Scrutiny of Einstein's Geodesic and Field Equations

Mohamed E. Hassani
Institute for Fundamental Research
BP.197, CTR, Ghardaïa 47800, Algeria
E-mail: hassani641@gmail.com

Abstract: Since its final version and publication in 1916, it is widely reported in several specialized textbooks and research articles that General Relativity theory reduces to Newton's theory of gravity in the limit of a weak gravitational field and slowly moving material bodies. In the present paper, the so-called reducibility of Einstein's geodesic and field equations, to Newton's equation of motion and Poisson's gravitational potential equation, respectively, is scrutinized and proven to be mathematically, physically and dimensionally incorrect, and that the geometrization of gravity is unnecessary.

Keywords: General Relativity theory, Newton's theory of gravity, physico-logical argumentation, dimensional analysis

1. Introduction

It is widely alleged and blindly believed that Newton's theory of gravity is manifestly a limiting case of General Relativity Theory (GRT), for so-called weak gravitational fields and slow motion, despite the fact that, strictly speaking, Newton's theory of gravity is not physically and/or even mathematically contained in GRT. Hence, the main aim of the present paper is to prove conclusively that the Einstein's geodesic and field equations cannot be reduced to Newton's equation of motion and Poisson's gravitational potential equation, respectively, and that the geometrization of gravity is unnecessary for physics. To clarify the problem, let us first inspect the following *physico-logical viewpoints* which are in fact tangible evidence.

(a) *First physico-logical viewpoint*: It was clearly shown in [1] that the exact equations of motion of matter and Einstein's field equations lead to the existence of only vanishing integrals of the motion, so that at each stage of approximate calculations, including the Newtonian approximation, these integrals of motion in GRT must vanish. Hence, it follows, in particular, that GRT does not really have a classical Newtonian limit, since the integrals of the motion of the two theories are not equal.

(b) *Second physico-logical viewpoint*: Since gravity is a universal attractive force in Newton's theory but not a force in GRT, rather space-time warped by the presence of mass and energy, in such a situation, *i.e*., phenomenologically, how can Newton's theory become a limiting case of GRT even for so-called weak gravitational fields and slow motion?

(c) *Third physico-logical viewpoint*: It is frequently asserted that the stress-energy tensor plays a role in GRT very similar to that of mass distribution in Newton's theory; more precisely, it tells space-time how to deform, creating what we observe as gravity. Therefore, space-time is not an inert entity. It acts on matter and can be acted upon. Consequently, curved space-time itself behaves like *a sort of matter*, not merely a geometrical seat in which arise physical phenomena without specific dynamical properties.

(d) *Fourth physico-logical viewpoint*: Phenomenologically speaking, the geometrization of gravity implies the materialization of curved space-time itself, and as a direct result the usual principle of causality is violated because the causal source of such *materialization* is absolutely without existence.

(e) *Fifth physico-logical viewpoint*: The presence of the speed of light in solutions to Einstein's field equations is not a purely physical consequence, but a geometrical one, historically speaking, due to Minkowski's space-time geometry. Mindful of this, the Schwarzschild solution has been framed in the context of Minkowski space-time geometry, because at infinity the Riemannian geometry reduces to space-time with the Minkowski metric. Therefore, the presence of the speed of light in solutions to Einstein's field equations does not necessarily mean that gravitational fields propagate at the speed of light.

2. GRT's Irreducibility

2.1. Geometrization of gravity is Unnecessary

According to Lehmkuhl [2], it seems that even Einstein did not seriously accept the geometrization of gravity as an indisputable physical reality. In 1926 Einstein's clarified his view about the fact that GRT should not be understood as a pure reduction of physics to geometry. This attitude was motivated by a letter from the philosopher of science, Hans Reichenbach, who was at the time cognizant of the theories of Weyl and Eddington, and wrote Einstein that he thought that seeing electricity as geometrical in Weyl's theory is no more than an illusion; one that, he argued, is equally possible (and equally trivial) in GRT. Einstein approved enthusiastically, and wrote [2]:

"*You are completely right. It is wrong to think that 'geometrization' is something essential. It is only a kind of crutch for the finding of numerical laws. Whether one links 'geometrical' intuitions with a theory is a ... private matter.*"

