The Dirac equation in quaternionic format

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ABSTRACT

In its original form the Dirac equation for the free electron and the free positron is formulated by using complex number based spinors and matrices. That equation can be split into two equations, one for the electron and one for the positron. These equations appear to apply different parameter spaces. The equation for the electron and the equation for the positron differ in the symmetry flavor of their parameter spaces. This results in special considerations for the corresponding quaternionic second order partial differential equation.

Contents

1 The Dirac equation in original format

In its original form the Dirac equation is a complex equation that uses spinors, matrices and partial derivatives.

Instead of the usual $\left\{\frac{\partial f}{\partial t}, \boldsymbol{i}\frac{\partial f}{\partial x}, \boldsymbol{j}\frac{\partial f}{\partial y}, \boldsymbol{k}\frac{\partial f}{\partial z}\right\}$ we want to use operators $\nabla = \{\nabla_0, \boldsymbol{\nabla}\}$

The subscript $_0$ indicates the scalar part. Bold face indicates the vector part.

The operator ∇ relates to the applied parameter space. This means that the parameter space is also configured of combinations $x = \{x_0, \boldsymbol{x}\}$ of a scalar x_0 and a vector $\boldsymbol{x}$. Also the functions $f = \{f_0, \boldsymbol{f}\}$ can be split in scalar functions f_0 and vector functions $\boldsymbol{f}$.

The local parameter $t = x_0$ represents the scalar part of the applied parameter space.

Dirac was searching for a split of the Klein-Gordon equation into two first order differential equations.

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} = -m^2 f \tag{1}$$

$$(\nabla_0\nabla_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle) f = \mathfrak{D} f = -m^2 f \tag{2}$$

Here $\mathfrak{D} = \nabla_0\nabla_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle$ is the d'Alembert operator.

Dirac used a combination of matrices and spinors in order to reach this result. He applied the Pauli matrices in order to simulate the behavior of vector functions under differentiation.

The unity matrix I and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are given by [3]:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

For one of the potential orderings of the quaternionic number system, the Pauli matrices together with the unity matrix I relate to the quaternionic base vectors $1, \boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

$$1 \mapsto I, \quad \boldsymbol{i} \mapsto i_0\, \sigma_1, \quad \boldsymbol{j} \mapsto i_0\, \sigma_2, \quad \boldsymbol{k} \mapsto i_0\, \sigma_3 \tag{4}$$

$$\sigma_1\sigma_2 - \sigma_2\sigma_1 = 2\, i\, \sigma_3;\ \sigma_2\sigma_3 - \sigma_3\sigma_2 = 2\, i\, \sigma_1;\ \sigma_3\sigma_1 - \sigma_1\sigma_3 = 2\, i\, \sigma_2 \tag{5}$$

$$\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = I \tag{6}$$

The different ordering possibilities of the quaternionic number system correspond to different symmetry flavors. Half of these possibilities offer a right handed external vector product. The other half offer a left handed external vector product.

We will regularly use:

$$\langle i_0\boldsymbol{\sigma}, \boldsymbol{\nabla}\rangle = \boldsymbol{\nabla}\,; i_0 = \sqrt{-1} \tag{7}$$

With

$$p_\mu = -i_0\nabla_\mu \tag{8}$$

follow

$$p_\mu \sigma_\mu = -i_0 e_\mu \nabla_\mu \tag{9}$$

$$\langle \boldsymbol{\sigma}, \boldsymbol{p} \rangle \leftrightarrow -i_0 \boldsymbol{\nabla} \tag{10}$$

1.1 Dirac's formulation

The original Dirac equation uses 4x4 matrices $\boldsymbol{\alpha}$ and β. [1]:
$\boldsymbol{\alpha}$ and β are matrices that implement the quaternion arithmetic behavior including the possible symmetry flavors of quaternionic number systems and continuums.

