Distributional SAdS BH spacetime-induced vacuum dominance

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Abstract: The vacuum energy density of free scalar quantum field Φ with a distributional background spacetime also is considered.It has been widely believed that, except in very extreme situations, the influence of gravity on quantum fields should amount to just small, sub-dominant contributions. Here we argue that this belief is false by showing that there exist well-behaved spacetime evolutions where the vacuum energy density of free quantum fields is forced, by the very same background distributional spacetime such BHs, to become dominant over any classical energydensity component. This semiclassical gravity effect finds its roots in the singular behavior of quantum fields on curved spacetimes. In particular we obtain that the vacuum fluctuations $\langle\Phi^2\rangle$ has a singular behavior on BHs horizon r_+: $\langle\Phi^2(r)\rangle\sim|r-r_+|^{-2}$. We argue that this vacuum dominance may bear importent astrophysical implications.

Keywords: Colombeau nonlinear generalized functions,Distributional Riemannian Geometry,Distributional Schwarzschild Geometry,Schwarzschild singularity,Schwarzschild Horizon,smooth regularization,nonsmooth

regularization,quantum fields on curved spacetime,vacuum fluctuations,vacuum dominance

I.Introduction

I.1.The breakdown of canonical formalism of Riemann

geometry for the singular solutions of the Einstein field equations

Einstein field equations was originally derived by Einstein in 1915 in respect with canonical formalism of Riemann geometry,i.e. by using the classical sufficiently smooth metric tensor, smooth Riemann curvature tensor, smooth Ricci tensor,smooth scalar curvature, etc.. However have soon been found singular solutions of the Einstein field equations with singular metric tensor and singular Riemann curvature tensor.

These singular solutions was formally accepted beyond rigorous canonical formalism of Riemannian geometry.

Remark 1.1.Note that if some components of the Riemann curvature tensor $R^i_{klm}(\hat{x})$ become infinite at point $\hat{x}^0$ one obtain the breakdown of canonical formalism of Riemann geometry in a sufficiently small neighborhood Ω of the point $\hat{x}^0 \in \Omega$, i.e. in such neighborhood Ω Riemann curvature tensor $R^i_{klm}(\hat{x})$ will be changed by formula (1.7) see remark 1.2.

Remark 1.2.Let Γ be infinitesimal closed contour and let Σ_Γ be the corresponding surface spanning by Γ, see Pic.1. We assume now that: (i) christoffel symbol $\Gamma^i_{kl}(\hat{x})$ become infinite at singular point $\hat{x}^0$ by formulae

$$\begin{cases} \Gamma^i_{kl}(\hat{x}) \asymp \Xi_{kl}(\hat{x})(x_i - x^0_i)^{-\delta}, \delta \geq 1 \\ \Xi_{kl}(\hat{x}) \in C^\infty(\Sigma_\Gamma) \end{cases} \tag{1.1}$$

and (ii) $\hat{x}^0 \in \Sigma_\Gamma$. Let us derive now to similarly canonical calculation [3]-[4] the general formula for the *regularized* change $\widetilde{\Delta A_k}$ in a vector $A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour Γ. This regularized change $\widetilde{\Delta A_k}$ can clearly be written in the form

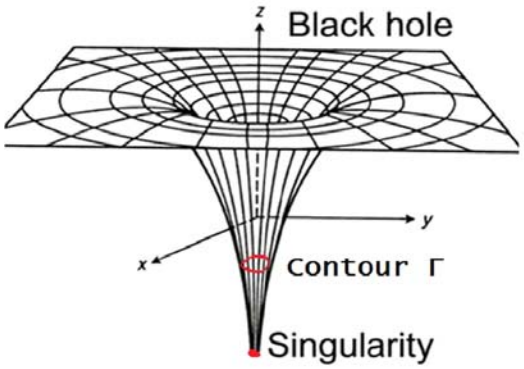
$$\widetilde{\Delta A_k} = \oint_\Gamma \Phi(\hat{x} - \hat{x}^0)\delta A_k, \tag{1.2}$$

where $\Phi(\hat{x} - \hat{x}^0) = \prod_{i=0}^{4}(x_i - x^0_i)^{2\delta}, \delta \geq 1$ and where the integral is taken over the given contour Γ. Substituting in place of δA_k the canonical expression $\delta A_k = \Gamma^i_{kl}(\hat{x})A_k dx^l$ (see [4],Eq.(85.5)) we obtain

$$\widetilde{\Delta A_k} = \oint_\Gamma \Phi(\hat{x} - \hat{x}^0)\delta A_k = \oint_\Gamma \Phi(\hat{x} - \hat{x}^0)\, \Gamma^i_{kl}(\hat{x})A_k dx^l\,, \tag{1.3}$$

where

$$\frac{\partial A_i}{\partial x^l} = \Gamma^i_{kl}(\hat{x})A_k. \tag{1.4}$$



Pic.1.Infinitesimal closed contour Γ
and corresponding singular surface $\Sigma_\Gamma \ni \hat{x}^0$
spanning by Γ.

Now applying Stokes' theorem (see [4],Eq.(6.19)) to the integral (1.3) and considering that the area enclosed by the contour has the infinitesimal value Δf^{lm}, we get

$$\widetilde{\Delta A_k} = \oint_\Gamma \Phi(\hat{x}-\hat{x}^0)\,\Gamma^i_{kl}(\hat{x})A_k dx^l =$$

$$= \frac{1}{2}\int_{\Sigma_\Gamma}\left[\frac{\partial(\Gamma^i_{km}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^m}\right]df^{lm} \approx$$

$$\approx \left[\frac{\partial(\Gamma^i_{km}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^m}\right]\frac{\Delta f^{lm}}{2} =$$

$$\left[\Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{km}(\hat{x}-\hat{x}^0)A_i)}{\partial x^l} + (\Gamma^i_{km}(\hat{x})A_i)\frac{\partial\Phi(\hat{x}-\hat{x}^0)}{\partial x^l} - \right. \tag{1.5}$$

$$\left. -\Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{kl}(\hat{x})A_i)}{\partial x^m} - (\Gamma^i_{kl}(\hat{x})A_i)\frac{\partial\Phi(\hat{x}-\hat{x}^0)}{\partial x^m}\right]\frac{\Delta f^{lm}}{2} =$$

$$\left[\Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{km}(\hat{x})A_i)}{\partial x^l} - \Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{kl}(\hat{x})A_i)}{\partial x^m} - \right.$$

$$\left. A_i(\hat{x})\Phi(\hat{x}-\hat{x}^0)\frac{2\delta\Gamma^i_{km}(\hat{x})}{x_l - x^0_l} - A_i(\hat{x})\Phi(\hat{x}-\hat{x}^0)\frac{2\delta\Gamma^i_{kl}(\hat{x})}{x_m - x^0_m}\right]\frac{\Delta f^{lm}}{2}.$$

Substituting the values of the derivatives (1.4) into Eq.(1.5), we get finally:

$$\widetilde{\Delta A_k} = \widetilde{R}^i_{klm}\frac{A_i(\hat{x})\Phi(\hat{x}-\hat{x}^0)\Delta f^{lm}}{2}, \tag{1.6}$$

where $\widetilde{R}^i_{klm}$, is a tensor of the fourth rank

$$\widetilde{R}^i_{klm} = R^i_{klm} + 2\delta\left[\frac{\Gamma^i_{km}(\hat{x})}{x_l - x^0_l} - \frac{\Gamma^i_{kl}(\hat{x})}{x_m - x^0_m}\right]. \tag{1.7}$$

Here R^i_{klm} is the classical Riemann curvature tensor.That $\widetilde{R}^i_{klm}$ is a tensor is clear from the fact that in (1.6) the left side is a vector—the difference $\widetilde{\Delta A_k}$ between the

values of vectors at one and the same point.

Remark 1.3. Note that similar result was obtained by many authors [5]-[17] by using Colombeau nonlinear generalized functions [1]-[2].

Definition1.1. The tensor $\widetilde{R}^{i}_{klm}$ is called the generalized curvature tensor or the generalized Riemann tensor.

Definition1.2. The generalized Ricci curvature tensor $\widetilde{R}_{km}$ is defined as

$$\widetilde{R}_{km} = \widetilde{R}^{i}_{kim}. \tag{1.8}$$

Definition1.3. The generalized Ricci scalar $\widetilde{R}$ is defined as

$$\widetilde{R} = g^{km}\widetilde{R}_{km}. \tag{1.9}$$

Definition1.3. The generalized Einstein tensor $\widetilde{G}_{km}$ is defined as

$$\widetilde{G}_{km} = \widetilde{R}_{km} - \frac{1}{2}g_{km}\widetilde{R}. \tag{1.10}$$

Remark 1.4. Note that in physical literature the spacetime singularity usually is defined as location where the quantities that are used to measure the gravitational field become infinite in a way that does not depend on the coordinate system. These quantities are the classical scalar invariant curvatures of singular spacetime, which includes a measure of the density of matter.

Remark 1.5. In general relativity, many investigations have been derived with regard to singular exact *vacuum solutions* of the Einstein equation and the singularity structure of space-time. Such solutions have been formally derived under condition

$$\mathbf{T}^{\nu}_{\mu}(x) = 0, \tag{1.11}$$

where $\mathbf{T}^{\nu}_{\mu}(x) = 0$ represent the energy-momentum densities of the gravity source. This for example is the case for the well-known Schwarzschild solution, which is given by, in the Schwarzschild coordinates (x^0, r, θ, ϕ),

$$ds^2 = -h(r)(dx^0)^2 + h^{-1}(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], h(r) = 1 - \frac{r_s}{r}, \tag{1.12}$$

where, r_s is the Schwarzschild radius $r_s = 2GM/c^2$ with G, M and c being the Newton gravitational constant, mass of the source, and the light velocity in vacuum Minkowski space-time, respectively. The metric (1.12) describe the gravitational field produced by a point-like particle located at $r = 0$.

Remark 1.6. Note that when we say, on the basis of the canonical expression of the curvature square

$$\hat{R}^{\rho\sigma\mu\nu}(r)\hat{R}_{\rho\sigma\mu\nu}(r) = 12r_s^2\left[\frac{1}{r^6}\right] \tag{1.13}$$

formally obtained from the metric (1.12), that $r = 0$ is a singularity of the Schwarzschild space-time, the source is considered to be point-like and this metric is regarded as *meaningful everywhere* in space-time.

Remark 1.7. From the metric (1.12), the calculation of the canonical Einstein tensor proceeds in a straighforward manner gives for $r \neq 0$

$$G_t^t(r) = G_r^r(r) = -\frac{h'(r)}{\hat{r}} - \frac{1+h(r)}{\hat{r}^2} \equiv 0 \ , G_\theta^\theta(r) = G_\varphi^\varphi(r) = -\frac{h''(r)}{2} - \frac{h(r)}{\hat{r}^2} \equiv 0, \quad (1.14)$$

where $h(r) = -1 + r_s/r.$ Using Eq.(1.14) one formally obtain boundary conditions

$$\begin{cases} G_t^t(0) \triangleq \lim\limits_{r\to 0} G_t^t(r) = 0, G_r^r(0) \triangleq \lim\limits_{r\to 0} G_r^r(r) = 0, \\ G_\theta^\theta(0) \triangleq \lim\limits_{r\to 0} G_\theta^\theta(r) = 0, G_\varphi^\varphi(0) \triangleq \lim\limits_{r\to 0} G_\varphi^\varphi(r) = 0. \end{cases} \quad (1.15)$$

However as pointed out above the canonical expression of the Einstein tensor in a sufficiently small neighborhood Ω of the point $r = 0$ and must be replaced by the generalized Einstein tensor $\widetilde{G}_{km}$ (1.10). By simple calculation easy to see that

$$\begin{cases} \widetilde{G}_t^t(0) \triangleq \lim\limits_{r\to 0} \widetilde{G}_t^t(r) = -\infty, \widetilde{G}_r^r(0) \triangleq \lim\limits_{r\to 0} \widetilde{G}_r^r(r) = -\infty, \\ \widetilde{G}_\theta^\theta(0) \triangleq \lim\limits_{r\to 0} \widetilde{G}_\theta^\theta(r) = -\infty, \widetilde{G}_\varphi^\varphi(0) \triangleq \lim\limits_{r\to 0} \widetilde{G}_\varphi^\varphi(r) = -\infty. \end{cases} \quad (1.16)$$

and therefore the boundary conditions (1.15) is completely wrong. But other hand as pointed out by many authors [5]-[17] that the canonical representation of the Einstein tensor, valid only in a weak (distributional) sense,i.e. [12]:

$$G_b^a(\hat{x}) = -8\pi m \delta_0^a \delta_b^0 \delta^3(\hat{x}) \quad (1.17)$$

and therefore again we obtain $G_b^a(0) = -\infty \times (\delta_0^a \delta_b^0).$ Thus canonical definition of the Einstein tensor is breakdown in rigorous mathematical sense for the Schwarzschild solution at origin $r = 0.$

I.2.The distributional Schwarzschild geometry

General relativity as a physical theory is governed by particular physical equations; the focus of interest is the breakdown of physics which need not coincide with the breakdown of geometry. It has been suggested to describe singularity at the origin as *internal* point of the Schwarzschild spacetime, where the Einstein field equations are satisfied in a weak (distributional) sense [5]-[22].

