The Dirac equation in quaternionic format

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ABSTRACT
In its original form the Dirac equation for the free electron and the free positron is formulated by using complex number based spinors and matrices. That equation can be split into two equations, one for the electron and one for the positron. If we use proper time rather than coordinate time, then these equations can easily be converted to their quaternionic format. The equation for the electron and the equation for the positron differ in the sign of a curl term. This means that the solutions differ in the handedness of the external vector product. This results in special considerations for the corresponding quaternionic wave equation.

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The Dirac equation in original format

In its original form the Dirac equation is a complex equation that uses spinors, matrices and partial derivatives.

Instead of the usual $\left\{i\frac{\partial f}{\partial \tau}, i\frac{\partial f}{\partial x}, i\frac{\partial f}{\partial y}, i\frac{\partial f}{\partial z}\right\}$ we use operators $\nabla = \{\nabla_{0,}\boldsymbol{\nabla}\}$

Subscript $_0$ indicates the real part. Bold face indicates the imaginary part.

Here τ is representing local proper time rather than local coordinate time t.
In quaternionic format a coordinate time step Δt is the sum of a proper time step $\Delta\tau$ and a space step. The proper time step $\Delta\tau$ is a real number. The space step is an imaginary quaternionic number. The original Dirac equation does not pay attention to the difference between coordinate time and proper time.

With these ingredients, the Dirac equation runs

$$\nabla_0\{\psi\} + \boldsymbol{\nabla\alpha}\{\psi\} = m\beta\{\psi\} \tag{1}$$

$\boldsymbol{\alpha}$ and β represent the matrices that implement the quaternion behavior including the sign flavors of quaternionic number systems and continuums.

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{4}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

The Pauli[1] matrices $\sigma_1, \sigma_2, \sigma_3$ are given by:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

They relate to the quaternionic base vectors 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto \sigma_1, \qquad \boldsymbol{j} \mapsto \sigma_2, \qquad \boldsymbol{k} \mapsto \sigma_3 \tag{7}$$

Splitting into two equations

The four component spinors $\{\psi\}$ can be converted in two component spinors $\{\psi_R\}$ and $\{\psi_L\}$.
Transferring the matrix form of the Dirac equation into quaternionic format delivers two

[1] http://en.wikipedia.org/wiki/Pauli_matrices

quaternionic fields ψ_e and ψ_p that replace the spinors and that couple two equations of motion. One of these fields is right handed and the other is left handed.

$$\nabla_0\psi_{eR} + \boldsymbol{\nabla}\psi_{eR} = m\psi_{eL} \tag{1}$$

$$\nabla_0\psi_{pL} - \boldsymbol{\nabla}\psi_{pL} = m\psi_{pR} \tag{2}$$

The factor m couples ψ_e and ψ_p.
These fields are each other's quaternionic conjugate. At the same time they differ in handedness of the external vector product.

$$\psi_R = \psi_L^* = \psi_0 + \boldsymbol{\psi} \tag{3}$$

Due to the four dimensions of quaternions, these fields represent two different sign flavors of one and the same quaternionic field that exists in 16 versions that only differ in their discrete symmetry set.

The quaternionic format

Reformulating the quaternionic equation for the free electron gives

$$\nabla\psi_e = m\,\psi_e^* \tag{1}$$

$$\nabla_0\psi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\psi_e}\rangle = m\,\psi_0 \tag{2}$$

$$\nabla_0\boldsymbol{\psi_e} + \boldsymbol{\nabla}\psi_0 \pm \boldsymbol{\nabla}\times\boldsymbol{\psi_e} = -m\,\boldsymbol{\psi_e} \tag{3}$$

$$m\psi_e = (\nabla\psi_e)^* = \nabla_0\psi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\psi_e}\rangle - \nabla_0\boldsymbol{\psi_e} - \boldsymbol{\nabla}\psi_0 \mp \boldsymbol{\nabla}\times\boldsymbol{\psi_e} \tag{4}$$