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \leftrightarrow -i_0 \begin{bmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \leftrightarrow -i_0 \begin{bmatrix} 0 & \boldsymbol{j} \\ \boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix} \leftrightarrow -i_0 \begin{bmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

$$\beta\beta = I \tag{5}$$

The interpretation of the Pauli matrices as representation of a special kind of angular momentum has led to the half integer eigenvalue of the corresponding spin operator. Dirac's selection leads to

$$(p_0 - \langle \boldsymbol{\alpha}, \boldsymbol{p} \rangle - \beta mc)\{\varphi\} = 0 \tag{6}$$

$\{\varphi\}$ is a four component spinor.
Which splits into

$$(p_0 - \langle \boldsymbol{\sigma}, \boldsymbol{p} \rangle - mc)\varphi_A = 0 \tag{7}$$

and

$$(p_0 - \langle \boldsymbol{\sigma}, \boldsymbol{p} \rangle + mc)\varphi_B = 0 \tag{8}$$

φ_A and φ_B are two component spinors. Thus the original Dirac equation splits into:

$$(\nabla_0 - \boldsymbol{\nabla} - i_0 mc)\varphi_A = 0 \tag{12}$$

$$(\nabla_0 - \boldsymbol{\nabla} + i_0 mc)\varphi_B = 0 \tag{13}$$

This split does not lead easily to a second order partial differential equation that looks like the Klein Gordon equation.

1.2 Relativistic formulation

Instead of Dirac's original formulation, usually the relativistic formulation is used [2]. That formulation applies gamma matrices, instead of the alpha and beta matrices. This different choice influences the form of the equations that result for the two component spinors.

$$\gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} \leftrightarrow -i_0 \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix} \leftrightarrow -i_0 \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} \leftrightarrow -i_0 \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

Thus

$$\gamma_\mu = \gamma_0 \alpha_\mu;\ \mu = 1,2,3;\ \gamma_0 = \beta \tag{5}$$

Further

$$\gamma_5 = i_0 \gamma_0 \gamma_1 \gamma_2 \gamma_3 \tag{6}$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The matrix γ_5 anti-commutes with all other gamma matrices.
Several different sets of gamma matrices are possible. The choice above leads to a "Dirac equation" of the form

$$(i_0 \gamma^\mu \nabla_\mu - mc)\varphi = 0 \tag{7}$$

More extended:

$$\left(\gamma_0 \frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} - \frac{m}{i_0 \hbar}\right)\{\psi\} = 0 \tag{8}$$

$$\left(\gamma_0 \frac{\partial}{\partial t} + \langle \boldsymbol{\gamma}, \boldsymbol{\nabla} \rangle - \frac{m}{i_0 \hbar}\right)\{\psi\} = 0 \tag{9}$$

(10)

$$\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{\partial}{\partial t}+\begin{bmatrix}0 & \langle\boldsymbol{\sigma},\boldsymbol{\nabla}\rangle\\ -\langle\boldsymbol{\sigma},\boldsymbol{\nabla}\rangle & 0\end{bmatrix}-\frac{m}{i_0\hbar}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}\varphi_A\\ \varphi_B\end{bmatrix}=0$$

$$\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{\partial}{\partial t}-i_0\begin{bmatrix}0 & \boldsymbol{\nabla}\\ -\boldsymbol{\nabla} & 0\end{bmatrix}-\frac{m}{i_0\hbar}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}\varphi_A\\ \varphi_B\end{bmatrix}=0 \tag{11}$$

$$\left(i_0\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{\partial}{\partial t}+\begin{bmatrix}0 & \boldsymbol{\nabla}\\ -\boldsymbol{\nabla} & 0\end{bmatrix}-\frac{m}{\hbar}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}\varphi_A\\ \varphi_B\end{bmatrix}=0 \tag{12}$$

$$i_0\frac{\partial}{\partial t}\varphi_A+\boldsymbol{\nabla}\varphi_B-\frac{m}{i_0\hbar}\varphi_A=0 \tag{13}$$

$$-i_0\frac{\partial}{\partial t}\varphi_B-\boldsymbol{\nabla}\varphi_A-\frac{m}{i_0\hbar}\varphi_B=0 \tag{14}$$

Also this split does not easily lead to a second order partial differential equation that looks like the Klein Gordon equation.