1.2.1.The smooth regularization of the singularity at the origin.

The two singular functions we will work with throughout this paper (namely the singular components of the Schwarzschild metric) are $\frac{1}{r}$ and $\frac{1}{r-r_s}$, $r_s \geq 0.$ Since $\frac{1}{r} \in L^1_{loc}(\mathbb{R}^3),$ it obviously gives the regular distribution $\frac{1}{r} \in D'(\mathbb{R}^3)$. By convolution with a mollifier $\rho(x)$ (adapted to the symmetry of the spacetime, i.e. chosen radially symmetric) we embed it into the Colombeau algebra $\mathcal{G}(\mathbb{R}^3)$ [22]:

$$\frac{1}{r} \xrightarrow{\iota} \iota\left(\frac{1}{r}\right) \triangleq \left(\frac{1}{r}\right) * \rho_\varepsilon \triangleq \left(\frac{1}{r}\right)_\varepsilon, \rho_\varepsilon = \frac{1}{\varepsilon^3}\rho(\frac{r}{\varepsilon}), \varepsilon \in (0,1]. \quad (1.18)$$

Inserting (1.18) into (1.12) we obtain a generalized Colombeau object modeling the

singular Schwarzschild spacetime [22]:

$$\begin{cases} (ds_\varepsilon^2)_\varepsilon = (h_\varepsilon(r)(dt)^2)_\varepsilon - (h_\varepsilon^{-1}(r)(dr)^2)_\varepsilon + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \\ h_\varepsilon(r) = -1 + r_s\left(\frac{1}{r}\right)_\varepsilon, \varepsilon \in (0,1]. \end{cases} \tag{1.19}$$

Remark 1.8.Note that under regularization (1.18) for any $\varepsilon \in (0,1]$ the metric

$$ds_\varepsilon^2 = h_\varepsilon(r)(dt)^2 - h_\varepsilon^{-1}(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]$$

obviously is a classical Riemannian object and there no exist an the breakdown of canonical formalism of Riemannian geometry for these metrics, even at origin $r = 0$. It has been suggested by many authors to describe singularity at the origin as an *internal point*, where the Einstein field equations are satisfied in a distributional sense [5]-[22]. From the Colombeau metric (1.19) one obtain in a distributional sense [22]:

$$\begin{aligned} (R_2^2(r,\varepsilon))_\varepsilon = (R_3^3)_\varepsilon = \left(\frac{h_\varepsilon'(r)}{r} + \frac{1+h_\varepsilon(r)}{r^2}\right)_\varepsilon = 8\pi m\frac{\delta(r)}{r^2}, \\ (R_0^0(r,\varepsilon))_\varepsilon = (R_1^1)_\varepsilon = \frac{1}{2}\left(\frac{h_\varepsilon''(r)}{2} + \frac{h_\varepsilon'(r)}{r}\right)_\varepsilon = -4\pi m\delta\frac{\delta(r)}{r^2}. \end{aligned} \tag{1.20}$$

Hence, the distributional Ricci tensor and the distributional curvature scalar $(R_\varepsilon(r))_\varepsilon$ are of δ-type, i.e. $(R_\varepsilon(r))_\varepsilon = \pi m\frac{\delta(r)}{r^2}$.

Remark 1.9. Note that the formulae (1.20) should be contrasted with what is the expected result $G_b^a(x) = -8\pi m\delta_0^a\delta_b^0\delta^3(x)$ given by Eq.(1.17). However the equations (1.20) are obviously given in spherical coordinates and therefore strictly speaking this is not correct, because the basis fields $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta}\right\}$ are not globally defined. Representing distributions concentrated at the origin requires a basis regular at the origin. Transforming the formulae for $(R_{ij}(\varepsilon))_\varepsilon$ into Cartesian coordinates associated with the spherical ones, i.e., $\{r,\theta,\varphi\} \leftrightarrow \{x^i\}$, we obtain, e.g., for the Einstein tensor the expected result $G_b^a(x) = -8\pi m\delta_0^a\delta_b^0\delta^3(x)$ given by Eq.(1.17), see [22].

1.2.2.The nonsmooth regularization of the singularity at the origin.

The nonsmooth regularization of the Schwarzschild singularity at the origin $r = 0$ is considered by N. R. Pantoja and H. Rago in paper [12]. Pantoja non smooth regularization regularization of the Schwarzschild singularity are

$$(h_\varepsilon(r))_\varepsilon = -1 + \left(\frac{r_s}{r}\Theta(r-\varepsilon)\right)_\varepsilon, \varepsilon \in (0,1], r < r_s. \tag{1.21}$$

Here $\Theta(u)$ is the Heaviside function and the limit $\varepsilon \to 0$ is understood in a distributional sense.Equation (1.19) with h_ε as given in (1.21) can be considered as an regularized version of the Schwarzschild line element in curvature coordinates. From equation (1.21), the calculation of the distributional Einstein tensor proceeds

in a straighforward manner. By simple calculation it gives [12]:

$$\begin{cases} (G_t^t(r,\varepsilon))_\varepsilon = (G_r^r(r,\varepsilon))_\varepsilon = -\left(\frac{h_\varepsilon'(r)}{r}\right)_\varepsilon - \left(\frac{1+h_\varepsilon(r)}{r^2}\right)_\varepsilon = \\ = -r_s\left(\frac{\delta(r-\varepsilon)}{r^2}\right)_\varepsilon = -r_s\frac{\delta(r)}{r^2} \end{cases} \tag{1.22}$$

and

$$\begin{cases} (G_\theta^\theta(r,\varepsilon))_\varepsilon = (G_\varphi^\varphi(r,\varepsilon))_\varepsilon = -\left(\frac{h_\varepsilon''(r)}{2}\right)_\varepsilon - \left(\frac{h_\varepsilon(r)}{r^2}\right)_\varepsilon = \\ r_s\left(\frac{\delta(r-\varepsilon)}{r^2}\right)_\varepsilon - r_s\left(\frac{\varepsilon}{r^2}\frac{d}{dr}\delta(r-\varepsilon)\right)_\varepsilon = -r_s\frac{\delta(r)}{r^2}. \end{cases} \tag{1.23}$$

which is exactly the result obtained in Ref. [9] using smoothed versions of the Heaviside function $\Theta(r-\varepsilon)$. Transforming now the formulae for $(G_b^a(r,\varepsilon))_\varepsilon$ into Cartesian coordinates associated with the spherical ones, i.e., $\{r,\theta,\varphi\} \leftrightarrow \{x^i\}$, we obtain for the generalized Einstein tensor the expected result given by Eq.(1.17)

$$G_b^a(x) = -8\pi m\delta_0^a\delta_b^0\delta^3(x), \tag{1.24}$$

see Remark 1.9.

1.2.3.The smooth regularization via Horizon.

The smooth regularization via Horizon is considered by J.M.Heinzle and R.Steinbauer in paper [22]. Note that $\frac{1}{r-r_s} \notin L^1_{loc}(\mathbb{R}^3)$. An canonical regularization is the principal value $\mathbf{vp}\left(\frac{1}{r-r_s}\right) \in D'(\mathbb{R}^3)$ which can be embedded into $\mathcal{G}(\mathbb{R}^3)$ [22]:

$$\frac{1}{r-r_s} \overset{\mathbf{vp}}{\to} \mathbf{vp}\left(\frac{1}{r-r_s}\right) \overset{\iota}{\to} \iota\left[\rho_\varepsilon * \mathbf{vp}\left(\frac{1}{r-r_s}\right)\right] \triangleq \left(\frac{1}{r-r_s}\right)_\varepsilon \in \mathcal{G}(\mathbb{R}^3). \tag{1.25}$$

Inserting now (1.25) into (1.12) we obtain a generalized Colombeau object modeling the singular Schwarzschild spacetime [22]:

$$\begin{cases} (ds_\varepsilon^2)_\varepsilon = (h(r)(dt)^2)_\varepsilon - (h_\varepsilon^{-1}(r)(dr)^2)_\varepsilon + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \\ h(r) = -1 + \frac{r_s}{r}, h_\varepsilon^{-1}(r) = -1 - r_s\left(\frac{1}{r-r_s}\right)_\varepsilon, \varepsilon \in (0,1]. \end{cases} \tag{1.26}$$

Remark 1.10.Note that obviously Colombeau object, (1.26) is degenerate at $r = r_s$, because $h(r)$ is zero at the horizon. However, this does not come as a surprise. Both $h(r)$ and $h^{-1}(r)$ are positive outside of the black hole and negative in the interior. As a consequence any *smooth* regularization of $h(r)$ (or h^{-1}) must pass through zero somewhere and, additionally, this zero must converge to $r = r_s$ as the regularization parameter goes to zero.

Remark 1.11.Note that due to the degeneracy of Colombeau object (1.26), even the distributional Levi-Civitá connection obviously is *not available.*

1.2.4.The nonsmooth regularization via Gorizon

In this paper we leave the neighborhood of the singularity at the origin and turn to the singularity at the horizon. The question we are aiming at is the following: using distributional geometry (thus without leaving Schwarzschild coordinates), is it possible to show that the horizon singularity of the Schwarzschild metric is not merely a coordinate singularity. In order to investigate this issue we calculate the distributional curvature at the horizon in Schwarzschild coordinates.

The main focus of this work is a (nonlinear) *superdistributional* description of the Schwarzschild spacetime. Although the nature of the Schwarzschild singularity is much “worse” than the quasi-regular conical singularity, there are several distributional treatments in the literature [8]-[29], mainly motivated by the following considerations: the physical interpretation of the Schwarzschild metric is clear as long as we consider it merely as an exterior (vacuum) solution of an extended (sufficiently large) massive spherically symmetric body. Together with the interior solution it describes the entire spacetime. The concept of point particles—well understood in the context of linear field theories—suggests a mathematical idealization of the underlying physics: one would like to view the Schwarzschild solution as defined on the entire spacetime and regard it as generated by a point mass located at the origin and acting as the gravitational source.

This of course amounts to the question of whether one can reasonably ascribe distributional curvature quantities to the Schwarzschild singularity at the horizon.