Once again, from Einstein's response, we can affirm that the geometrization of gravity *was* and *is* in principle unnecessary. Furthermore, skeptics and opponents of the geometrization of gravity are most notably represented by the two Nobel laureates Feynman and Weinberg. At the time of his lectures delivered at Caltech, Feynman was struggling to quantize gravity – that is, to forge a synthesis of General Relativity and the fundamental principles of Quantum Mechanics. Feynman's whole approach to General Relativity was shaped by his desire to obtain a quantum theory of gravitation as straightforwardly as possible. For this purpose, geometrical subtleties seem a distraction; in particular, the conventional geometrical approach to gravitation obscures the telling analogy between gravitation and electrodynamics. Feynman preserved the concepts of force and motion, with their well-known classical meanings, and commented [3]:

" *It is one of the peculiar aspects of the theory of gravitation, that it has both a field interpretation and a geometrical interpretation ... the fact is that a spin-two field has this is just marvelous. The geometrical interpretation is not really necessary or essential to physics.*"

In reinforcement of the above comments, Weinberg wrote [4]:

"*In learning general relativity, and then in teaching it to classes at Berkeley and M.I.T., I became dissatisfied with what seemed to be the usual approach to the subject. I found that in most textbooks geometric ideas were given a starring role, so that a student who asked why the gravitational field is represented by a metric tensor, or why freely falling particles move on geodesics, or why the field equations are generally covariant would come away with an impression that this had something to do with the fact that space-time is a Riemannian manifold.*

Of course, this was Einstein's point of view, and his preeminent genius necessarily shapes our understanding of the theory he created. [...] Einstein did hope, that matter would eventually be understood in geometrical terms [...]. [I believe that] too great an emphasis on geometry can only obscure the deep connections between gravitation and the rest of physics."

As we have seen, even the preeminent scholars were, explicitly, in total disagreement about the idea of geometrizing physics in general and gravity in particular.

2.2. Geodesic equation is irreducible to Newton's equation of motion

2.3. Physico-logical argumentation

In order to show, without calculations, that the geodesic equation is irreducible to Newton's equation of motion, due to the fact that Newtonian gravity is not contained in GRT, we begin by recalling the profound difference between mathematics and physics. Such a realisation is indispensable for the reason that in the framework of GRT there is no clear and explicit distinction between a physical equation (mathematical equation written in a purely physical context) and a mathematical equation (any equation written in a purely mathematical context).

First, mathematics is not physics and physics is not mathematics. The inhabitants of the mathematical world are purely abstract objects characterized by an absolute freedom. However, the inhabitants of the physical world are purely concrete objects – in the theoretical sense and/or in the experimental/observational sense – and are characterized by very relative and restricted freedom.

When applied outside its proper context, mathematics should play the role of an accurate language and useful tool, and gradually should lose its abstraction. Let us illustrate these considerations by writing the following ordinary differential equation (ODE) in a purely mathematical context:

$$a\,y' + b\,y = 0\,, \tag{1}$$

where $y \equiv y(x)$; $x, a, b \in \boldsymbol{R}$, and $(a,b) \neq (0,0)$ are constants that may be very small or very large. By virtue of the abstraction and freedom that characterise Eq.(1), we can rewrite it, without any major or minor worries, as follows,

$$y' + \tfrac{b}{a}\,y = 0\,. \tag{2}$$

Furthermore, we can rewrite Eq.(1), after taking a second derivative with respect to *x*, as:

$$a\,y''+b\,y'=0\,. \tag{3}$$

All that we have done here is pure *abstraction*, reinforced by a high degree of *freedom*, that is, without any direct contact with tangible reality. However, when we are dealing with physical equations, abstraction and freedom together lose their absolutism and become very relative and thus restricted, because each parameter contained in the physical equation has a well-defined role, fixed by its own physical dimensions, as we shall see. To this end, consider the following physical equation,

$$\frac{d^2\mathbf{r}}{dt^2}=-\nabla U\,, \tag{4}$$

in the pure context of classical gravitational physics, with $\|\mathbf{r}\|\equiv r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$ and U the gravitational potential defined by,

$$U\equiv U(r)=\begin{cases}\frac{2}{3}\pi G\rho\left(r^2-3R^2\right), & r<R\\ -\dfrac{GM}{r}, & r>R\end{cases}\,, \tag{5}$$

where G, ρ, M, R and r are, respectively, the gravitational constant, density, mass of gravitational source (which is supposed to be spherically symmetric), radius of source, and the relative distance between the centre of the source and the centre of another body of mass m $(m<M)$. In classical gravitational physics we can interpret the two above equations as follows: (*i*) Eq.(4) describes the motion of a small body of mass *m* under the action of a gravitational field represented by $-\nabla U$; (*ii*) according to (5), the small body is inside the gravitational source if $r<R$ and outside it if $r>R$; (*iii*) since we can write $d^2\mathbf{r}/dt^2=d\mathbf{v}/dt$ this means that the small body may be in relative motion with respect to the centre of the source, and consequently Eq.(4) may be rewritten as,

