The Dirac equation in quaternionic format

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ABSTRACT
In its original form the Dirac equation for the free electron and the positron is formulated by using complex number based spinors and matrices. That equation can be split into two equations, one for the electron and one for the positron. These equations can easily be converted to their quaternionic format. The corresponding wave equation contains a striking curl term.

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The Dirac equation in original format

In its original form the Dirac equation is a complex equation that uses spinors and partial derivatives. Instead of the usual $\left\{i\frac{\partial f}{\partial \tau}, i\frac{\partial f}{\partial x}, i\frac{\partial f}{\partial y}, i\frac{\partial f}{\partial z}\right\}$ we use operators $\nabla = \{\nabla_0, \boldsymbol{\nabla}\}$

In that case the Dirac equation runs

$$\nabla_0\{\psi\} + \boldsymbol{\nabla}\boldsymbol{\alpha}\{\psi\} = m\beta\{\psi\} \tag{1}$$

$\boldsymbol{\alpha}$ and β represent the matrices that implement the quaternion behavior including the sign flavors of quaternionic number systems and continuums.

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{4}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

The Pauli[1] matrices $\sigma_1, \sigma_2, \sigma_3$ are given by:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto \sigma_1, \qquad \boldsymbol{j} \mapsto \sigma_2, \qquad \boldsymbol{k} \mapsto \sigma_3 \tag{7}$$

Splitting into two equations

Transferring the matrix form of the Dirac equation into quaternionic format delivers two quaternionic fields ψ_R and ψ_L that couple two equations of motion.

$$\nabla_0\psi_R + \boldsymbol{\nabla}\psi_R = m\psi_L \tag{1}$$

$$\nabla_0\psi_L - \boldsymbol{\nabla}\psi_L = m\psi_R \tag{2}$$

The factor m couples ψ_{L} and ψ_{R}.
These fields are each other's quaternionic conjugate.

$$\psi_R = \psi_L^* = \psi_0 + \boldsymbol{\psi} \tag{3}$$

The quaternionic format

Reformulating the quaternionic equation for the free electron gives

$$\nabla\psi = m\,\psi^* \tag{1}$$

$$\nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi}\rangle = m\,\psi_0 \tag{2}$$

$$\nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 + \boldsymbol{\nabla} \times \boldsymbol{\psi} = -m\,\boldsymbol{\psi} \tag{3}$$

For the antiparticle holds

$$\nabla^*\psi^* = m\,\psi \tag{4}$$

This is not just the conjugate of equation 1!

The wave equation

In general the non-homogeneous wave equation is given by

$$\nabla^*\nabla\chi \equiv \nabla_0^2\,\chi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\,\chi = \rho \tag{1}$$

Here χ represents the embedding continuum and ρ represents a location density distribution of triggers.
Taking the conjugate $(\nabla\psi)^*$ of $\nabla\psi$ is not straightforward.

[1] http://en.wikipedia.org/wiki/Pauli_matrices

$$\nabla\psi = \nabla_0(\psi_0 + \boldsymbol{\psi}) + \boldsymbol{\nabla}(\psi_0 + \boldsymbol{\psi}) = \nabla_0\psi_0 + \nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\psi}\rangle + \boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{2}$$

$$\nabla^*\psi^* = \nabla_0(\psi_0 - \boldsymbol{\psi}) - \boldsymbol{\nabla}(\psi_0 - \boldsymbol{\psi}) = \tag{3}$$

$$\nabla_0\psi_0 - \nabla_0\boldsymbol{\psi} - \boldsymbol{\nabla}\psi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\psi}\rangle + \boldsymbol{\nabla}\times\boldsymbol{\psi}$$

$$\nabla^*\psi^* = (\nabla\psi)^* + 2\boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{4}$$

$$(\nabla^*\psi^*)^* = \nabla\psi - 2\boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{5}$$

Thus, the non-homogeneous wave equation for the electron is given by:

$$\nabla^*\nabla\psi \equiv \nabla_0^2\,\psi\ + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\,\psi = m\nabla^*\psi^* = m^2\psi + 2m\boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{6}$$

The non-homogeneous wave equation for the positron is given by:

$$\nabla\nabla^*\psi^* \equiv \nabla_0^2\,\psi^*\ + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\,\psi^* = m\nabla\psi = m^2\psi^* - 2m\boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{7}$$

Equations 6 and 7 are equivalent. They contain a striking curl term.

The coupling equation

The Dirac equation is a more specific form of the coupling equation that holds more generally:

$$\phi = \nabla\chi = m\,\varphi;\ \|\chi\| = \|\varphi\| = 1 \tag{1}$$

By adapting φ the coupling factor m can become a real positive number.