The emphasis of the present work lies on mathematical rigor. We derive the “physically expected” result for the distributional energy momentum tensor of the Schwarzschild geometry, i.e., $T_0^0 = 8\pi m\delta^{(3)}(\vec{x})$, in a conceptually satisfactory way. Additionally, we set up a unified language to comment on the respective merits of some of the approaches taken so far. In particular, we discuss questions of differentiable structure as well as smoothness and degeneracy problems of the regularized metrics, and present possible refinements and workarounds.These aims are accomplished using the framework of nonlinear supergeneralized functions (supergeneralized Colombeau algebras $\widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$).Examining the Schwarzschild metric (1.12) in a neighborhood of the horizon, we see that, whereas $h(r)$ is smooth, $h^{-1}(r)$ is not even L^1_{loc} (note that the origin is now always excluded from our considerations; the space we are working on is $\mathbb{R}^3\backslash\{0\}$). Thus, regularizing the Schwarzschild metric amounts to embedding h^{-1} into $\widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$ (as done in (3.2)).Obviously, (3.1) is degenerate at $r = 2m$, because $h(r)$ is zero at the horizon. However, this does not come as a surprise. Both $h(r)$ and $h^{-1}(r)$ are positive outside of the black hole and negative in the interior. As a consequence any (smooth) regularization $h^+_\epsilon(r)$ ($h^-_\epsilon(r)$) [above (below) horizon] of $h(r)$ must pass through small enough vicinity $O^+_\epsilon(2m) = \{\vec{x} \in \mathbb{R}^3 | \|\vec{x}\| > 2m, \|\vec{x} - 2m\| \leqslant \epsilon\}$ ($O^-_\epsilon(2m) = \{\vec{x} \in \mathbb{R}^3 | \|\vec{x}\| < 2m, \|\vec{x} - 2m\| \leqslant \epsilon\}$) of zeros set $O_0(2m) = \{\vec{y} \in \mathbb{R}^3 | \|\vec{y}\| = 2m\}$ somewhere and, additionally, this vicinity $O^+_\epsilon(2m)$ ($O^-_\epsilon(2m)$) must converge to $O_0(2m)$ as the regularization parameter ϵ goes to

zero.Due to the degeneracy of (1.12), the Levi-Cività connection is not available. Consider, therefore, the following connections $\Gamma^{+l}_{kj}(\epsilon) = \Gamma^{+l}_{kj}[h^+_\epsilon] \in \widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$ and $\Gamma^{-l}_{kj}(\epsilon) = \Gamma^{-l}_{kj}[h^+_\epsilon] \in \widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$:

$$\begin{aligned}\Gamma^{+l}_{kj}(\epsilon) &= \frac{1}{2}[((g^+_\epsilon)^{-1})]^{lm}[(g^+_\epsilon)_{mk,j} + (g^+_\epsilon)_{mj,k} - (g^+_\epsilon)_{kj,m}],\\ \Gamma^{-l}_{kj}(\epsilon) &= \frac{1}{2}[((g^-_\epsilon)^{-1})]^{lm}[(g^-_\epsilon)_{mk,j} + (g^-_\epsilon)_{mj,k} - (g^-_\epsilon)_{kj,m}].\end{aligned} \tag{1.27}$$

$\Gamma^{+l}_{kj}(0), \Gamma^{-l}_{kj}(0)$ coincides with the Levi-Cività connection on $\mathbb{R}^3\backslash\{r=0\}$, as $(g^+_0) = g$, $(g^-_0) = g$ and $(g^+_0)^{-1} = g^{-1}, (g^-_0)^{-1} = g^{-1}$ there.Clearly,connections $\Gamma^{+l}_{kj}(\epsilon), \Gamma^{-l}_{kj}(\epsilon)$ respect the regularized metric $g^\pm_\epsilon$,i.e., $(g^\pm_\epsilon)_{ij;k} = 0$. Proceeding in this manner, we obtain the nonstandard result

$$\begin{cases} \left([\mathbf{R}^+_\epsilon]^1_1\right)_\epsilon = \left([\mathbf{R}^+_\epsilon]^0_0\right)_\epsilon \approx -m\tilde{\Phi}(2m),\\ \left([\mathbf{R}^-_\epsilon]^1_1\right)_\epsilon = \left([\mathbf{R}^-_\epsilon]^0_0\right)_\epsilon \simeq m\tilde{\Phi}(2m).\end{cases} \tag{1.28}$$

Investigating the weak limit of the angular components of the generalized Ricci tensor using the abbreviation

$$\tilde{\Phi}(r) = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \Phi(\vec{x})$$

and let $\Phi(\vec{x})$ be the function $\Phi(\vec{x}) \in \mathcal{S}_{2m}(\mathbb{R}^3,k),$ where by $\mathcal{S}_{2m}(\mathbb{R}^3,k)$ we denote the class of all functions $\Phi(\vec{x})$ such that (i) $\tilde{\Phi}(r)\sim(r-2m)^k,\ r\sim 2m, k \geqslant 2$ and (ii) $\Phi(\vec{x}) \in C^\infty(\mathbb{R}^3)$. Then for any function $\Phi(\vec{x}) \in \mathcal{S}_{2m}(\mathbb{R}^3,2)$ with compact support we get:

$$\begin{cases} w\text{-}\lim\limits_{\epsilon\to 0}[\mathbf{R}^+_\epsilon]^1_1 = w\text{-}\lim\limits_{\epsilon\to 0}[\mathbf{R}^+_\epsilon]^0_0 = m\left\langle\tilde{\delta}|\tilde{\Phi}\right\rangle = -m\tilde{\Phi}(2m),\\ w\text{-}\lim\limits_{\epsilon\to 0}[\mathbf{R}^-_\epsilon]^1_1 = w\text{-}\lim\limits_{\epsilon\to 0}[\mathbf{R}^-_\epsilon]^0_0 = m\left\langle\tilde{\delta}|\tilde{\Phi}\right\rangle = m\tilde{\Phi}(2m),\end{cases} \tag{1.29}$$

i.e., the Schwarzschild spacetime is weakly *Ricci-nonflat* (the origin was excluded from our considerations).
Furthermore,the Tolman formula [3],[4] for the total energy of a static and asymptotically flat spacetime with g the determinant of the four dimensional metric

and d^3x the coordinate volume element, gives

$$E_T = \int\left(\mathbf{T}^r_r + \mathbf{T}^\theta_\theta + \mathbf{T}^\phi_\phi + \mathbf{T}^t_t\right)\sqrt{-g}\, d^3x = m, \tag{1.30}$$

as it should be.

The paper is organized in the following way: in section II we discuss the conceptual as well as the mathematical prerequisites. In particular we comment on geometrical matters (differentiable structure, coordinate invariance) and recall the basic facts of nonlinear superdistributional geometry in the context of algebras $\widetilde{\mathcal{G}}(M,\Sigma)$ of supergeneralized functions. Moreover, we derive sensible nonsmooth

regularizations of the singular functions to be used throughout the paper. Section III is devoted to these approach to the problem. We present a new conceptually satisfactory method to derive the main result. In these final section III we investigate the horizon and describe its distributional curvature. Using nonlinear superdistributional geometry and supergeneralized functions it seems possible to show that the horizon singularity is not only a coordinate singularity without leaving Schwarzschild coordinates.

II. Generalized Colombeau Calculus

II.1.Notation and basic notions from standard Colombeau theory

We use [1],[2],[7] as standard references for the foundations and various applications of standard Colombeau theory. We briefly recall the basic Colombeau construction. Throughout the paper Ω will denote an open subset of $\mathbb{R}^n$. Stanfard Colombeau generalized functions on Ω are defined as equivalence classes $u = [(u_\varepsilon)_\varepsilon]$ of nets of smooth functions $u_\varepsilon \in C^\infty(\Omega)$ (regularizations) subjected to asymptotic norm conditions with respect to $\varepsilon \in (0,1]$ for their derivatives on compact sets.

The basic idea of *classical Colombeau's theory of nonlinear generalized functions* [1],[2] is regularization by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularization parameter ε. Let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ with $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)$ for all $\varepsilon \in \mathbb{R}_+$, where M a separable, smooth orientable Hausdorff manifold of dimension n.

Definition 2.1.The classical Colombeau's algebra of generalized functions on M is defined as the quotient:

$$\mathcal{G}(M) \triangleq \mathcal{E}_M(M)/\mathcal{N}(M) \tag{2.1}$$

of the space $\mathcal{E}_M(M)$ of sequences of moderate growth modulo the space $\mathcal{N}(M)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates (where $\mathfrak{X}(M)$ denoting the space of smooth vector fields on M):

$$\left\{ \begin{array}{c} \mathcal{E}_M(M) \triangleq \Big\{(u_\varepsilon)_\varepsilon |\ \forall K(K \subsetneqq M)\, \forall k(k \in \mathbb{N})\, \exists N(N \in \mathbb{N}) \\ \forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M)) \left[\sup_{p\in K} |L_{\xi_1}\ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0 \right] \Big\}, \end{array} \right. \tag{2.2}$$

$$\left\{ \begin{array}{c} \mathcal{N}(M) \triangleq \Big\{(u_\varepsilon)_\varepsilon |\ \forall K(K \subsetneqq M),\, \forall k(k \in \mathbb{N}_0) \forall q(q \in N) \\ \forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M)) \left[\sup_{p\in K} |L_{\xi_1}\ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0 \right] \Big\}. \end{array} \right. \tag{2.3}$$

Remark 2.1. In the definition the Landau symbol $a_\varepsilon = O(\psi(\varepsilon))$ appears, having the following meaning: $\exists C(C > 0) \exists \varepsilon_0(\varepsilon_0 \in (0,1]) \forall \varepsilon(\varepsilon < \varepsilon_0)[a_\varepsilon \leq C\psi(\varepsilon)]$.

Definition 2.3. Elements of $\mathcal{G}(M)$ are denoted by:

$$u = \mathbf{cl}[(u_\varepsilon)_\varepsilon] \triangleq (u_\varepsilon)_\varepsilon + \mathcal{N}(M). \tag{2.4}$$

Remark 2.2.With componentwise operations $(\bullet, \pm)$ $\mathcal{G}(M)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by $L_\xi u \triangleq \mathbf{cl}[(L_\xi u_\varepsilon)_\varepsilon]$.

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $u \in \mathcal{G}(M)$ iff $u \circ \psi_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ for all charts (V_α, ψ_α), where on the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives.

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $u \in \mathcal{G}(M)$ iff $u \circ \psi_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ for all charts (V_α, ψ_α), where on the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives.

Remark 2.4.Smooth functions $f \in \mathcal{C}^\infty(M)$ are embedded into $\mathcal{G}(M)$ simply by the "constant" embedding σ, i.e., $\sigma(f) = \mathbf{cl}[(f)_\varepsilon]$, hence $\mathcal{C}^\infty(M)$ is a faithful subalgebra of $\mathcal{G}(M)$.

Point Values of a Generalized Functions on M. Generalized Numbers.

Within the classical distribution theory, distributions cannot be characterized by their point values in any way similar to classical functions. On the other hand, there is a very natural and direct way of obtaining the point values of the elements of Colombeau's algebra: points are simply inserted into representatives. The objects so obtained are sequences of numbers, and as such are not the elements in the field $\mathbb{R}$ or $\mathbb{C}$. Instead, they are the representatives of *Colombeau's generalized numbers.* We give the exact definition of these "numbers".

Definition 2.5.Inserting $p \in M$ into $u \in \mathcal{G}(M)$ yields a well defined element of the ring of constants (also called generalized numbers) $\mathcal{K}$ (corresponding to $\mathbf{K} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers $((r_\varepsilon)_\varepsilon \in \mathbf{K}^{(0,1]}$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some N) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each m); componentwise insertion of points of M into elements of $\mathcal{G}(M)$ yields well-defined generalized numbers, i.e.,elements of the ring of constants:

$$\mathcal{K} = \mathcal{E}_\mathbf{c}(M)/\mathcal{N}_\mathbf{c}(M) \tag{2.5}$$

(with $\mathcal{K} = \widetilde{\mathbb{R}}$ or $\mathcal{K} = \widetilde{\mathbb{C}}$ for $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$), where

$$\begin{cases} \mathcal{E}_\mathbf{c}(M) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \exists n (n \in \mathbb{N}) \left[|r_\epsilon| = O(\epsilon^{-n}) \text{ as } \varepsilon \to 0 \right]\right\} \\ \mathcal{N}_\mathbf{c}(M) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \forall m (m \in \mathbb{N}) \left[|r_\epsilon| = O(\epsilon^{m}) \text{ as } \varepsilon \to 0 \right]\right\} \\ I = (0,1]. \end{cases} \tag{2.6}$$

Generalized functions on M are characterized by their generalized point values, i.e., by their values on points in $\tilde{M}_c$, the space of equivalence classes of compactly supported nets $(p_\varepsilon)_\varepsilon \in M^{(0,1]}$ with respect to the relation

$p_\varepsilon \sim p'_\varepsilon :\Leftrightarrow d_h(p_\varepsilon, p'_\varepsilon) = O(\varepsilon^m)$ for all m, where d_h denotes the distance on M induced by any Riemannian metric.

Definition 2.6. For $u \in \mathcal{G}(M)$ and $x_0 \in M$, the point value of u at the point $x_0, u(x_0)$, is defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\mathcal{K}$.