1.3 A better choice

Another interpretation of the Dirac approach replaces γ_0 with γ_5 [4]:

$$\left(\gamma_5\frac{\partial}{\partial t}-\gamma_1\frac{\partial}{\partial x}-\gamma_2\frac{\partial}{\partial y}-\gamma_3\frac{\partial}{\partial z}-\frac{m}{i_0\hbar}\right)\{\psi\}=0 \tag{1}$$

$$\left(\gamma_5\frac{\partial}{\partial t}-\langle\boldsymbol{\gamma},\boldsymbol{\nabla}\rangle-\frac{m}{i_0\hbar}\right)\{\psi\}=0 \tag{2}$$

$$\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{\partial}{\partial t}-\begin{bmatrix}0 & \langle\boldsymbol{\sigma},\boldsymbol{\nabla}\rangle\\ -\langle\boldsymbol{\sigma},\boldsymbol{\nabla}\rangle & 0\end{bmatrix}-\frac{m}{i_0\hbar}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}\psi_A\\ \psi_B\end{bmatrix}=0 \tag{3}$$

This invites splitting of the four component spinor equation into two equations for two component spinors:

$$i_0\nabla_0\psi_A+i_0\langle\boldsymbol{\sigma},\boldsymbol{\nabla}\rangle\psi_A=\frac{m}{\hbar}\psi_B \tag{4}$$

$$i_0\nabla_0\varphi_B-i_0\langle\boldsymbol{\sigma},\boldsymbol{\nabla}\rangle\psi_B=\frac{m}{\hbar}\psi_A \tag{5}$$

$$(i_0\nabla_0+\boldsymbol{\nabla})\psi_A=\frac{m}{\hbar}\psi_B \tag{6}$$

$$(i_0\nabla_0-\boldsymbol{\nabla})\psi_B=\frac{m}{\hbar}\psi_A \tag{7}$$

This looks far more promising. We can insert the right part of the first equation into the left part of the second equation.

$$(i_0\nabla_0-\boldsymbol{\nabla})(i_0\nabla_0+\boldsymbol{\nabla})\psi_A=(-\nabla_0\nabla_0-\boldsymbol{\nabla}\boldsymbol{\nabla})\psi_A=(\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle-\nabla_0\nabla_0)\psi_A \tag{8}$$

$$= \frac{m}{\hbar}(i_0 \nabla_0 - \boldsymbol{\nabla})\, \psi_B = \frac{m^2}{\hbar^2}\, \psi_A$$

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \nabla_0 \nabla_0) \psi_A = \frac{m^2}{\hbar^2}\, \psi_A \tag{9}$$

$$(i_0 \nabla_0 + \boldsymbol{\nabla})(i_0 \nabla_0 - \boldsymbol{\nabla}) \psi_B = (-\nabla_0 \nabla_0 - \boldsymbol{\nabla}\boldsymbol{\nabla}) \psi_B = (\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \nabla_0 \nabla_0) \psi_B \tag{10}$$
$$= \frac{m}{\hbar}(i_0 \nabla_0 + \boldsymbol{\nabla})\, \psi_A = \frac{m^2}{\hbar^2}\, \psi_B$$

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \nabla_0 \nabla_0) \psi_B = \frac{m^2}{\hbar^2}\, \psi_B \tag{11}$$

This is what Dirac wanted to achieve. The two first order differential equations couple into a second order differential equation that is equivalent to a Klein Gordon equation. The homogeneous version of this second order partial differential equation is a wave equation and offers solutions that are waves.
The nabla operator acts differently onto the two component spinors ψ_A and ψ_B.

2 Wave equations

A wave equation is a second order partial differential equation that has waves as part of its solutions. For example, the following equation is a wave equation:

$$\mathfrak{D} f = (\nabla_0 \nabla_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\, f = g \tag{1}$$

The operator $\mathfrak{D} \equiv \nabla_0 \nabla_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle$ is the d'Alembert operator.
A similar equation exists for spherical coordinates.

3 The quaternionic nabla and the Dirac nabla

The modified Pauli matrices together with a 2×2 identity matrix implement the equivalent of a quaternionic number system with a selected symmetry flavor.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ i_0 \sigma_1 = \begin{bmatrix} 0 & i_0 \\ i_0 & 0 \end{bmatrix};\ i_0 \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};\ i_0 \sigma_3 = \begin{bmatrix} i_0 & 0 \\ 0 & -i_0 \end{bmatrix} \tag{1}$$

The modified Pauli matrices together with the I_0 matrix implements another structure, which is not a version of a quaternionic number system.

