The Dirac equation in quaternionic format

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ABSTRACT

In its original form the Dirac equation for the free electron and the free positron is formulated by using complex number based spinors and matrices. That equation can be split into two equations, one for the electron and one for the positron. If we use proper time rather than coordinate time, and apply the existence of different versions of quaternionic number systems, then these equations can easily be converted to their quaternionic format. The equation for the electron and the equation for the positron differ in the symmetry flavor of their parameter spaces. This results in special considerations for the corresponding quaternionic wave equation.

Contents

1 The Dirac equation in original format

In its original form the Dirac equation is a complex equation that uses spinors, matrices and partial derivatives.

Instead of the usual $\left\{\frac{\partial f}{\partial t}, \boldsymbol{i}\frac{\partial f}{\partial x}, \boldsymbol{j}\frac{\partial f}{\partial y}, \boldsymbol{k}\frac{\partial f}{\partial z}\right\}$ we want to use operators $\nabla = \{\nabla_0, \boldsymbol{\nabla}\}$

The operator ∇ relates to the applied parameter space. This means that the parameter space is also configured of combinations $x = \{t, \boldsymbol{x}\}$ of a scalar t and a vector $\boldsymbol{x}$. Also the functions can be split in scalar functions and vector functions. The subscript $_0$ indicates the scalar part. Bold face indicates the vector part.

Here t represents a local scalar, which is defined as the scalar part of the applied parameter space.

The original Dirac equation uses 4x4 matrices $\boldsymbol{\alpha}$ and β. [1] [2]:

$\boldsymbol{\alpha}$ and β *are* matrices that implement the quaternion arithmetic behavior including the possible symmetry flavors of quaternionic number systems and continuums.

$$\alpha_1 = \gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} \leftrightarrow -i\begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\alpha_2 = \gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix} \leftrightarrow -i\begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_3 = \gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} \leftrightarrow -i\begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\beta = \gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

The unity matrix I and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are given by [3]:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

For one of the potential orderings of the quaternionic number system, the Pauli matrices together with the unity matrix I relate to the quaternionic base vectors $1, \boldsymbol{i}, \boldsymbol{j}$ *and* $\boldsymbol{k}$

$$1 \longmapsto I, \quad \boldsymbol{i} \longmapsto i\,\sigma_1, \quad \boldsymbol{j} \longmapsto i\,\sigma_2, \quad \boldsymbol{k} \longmapsto i\,\sigma_3 \tag{6}$$

$$\sigma_1\sigma_2 - \sigma_2\sigma_1 = 2\,i\,\sigma_3;\ \sigma_2\sigma_3 - \sigma_3\sigma_2 = 2\,i\,\sigma_1;\ \sigma_3\sigma_1 - \sigma_1\sigma_3 = 2\,i\,\sigma_2 \tag{7}$$

$$\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = \beta\beta = I \tag{8}$$

Together with the $\boldsymbol{\alpha}$ matrices, the matrix β represents quaternionic conjugation. As a consequence, it switches the handedness of the external vector product.

The interpretation of the Pauli matrices as representation of a special kind of angular momentum has led to the half integer eigenvalue of the corresponding spin operator.

2 Splitting into two equations

One interpretations of the Dirac equation is [4]:

$$\left(\gamma_0 \frac{\partial}{\partial t} - \gamma_1 \frac{\partial}{\partial x} - \gamma_2 \frac{\partial}{\partial y} - \gamma_3 \frac{\partial}{\partial z} - \frac{m}{i\hbar}\right)\{\psi\} = 0 \tag{1}$$