Definition 2.7.We say that an element $r \in \mathcal{K}$ is *strictly nonzero* if there exists a representative $(r_\varepsilon)_\varepsilon$ and a $q \in \mathbb{N}$ such that $|r_\varepsilon| \geqslant \varepsilon^q$ for ε sufficiently small. If r is strictly nonzero, then it is also invertible with the inverse $[(1/r_\varepsilon)_\varepsilon]$. The converse is true as well.

Treating the elements of Colombeau algebras as a generalization of classical functions, the question arises whether the definition of point values can be extended in such a way that each element is characterized by its values. Such an extension is indeed possible.

Definition 2.8. Let Ω be an open subset of $\mathbb{R}^n$. On a set $\hat{\Omega}$:

$$\begin{cases} \hat{\Omega} = \{(x_\varepsilon)_\varepsilon \in \Omega^I | \exists p(p > 0)[|x_\varepsilon| = O(\varepsilon^p)]\} = \\ \left\{(x_\varepsilon)_\varepsilon \in \Omega^I | \exists p(p > 0) \exists \varepsilon_0(\varepsilon_0 > 0)\left[|x_\varepsilon| \leq \varepsilon^p, \text{ for } 0 < \varepsilon < \varepsilon_0\right]\right\} \end{cases} \tag{2.7}$$

we introduce an equivalence relation:

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall q(q > 0) \forall \varepsilon(\varepsilon > 0)\left[|x_\varepsilon - y_\varepsilon| \leq \varepsilon^q, \text{ for } 0 < \varepsilon < \varepsilon_0\right] \tag{2.8}$$

and denote by $\widetilde{\Omega} = \hat{\Omega}/\sim$ the set of generalized points. The set of points with compact support is

$$\widetilde{\Omega}_{\mathbf{c}} = \left\{\tilde{x} = \mathbf{cl}[(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega} | \exists K(K \subset \Omega) \exists \varepsilon_0(\varepsilon_0 > 0)\left[x_\varepsilon \in K \text{ for } 0 < \varepsilon < \varepsilon_0\right]\right\} \tag{2.9}$$

Definition 2.9. A generalized function $u \in \mathcal{G}(M)$ is called associated to zero, $u \approx 0$ on $\Omega \subseteq M$ in L.Schwartz sense if one (hence any) representative $(u_\epsilon)_\epsilon$ converges to zero weakly,i.e.

$$w\text{-}\lim_{\epsilon \to 0} u_\epsilon = 0 \tag{2.10}$$

We shall often write:

$$u \underset{\mathbf{Sch}}{\approx} 0. \tag{2.11}$$

The $\mathcal{G}(M)$-module of generalized sections in vector bundles-especially the space of generalized tensor fields $\mathcal{T}_s^r(M)$-is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. However, it is more convenient to use the following algebraic description of generalized tensor fields

$$\mathcal{G}_s^r(M) = \mathcal{G}(M) \otimes \mathcal{T}_s^r(M), \tag{2.12}$$

where $\mathcal{T}_s^r(M)$ denotes the space of smooth tensor fields and the tensor product is taken over the module $C^\infty(M)$. Hence generalized tensor fields are just given by classical ones with generalized coefficient functions. Many concepts of classical

tensor analysis carry over to the generalized setting [1]-[2], in particular Lie derivatives with respect to both classical and generalized vector fields, Lie brackets, exterior algebra, etc. Moreover, generalized tensor fields may also be viewed as $\mathcal{G}(M)$-multilinear maps taking generalized vector and covector fields to generalized functions, i.e., as $\mathcal{G}(M)$-modules we have

$$\mathcal{G}^r_s(M) \cong L_{(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)). \tag{2.13}$$

In particular a generalized metric is defined to be a symmetric, generalized $(0,2)$-tensor field $g_{ab} = [((g_\epsilon)_{ab})_\epsilon]$ (with its index independent of ε and) whose determinant $\det(g_{ab})$ is invertible in $\mathcal{G}(M)$. The latter condition is equivalent to the following notion called strictly nonzero on compact sets: for any representative $\det((g_\epsilon)_{ab})_\epsilon$ of $\det(g_{ab})$ we have $\forall K \subset M\, \exists m \in \mathbb{N}[\inf_{p\in K}|\det(g_{ab}(\epsilon))| \geq \epsilon^m]$ for all ϵ small enough. This notion captures the intuitive idea of a generalized metric to be a sequence of classical metrics approaching a singular limit in the following sense: g_{ab} is a generalized metric iff (on every relatively compact open subset V of M) there exists a representative $((g_\epsilon)_{ab})_\epsilon$ of g_{ab} such that for fixed ϵ (small enough)$(g_\epsilon)_{ab} = g_{ab}(\epsilon)$ (resp. $(g_\epsilon)_{ab}|_V$) is a classical pseudo-Riemannian metric and $\det(g_{ab})$ is invertible in the algebra of generalized functions. A generalized metric induces a $\mathcal{G}(M)$-linear isomorphism from $\mathcal{G}^1_0(M)$ to $\mathcal{G}^0_1(M)$ and the inverse metric $g^{ab} \triangleq [(g^{-1}_{ab}(\epsilon))_\epsilon]$ is a well defined element of $\mathcal{G}^2_0(M)$ (i.e., independent of the representative $((g_\epsilon)_{ab})_\epsilon$). Also the generalized Levi-Civita connection as well as the generalized Riemann-, Ricci- and Einstein tensor of a generalized metric are defined simply by the usual coordinate formulae on the level of representatives.

II.2. Generalized Colombeau Calculus.

We briefly recall the basic generalized Colombeau construction. Colombeau supergeneralized functions on $\Omega \subseteq \mathbb{R}^n$, where $\dim(\Omega) = n$ are defined as equivalence classes $u = [(u_\varepsilon)_\varepsilon]$ of nets of smooth functions $u_\varepsilon \in C^\infty(\Omega\backslash\Sigma)$, where $\dim(\Sigma) < n$ (regularizations) subjected to asymptotic norm conditions with respect to $\varepsilon \in (0,1]$ for their derivatives on compact sets.

The basic idea of generalized *Colombeau's theory of nonlinear supergeneralized functions* [1],[2] is regularization by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularization parameter ε. Let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ with u_ε such that: (i) $u_\varepsilon \in \mathcal{C}^\infty(M\backslash\Sigma)$ and (ii) $u_\varepsilon \in D'(M)$, for all $\varepsilon \in (0,1]$, where M a separable, smooth orientable Hausdorff manifold of dimension n.

Definition 2.10.The supergeneralized Colombeau's algebra $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(M,\Sigma)$ of supergeneralized functions on M, where $\Sigma \subset M, \dim(M) = n, \dim(\Sigma) < n$, is defined as the quotient:

$$\widetilde{\mathcal{G}}(M,\Sigma) \triangleq \mathcal{E}_M(M,\Sigma)/\mathcal{N}(M,\Sigma) \tag{2.14}$$

of the space $\mathcal{E}_M(M,\Sigma)$ of sequences of moderate growth modulo the space $\mathcal{N}(M,\Sigma)$ of negligible sequences. More precisely the notions of moderateness resp.

negligibility are defined by the following asymptotic estimates (where $\mathfrak{X}(M\backslash\Sigma)$ denoting the space of smooth vector fields on $M\backslash\Sigma$):

$$\begin{aligned}
&\mathcal{E}_M(M,\Sigma) \triangleq \Big\{(u_\varepsilon)_\varepsilon |\ \forall K(K \subsetneqq M\backslash\Sigma)\,\forall k(k\in\mathbb{N})\,\exists N(N\in\mathbb{N})\\
&\forall \xi_1,\dots,\xi_k(\xi_1,\dots,\xi_k \in \mathfrak{X}(M\backslash\Sigma))\left[\sup_{p\in K}|L_{\xi_1}\dots L_{\xi_k}u_\varepsilon(p)| = O(\varepsilon^{-N}),\varepsilon\to 0\right]\ \&\\
&\forall K(K\subsetneqq M)\,\forall k(k\in\mathbb{N})\,\exists N(N\in\mathbb{N})\forall(f\in\mathcal{C}^\infty(M))\forall\xi_1,\dots,\xi_k(\xi_1,\dots,\xi_k\in\mathfrak{X}(M))\\
&\left[|L^w_{\xi_1}\dots L^w_{\xi_k}u_\varepsilon(f)| = \left(\sup_{p\in K}L_{\xi_1}\dots L_{\xi_k}f(p)\right)O(\varepsilon^{-N}),\varepsilon\to 0\right]\Big\},
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
&\mathcal{N}(M,\Sigma) \triangleq \Big\{(u_\varepsilon)_\varepsilon |\ \forall K(K \subsetneqq M\backslash\Sigma),\,\forall k(k\in\mathbb{N}_0)\forall q(q\in N)\\
&\forall \xi_1,\dots,\xi_k(\xi_1,\dots,\xi_k \in \mathfrak{X}(M\backslash\Sigma))\left[\sup_{p\in K}|L_{\xi_1}\dots L_{\xi_k}u_\varepsilon(p)| = O(\varepsilon^{q}),\varepsilon\to 0\right]\Big\}\ \&\\
&\forall K(K\subsetneqq M)\,\forall k(k\in\mathbb{N})\,\exists N(N\in\mathbb{N})\forall(f\in\mathcal{C}^\infty(M))\forall\xi_1,\dots,\xi_k(\xi_1,\dots,\xi_k\in\mathfrak{X}(M))\\
&\left[|L^w_{\xi_1}\dots L^w_{\xi_k}u_\varepsilon(f)| = \left(\sup_{p\in K}L_{\xi_1}\dots L_{\xi_k}f(p)\right)O(\varepsilon^{q}),\varepsilon\to 0\right]\Big\},
\end{aligned}\tag{2.16}$$

where $L^w_{\xi_k}$ denoting the weak Lie derivative in L.Schwartz sense.In the definition the Landau symbol $a_\varepsilon = O(\psi(\varepsilon))$ appears, having the following meaning:
$\exists C(C>0)\exists\varepsilon_0(\varepsilon_0\in(0,1])\forall\varepsilon(\varepsilon<\varepsilon_0)[a_\varepsilon \le C\psi(\varepsilon)]$.

Definition 2.11. Elements of $\widetilde{\mathcal{G}}(M,\Sigma)$ are denoted by:

$$u = \mathbf{cl}[(u_\varepsilon)_\varepsilon] \triangleq (u_\varepsilon)_\varepsilon + \mathcal{N}(M,\Sigma).\tag{2.17}$$

Remark 2.5.With componentwise operations $(\bullet,\pm)$ $\widetilde{\mathcal{G}}(M,\Sigma)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by $L_\xi u \triangleq \mathbf{cl}[(L_\xi u_\varepsilon)_\varepsilon]$.

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $u\in\widetilde{\mathcal{G}}(M,\Sigma)$ iff $u\circ\psi_\alpha\in\widetilde{\mathcal{G}}(\psi_\alpha(V_\alpha))$ for all charts (V_α,ψ_α), where on the open set $\psi_\alpha(V_\alpha)\subset\mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives.

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $u\in\widetilde{\mathcal{G}}(M,\Sigma)$ iff $u\circ\psi_\alpha\in\widetilde{\mathcal{G}}(\psi_\alpha(V_\alpha))$ for all charts (V_α,ψ_α), where on the open set $\psi_\alpha(V_\alpha)\subset\mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives.

Remark 2.6.Smooth functions $f\in\mathcal{C}^\infty(M\backslash\Sigma)$ are embedded into $\widetilde{\mathcal{G}}(M,\Sigma)$ simply by the “constant” embedding σ, i.e., $\sigma(f) = \mathbf{cl}[(f)_\varepsilon]$, hence $\mathcal{C}^\infty(M\backslash\Sigma)$ is a faithful subalgebra of $\widetilde{\mathcal{G}}(M,\Sigma)$.

Point Values of a Supergeneralized Functions on M.

Supergeneralized Numbers

Within the classical distribution theory, distributions cannot be characterized by their point values in any way similar to classical functions. On the other hand, there is a very natural and direct way of obtaining the point values of the elements of Colombeau's algebra: points are simply inserted into representatives. The objects so obtained are sequences of numbers, and as such are not the elements in the field $\mathbb{R}$ or $\mathbb{C}$. Instead, they are the representatives of *Colombeau's generalized numbers.* We give the exact definition of these "numbers".

