}$$

$$\boldsymbol{\rho} \equiv \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \nabla_t \boldsymbol{\mathcal{E}} \tag{9}$$

If we add contributions of the new scalar potential to the distributions, then we get:

$$\phi_0 = \varkappa \tag{10}$$

$$\boldsymbol{\phi} = -\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{B}} \tag{11}$$

$$\nabla_t \boldsymbol{\mathcal{B}} = -\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = \boldsymbol{\nabla} \times \nabla_t \boldsymbol{\varphi} \tag{12}$$

$$\nabla_t \varkappa = \nabla_t \nabla_t \varphi_0 + \langle \boldsymbol{\nabla}, \nabla_t \boldsymbol{\varphi} \rangle \tag{13}$$

$$\boldsymbol{\nabla} \varkappa \equiv \nabla_t \boldsymbol{\nabla} \varphi_0 + \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{14}$$

$$\begin{aligned} \boldsymbol{\rho} \equiv \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \nabla_t \boldsymbol{\mathcal{E}} &= \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\varphi} + \nabla_t \boldsymbol{\nabla} \varphi_0 + \nabla_t \nabla_t \boldsymbol{\varphi} \\ &= \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} + \nabla_t \boldsymbol{\nabla} \varphi_0 + \nabla_t \nabla_t \boldsymbol{\varphi} \\ &= \boldsymbol{\nabla} \varkappa - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} + \nabla_t \nabla_t \boldsymbol{\varphi} \end{aligned} \tag{15}$$

$$\begin{aligned} \rho_0 \equiv \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle &= -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \varphi_0 \rangle - \langle \boldsymbol{\nabla}, \nabla_t \boldsymbol{\varphi} \rangle \\ &= -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 - \langle \boldsymbol{\nabla}, \nabla_t \boldsymbol{\varphi} \rangle = \nabla_t \varkappa - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 + \nabla_t \nabla_t \varphi_0 \end{aligned} \tag{16}$$

$$\boldsymbol{\xi} \equiv \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \nabla_t \boldsymbol{\mathcal{E}} + \boldsymbol{\nabla} \varkappa = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} + \nabla_t \nabla_t \boldsymbol{\varphi} \tag{17}$$

$$\xi_0 \equiv \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle + \nabla_t \varkappa = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 + \nabla_t \nabla_t \varphi_0 \tag{18}$$

$$\xi = \{\nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi \tag{19}$$

7.1.1 Poisson equations

The Poisson equations for the Maxwell based differential calculus are similar to the Poisson equations for the quaternionic differential calculus.

7.1.2 The Maxwell-Huygens wave equation

In Maxell format the wave equation uses coordinate time t. It runs as:

$$\partial^2 \varphi / \partial t^2 - \partial^2 \varphi / \partial x^2 - \partial^2 \varphi / \partial y^2 - \partial^2 \varphi / \partial z^2 = \xi \tag{1}$$

Papers on Huygens principle work with the homogeneous version of this formula or it uses the version with polar coordinates.

For isotropic conditions in three participating dimensions the general solution runs:

$$\psi = f(r - ct)/r, \text{ where } c = \pm 1; f \text{ is real} \tag{2}$$

In a single participating dimension the general solution runs:

$\psi = f(x - ct)$, where $c = \pm 1; f$ is real (3)

8 Conclusion

Great similarities, but also essential differences exist between quaternionic differential equations and Maxwell based differential equations. This results mainly from the definition of the four components of the differential $\phi = \nabla\varphi$. In the quaternionic differential calculus this differential can be seen as a product between the four-component differential operator ∇ and the four-component field φ. That simple interpretation is not possible in Maxwell based differential calculus. It is not possible to interpret the Maxwell field as a function of a number system that offers target values from that number system. In the Maxwell approach, the parameter space has a Minkowski signature and does not form a division ring. In quaternionic function theory the parameter space has a Euclidean signature. This shows in the structure of the wave equations of the two approaches. These wave equations have solutions that differ between the two approaches. However, the Poisson part of the two wave equations is similar. As a consequence the corresponding Green's functions are also similar. Both homogeneous wave equations have solutions in the form of one dimensional and three dimensional wave fronts. But these wave fronts have different mathematical representations. In applications, the wave fronts can act as carriers of information or energy.

In Maxwell based differential calculus the usage of gauges can be replaced by the introduction of an extra scalar field $\varkappa$. This results in the same form of the Maxwell based wave equation as the Lorentz gauge delivers, but the non-homogeneous equation applies different charge and current distributions. The impact of the difference in charges and currents is not treated here.

The speed of information transfer c differs between the two approaches.

8.1 Extra

Maxwell based differential calculus can be implemented with complex numbers. In that way it does not have to cope so intensively with non-commutative operators. As a consequence, gauges can be implemented easily. A disadvantage is that spacetime based dynamic geometric data must first be dismantled into real numbers before Hilbert spaces can handle them.

Quaternionic differential calculus must circumvent most gauges. On the other hand, the quaternionic approach offers compensating advantages.

Hilbert spaces can cope with quaternions as eigenvalues of operators. This holds for separate quaternions and for quaternionic continuums [1][4].

Since proper time is Lorentz invariant, all quaternionic differential equations are inherently Lorentz invariant.

Due to the fact that quaternions form a number system with a non-commutative product, they can implement rotations:

$$c = a\,b/a$$

In this way they can implement the functionality of gluons [4]. This is not possible with members of the Maxwell based "field".

9 References

[1] “Quaternions and quaternionic Hilbert spaces”; http://vixra.org/abs/1411.0178

[2] The vector differential calculus formulas and the Maxwell based equations that are used here are taken from the online textbook Bo Thidé; “Electromagnetic field theory”; second edition; http://www.plasma.uu.se/CED/Book/

[3] The relation between spinor notation and quaternionic formulation is shown in “The Dirac Equation in Quaternionic Format”; http://vixra.org/abs/1505.0149

[4] “Foundations of a Mathematical Model of Physical Reality”; http://vixra.org/abs/1502.0186