$$\frac{d\mathbf{v}}{dt}=-\nabla U\,, \tag{6}$$

that is to say, we have also an equation of motion; (i*v*) the quantity $-\nabla U$ has the physical dimensions of gravitational acceleration, therefore, we can rewrite Eq.(4) by using the usual notation, *viz*, $\mathbf{g}=d^2\mathbf{r}/dt^2$ and get the well-known classical equation for the gravitational field,

$$\mathbf{g}=-\nabla U\,. \tag{7}$$

Finally, if we multiply both sides of (7) by the scalar *m*, *i.e.*, the mass of a small body, we obtain the expression for the gravitational force acting on the body,

$$\mathbf{F} = m\mathbf{g} = -m\nabla U \,. \tag{8}$$

Phenomenologically, Eq.(7) is completely different from Eq.(8), but this is not the case with Eqs.(1) and (2). In passing, it is worthwhile to note that the physical equations are permanently subject to dimensional analysis (DA). The principal role of DA is to check and verify the correctness and the coherence of the physical equations in their proper context. Unfortunately, many physics students and professional physicists ignore or neglect the veritable goal and usefulness of DA. For example, the ignorance or negligence of DA is well reflected by the fact that we can find in many specialised textbooks and research articles some fundamental physical equations written without c^2 and G, which are the speed of light squared and the Newtonian gravitational constant respectively. They are, by common ill-convention, supposed to be $c = 1$ and $G = 1$, which, perhaps, is *mathematically* an acceptable trick to facilitate calculations. But *physically*, it represents a loss of information and can lead to confusion, and such equations cannot be checked by DA. Let us turn now to the geodesic equation and apply the above ideas to Einstein's approach.

At the beginning of the fourth Princeton lecture (1921), Einstein commenced with a discussion of the motion of material points, stating that in GRT the law of inertia has to be generalised by generalising the old concept of a straight line [2]:

"*According to the principle of inertia, the motion of a material point in the absence of forces is straight and uniform. In the four dimensional continuum of special relativity, this is a real straight line. The natural, i.e. the simplest, generalization of the straight line making sense in the conceptual scheme of the general (Riemannian) theory of invariants is the straightest (geodesic) line.*

Following the equivalence principle, we will have to assume that the motion of a material point subject only to inertia and gravity is described by the equation

$$\frac{d^2 x_\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0 \,. \tag{i}$$

Indeed, this equation becomes that of a straight line if the components $\Gamma^\nu_{\mu\sigma}$ of the gravitational field vanish.

Einstein then showed, without any explicit physical support, that in the Newtonian limit the geodesic equation becomes (with $dl^2 = dt^2 = ds^2$),

$$\frac{d^2 x_\mu}{dl^2} = \frac{\partial}{\partial x_\mu}\left(\frac{\gamma_{44}}{2}\right). \tag{ii}$$

where γ_{44} is defined by $g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}$.

It seems even Einstein had adopted the common ill-convention $c = 1$ and $G = 1$. Furthermore, there is a problem behind Eq.(ii), since the quantity γ_{44} is not previously well-defined, contrary to Eq.(4)

in which the leading term U is well-defined by (5) in advance. Therefore, we cannot apply DA to Eq.(*ii*) so, consequently, Eq.(*ii*) is a purely mathematical equation.

In order to dodge this inevitable impasse, Einstein simply and verbally pointed out a link between the geodesic equation and the Newtonian equation of motion for particles subject to gravitational fields:

This equation (ii) is identical with Newton's equation of motion of a point in a gravitational field if one identifies $-\gamma_{44}/2$ *with the gravitational potential. ... One look at Eqs.(i) and (ii) shows that the quantities* $\Gamma^{\nu}_{\mu\sigma}$ *play the role of the field strength of the gravitational field. These quantities are not tensorial.*

The above passage reveals a major problem. First, there is not a single physico-mathematical proof concerning the so-called reduction of the geodesic equation to Newton's equation of motion. Instead of an explicit physico-mathematical proof, there is a sort of 'word-game' contained in the expression:

This equation (ii) is identical with Newton's equation of motion of a point in a gravitational field if one identifies $-\gamma_{44}/2$ *with the gravitational potential.*

Because if we substitute '*if one identifies*' with '*if one replaces*', we get an equivalent expression:

This equation (ii) is identical with Newton's equation of motion of a point in a gravitational field if one <u>replaces</u> $-\gamma_{44}/2$ *with the gravitational potential.*

Hence, the geodesic equation is *unphysically* reduced to Newton's equation of motion only *via* a replacement/substitution, not *via* an explicit calculation carried out in a purely physical context. Accordingly, and strictly speaking, we cannot consider the geodesic equation as a physical equation.