Definition 2.12.Inserting $p \in M$ into $u \in \widetilde{\mathcal{G}}(M,\Sigma)$ yields a well defined element of the ring of constants (also called generalized numbers) $\widetilde{\mathcal{K}}$ (corresponding to $\mathbf{K} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers ($(r_\varepsilon)_\varepsilon \in \mathbf{K}^{(0,1]}$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some N) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each m); componentwise insertion of points of M into elements of $\widetilde{\mathcal{G}}(M,\Sigma)$ yields well-defined generalized numbers, i.e.,elements of the ring of constants:

$$\widetilde{\mathcal{K}}_\Sigma = \mathcal{E}_\mathbf{c}(M,\Sigma)/\mathcal{N}_\mathbf{c}(M,\Sigma) \tag{2.18}$$

(with $\widetilde{\mathcal{K}}_\Sigma = \widetilde{\mathbb{R}}_\Sigma$ or $\mathcal{K} = \widetilde{\mathbb{C}}_\Sigma$ for $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$), where

$$\begin{cases} \mathcal{E}_\mathbf{c}(M,\Sigma) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \exists n (n \in \mathbb{N}) \left[|r_\varepsilon| = O(\varepsilon^{-n}) \text{ as } \varepsilon \to 0 \right]\right\}, \\ \mathcal{N}_\mathbf{c}(M,\Sigma) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \forall m (m \in \mathbb{N}) \left[|r_\varepsilon| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0 \right]\right\} \\ I = (0,1]. \end{cases} \tag{2.19}$$

Supergeneralized functions on M are characterized by their generalized point values, i.e., by their values on points in $\tilde{M}_c$, the space of equivalence classes of compactly supported nets $(p_\varepsilon)_\varepsilon \in (M\backslash\Sigma)^{(0,1]}$ with respect to the relation $p_\varepsilon \sim p'_\varepsilon :\Leftrightarrow d_h(p_\varepsilon, p'_\varepsilon) = O(\varepsilon^m)$ for all m, where d_h denotes the distance on $M\backslash\Sigma$ induced by any Riemannian metric.

Definition 2.13. For $u \in \widetilde{\mathcal{G}}(M,\Sigma)$ and $x_0 \in M$,the point value of u at the point $x_0, u(x_0)$, is defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\widetilde{\mathcal{K}}$.

Definition 2.14.We say that an element $r \in \widetilde{\mathcal{K}}$ is *strictly nonzero* if there exists a representative $(r_\varepsilon)_\varepsilon$ and a $q \in \mathbb{N}$ such that $|r_\varepsilon| \geqslant \varepsilon^q$ for ε sufficiently small. If r is strictly nonzero, then it is also invertible with the inverse $[(1/r_\varepsilon)_\varepsilon]$. The converse is true as well.

Treating the elements of Colombeau algebras as a generalization of classical functions, the question arises whether the definition of point values can be extended in such a way that each element is characterized by its values. Such an extension is indeed possible.

Definition 2.15. Let Ω be an open subset of $\mathbb{R}^n\backslash\Sigma$. On a set $\hat{\Omega}_\Sigma$:

$$\begin{cases} \hat{\Omega}_\Sigma = \{(x_\varepsilon)_\varepsilon \in (\Omega\backslash\Sigma)^I | \exists p(p>0)[|x_\varepsilon| = O(\varepsilon^p)]\} = \\ \left\{(x_\varepsilon)_\varepsilon \in (\Omega\backslash\Sigma)^I | \exists p(p>0)\exists\varepsilon_0(\varepsilon_0>0)\left[|x_\varepsilon| \le \varepsilon^p, \text{ for } 0<\varepsilon<\varepsilon_0\right]\right\} \end{cases} \tag{2.20}$$

we introduce an equivalence relation:

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall q(q>0)\forall\varepsilon(\varepsilon>0)\left[|x_\varepsilon - y_\varepsilon| \le \varepsilon^q, \text{ for } 0<\varepsilon<\varepsilon_0\right] \tag{2.21}$$

and denote by $\widetilde{\Omega}_\Sigma = \hat{\Omega}_\Sigma/\sim$ the set of supergeneralized points. The set of points with compact support is

$$\widetilde{\Omega}_{\Sigma,\mathbf{c}} = \left\{\tilde{x} = \mathbf{cl}[(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_\Sigma | \exists K(K \subset \Omega\backslash\Sigma)\exists\varepsilon_0(\varepsilon_0>0)\left[x_\varepsilon \in K \text{ for } 0<\varepsilon<\varepsilon_0\right]\right\} \tag{2.22}$$

Definition 2.16. A supergeneralized function $u \in \widetilde{\mathcal{G}}(M,\Sigma)$ is called associated to zero, $u \approx 0$ on $\Omega \subseteq M$ in *L. Schwartz's sense* if one (hence any) representative $(u_\varepsilon)_\varepsilon$ converges to zero weakly,i.e.

$$w\text{-}\lim_{\varepsilon\to 0} u_\varepsilon = 0 \tag{2.23}$$

We shall often write:

$$u \underset{\mathbf{Sch}}{\approx} 0. \tag{2.24}$$

Definition 2.17.The $\widetilde{\mathcal{G}}(M,\Sigma)$-module of supergeneralized sections in vector bundles- especially the space of generalized tensor fields $\mathcal{T}_s^{\ r}(M\backslash\Sigma)$-is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. However, it is more convenient to use the following algebraic description of generalized tensor fields

$$\widetilde{\mathcal{G}}_s^r(M,\Sigma) = \widetilde{\mathcal{G}}(M,\Sigma) \otimes \mathcal{T}_s^{\ r}(M\backslash\Sigma), \tag{2.25}$$

where $\mathcal{T}_s^{\ r}(M\backslash\Sigma)$ denotes the space of smooth tensor fields and the tensor product is taken over the module $C^\infty(M\backslash\Sigma)$. Hence generalized tensor fields are just given by classical ones with generalized coefficient functions. Many concepts of classical tensor analysis carry over to the generalized setting [], in particular Lie derivatives with respect to both classical and generalized vector fields, Lie brackets, exterior algebra, etc. Moreover, generalized tensor fields may also be viewed as $\widetilde{\mathcal{G}}(M,\Sigma)$-multilinear maps taking generalized vector and covector fields to generalized functions, i.e., as $\widetilde{\mathcal{G}}(M,\Sigma)$-modules we have

$$\widetilde{\mathcal{G}}_s^r(M,\Sigma) \cong L_{(M)}(\widetilde{\mathcal{G}}_1^0(M,\Sigma)^r, \widetilde{\mathcal{G}}_0^1(M,\Sigma)^s; \widetilde{\mathcal{G}}(M,\Sigma)). \tag{2.26}$$

In particular a supergeneralized metric is defined to be a symmetric, supergeneralized $(0,2)$-tensor field $g_{ab} = [((g_\varepsilon)_{ab})_\varepsilon]$ (with its index independent of ε and) whose determinant $\det(g_{ab})$ is invertible in $\widetilde{\mathcal{G}}(M\backslash\Sigma)$. The latter condition is equivalent to the following notion called strictly nonzero on compact sets: for any representative $\det((g_\varepsilon)_{ab})_\varepsilon$ of $\det(g_{ab})$ we have
$\forall K \subset M\backslash\Sigma\ \exists m \in \mathbb{N}[\inf_{p\in K}|\det(g_{ab}(\varepsilon))| \ge \varepsilon^q]$ for all ε small enough. This notion captures the intuitive idea of a generalized metric to be a sequence of classical

metrics approaching a singular limit in the following sense: g_{ab} is a generalized metric iff (on every relatively compact open subset V of M) there exists a representative $((g_\varepsilon)_{ab})_\varepsilon$ of g_{ab} such that for fixed ε (small enough)$(g_\varepsilon)_{ab} = g_{ab}(\varepsilon)$ (resp. $(g_\varepsilon)_{ab}|_V$) is a classical pseudo-Riemannian metric and $\det(g_{ab})$ is invertible in the algebra of generalized functions. A generalized metric induces a $\tilde{\mathcal{G}}(M,\Sigma)$-linear isomorphism from $\tilde{\mathcal{G}}_0^1(M,\Sigma)$ to $\tilde{\mathcal{G}}_1^0(M,\Sigma)$ and the inverse metric $g^{ab} \triangleq [(g_{ab}^{-1}(\varepsilon))_\varepsilon]$ is a well defined element of $\tilde{\mathcal{G}}_0^2(M,\Sigma)$ (i.e., independent of the representative $((g_\varepsilon)_{ab})_\varepsilon$). Also the supergeneralized Levi-Civita connection as well as the supergeneralized Riemann, Ricci and Einstein tensor of a supergeneralized metric are defined simply by the usual coordinate formulae on the level of representatives.

II.3.Superdistributional general relativity

We briefly summarize the basics of superdistributional general relativity, as a preliminary to latter discussion.In the classical theory of gravitation one is led to consider the Einstein field equations which are,in general,quasilinear partial differential equations involving second order derivatives for the metric tensor. Hence, continuity of the first fundamental form is expected and at most, discontinuities in the second fundamental form, the coordinate independent statements appropriate to consider 3-surfaces of discontinuity in the spacetime manifolfd of General Relativity.

In standard general relativity, the space-time is assumed to be a four-dimensional differentiable manifold M endowed with the Lorentzian metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ $(\mu,\nu = 0,1,2,3)$. At each point p of space-time M, the metric can be diagonalized as $ds_p^2 = \eta_{\mu\nu}(dX^\mu)_p(dX^\nu)_p$ with $\eta_{\mu\nu} \triangleq (-1,1,1,1)$, by choosing the coordinate system $\{X^\mu;\mu = 0,1,2,3\}$ appropriately.

In superdistributional general relativity the space-time is assumed to be a four-dimensional differentiable manifold $M\backslash\Sigma$, where $\dim(M) = 4, \dim(\Sigma) \leqslant 3$ endowed with the Lorentzian supergeneralized metric

$$(ds_\epsilon^2)_\epsilon = (g_{\mu\nu}(\epsilon)dx^\mu dx^\nu)_\epsilon \ ;\mu,\nu = 0,1,2,3). \tag{2.27}$$

At each point $p \in M\backslash\Sigma$, the metric can be diagonalized as

$$(ds_p^2(\epsilon))_\epsilon = (\eta_{\mu\nu}(dX_\epsilon^\mu)_p(dX_\epsilon^\nu)_p)_\epsilon \text{ with } \eta_{\mu\nu} \triangleq (-1,1,1,1), \tag{2.28}$$

by choosing the generalized coordinate system $\{(X_\epsilon^\mu)_\epsilon;\mu = 0,1,2,3\}$ appropriately.