This invites splitting of the four component spinor equation into two equations for two component spinors:

$$i\frac{\partial \varphi_A}{\partial t} - i\sigma_1 \frac{\partial \varphi_A}{\partial x} - i\sigma_2 \frac{\partial \varphi_A}{\partial y} - i\sigma_3 \frac{\partial \varphi_A}{\partial z} = \frac{m}{\hbar}\, \varphi_B \tag{2}$$

$$i\frac{\partial \varphi_B}{\partial t} + i\sigma_1 \frac{\partial \varphi_B}{\partial x} + i\sigma_2 \frac{\partial \varphi_B}{\partial y} + i\sigma_3 \frac{\partial \varphi_B}{\partial z} = \frac{m}{\hbar}\, \varphi_A \tag{3}$$

$$i\frac{\partial \varphi_A}{\partial t} - \boldsymbol{i}\frac{\partial \varphi_A}{\partial x} - \boldsymbol{j}\frac{\partial \varphi_A}{\partial y} - \boldsymbol{k}\frac{\partial \varphi_A}{\partial z} = (i\nabla_0 - \boldsymbol{\nabla})\varphi_A = \frac{m}{\hbar}\, \varphi_B \tag{4}$$

$$i\frac{\partial \varphi_B}{\partial t} + \boldsymbol{i}\frac{\partial \varphi_B}{\partial x} + \boldsymbol{j}\frac{\partial \varphi_B}{\partial y} + \boldsymbol{k}\frac{\partial \varphi_B}{\partial z} = (i\nabla_0 + \boldsymbol{\nabla})\varphi_B = \frac{m}{\hbar}\, \varphi_A \tag{5}$$

$$\begin{aligned}(i\nabla_0 + \boldsymbol{\nabla})(i\nabla_0 - \boldsymbol{\nabla})\varphi_A &= (-\nabla_0\nabla_0 - \boldsymbol{\nabla}\boldsymbol{\nabla})\varphi_A = (\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_A \\ &= \frac{m}{\hbar}(i\nabla_0 + \boldsymbol{\nabla})\, \varphi_B = \frac{m^2}{\hbar^2}\, \varphi_A\end{aligned} \tag{6}$$

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_A = \frac{m^2}{\hbar^2}\, \varphi_A \tag{7}$$

$$\begin{aligned}(i\nabla_0 - \boldsymbol{\nabla})(i\nabla_0 + \boldsymbol{\nabla})\varphi_B &= (-\nabla_0\nabla_0 - \boldsymbol{\nabla}\boldsymbol{\nabla})\varphi_B = (\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_B \\ &= \frac{m}{\hbar}(i\nabla_0 - \boldsymbol{\nabla})\, \varphi_A = \frac{m^2}{\hbar^2}\, \varphi_B\end{aligned} \tag{8}$$

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_B = \frac{m^2}{\hbar^2}\, \varphi_B \tag{9}$$

Thus the four component spinors $\{\psi\}$ can be converted in two component spinors $\{\varphi_A\}$ and $\{\varphi_B\}$. Quaternionic functions are not complex number based spinors, but the form of equations (7) and (9) offer sufficient info for the conversion. With respect to second order differentiation, the two component spinors and the quaternionic functions show similar behavior.

Transferring the matrix form of the Dirac equation into quaternionic format delivers two quaternionic functions χ_A and χ_B that replace the spinors $\{\varphi_A\}$ and $\{\varphi_B\}$. These functions have different parameter spaces. As a consequence the nabla operators act differently onto χ_A and χ_B. This results into two coupled first order partial differential equations.

$$(\nabla_0 - \boldsymbol{\nabla})\chi_A = \nabla^*\chi_A = \frac{m}{\hbar}\, \chi_B \tag{10}$$

$$(\nabla_0 + \boldsymbol{\nabla})\chi_B = \nabla\chi_B = \frac{m}{\hbar}\,\chi_A \tag{11}$$

This also corresponds to two quite similar second order partial differential equations:

$$(\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\chi_A = \frac{m^2}{\hbar^2}\chi_A \tag{12}$$

$$(\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\chi_B = \frac{m^2}{\hbar^2}\chi_B \tag{13}$$

