Anti Aristotle - The Division Of Zero By Zero

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Abstract. Today, the division of zero by zero (0/0) is a concept in philosophy, mathematics and physics without a definite solution. On this view, we are left with an inadequate and unsatisfactory situation that we are not allowed to divide zero by zero while the need to divide zero by zero (i. e. divide a tensor component which is equal to zero by another tensor component which is equal to zero) is great. A solution of the philosophically, logically, mathematically and physically far reaching problem of the division of zero by zero (0/0) is still not in sight. The aim of this contribution is to solve the problem of the division of zero by zero (0/0) while relying on Einstein's theory of special relativity. In last consequence, Einstein's theory of special relativity demands the division of zero by zero. Due to Einstein's theory of special relativity it is (0/0) = 1. As we will see, either we must accept the division of zero by zero as possible and defined or we must abandon Einstein's theory of special relativity as refuted.

Key words: Number theory, Relativity theory, causality.

1. Introduction

The development of mathematical science is full of contradictions and serious misrepresentations, especially concerning the division of zero (denoted by the sign 0) by zero. In contemporary mathematics a division of zero by zero (0/0) is called an indeterminate form and still, it is customary to claim that a division of zero by zero (0/0) has no defined value. Historically, some kind of symbols for zero or empty places corresponding in this respect to our zero in the positional representation of numbers were already used by the Babylonians, the Greeks, and the Mayas too. Nevertheless, in many reference works in mathematics, the arithmetic of zero is credited entirely to the Hindu contribution and especially to Brahmagupta.

"the arithmetic of zero is entirely the Hindu contribution to the development of mathematical science. With no other early nations do we find any treatment of zero." [1]

In contrast to the statement above, especially Aristotle (384 BC - 322 BC), a pupil of Plato, contributed some very important positions concerning the numerical notion of zero and to the result of division by zero. Moreover, Aristotle himself explicitly stated the impossibility of the division by zero just about fifteen hundred years before the time of Bhaskara. A significant passage in Aristotle's Physics is related to the numerical notion of zero and the result of the division by zero too. In Physica, Aristotle [2] wrote:

"Now there is no ratio in which the void is exceeded by body, as there is no ratio of 0 to a number. For if 4 exceeds 3 by 1, and 2 by more than 1, and 1 by still more than it exceeds 2, still there is no ratio by which it exceeds 0; for that which exceeds must be divisible into the excess + that which is exceeded, so that 4 will be what it exceeds 0 by +0. For this reason, too, a line does not exceed a point-unless it is composed of points." [3]

Clearly, in this quotation Aristotle did not look upon zero as a number in the strict sense of the word but had the arithmetical zero in mind. Aristotle excluded the division by zero by using the traditional meanings of words. According to Aristotle, if a division by zero were possible, then the result would exceed every possible integer.

To proceed further, some arithmetic operations with zero are allowed. Nicomachus (~60 - ~120 AD), born in Gerasa (the ancient Roman province of Syria) was influenced by Aristotle's work. In his publication *Introduction to Arithmetic*, Nicomachus [4] claimed that the sum of nothing added to nothing is nothing. According to Nicomachus, be sure that

$$0+0=0 \tag{1}$$

However, until the sixteenth and seventeenth centuries, zero as such was not fully accepted in algebra. Nevertheless, the earliest, but quite inadequate, consideration and extant reference to the division by zero is ascribed to Brahmagupta (597-668 AD), an Indian mathematician and astronomer. As a matter of fact, Brahmagupta's earliest recorded Indian (Hindu) contributions to explain division by zero is due to his writing of the Brahmasphutasiddhanta in 628 A.D. Brahmagupta wrote

"Positive, divided by positive, or negative by negative, is affirmative. Cipher, divided by cipher, is nought. Positive, divided by negative, is negative. Negative, divided by affirmative, is negative. Positive, or negative, divided by cipher, is a fraction with that for denominator: or cipher divided by negative or affirmative." [5]