The classical smooth curvature tensor is given by

$$R^\rho{}_{\sigma\mu\nu} \triangleq \partial_\mu\left\{\frac{\rho}{\sigma\,\nu}\right\} - \partial_\nu\left\{\frac{\rho}{\sigma\,\mu}\right\} + \left\{\frac{\rho}{\lambda\,\mu}\right\}\left\{\frac{\lambda}{\sigma\,\nu}\right\} - \left\{\frac{\rho}{\lambda\,\nu}\right\}\left\{\frac{\lambda}{\sigma\,\mu}\right\} \tag{2.29}$$

with $\left\{\frac{\rho}{\sigma\,\nu}\right\}$ being the smooth Christoffel symbol.The supergeneralized nonsmooth curvature tensor is given by

$$\begin{cases} (R^{\rho}{}_{\sigma\mu\nu}(\epsilon))_{\epsilon} \triangleq \partial_{\mu}\left(\left\{\frac{\rho}{\sigma\,\nu}\right\}_{\epsilon}\right)_{\epsilon} - \partial_{\nu}\left(\left\{\frac{\rho}{\sigma\,\mu}\right\}_{\epsilon}\right)_{\epsilon} + \left(\left\{\frac{\rho}{\lambda\,\mu}\right\}_{\epsilon}\right)_{\epsilon}\left(\left\{\frac{\lambda}{\sigma\,\nu}\right\}_{\epsilon}\right)_{\epsilon} - \\ -\left(\left\{\frac{\rho}{\lambda\,\nu}\right\}_{\epsilon}\right)_{\epsilon}\left(\left\{\frac{\lambda}{\sigma\,\mu}\right\}_{\epsilon}\right)_{\epsilon} \end{cases} \tag{2.30}$$

with $\left(\left\{\frac{\rho}{\sigma\,\nu}\right\}_{\epsilon}\right)_{\epsilon}$ being the supergeneralized Christoffel symbol.The fundamental classical action integral $\mathbb{I}$ is

$$\mathbb{I} = \frac{1}{c}\int(\bar{\mathbb{L}}_G + \mathbb{L}_M)d^4x, \tag{2.31}$$

where $\mathbb{L}_M$ is the Lagrangian density of a gravitational source and $\bar{\mathbb{L}}_G$ is the gravitational Lagrangian density given by

$$\bar{\mathbb{L}}_G = \frac{1}{2\kappa}\mathbb{G}\ . \tag{2.32}$$

Here κ is the Einstein gravitational constant $\kappa = 8\pi G/c^4$ and $\mathbb{G}$ is defined by

$$\mathbb{G} = \sqrt{-g}\,g^{\mu\nu}\left(\left\{\frac{\lambda}{\mu\,\rho}\right\}\left\{\frac{\rho}{\nu\,\lambda}\right\} - \left\{\frac{\lambda}{\mu\,\nu}\right\}\left\{\frac{\rho}{\lambda\,\rho}\right\}\right) \tag{2.33}$$

with $g = \det(g_{\mu\nu})$. There exists the relation

$$\sqrt{-g}\,R = \mathbb{G} + \partial_{\mu}\mathbb{D}^{\mu}\ , \tag{2.34}$$

with

$$\mathbb{D}^{\mu} = -\sqrt{-g}\left(g^{\mu\nu}\left\{\frac{\lambda}{\nu\,\lambda}\right\} - g^{\nu\lambda}\left\{\frac{\mu}{\nu\,\lambda}\right\}\right)\ . \tag{2.35}$$

Thus the supergeneralized fundamental action integral $(\mathbb{I}_{\epsilon})_{\epsilon}$ is

$$(\mathbb{I}_{\epsilon})_{\epsilon} = \frac{1}{c}\int((\bar{\mathbb{L}}_G(\epsilon))_{\epsilon} + (\mathbb{L}_M(\epsilon))_{\epsilon})d^4x\ , \tag{2.36}$$

where $(\mathbb{L}_M(\epsilon))_{\epsilon}$ is the supergeneralized Lagrangian density of a gravitational source and $(\bar{\mathbb{L}}_G(\epsilon))_{\epsilon}$ is the supergeneralized gravitational Lagrangian density given by

$$(\bar{\mathbb{L}}_G(\epsilon))_{\epsilon} = \frac{1}{2\kappa}(\mathbb{G}_{\epsilon})_{\epsilon}\ . \tag{2.37}$$

Here κ is the Einstein gravitational constant $\kappa = 8\pi G/c^4$ and $(\mathbb{G}_{\epsilon})_{\epsilon}$ is defined by

$$(\mathbb{G}_{\epsilon})_{\epsilon} = \sqrt{-(g_{\epsilon})_{\epsilon}}\,(g^{\mu\nu}_{\epsilon})_{\epsilon}\left(\left(\left\{\frac{\lambda}{\mu\,\rho}\right\}_{\epsilon}\right)\left(\left\{\frac{\rho}{\nu\,\lambda}\right\}_{\epsilon}\right) - \left(\left\{\frac{\lambda}{\mu\,\nu}\right\}_{\epsilon}\right)\left(\left\{\frac{\rho}{\lambda\,\rho}\right\}_{\epsilon}\right)\right) \tag{2.38}$$

with $g_{\epsilon} = \det[(g_{\mu\nu}(\epsilon))_{\epsilon}]$. There exists the relation

$$\sqrt{-(g_{\epsilon})_{\epsilon}}\,(\mathbf{R}_{\epsilon})_{\epsilon} = (\mathbb{G}_{\epsilon})_{\epsilon} + \partial_{\mu}(\mathbb{D}^{\mu}_{\epsilon})_{\epsilon}\ , \tag{2.39}$$

with

$$(\mathbb{D}^{\mu}_{\epsilon})_{\epsilon} = -\sqrt{-(g_{\epsilon})_{\epsilon}}\left((g^{\mu\nu}_{\epsilon})_{\epsilon}\left(\left\{\frac{\lambda}{\nu\,\lambda}\right\}_{\epsilon}\right)_{\epsilon} - (g^{\nu\lambda}_{\epsilon})_{\epsilon}\left(\left\{\frac{\mu}{\nu\,\lambda}\right\}_{\epsilon}\right)_{\epsilon}\right)\ . \tag{2.40}$$

Also, we have defined the classical scalar curvature by

$$\mathbf{R} = R^{\mu}{}_{\mu} \tag{2.41}$$

with the smooth Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \ . \tag{2.42}$$

From the action $\mathbb{I}$, the classical Einstein equation

$$G_{\mu}{}^{\nu} = R_{\mu}{}^{\nu} - \frac{1}{2}\delta_{\mu}{}^{\nu}R = \kappa T_{\mu}{}^{\nu} \ , \tag{2.43}$$

follows, where $T_{\mu}{}^{\nu}$ is defined by

$$T_{\mu}{}^{\nu} = \frac{\widetilde{\mathbf{T}}_{\mu}{}^{\nu}}{\sqrt{-g}} \tag{2.44}$$

with

$$\widetilde{\mathbf{T}}_{\mu}^{\ \nu} \triangleq 2g_{\mu\lambda}\frac{\delta\mathbb{L}_M}{\delta g_{\lambda\nu}} \tag{2.45}$$

being the energy-momentum density of the classical gravity source. Thus we have defined the supergeneralized scalar curvature by

$$(\mathbf{R}_{\epsilon})_{\epsilon} = (R^{\mu}{}_{\mu}(\epsilon))_{\epsilon} \tag{2.46}$$

with the supergeneralized Ricci tensor

$$(\mathbf{R}_{\mu\nu}(\epsilon))_{\epsilon} = (R^{\lambda}{}_{\mu\lambda\nu}(\epsilon))_{\epsilon} \ . \tag{2.47}$$

From the action $(\mathbb{I}_{\epsilon})_{\epsilon}$, the generalized Einstein equation

$$(G_{\mu}{}^{\nu}(\epsilon))_{\epsilon} = (R_{\mu}{}^{\nu}(\epsilon))_{\epsilon} - \frac{1}{2}\delta_{\mu}{}^{\nu}(\mathbf{R}_{\epsilon})_{\epsilon} = \kappa(T_{\mu}{}^{\nu}(\epsilon))_{\epsilon} \ , \tag{2.48}$$

follows, where $(T_{\mu}{}^{\nu}(\epsilon))_{\epsilon}$ is defined by

$$(T_{\mu}{}^{\nu}(\epsilon))_{\epsilon} = \frac{\left(\widetilde{\mathbf{T}}_{\mu}{}^{\nu}(\epsilon)\right)_{\epsilon}}{\sqrt{-(g_{\epsilon})_{\epsilon}}} \tag{2.49}$$

with

$$\left(\widetilde{\mathbf{T}}_{\mu}^{\ \nu}(\epsilon)\right)_{\epsilon} \triangleq 2(g_{\mu\lambda}(\epsilon))_{\epsilon}\frac{\delta(\mathbb{L}_M(\epsilon))_{\epsilon}}{\delta(g_{\lambda\nu}(\epsilon))_{\epsilon}} \tag{2.50}$$

being the supergeneralized energy-momentum density of the supergeneralized gravity source.The classical energy-momentum pseudo-tensor density $\widetilde{\mathbf{t}}_{\mu}^{\ \nu}$ of the gravitational field is defined by

$$\widetilde{\mathbf{t}}_{\mu}^{\ \nu} = \delta_{\mu}{}^{\nu}\bar{\mathbb{L}}_G - \frac{\partial\bar{\mathbb{L}}_G}{\partial g_{\sigma\tau,\nu}}g_{\sigma\tau,\mu} \tag{2.51}$$

with $g_{\sigma\tau,\nu} = \partial g_{\sigma\tau}/\partial x^{\nu}$.The supergeneralized energy-momentum pseudo-tensor density $\widetilde{\mathbf{t}}_{\mu}^{\ \nu}$ of the gravitational field is defined by

$$\left(\widetilde{\mathbf{t}}_{\mu}^{\ \nu}(\epsilon)\right)_{\epsilon} = \delta_{\mu}{}^{\nu}(\bar{\mathbb{L}}_G(\epsilon))_{\epsilon} - \left(\frac{\partial\bar{\mathbb{L}}_G(\epsilon)}{\partial g_{\sigma\tau,\nu}(\epsilon)}\right)_{\epsilon}(g_{\sigma\tau,\mu}(\epsilon))_{\epsilon} \tag{2.52}$$

with $(g_{\sigma\tau,\nu}(\epsilon))_{\epsilon} = (\partial g_{\sigma\tau}(\epsilon)/\partial x^{\nu})_{\epsilon}$.

III.Distributional Schwarzschild Geometry from nonsmooth regularization via Horizon

In this last section we leave the neighborhood of the singularity at the origin and turn to the singularity at the horizon. The question we are aiming at is the following: using distributional geometry (thus without leaving Schwarzschild coordinates), is it possible to show that the horizon singularity of the Schwarzschild metric is not merely only a coordinate singularity. In order to investigate this issue we calculate the distributional curvature at horizon (in Schwarzschild coordinates). In the usual Schwarzschild coordinates $(t, r > 0, \theta, \phi)$ the metric takes the form

$$\begin{cases} ds^2 = h(r)dt^2 - h(r)^{-1}dr^2 + r^2 d\Omega^2, \\ h(r) = -1 + \frac{2m}{r}. \end{cases} \tag{3.1}$$

Following the above discussion we consider the singular metric coefficient $h(r)$ as an element of $\mathcal{D}'(\mathbb{R}^3)$ and embed it into $(\mathcal{G}(\mathbb{R}^3))$ by replacement

$$r - 2m \mapsto \sqrt{(r-2m)^2 + \epsilon^2}\,.$$

Note that, accordingly, we have fixed the differentiable structure of the manifold: the Cartesian coordinates associated with the spherical Schwarzschild coordinates in (3.1) are extended through the origin. We have above $r \geqslant 2m$ (below ($r \leqslant 2m$)) horizon

$$h(r)=\left\{\begin{array}{c}-\frac{r-2m}{r}\text{ if }r\geqslant 2m\\ 0\text{ if }r\leqslant 2m\end{array}\right\}\mapsto (h_\epsilon^+(r))_\epsilon=\left(-\frac{\sqrt{(r-2m)^2+\epsilon^2}}{r}\right)_\epsilon,$$

where $(h_\epsilon^+(r))_\epsilon\in\widetilde{\mathcal{G}}(\mathbb{R}^3,B^+(2m,R)),B^+(2m,R)=\{x\in\mathbb{R}^3|2m\leqslant\|x\|\leqslant R\}.$

$$h^{-1}(r)=\left\{\begin{array}{c}-\frac{r}{r-2m},r>2m\\ \infty,r=2m\end{array}\right\}\mapsto (h_\epsilon^+)^{-1}(r)=$$

$$h^-(r)=\left\{\begin{array}{c}-\frac{r-2m}{r}\text{ if }r\leqslant 2m\\ 0\text{ if }r\geq 2m\end{array}\right\}\mapsto h_\epsilon^-(r)= \tag{3.2}$$

$$=\left(\frac{\sqrt{(2m-r)^2+\epsilon^2}}{r}\right)_\epsilon\in\widetilde{\mathcal{G}}(\mathbb{R}^3,B^-(0,2m)),$$

where $B^-(0,2m)=\{x\in\mathbb{R}^3|0<\|x\|\leqslant 2m\}$

$$\left\{\begin{array}{c}-\frac{r}{r-2m},r<2m\\ \infty,r=2m\end{array}\right\}\mapsto (h_\epsilon^-)^{-1}(r)=$$

$$=\left(\frac{r}{\sqrt{(r-2m)^2+\epsilon^2}}\right)_\epsilon\in\widetilde{\mathcal{G}}(\mathbb{R}^3,B^-(0,2m))$$