$$I_0 = \begin{bmatrix} i_0 & 0 \\ 0 & i_0 \end{bmatrix};\ i_0 \sigma_1 = \begin{bmatrix} 0 & i_0 \\ i_0 & 0 \end{bmatrix};\ i_0 \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};\ i_0 \sigma_3 = \begin{bmatrix} i_0 & 0 \\ 0 & -i_0 \end{bmatrix} \tag{2}$$

Both the quaternionic nabla and the Dirac nabla implement a way to let these differential operators act as multipliers.
The quaternionic nabla is defined as

$$\nabla = \nabla_0 + \boldsymbol{\nabla} = e^{\mu}\nabla_{\mu} = \nabla_0 + i_0\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}\rangle \tag{3}$$

$$\nabla^* = \nabla_0 - \boldsymbol{\nabla} \tag{4}$$

For scalar functions and for vector functions hold:

$$\nabla^*\nabla = \nabla\nabla^* = \nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \tag{5}$$

The Dirac nabla is defined as

$$\mathcal{D} = i_0\nabla_0 + \boldsymbol{\nabla} = i_0\nabla_0 + i_0\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}\rangle \tag{6}$$

$$\mathcal{D}^* = i_0\nabla_0 - \boldsymbol{\nabla} \qquad 7)$$

$$\mathcal{D}^*\mathcal{D} = \mathcal{D}\,\mathcal{D}^* = -\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \tag{8}$$

3.1.1 Prove

We use

$$\nabla_0\boldsymbol{\nabla} f_0 = \boldsymbol{\nabla}\nabla_0 f_0 \tag{1}$$

$$\nabla_0\boldsymbol{\nabla}\boldsymbol{f} = \boldsymbol{\nabla}\nabla_0\boldsymbol{f} = -\nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{f}\rangle + \nabla_0\boldsymbol{\nabla}\times\boldsymbol{f} \tag{2}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla} f_0 = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle f_0 + \boldsymbol{\nabla}\times\boldsymbol{\nabla} f_0 = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle f_0 \tag{3}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{f}) = -\boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{f}\rangle + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{f} = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{f} = (\boldsymbol{\nabla}\boldsymbol{\nabla})\boldsymbol{f} \tag{4}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{f} = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{f}\rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{f} \tag{5}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\times\boldsymbol{f}\rangle = 0 \tag{6}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\, f_0 = \boldsymbol{0} \tag{7}$$

This results in

$$(\alpha\nabla_0 + \boldsymbol{\nabla})f_0 = \alpha\nabla_0 f_0 + \boldsymbol{\nabla} f_0 \tag{8}$$

$$(\alpha\nabla_0 - \boldsymbol{\nabla})(\alpha\nabla_0 + \boldsymbol{\nabla})f_0 \tag{9}$$

$$= \alpha^2\nabla_0\nabla_0 + \alpha\nabla_0\boldsymbol{\nabla} f_0 - \alpha\boldsymbol{\nabla}\nabla_0 f_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle f_0 - \boldsymbol{\nabla}\times\boldsymbol{\nabla} f_0$$

$$= \alpha^2\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle f_0$$

$$(\alpha\nabla_0 + \boldsymbol{\nabla})\boldsymbol{f} = \alpha\nabla_0\boldsymbol{f} - \langle \boldsymbol{\nabla}, \boldsymbol{f}\rangle + \boldsymbol{\nabla}\times\boldsymbol{f} \tag{10}$$

$$(\alpha\nabla_0 - \alpha\nabla_0\boldsymbol{f} - \langle \boldsymbol{\nabla}, \boldsymbol{f}\rangle + \boldsymbol{\nabla}\times\boldsymbol{f})(\alpha\nabla_0 + \boldsymbol{\nabla})\boldsymbol{f}$$

$$(\alpha\nabla_0 - \boldsymbol{\nabla})(\alpha\nabla_0 + \boldsymbol{\nabla})f_0 \tag{11}$$

$$= \alpha^2 \nabla_0 \nabla_0 \boldsymbol{f} - \alpha \nabla_0 \langle \boldsymbol{\nabla}, \boldsymbol{f} \rangle + \alpha \nabla_0 \boldsymbol{\nabla} \times \boldsymbol{f} + \alpha \nabla_0 \langle \boldsymbol{\nabla}, \boldsymbol{f} \rangle$$

$$-\alpha \nabla_0 \boldsymbol{\nabla} \times \boldsymbol{f} + \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{f} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{f} \rangle - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f}$$

$$= \alpha^2 \nabla_0 \nabla_0 \boldsymbol{f} + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{f}$$

3.1.2 Discussion

For $\alpha = 1$ the equations

$$(\nabla^* \nabla\, f_0 = \nabla \nabla^*\, f_0 = \nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\, f_0 \tag{1}$$

$$(\nabla^* \nabla\, \boldsymbol{f} = \nabla \nabla^*\, \boldsymbol{f} = \nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\, \boldsymbol{f} \tag{2}$$

work for both parts of a quaternionic function $f = f_0 + \boldsymbol{f}$.