It is well known that the Brahmasphutasiddhanta of Brahmagupta lead to some algebraic absurdities. Consequently, around 200 years after Brahmagupta, Mahavira (Mysore, India) tried to revise the Brahmasphutasiddhanta of Brahmagupta. Bhaskara (over 500 years after Brahmagupta) worked on the division by zero too. In contrast to Aristotle's claim of the impossibility of division by zero, the division by zero is given by Bhaskara in 1152 as follows:

"Statement: Dividend 3. Divisor 0. Quotient the fraction 3/0. This fraction, of which the denominator is cipher, is termed an infinite quantity." [6]

In particular, Bhaskara himself did not assert the impossibility of the division by zero. The work of the Hindu mathematicians spread west to the Arabic mathematicians as well as east to China and later to Europe too. In 1247, the Chinese mathematician Qin Jiu-shao (also known as Ch'in Chiu-Shao) introduced the symbol O for zero in his mathematical text "Mathematical treatise in nine sections". The number zero is related to infinity. The

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contemporary viewpoint of infinity is associated with the name John Wallis. In 1655, John Wallis (1616-1703), an English mathematician, introduced the symbol ∞ for infinity. According to John Wallis,

"esto enim ∞ nota numeri infiniti" [7].

Translated in English: "let the symbol ∞ denote infinity". In particular, Wallis himself claimed in 1656

"1/∞ ... habenda erit pro nihilo" [8] or

$$\frac{+1}{+\infty} = +0 \tag{2}$$

Thus far, according to Wallis [9], it is

$$\frac{+1}{+\infty} \times +\infty = +1 \tag{3}$$

George Berkeley (1685 - 1753), Bishop of Cloyne, claimed that reality (no longer objective) consists exclusively of minds. Another recorded reference to the mathematical impossibility of assigning a value to a division by 0 is credited to George Berkeley's criticism of infinitesimal calculus in The Analyst.

"They are neither finite Quantities, nor Quantities infinitely small, nor yet nothing. May we not call them the Ghofts of departed Quantities?" [10]

Berkeley's Analyst was a direct attack on the foundations and principles of the infinitesimal calculus as developed by Newton and Leibniz. Finally, a rigorous foundation for the principles of the infinitesimal calculus was given through the work of the prolific mathematician, Augustin-Louis Cauchy. Cauchy formalized the concept of a limit and created the specialism now called analysis.

Many great mathematicians tried to put an end to the debate concerning the division of zero by zero. But still, we are no closer to finding a solution. Today, the division of zero by zero is a concept in mathematics without a definitive answer. We may ask ourselves, can Einstein's theory of special relativity bring us to the point of admitting or disabling the division of zero by zero, definitely?

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2. Definitions

2.1. **Thought Experiments**

The general acceptance, importance and enormous influence of properly constructed (real or) thought experiments (as devices of scientific investigation) is backgrounded by many common features. Especially, the possibility to investigate some basic properties of the nature even under conditions when it is difficult or too expensive to run a real experiment, is worth being mentioned. Furthermore, a thought experiment can draw out a contradiction in a theory and thereby refuting the same. Again, it is necessary to highlight the possibility of a thought experiments to provide evidence against or in favor of a theory. However, thought experiments used for diverse reasons in a variety of areas are at the end no substitute for a real experiment. Thus far, real or thought experiments can help us to solve the problem of the division of zero by zero.

2.2. **Definition. Einstein's Mass-Energy Equivalence Relation**

Einstein's discovery of the equivalence of matter/mass and energy [11] in the year 1905 lies at the core of today's modern physics. According to Albert Einstein [12], the rest-mass $_{0}m$, a measure of the inertia of a (quantum mechanical) object is related to the relativistic mass $_{R}m$ by the equation

$$_{0}m = _{R}m \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{4}$$

Without loss of generality, the total energy of a physical system $_{R}E$ is numerically equal to the product of its matter/mass $_{R}m$ and the speed of light c squared with

$$\frac{_{0}E}{_{R}E} = \frac{_{0}m \times c^2}{_{R}m \times c^2} = \sqrt[2]{1-\frac{v^2}{c^2}} \tag{5}$$

where $_{0}m$ denotes the “rest” mass, $_{R}m$ denotes the “relativistic” mass, *v* denotes the relative velocity and *c* denotes the speed of light in vacuum.