Inserting (3.2) into (3.1) we obtain a generalized object modeling the singular Schwarzschild metric above (below) gorizon, i.e.,

$$\left\{\begin{array}{l}(ds_\epsilon^{+2})_\epsilon=(h_\epsilon^+(r)dt^2)_\epsilon-\left([h_\epsilon^+(r)]^{-1}dr^2\right)_\epsilon+r^2d\Omega^2,\\ (ds_\epsilon^{-2})_\epsilon=(h_\epsilon^-(r)dt^2)_\epsilon-\left([h_\epsilon^-(r)]^{-1}dr^2\right)_\epsilon+r^2d\Omega^2\end{array}\right. \tag{3.3}$$

The generalized Ricci tensor above horizon $[\mathbf{R}^+]_\alpha^\beta$ may now be calculated componentwise using the classical formulae

$$\left\{\begin{array}{l}\left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon=\left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon=\frac{1}{2}\left((h_\epsilon^{+\prime\prime})_\epsilon+\frac{2}{r}(h_\epsilon^{+\prime})_\epsilon\right)\\ \left([\mathbf{R}_\epsilon^+]_2^2\right)_\epsilon=\left([\mathbf{R}_\epsilon^+]_3^3\right)_\epsilon=\frac{(h_\epsilon^{+\prime})_\epsilon}{r}+\frac{1+(h_\epsilon^+)_\epsilon}{r^2}.\end{array}\right. \tag{3.4}$$

From (3.2) we obtain

$$h_\epsilon^{+\prime}(r) = -\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2},$$

$$r(h_\epsilon^{+\prime})_\epsilon + 1 + (h_\epsilon^{+})_\epsilon =$$

$$r\left\{-\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right\} + 1 - \frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} =$$

$$-\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} + 1 - \frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} =$$

$$-\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + 1.$$

$$h_\epsilon^{+\prime\prime}(r) = -\left(\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}}\right)' + \left(\frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right)' =$$

$$= -\frac{1}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{(r-2m)^2}{r\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +$$

$$+\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}.$$

$$r^2(h_\epsilon^{+\prime\prime})_\epsilon + 2r(h_\epsilon^{+\prime})_\epsilon = \tag{3.5}$$

$$r^2\left\{-\frac{1}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{(r-2m)^2}{r\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +\right.$$

$$\left.+\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}\right\} +$$

$$+2r\left\{-\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right\} =$$

$$-\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +$$

$$+\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} +$$

$$-\frac{2(r-2m)}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} =$$

$$-\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}.$$

Investigating the weak limit of the angular components of the Ricci tensor (using the abbreviation

$$\tilde{\Phi}(r) = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \Phi(\vec{x})$$

and let $\Phi(\vec{x})$ be the function $\Phi(\vec{x}) \in \mathcal{S}_{2m}(\mathbb{R}^3, k),$ where by $\mathcal{S}_{2m}(\mathbb{R}^3, k)$ we denote the class of all functions $\Phi(\vec{x})$ such that (i) $\tilde{\Phi}(r) \sim (r-2m)^k,\ r \sim 2m, k \geqslant 2$ and (ii) $\Phi(\vec{x}) \in C^{\infty}(\mathbb{R}^3).$ Then for any function $\Phi(\vec{x}) \in \mathcal{S}_{2m}(\mathbb{R}^3, 2)$ with compact support we get:

$$\begin{gathered}\int_K \left([\mathbf{R}_{\epsilon}^{+}]_2^2\right)_{\epsilon} \Phi(\vec{x}) d^3x = \int_K \left([\mathbf{R}_{\epsilon}^{+}]_3^3\right)_{\epsilon} \Phi(\vec{x}) d^3x = \\ \int_{2m}^{R} (r(h_{\epsilon}^{+\prime})_{\epsilon} + 1 + (h_{\epsilon}^{+})_{\epsilon}) \tilde{\Phi}(r) dr = \\ \int_{2m}^{R} \left\{ -\frac{r-2m}{\left[(r-2m)^2 + \epsilon^2\right]^{1/2}} \right\} \tilde{\Phi}(r) dr + \int_{2m}^{R} \tilde{\Phi}(r) dr.\end{gathered} \tag{3.6}$$

By replacement $r - 2m = u,$ from (3.6) we obtain

$$\int_K \left([\mathbf{R}_{\epsilon}^{+}]_2^2\right)_{\epsilon} \Phi(x) d^3x = \int_K \left([\mathbf{R}_{\epsilon}^{+}]_3^3\right)_{\epsilon} \Phi(x) d^3x = -\int_0^{R-2m} \frac{u\tilde{\Phi}(u+2m)du}{(u^2+\epsilon^2)^{1/2}} + \int_0^{R-2m} \tilde{\Phi}(u+2m)du. \tag{3.7}$$

By replacement $u = \epsilon\eta,$ from (3.7) we obtain the expression

$$\begin{cases} \mathbf{I}_3^{+}(\epsilon) = \int_K \left([\mathbf{R}_{\epsilon}^{+}]_3^3\right)_{\epsilon} \Phi(x) d^3x = \mathbf{I}_2^{+}(\epsilon) = \int_K \left([\mathbf{R}_{\epsilon}^{+}]_2^2\right)_{\epsilon} \Phi(\vec{x}) d^3x = \\ -\epsilon \times \left(\int_0^{\frac{R-2m}{\epsilon}} \frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} - \int_0^{\frac{R-2m}{\epsilon}} \tilde{\Phi}(\epsilon\eta+2m)d\eta \right). \end{cases} \tag{3.8}$$

From Eq.(3.8) we obtain

$$\begin{gathered}\mathbf{I}_3^{+}(\epsilon) = \mathbf{I}_2^{+}(\epsilon) = -\epsilon\frac{\tilde{\Phi}(2m)}{0!} \int_0^{\frac{R-2m}{\epsilon}} \left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1 \right] d\eta + \\ -\frac{\epsilon^2}{1!} \int_0^{\frac{R-2m}{\epsilon}} \left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1 \right] \tilde{\Phi}^{(1)}(\xi)\eta d\eta = \\ -\epsilon\tilde{\Phi}(2m) \left[\sqrt{\left(\frac{R-2m}{\epsilon}\right)^2 + 1} - 1 - \frac{R-2m}{\epsilon} \right] + \\ -\frac{\epsilon^2}{1} \int_0^{\frac{R-2m}{\epsilon}} \left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1 \right] \tilde{\Phi}^{(1)}(\xi)\eta d\eta,\end{gathered} \tag{3.9}$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta + 2m)$ as

$$\begin{cases} \widetilde{\Phi}(\epsilon\eta+2m)=\sum_{l=0}^{n-1}\dfrac{\Phi^{(l)}(2m)}{l!}(\epsilon\eta)^l+\dfrac{1}{n!}(\epsilon\eta)^n\Phi^{(n)}(\xi)\ , \\ \xi\triangleq\theta\epsilon\eta+2m\ ,\quad 1>\theta>0\ ,\quad n=1 \end{cases} \tag{3.10}$$

with $\tilde{\Phi}^{(l)}(\xi)\triangleq d^l\tilde{\Phi}/d\xi^l$. Equations (3.9)-(3.10) gives

$$\begin{cases} \lim\limits_{\epsilon\to0}\mathbf{I}_3^+(\epsilon)=\lim\limits_{\epsilon\to0}\mathbf{I}_2^+(\epsilon)= \\ \lim\limits_{\epsilon\to0}\left\{-\epsilon\tilde{\Phi}(2m)\left[\sqrt{\left(\dfrac{R-2m}{\epsilon}\right)^2+1}-1-\dfrac{R-2m}{\epsilon}\right]\right\}+ \\ +\lim\limits_{\epsilon\to0}\left\{-\dfrac{\epsilon^2}{1}\int\limits_0^{\frac{R-2m}{\epsilon}}\left[\dfrac{\eta}{(\eta^2+1)^{1/2}}-1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta\right\}=0 \end{cases} \tag{3.11}$$

Thus in $\mathcal{S}'_{2m}(B_R^+(2m),k)\subset\mathcal{S}'_{2m}(\mathbb{R}^3,k)\subset\mathcal{D}'(\mathbb{R}^3)$, where $B^+(2m,R)=\{x\in\mathbb{R}^3|2m\leqslant\|x\|\leqslant R\}$ from Eq.(3.11) we obtein

$$\begin{cases} w\text{–}\lim\limits_{\epsilon\to0}[\mathbf{R}_\epsilon^+]_3^3=\lim\limits_{\epsilon\to0}\mathbf{I}_3^+(\epsilon)=0, \\ w\text{–}\lim\limits_{\epsilon\to0}[\mathbf{R}_\epsilon^+]_2^2=\lim\limits_{\epsilon\to0}\mathbf{I}_2^+(\epsilon)=0. \end{cases} \tag{3.12}$$

For $\left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon,\left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon$ we get:

$$\begin{cases} 2\int_K\left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon\Phi(x)d^3x=2\int_K\left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon\Phi(x)d^3x= \\ \int\limits_{2m}^R(r^2(h_\epsilon^{+\prime\prime})_\epsilon+2r(h_\epsilon^{+\prime})_\epsilon)\tilde{\Phi}(r)dr= \\ =\int\limits_{2m}^R\left\{-\dfrac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}+\dfrac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}\right\}\tilde{\Phi}(r)dr. \end{cases} \tag{3.13}$$

By replacement $r-2m=u,$ from (3.13) we obtain

$$\begin{cases} \mathbf{I}_1^+(\epsilon)=2\int_K\left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon\Phi(x)d^3x=I_2^+(\epsilon)=2\int_K\left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon\Phi(x)d^3x \\ =\int\limits_{2m}^R(r^2(h_\epsilon^{+\prime\prime})_\epsilon+2r(h_\epsilon^{+\prime})_\epsilon)\tilde{\Phi}(r)dr= \\ =\int\limits_0^{R-2m}\left\{-\dfrac{u+2m}{(u^2+\epsilon^2)^{1/2}}+\dfrac{u^2(u+2m)}{(u^2+\epsilon^2)^{3/2}}\right\}\tilde{\Phi}(u+2m)du. \end{cases} \tag{3.14}$$

By replacement $u=\epsilon\eta,$ from (3.14) we obtain

$$2\int_K \left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon \Phi(x)d^3x = 2\int_K \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon \Phi(x)d^3x =$$

$$= \int_{2m}^{R} (r^2(h_\epsilon^{+\prime\prime})_\epsilon + 2r(h_\epsilon^{+\prime})_\epsilon)\tilde{\Phi}(r)dr =$$

$$= \epsilon \int_0^{\frac{R-2m}{\epsilon}} \left\{ -\frac{\epsilon\eta + 2m}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} + \frac{\epsilon^2\eta^2(\epsilon\eta+2m)}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} \right\} \tilde{\Phi}(\epsilon\eta+2m)d\eta =$$

$$- \int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon^2\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} - 2m \int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} +$$

$$\int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon^4\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} + 2m \int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon^3\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} = \tag{3.15}$$

$$\epsilon\left[- \int_0^{\frac{R-2m}{\epsilon}} \frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} + \int_0^{\frac{R-2m}{\epsilon}} \frac{\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}} \right] +$$

$$2m\left[- \int_0^{\frac{R-2m}{\epsilon}} \frac{\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} + \int_0^{\frac{R-2m}{\epsilon}} \frac{\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}} \right].$$