For $\alpha = i_0$ the equations

$$(\mathcal{D}^* \mathcal{D}\, f_0 = \mathcal{D} \mathcal{D}^*\, f_0 = -\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\, f_0 \tag{3}$$

$$(\mathcal{D}^* \mathcal{D}\, \boldsymbol{f} = \mathcal{D} \mathcal{D}^* \boldsymbol{f} = -\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\, \boldsymbol{f} \tag{4}$$

work separately for scalar function f_0.and vector function $\boldsymbol{f}$.

The nabla operators reflects the structure of the parameter space of the functions on which they work. Thus the quaternionic nabla operator reflects a quaternionic number system. The Dirac nabla operator reflects the structure of the parameters of the two component spinors that figure in the modified Dirac equation.
Between the two component spinors ψ_A and ψ_B, the scalar part of the parameter space appears to change sign with respect to the vector part.
Applied to a quaternionic function, the quaternionic nabla results again in a ***quaternionic*** function.

$$\phi = \phi_0 + \boldsymbol{\phi} = (\nabla_0 + \boldsymbol{\nabla})(f_0 + \boldsymbol{f}) = \nabla_0 f_0 - \langle \boldsymbol{\nabla}, \boldsymbol{f} \rangle + \boldsymbol{\nabla} f_0 + \nabla_0 \boldsymbol{f} + \boldsymbol{\nabla} \times \boldsymbol{f} \tag{5}$$

Applied to a quaternionic function, the Dirac nabla results in a ***biquaternionic*** function.

$$(i_0\, \nabla_0 + \boldsymbol{\nabla})(f_0 + \boldsymbol{f}) = \nabla_0 i_0 f_0 - \langle \boldsymbol{\nabla}, \boldsymbol{f} \rangle + \boldsymbol{\nabla} f_0 + i_0\, \nabla_0 \boldsymbol{f} + \boldsymbol{\nabla} \times \boldsymbol{f} \tag{6}$$

When applied to a quaternionic function, the $\nabla^* \nabla$ operator results again in a quaternionic function.

$$\nabla^* \nabla (f_0 + \boldsymbol{f}) = (\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)(f_0 + \boldsymbol{f}) \tag{7}$$

When applied to a quaternionic function, the d’Alembert operator $\mathfrak{D} = \mathcal{D}^* \mathcal{D}$ results again in a quaternionic function.

$$\mathcal{D}^*\mathcal{D}(f_0 + \boldsymbol{f}) = (\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)(f_0 + \boldsymbol{f}) \tag{8}$$

Neither the Dirac nabla $\mathcal{D}$ nor its conjugate $\mathcal{D}^*$ delivers quaternionic functions from quaternionic functions.

Thus the d'Alembert operator cannot be split into two operators that map quaternionic functions onto quaternionic functions.

4 Quaternionic format of Dirac equation

The initial goal of Dirac was to split the Klein Gordon equation into two first order differential equations. He tried to achieve this via the combination of matrices and spinors. This leads to a result that does not lead to an actual second order differential equation, but instead it leads to two different first order differential equations for two different spinors that can be coupled into a second order partial differential equation that looks like a Klein Gordon equation. The homogeneous version of the Klein Gordon equation is a wave equation.

The quaternionic differential calculus supports first order differential equations that in a natural way lead to a second order partial differential equation that differs significantly from a wave equation.
The closest quaternionic equivalents of the first order Dirac equations for the electron and the positron are:

$$\nabla f = (\nabla_0 + \boldsymbol{\nabla})(f_0 + \boldsymbol{f}) = mg \tag{1}$$

$$\nabla^* g = (\nabla_0 - \boldsymbol{\nabla})(g_0 + \boldsymbol{g}) = mf \tag{2}$$

$$\nabla^* \nabla f = (\nabla_0 - \boldsymbol{\nabla})(\nabla_0 + \boldsymbol{\nabla})(f_0 + \boldsymbol{f}) = m^2 f \tag{3}$$

$$\nabla^* \nabla f = \nabla^* \nabla f = (\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\, f = m^2 f \tag{4}$$

$$\nabla \nabla^* f = \nabla^* \nabla g = (\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\, g = m^2 g \tag{5}$$

A similar equation exists for spherical coordinates.