2.3. **The Normalized Relativistic Energy-Momentum Relation**

Before going on to discuss the relationship between Einstein's special relativity theory and the problem of the division of zero by zero in more detail, it is only slightly more complicated to derive the general form of *the normalized relativistic energy-momentum relation* [13] as

$$\frac{_{0}m \times _{0}m}{_{R}m \times _{R}m} + \frac{v \times v}{c \times c} = 1 \tag{6}$$

from which we find that

$$\frac{v \times v}{c \times c \times \left(1-\frac{_{0}m \times _{0}m}{_{R}m \times _{R}m}\right)} = 1 \,. \tag{7}$$

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3. Results

Albert Einstein's (1879-1955) theory of special relativity when published 1905, superseded the 200-year-old theory of mechanics created by Isaac Newton [14]. One of the features of Einstein's theory of special relativity is that all observers will measure exactly the same speed of light in a vacuum, independent of photon energy (Lorentz invariance). Meanwhile, Einstein's theory of special relativity has passed a lot of observational and experimental tests, opportunities to test the validity of Einstein's theory of special relativity are increasing. In particular, the predictions of Einstein's theory of special relativity are still consistent with experimental data.

3.1. Theorem. Einstein's Relativistic Energy-Momentum Relation Under Conditions Where $_0m=0$

Einstein's theory of special relativity allows that the rest-mass $_0m$ of a particle can be equal to zero. In this case energy as such is not destroyed but changes into pure energy of a wave.

Claim.
In reality and especially under conditions of special relativity (inertial frames of reference) there are circumstances, where the rest-mass (i.e. of a particle like photon) is $_0m = 0$. Under these conditions we must accept that

$$+v\times v=+c\times c \tag{8}$$

Proof.
In general, it is

$$_0m={}_Rm*\sqrt[2]{1-\frac{v^2}{c^2}} \tag{9}$$

or

$$\frac{_0m\times{}_0m}{_Rm\times{}_Rm}+\frac{v\times v}{c\times c}=1 \tag{10}$$

Under these experimental conditions, it is $_0m=0$. We obtain

$$\frac{0\times 0}{_Rm\times{}_Rm}+\frac{v\times v}{c\times c}=1 \tag{11}$$

or

$$0+\frac{v\times v}{c\times c}=1 \tag{12}$$

or

$$+v\times v=+c\times c \tag{13}$$

Quod erat demonstrandum.

3.2. **Theorem. Einstein's Relativistic Energy-Momentum Relation Under Conditions Where v=0**

Under conditions of Einstein's theory of special relativity, theoretically it is possible that the relative velocity v = 0. In this case wave energy as such is not destroyed but changes into pure energy of a particle.

Claim.
Especially under conditions of special relativity (inertial frames of reference) there are circumstances, where the relative velocity v = 0. Under these conditions we must accept too, that

$$ {}_0m \times {}_0m = {}_Rm \times {}_Rm \tag{14} $$

Proof.
In general, it is

$$ {}_0m = {}_Rm * \sqrt[2]{1-\frac{v^2}{c^2}} \tag{15} $$

or

$$ \frac{{}_0m \times {}_0m}{{}_Rm \times {}_Rm} + \frac{v \times v}{c \times c} = 1 \tag{16} $$

Under these experimental conditions, it is v=0. We obtain

$$ \frac{{}_0m \times {}_0m}{{}_Rm \times {}_Rm} + \frac{0 \times 0}{c \times c} = 1 \tag{17} $$

or

$$ \frac{{}_0m \times {}_0m}{{}_Rm \times {}_Rm} + 0 = 1 \tag{18} $$

or

$$ \frac{{}_0m \times {}_0m}{{}_Rm \times {}_Rm} = 1 \tag{19} $$

or

$$ {}_0m \times {}_0m = {}_Rm \times {}_Rm \tag{20} $$

Quod erat demonstrandum.

3.3. **Theorem. The Division Of Zero By Zero**

Thus far, let us perform a thought experiment under conditions of inertial frames of reference where v = 0. Thus far, two observers at rest relative to each other (the relative velocity v is equal to zero) are moving in deep space. Let us now consider the particular case of special relativity where the relative velocity between observers is equal to v= 0 in more detail.