And one homogeneous second order partial differential equation

$$(\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\chi = 0 \tag{14}$$

This equation is a wave equation. The set of its solutions includes waves.
Thus, the functions χ_A and χ_B describe two different solutions of the same Maxwell-like second order partial differential equation.
According to the Dirac matrices the natural parameter spaces of functions χ_A and χ_B concern two different quaternionic number systems that differ in the handedness of their external vector product.
One of these natural parameter spaces is right handed and the other natural parameter space is left handed.
The factor m couples χ_A and χ_B.
Since both χ_A and χ_B are quaternionic functions, they also obey other second order partial wave equations.

$$(\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle + \nabla_0\nabla_0)\chi_A = \xi_A \tag{15}$$

$$(\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle + \nabla_0\nabla_0)\chi_B = \xi_B \tag{16}$$

$$\left(2\,\nabla_0\nabla_0 + \frac{m^2}{\hbar^2}\right)\chi_A = \xi_A \tag{17}$$

$$\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\,\chi_A = \xi_A + \frac{m^2}{\hbar^2}\chi_A \tag{18}$$

Natural parameter spaces are spanned by a version of the quaternionic number system. Due to the four dimensions of quaternions, these natural parameter spaces represent two different sign flavors of one and the same quaternionic field that exists in 16 versions that only differ in their discrete symmetry set.
The fields that represent the two natural parameter spaces can be considered to be each other's quaternionic conjugate. As a consequence, they differ in the handedness of the external vector product. These fields relate to the symmetry centers from which the particle generating mechanisms take their resources.

In the direct environment of the free particle, apparently two kinds of potential embedding exist. Alongside the particle embedding with solution χ_A exists an antiparticle embedding with solution χ_B.
Function χ represents a curl free field.

$$\nabla\chi = \nabla^*\chi^* \tag{19}$$

3 Alternatives

3.1 Minkowski parameter space

In quaternionic differential calculus the local quaternionic distance can represent a scalar that is independent of the direction of progression. It corresponds to the notion of coordinate time. That means that a small coordinate time step Δt equals the sum of a small proper time step $\Delta\tau$ and a small pure space step $\Delta\boldsymbol{x}$. In quaternionic format the step $\Delta\tau$ is a real number. The space step $\Delta\boldsymbol{x}$ is an imaginary quaternionic number. The original Dirac equation does not pay attention to the difference between coordinate time and proper time, but the quaternionic presentation of these equations show that a progression independent scalar can be useful as the scalar part of the parameter space. This holds especially for solutions of the homogeneous wave equation.

3.2 Other natural parameter spaces

The Dirac equation in quaternionic format treats a coupling of parameter spaces that are each other's quaternionic conjugate. This can also be applied when anisotropic conjugation is applied. This concerns conjugations in which only one or two dimensions get a reverse ordering. In that case the equations handle the dynamic behavior of anisotropic particles such as quarks.

4 The coupling equation

The Dirac equation is a more specific form of the coupling equation [5]. The coupling equation holds for quaternionic functions for which the nabla based differential can be normalized:

$$\phi = \nabla\chi = m\,\varphi;\ \|\chi\| = \|\varphi\| = 1 \tag{1}$$

By adapting φ, the coupling factor m can become a real positive number.

The quaternionic second order partial differential equation corresponds to two coupling equations:

$$\phi = \nabla\chi = m_1\,\varphi \tag{2}$$

and

$$\nabla^*\varphi = m_2\psi \tag{3}$$

$$\nabla^*\nabla\chi = (\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\chi = m_1 m_2\psi \tag{4}$$

References

[1] http://en.wikipedia.org/wiki/Dirac_equation#Mathematical_formulation

[2] Dirac, P.A.M. (1982) [1958]. Principles of Quantum Mechanics. International Series of Monographs on Physics (4th ed.). Oxford University Press. p. 255. ISBN 978-0-19-852011-5.

[3] http://en.wikipedia.org/wiki/Pauli_matrices

[4] http://www.mathpages.com/home/kmath654/kmath654.htm; equation (6)

[5] Quaternionic differential calculus is treated in more detail in "Quaternions and quaternionic Hilbert spaces"; http://vixra.org/abs/1411.0178 .