These second order equations are not wave equations. Their set of solutions does not include waves.

5 Interpretation of the Dirac equation

The original Dirac equation can be split into two equations. One of them describes the behavior of the electron. The other equation describes the behavior of the positron.
The positron is the anti-particle of the electron. These particles feature the same rest mass, but other characteristics such as their electric charge differ in sign. The positron can be interpreted as an electron that moves back in time. Sometimes the positron is interpreted as a hole in a sea of electrons. These interpretations indicate that the functions that describe these particles feature different parameter spaces that differ in the sign of the scalar part.

5.1 Particle fields

In “On the Origins of Physical Fields” [5]the fields that characterize different types of particles can be related to parameter spaces that belong to different versions of quaternionic number system. These fields are coupled to an embedding field on which the particles and their private parameter spaces float.

The reverse bra-ket method [6] shows how fields can on the one hand be coupled to eigenspaces and eigenvectors of operators which reside in quaternionic non-separable

Hilbert spaces and on the other hand can be coupled to pairs of parameter spaces and quaternionic functions. Quaternionic functions can be split into scalar functions and vector functions. In a quaternionic Hilbert space several different natural parameter spaces can coexist. Natural parameter spaces are formed by versions of the quaternionic number system. These versions differ in the way that these number systems are ordered.
The original Dirac equations might represent this coupling between the particle field and the embedding field.

6 Alternatives

6.1 Minkowski parameter space

In quaternionic differential calculus the local quaternionic distance can represent a scalar that is independent of the direction of progression. It corresponds to the notion of coordinate time t. This means that a small coordinate time step Δt equals the sum of a small proper time step $\Delta\tau$ and a small pure space step $\Delta\boldsymbol{x}$. In quaternionic format the step $\Delta\tau$ is a real number. The space step $\Delta\boldsymbol{x}$ is an imaginary quaternionic number. The original Dirac equation does not pay attention to the difference between coordinate time and proper time, but the quaternionic presentation of these equations show that a progression independent scalar can be useful as the scalar part of the parameter space. This holds especially for solutions of the homogeneous wave equation.
In this way coordinate time is a function of proper time τ and distance in pure space $|\Delta\boldsymbol{x}|$.

$$|\Delta t|^2 = |\Delta\tau|^2 + |\Delta\boldsymbol{x}|^2$$

Together t and $\boldsymbol{x}$ deliver a spacetime model that has a Minkowski signature.

$$|\Delta\tau|^2 = |\Delta t|^2 - |\Delta\boldsymbol{x}|^2$$

6.2 Other natural parameter spaces

The Dirac equation in quaternionic format treats a coupling of parameter spaces that are each other's quaternionic conjugate. This can also be applied when anisotropic conjugation is applied. This concerns conjugations in which only one or two dimensions get a reverse ordering. In that case the equations handle the dynamic behavior of anisotropic particles such as quarks.

7 The coupling equation

The Dirac equation is a more specific form of the coupling equation [7]. The coupling equation holds for quaternionic functions for which the nabla based differential can be normalized:

$$\phi = \nabla\chi = m\,\varphi;\ \|\chi\| = \|\varphi\| = 1 \tag{1}$$

By adapting φ, the coupling factor m can become a real positive number.

The quaternionic second order partial differential equation corresponds to two coupling equations:

$$\phi = \nabla\chi = m_1 \varphi \quad (2)$$

and

$$\nabla^*\varphi = m_2\psi \quad (3)$$

$$\nabla^*\nabla\chi = (\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\chi = m_1 m_2 \psi \quad (4)$$

References

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[6] “The reverse bra-ket method”; http://vixra.org/abs/1511.0266 .
[7] Quaternionic differential calculus is treated in more detail in “Quaternions and quaternionic Hilbert spaces”; http://vixra.org/abs/1411.0178 .