Claim.
Under conditions of special relativity (inertial frames of reference) the division of zero by zero is possible and allowed. In particular, it is

$$\frac{0}{0}=1 \tag{21}$$

Proof.
Under conditions of inertial frames of reference, Einstein's special relativity is generally valid, which is equally the starting point of this proof by contradiction. In general, Einstein's generally valid relativistic energy-momentum relation is defined as

$${}_0m = {}_R m \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{22}$$

Squaring this equation, we obtain

$${}_0m^2 = {}_R m^2 \times \left(1-\frac{v^2}{c^2}\right). \tag{23}$$

Collecting and rearranging together the terms one then finds straightforwardly the probability theory consistent form of the normalized relativistic energy-momentum relation [15] as

$$\frac{{}_0m \times {}_0 m}{{}_R m \times {}_R m} + \frac{v \times v}{c \times c} = 1 \tag{24}$$

from which we find in general that

$$+\frac{v \times v}{c \times c} = 1 - \frac{{}_0m \times {}_0 m}{{}_R m \times {}_R m} \tag{25}$$

or that

$$+v \times v = c \times c\left(1 - \frac{{}_0m \times {}_0 m}{{}_R m \times {}_R m}\right) \tag{26}$$

or that

$$\frac{v \times v}{c \times c \times \left(1 - \frac{{}_0 m \times_0 m}{{}_R m \times_R m}\right)} = 1 \quad . \tag{27}$$

Under conditions of the theory of special relativity, this form of Einstein's relativistic energy momentum relation is unrestrictedly valid. In other words, by direct substitution, under these experimental conditions (v=0), we obtain

$$\frac{0 \times 0}{c \times c \times \left(1 - \frac{{}_0 m \times_0 m}{{}_R m \times_R m}\right)} = 1 \quad . \tag{28}$$

Due to our theorem above (under conditions where v=0) it is equally ${}_0m={}_Rm$ or ${}_0 m \times_0 m =_R m \times_R m$. Thus far, after substituting ${}_0m$ by ${}_Rm$ we obtain

$$\frac{0 \times 0}{c \times c \times \left(1 - \frac{{}_R m \times_R m}{{}_R m \times_R m}\right)} = 1 \tag{29}$$

or

$$\frac{0 \times 0}{c \times c \times (1 - 1)} = 1 \tag{30}$$

or

$$\frac{0 \times 0}{c \times c \times 0} = 1 \tag{31}$$

or

$$\frac{0}{0} = 1 \tag{32}$$

Quod erat demonstrandum.

4. Discussion

As it is, we are always and already linked to the historical development of science as such. Thus far, for thousands of years, at least since Aristotle, the division of zero by zero was not allowed. In general, mathematical expressions which are not definitively or precisely determined are said to be indeterminate. The term indeterminate forms was originally introduced by François-Napoléon-Marie Moigno (1804 – 1884), a student of Cauchy, in the middle of the 19th century. In principle, several types of indeterminate forms are distinguished. Some of the indeterminate forms typically considered in the literature are denoted by 0/0 or by ∞/∞ or by 0 × ∞ or by ∞ − ∞ or by 0^0 or by 1^∞ and by ∞^0.

With that brief sketch of the historical background of indeterminate forms, we resolved this vagueness by using Einstein’s special theory of relativity. Einstein’s special theory of relativity determines very precisely what happens, if 0 is divided by 0. Following Einstein’s special theory of relativity, the division of 0 by 0 is not indeterminate at all, the division of 0 by 0 is determinate as (0/0) = 1. As is known, Einstein's theory of special relativity has passed a lot of tests, the experiments supporting the validity of Einstein's theory of special relativity are increasing. In particular, the predictions of Einstein's theory of special relativity are still and without any contradiction consistent with experimental data. In this paper with have used Einstein's theory of special relativity to demonstrate that the division of zero by zero make sense and is defined without any contradictions. The thought experiments are properly constructed. Especially Eq. 27 is only a reformulation of Einstein’s relativistic energy momentum relation and is thus far defined at any event under conditions of special theory of relativity. Following the predictions of Einstein's theory of special relativity, it is (0/0) = 1.
There are, however, differences in the way how to treat the division of 0 by 0. Contrary to Einstein’s special theory of relativity, *L'Hospital's Rule* is undoubtedly founded on the assumption that it is just not clear what is happening in the limit. Thus far, *L'Hospital's Rule*, named after the 17th-century French mathematician Guillaume de l'Hôpital and published in his 1696 book *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* tell us all we need and all we have to do if we have an indeterminate form. As already pointed out the general form of L'Hôpital's rule may cover many cases. In the light of this publication, it appears to be possible that L'Hôpital's rule is not generally valid at all. The general validity of L'Hôpital's rule should be reviewed from the beginning.
Altogether, the division of zero by zero is possible, allowed and defined. But the division of zero by zero can lead to some paradoxes if some specific rules of precedence on which the division of zero by zero is grounded, are not respected.