Second, supposing that the gravitational potential U itself is completely unknown to Einstein and/or his successors, they would be unable to identify/replace $-\gamma_{44}/2$ with U .

2.4. Einstein's (gravitational) field equations are irreducible to Poisson's (gravitational) potential equation

As the title of the present subsection reveals clearly, many specialised textbooks and research articles incorrectly utilise the expression ‘Poisson's equation for/of a gravitational field’ and also consider the same equation as a generalization of Laplace's equation. However historically speaking, the French mathematician Siméon-Denis Poisson (1781-1840), who was one of the founders of mathematical physics, developed his equation within the framework of potential theory and published it in 1813 (in the Bulletin de la Société Philomatique, pp. 388-392). He never considered his equation as a generalization of Laplace's equation. In modern language and notation, Poisson said in his paper that Laplace’s equation,

$$\nabla^2 U = 0, \tag{9}$$

is applicable only if the material body is outside the gravitational source, and when the body is inside the source we should take into consideration the mass density $\rho \equiv \rho(r)$ inside the radius $r < R$ where Laplace's equation should be replaced with his equation,

$$\nabla^2 U = 4\pi G\rho . \tag{10}$$

Thus, according to (5), the gravitational potential function $U \equiv U(r)$ is at the same time a fundamental solution for Eqs. (9) and (10) for the cases $r > R$ and $r < R$, respectively. Curiously, it seems until now, that Poisson's equation has not been correctly understood because many specialised textbooks and research articles claim that Eq.(10) may be reduced to Eq.(9) when $\rho = 0$. However, the mass density ρ in Eq.(10) plays the role of a gravitational source and in Eq.(9) this role is played by the mass M which is contained in the expression of U when $r > R$. Therefore, physically, Eq.(10) cannot reduce to Eq.(9) since the mass density itself gives rise to the gravitational field *via* the gravitational potential U and once again this implies that the mass density must play the role as a gravitational source, which is why ρ is explicitly present in the expression of U for the case $r < R$.

Therefore, according to (5), when $\rho = 0 \Rightarrow U = 0$ and Eq.(10) becomes an identity of the form 0 = 0. For this reason Poisson himself did not consider his equation as a generalisation of Laplace's equation.

It is routinely claimed that Einstein's field equations (here without cosmological constant),

$$G_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{11}$$

reduce to Poisson's equation (10) in the limit of a weak gravitational field and slow motion of material bodies. In order to prove the irreducibility of Eqs.(11) to Eq.(10) we must apply DA. By means of DA, which is a very robust tool, we obtain the following dimensional expressions for the Laplace and Poisson equations, respectively:

$$\mathrm{T}^{-2} - \mathrm{T}^{-2} = 0, \tag{12}$$

and

$$\mathrm{T}^{-2} = \mathrm{T}^{-2}. \tag{13}$$

Eqs.(12) and (13) are dimensionally identical. Moreover, if we multiply Eq.(12) or Eq.(13) by the dimensional quantity L (length) we get acceleration, and if we multiply the same equations by L^2 we obtain speed squared. Applying the same DA to Einstein's field Eqs.(11) we find:

$$\mathrm{L}^{-2} = \mathrm{L}^{-2}. \tag{14}$$

Note that Eq.(14) is not dimensionally identical to Eqs.(12) and (13); hence, mathematically and physically, Einstein's field Eqs.(11) cannot reduce to Poisson's Eq.(10).

Finally, in contrast to Einstein's approach, the expression for the constant κ cannot be deduced by comparison with Poisson's Eq.(10). Therefore, we should have,

$$\kappa \neq 8\pi G/c^4 . \tag{15}$$

3. Conclusion

It has been proven herein that, contrary to folklore, General Relativity theory cannot be reduced to Newton's theory of gravity, even in the so-called limit of a weak gravitational field and slowly moving material bodies. With certain physico-logical argumentation, the geodesic equation is proven to be irreducible to Newton's equation of motion. Finally, by means of dimensional analysis, Einstein's field equations have been shown to be irreducible to Poisson's gravitational potential equation.

References

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