The $\pm$ sign indicates the choice between right handed and left handed versions of quaternions.
For the antiparticle holds

$$\nabla^*\psi_p^* = m\,\psi_p \tag{5}$$

$$\nabla_0\psi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\psi_p}\rangle = m\,\psi_0 \tag{6}$$

$$-\nabla_0\boldsymbol{\psi_p} - \boldsymbol{\nabla}\psi_0 \pm \boldsymbol{\nabla}\times\boldsymbol{\psi_p} = +m\,\boldsymbol{\psi_p} \tag{7}$$

$$m\,\psi_p = \nabla^*\psi_p^* = \nabla_0\psi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\psi_p}\rangle - \nabla_0\boldsymbol{\psi_p} - \boldsymbol{\nabla}\psi_0 \pm \boldsymbol{\nabla}\times\boldsymbol{\psi_p} \tag{8}$$

The equations 4 and 8 differ in the sign of the $\boldsymbol{\nabla}\times\boldsymbol{\psi}$ term. For example:

$$(\nabla\psi)^* = \nabla^*\psi^* - 2\,\boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{9}$$

The Dirac equation can only be valid when for the particle the external product of the solution ψ_e is right handed, while for the antiparticle the external product of the solution ψ_p is left handed. Thus, the two solutions belong to different symmetry flavors.

The wave equation

In general the non-homogeneous wave equation in quaternionic format is given by

$$\nabla^*\nabla\chi \equiv \nabla_0^2\,\chi\ + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\,\chi = \frac{\partial^2\chi}{\partial\tau^2} + \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2} = \rho \tag{1}$$

With $\nabla\chi = \varphi$ follows $\nabla^*\varphi = \rho$ (2)

Here χ represents the embedding continuum and ρ represents a location density distribution of triggers. Equation 2 represents two continuity equations.

By using the Dirac equation for the electron as the first continuity equation and the Dirac equation of the positron as the second continuity equation a sensible wave equation is obtained. This is a strange coupling between a right handed and a left handed field that is only justified in curl free conditions.
Thus, the non-homogeneous wave equation for the combined electron and positron in curl free conditions is given by:

$$\nabla^*\nabla\psi \equiv \nabla_0^2\,\psi\ + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\,\psi = \frac{\partial^2\chi}{\partial\tau^2} + \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2} = m\nabla^*\psi^* = m^2\psi \tag{3}$$

Any curl will add triggers to the right side of the equation.

Equation (3) is the quaternionic equivalent of the Klein-Gordon equation.
This quaternionic version has an Euclidean signature, where the Klein-Gordon equation represents a Minkowski signature. The Klein-Gordon equation (4) uses coordinate time t, where the quaternionic equivalent (3) uses proper time τ.

$$-\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = m^2\psi \tag{4}$$

This difference indicates a difference in corresponding space-progression models. The solutions of the corresponding homogeneous wave equations differ, but in odd numbers of participating dimensions, both equations offer wave fronts as solutions. These wave fronts can be considered as carriers of information.

In both space-progression models proper time is Lorentz invariant. A Lorentz transformation keeps proper time and proper time differences invariant.

The coupling equation

The Dirac equation is a more specific form of the coupling equation.

The coupling equation holds generally for differentiable normalizable quaternionic functions:

$$\phi = \nabla\chi = m\,\varphi;\ \|\chi\| = \|\varphi\| = 1 \tag{1}$$

By adapting φ, the coupling factor m can become a real positive number.

Quaternionic differential calculus

Since quaternionic differential calculus uses proper time as progression parameter, the corresponding equations are inherently Lorentz invariant. The symmetry flavors of quaternionic number systems and the symmetry flavors of quaternionic functions pose a more serious problem for properly interpreting the differential equations.

Quaternionic differential calculus is treated in more detail in “Quaternions and quaternionic Hilbert spaces”; http://vixra.org/abs/1411.0178 .