Example I.
Clearly, it is **incorrect** that

$$1 = 2. \tag{33}$$

Multiplying by 0, it is

$$1 \times 0 = 2 \times 0 \tag{34}$$

and we obtain

$$0 = 0 \tag{35}$$

which is **correct**. Dividing by zero, it is

$$\frac{0}{0} = \frac{0}{0} \tag{36}$$

or due to our finding

$$1 = 1 \tag{37}$$

which is of course correct but equally a contradiction too since we started with the claim that +1=+2. Under these circumstances, we may infer that we are not allowed to divide by zero since we obtained an erroneous result. Contrary to facts, the erroneous result obtained is due to the problem of the multiplication by zero. Consequently, a multiplication by 0 can lead to an erroneous result **since something obviously false** (i.e 1=2) **changes to something true** (i.e. 0=0). In other words, from something incorrect follows something correct. The multiplication by zero is much more problematic than the division by zero. Altogether, a division by zero appears to possess a greater priority then the multiplication by zero. Respecting this fact we get another picture.

Example II.
Again, it is **incorrect** that

$$1 = 2. \tag{38}$$

Multiplying by 0, it is

$$1 \times 0 = 2 \times 0 \tag{39}$$

Dividing by zero, it is

$$\frac{1 \times 0}{0} = \frac{2 \times 0}{0} \tag{40}$$

Under circumstances where the division by zero is performed prior to the multiplication by zero we obtain

$$1 \times \left(\frac{0}{0}\right) = 2 \times \left(\frac{0}{0}\right). \tag{41}$$

Since 0/0 = 1, it follows that

$$1 = 2 \tag{42}$$

which is equal to the (incorrect) starting point of our example II. In other words, as mentioned previously, the division of zero by zero is possible, allowed and defined. But to avoid some paradoxes while performing the division of zero by zero some **specific rules of precedence** on which the division of zero by zero is grounded, should be worked out in detail and respected. The multiplication by zero appears to be not less difficult then the division by zero.

Example III.
Again, it is **incorrect** that

$$1 = 2. \tag{43}$$

Multiplying by 0, it is

$$1 \times 0 = 2 \times 0 \tag{44}$$

Dividing by zero, it is

$$\frac{1\times 0}{0}=\frac{2\times 0}{0} \tag{45}$$

or

$$\frac{1}{0}\times 0=\frac{2}{0}\times 0 \tag{46}$$

This is equivalent with

$$\frac{1}{0}\times 0=2\times\frac{1}{0}\times 0 \tag{47}$$

Following Wallis himself, who claimed in 1656 that "1/∞ ... habenda erit pro nihilo" [16] we obtain another picture. Thus far, under circumstances where (1/0=∞ and 1 = 0*∞ and 0/0=1) we obtain

$$\infty\times 0=2\times\infty\times 0 \tag{48}$$

which is equivalent with

$$1=2\times 1 \tag{49}$$

or with

$$1=2. \tag{50}$$

5. Conclusion

The general problem of the division of zero by zero is solved. In general, under conditions of special relativity, it is $\frac{0}{0}=1$. Thus far, while the problem of the division of zero by zero is solved new problems are created too. It appears to be necessary to review the general validity of L'Hôpital's rule and to work out the rules of precedence, when performing some algebraic operations with zero.

Acknowledgment

None.

Appendix

None.

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