From Eq.(3.15) we obtain

$$\mathbf{I}_0^+(\epsilon) = \mathbf{I}_1^+(\epsilon) = 2m\frac{\tilde{\Phi}(2m)}{0!} \int_0^{\frac{R-2m}{\epsilon}} \left[-\frac{1}{(\eta^2+1)^{1/2}} + \frac{\eta^2}{(\eta^2+1)^{3/2}} \right] d\eta +$$

$$+\frac{\epsilon}{1!} \int_0^{\frac{R-2m}{\epsilon}} \tilde{\Phi}^{(1)}(\xi)\left[-\frac{1}{(\eta^2+1)^{1/2}} + \frac{\eta^2}{(\eta^2+1)^{3/2}} \right]\eta d\eta +$$

$$+\frac{\epsilon\tilde{\Phi}(2m)}{0!} \int_{\frac{-2m}{\epsilon}}^{\frac{R-2m}{\epsilon}} \left[-\frac{1}{(\eta^2+1)^{1/2}} + \frac{\eta^2}{(\eta^2+1)^{3/2}} \right] d\eta + \tag{3.16}$$

$$+\frac{\epsilon^2}{1!} \int_0^{\frac{R-2m}{\epsilon}} \tilde{\Phi}^{(1)}(\xi)\left[-\frac{1}{(\eta^2+1)^{1/2}} + \frac{\eta^2}{(\eta^2+1)^{3/2}} \right]\eta d\eta,$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta+2m)$ as

$$\widetilde{\Phi}(\epsilon\eta+2m) = \sum_{l=0}^{n-1} \frac{\Phi^{\alpha\beta(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n\Phi^{\alpha\beta(n)}(\xi) \ , \tag{3.17}$$

$$\xi \triangleq \theta\epsilon\eta + 2m \ , \quad 1 > \theta > 0 \ , \quad n = 1$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l\tilde{\Phi}/d\xi^l$.Equation (3.17) gives

$$w\text{-}\lim_{\epsilon\to 0}\mathbf{I}_0^+(\epsilon)=\ w\text{-}\lim_{\epsilon\to 0}\mathbf{I}_1^+(\epsilon)=$$
$$2m\tilde{\Phi}(2m)\lim_{\epsilon\to 0}\left\{\int_0^{\frac{R-2m}{\epsilon}}\left[-\frac{1}{(\eta^2+1)^{1/2}}+\frac{\eta^2}{(\eta^2+1)^{3/2}}\right]d\eta\right\}= \tag{3.18}$$
$$2m\tilde{\Phi}(2m)\lim_{s\to\infty}\left[\int_0^s\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}-\int_0^s\frac{d\eta}{(\eta^2+1)^{1/2}}\right]=$$
$$=-2m\tilde{\Phi}(2m).$$

where use is made of the relation

$$\lim_{s\to\infty}\left[\int_0^s\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}-\int_0^s\frac{d\eta}{(u^2+1)^{1/2}}\right]=-1 \tag{3.19}$$

Thus in $\mathcal{S}'_{2m}(B^+(2m,R),k)\subset\mathcal{S}'_{2m}(\mathbb{R}^3,k)$ we obtain

$$w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_1^1=w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_0^0=-m\tilde{\Phi}(2m). \tag{3.20}$$

The supergeneralized Ricci tensor below horizon $[\mathbf{R}_\epsilon^-]_\alpha^\beta=[\mathbf{R}_\epsilon^-]_\alpha^\beta$ may now be calculated componentwise using the classical formulae

$$\begin{cases}\left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon=\left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon=\frac{1}{2}\left((h_\epsilon^{-\prime\prime})_\epsilon+\frac{2}{r}(h_\epsilon^{-\prime})_\epsilon\right),\\ \left([\mathbf{R}_\epsilon^-]_2^2\right)_\epsilon=\left([\mathbf{R}_\epsilon^-]_3^3\right)_\epsilon=\frac{(h_\epsilon^{-\prime})_\epsilon}{r}+\frac{1+(h_\epsilon^-)_\epsilon}{r^2}.\end{cases} \tag{3.21}$$

From (3.2) we obtain

$$h_\epsilon^-(r)=-\frac{r-2m}{r}\ \mapsto\ h_\epsilon^-(r)=\left(\frac{\sqrt{(2m-r)^2+\epsilon^2}}{r}\right)=-h_\epsilon^+(r),r<2m.$$
$$h_\epsilon^{-\prime}(r)=-h_\epsilon^{+\prime}(r)=\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}}-\frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2},$$
$$r(h_\epsilon^{-\prime})_\epsilon+1+(h_\epsilon^-)_\epsilon=-r(h_\epsilon^{+\prime})_\epsilon+1-(h_\epsilon^+)_\epsilon=$$
$$\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}+1. \tag{3.22}$$
$$h_\epsilon^{-\prime\prime}(r)=-h_\epsilon^{+\prime\prime}(r)=$$
$$-\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}+\frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}.$$
$$r^2(h_\epsilon^{-\prime\prime})_\epsilon+2r(h_\epsilon^{-\prime})_\epsilon=-r^2(h_\epsilon^{+\prime\prime})_\epsilon-2r(h_\epsilon^{+\prime})_\epsilon=$$
$$\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}-\frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}.$$

Investigating the weak limit of the angular components of the Ricci tensor (using

the abbreviation $\tilde{\Phi}(r) = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \Phi(\vec{x})$ where

$\Phi(\vec{x}) \in \mathcal{S}_{2m}(\mathbb{R}^3, k)$, i.e. $\tilde{\Phi}(r) \sim (r-2m)^k$, $k \geqslant 2$ and $\Phi(\vec{x})$ is a function with compact support K of the class $C^{\infty}(\mathbb{R}^3)$, such that $K \subseteq B^{-}(0,2m) = \{x \in \mathbb{R}^3 | 0 \leqslant \|x\| \leqslant 2m\}$ we get:

$$\begin{aligned} &\int_K \left([\mathbf{R}_{\epsilon}^{-}]_2^2\right)_{\epsilon} \Phi(\vec{x}) d^3x = \int_K \left([\mathbf{R}_{\epsilon}^{-}]_3^3\right)_{\epsilon} \Phi(\vec{x}) d^3x = \\ &\int_0^{2m} (r(h_{\epsilon}^{-\prime})_{\epsilon} + 1 + (h_{\epsilon}^{-})_{\epsilon}) \tilde{\Phi}(r) dr = \\ &\int_0^{2m} \left\{ \frac{r-2m}{\left[(r-2m)^2 + \epsilon^2\right]^{1/2}} \right\} \tilde{\Phi}(r) dr + \int_0^{2m} \tilde{\Phi}(r) dr. \end{aligned} \tag{3.23}$$

By replacement $r - 2m = u$, from Eq.(3.23) we obtain

$$\begin{cases} \int_K \left([\mathbf{R}_{\epsilon}^{-}]_2^2\right)_{\epsilon} \Phi(x) d^3x = \int_K \left([\mathbf{R}_{\epsilon}^{-}]_3^3\right)_{\epsilon} \Phi(x) d^3x = \\ \int_{-2m}^{0} \frac{u\tilde{\Phi}(u+2m)du}{(u^2+\epsilon^2)^{1/2}} + \int_{-2m}^{0} \tilde{\Phi}(u+2m)du. \end{cases} \tag{3.24}$$

By replacement $u = \epsilon\eta$, from (3.23) we obtain

$$\begin{aligned} &\mathbf{I}_3^{-}(\epsilon) = \int_K \left([\mathbf{R}_{\epsilon}^{-}]_3^3\right)_{\epsilon} \Phi(x) d^3x = \mathbf{I}_2^{-}(\epsilon) = \int_K \left([\mathbf{R}_{\epsilon}^{-}]_2^2\right)_{\epsilon} \Phi(\vec{x}) d^3x = \\ &\epsilon \times \left(\int_{-\frac{2m}{\epsilon}}^{0} \frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} + \int_{-\frac{2m}{\epsilon}}^{0} \tilde{\Phi}(\epsilon\eta+2m)d\eta \right), \end{aligned} \tag{3.25}$$

which is calculated to give

$$\begin{aligned} &\mathbf{I}_3^{-}(\epsilon) = \mathbf{I}_2^{-}(\epsilon) = \epsilon \frac{\tilde{\Phi}(2m)}{0!} \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{\eta}{(\eta^2+1)^{1/2}} + 1 \right] d\eta + \\ &+ \frac{\epsilon^2}{1!} \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{\eta}{(\eta^2+1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta = \\ &\epsilon\tilde{\Phi}(2m) \left[1 - \sqrt{\left(\frac{2m}{\epsilon}\right)^2 + 1} + \frac{2m}{\epsilon} \right] + \\ &+ \frac{\epsilon^2}{1} \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{\eta}{(\eta^2+1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta, \end{aligned} \tag{3.26}$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta + 2m)$ as

$$\begin{cases} \widetilde{\Phi}(\epsilon\eta+2m)=\sum_{l=0}^{n-1}\dfrac{\Phi^{(l)}(2m)}{l!}(\epsilon\eta)^l+\dfrac{1}{n!}(\epsilon\eta)^n\Phi^{(n)}(\xi)\ , \\ \xi\triangleq\theta\epsilon\eta+2m\ ,\quad 1>\theta>0\ ,\quad n=1 \end{cases} \tag{3.27}$$

with $\tilde{\Phi}^{(l)}\triangleq d^l\tilde{\Phi}/dr^l$. Equation (3.27) gives

$$\begin{cases} \lim\limits_{\epsilon\to 0}\mathbf{I}_3^-(\epsilon)=\lim\limits_{\epsilon\to 0}\mathbf{I}_2^-(\epsilon)= \\ \lim\limits_{\epsilon\to 0}\left\{\epsilon\tilde{\Phi}(2m)\left[1-\sqrt{\left(\frac{2m}{\epsilon}\right)^2+1}+\frac{2m}{\epsilon}\right]\right\}+ \\ +\lim\limits_{\epsilon\to 0}\left\{\dfrac{\epsilon^2}{2}\int\limits_{-\frac{2m}{\epsilon}}^{0}\left[\dfrac{\eta}{(\eta^2+1)^{1/2}}+1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta\right\}=0. \end{cases} \tag{3.28}$$

Thus in $\mathcal{S}'_{2m}(B_R^-(2m),k)\subset\mathcal{S}'_{2m}(\mathbb{R}^3,k)$, where $B^-(0,2m)=\{x\in\mathbb{R}^3|0\leqslant\|x\|\leqslant 2m\}$ from Eq.(3.28) we obtain

$$\begin{cases} w\text{–}\lim\limits_{\epsilon\to 0}\left[\mathbf{R}_\epsilon^-\right]_3^3=\lim\limits_{\epsilon\to 0}\mathbf{I}_3^-(\epsilon)=0. \\ w\text{–}\lim\limits_{\epsilon\to 0}\left[\mathbf{R}_\epsilon^-\right]_2^2=\lim\limits_{\epsilon\to 0}\mathbf{I}_2^-(\epsilon)=0. \end{cases} \tag{3.29}$$

For $\left(\left[\mathbf{R}_\epsilon^-\right]_1^1\right)_\epsilon,\left(\left[\mathbf{R}_\epsilon^-\right]_0^0\right)_\epsilon$ we get:

$$\begin{gathered} 2\int_K\left(\left[\mathbf{R}_\epsilon^-\right]_1^1\right)_\epsilon\Phi(x)d^3x=2\int_K\left(\left[\mathbf{R}_\epsilon^-\right]_0^0\right)_\epsilon\Phi(x)d^3x= \\ \int\limits_0^{2m}(r^2(h_\epsilon^{-\prime\prime})_\epsilon+2r(h_\epsilon^{-\prime})_\epsilon)\tilde{\Phi}(r)dr= \\ =\int\limits_0^{2m}\left\{\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}-\frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}\right\}\tilde{\Phi}(r)dr. \end{gathered} \tag{3.30}$$

By replacement $r-2m=u,$ from (3.30) we obtain

$$\begin{gathered} I_1^+(\epsilon)=2\int\left(\left[\mathbf{R}_\epsilon^-\right]_1^1\right)_\epsilon\Phi(x)d^3x=I_2^+(\epsilon)=2\int\left(\left[\mathbf{R}_\epsilon^-\right]_0^0\right)_\epsilon\Phi(x)d^3x \\ =\int\limits_0^{2m}(r^2(h_\epsilon^{-\prime\prime})_\epsilon+2r(h_\epsilon^{-\prime})_\epsilon)\tilde{\Phi}(r)dr= \\ =\int\limits_{-2m}^{0}\left\{\frac{u+2m}{(u^2+\epsilon^2)^{1/2}}-\frac{u^2(u+2m)}{(u^2+\epsilon^2)^{3/2}}\right\}\tilde{\Phi}(u+2m)du. \end{gathered} \tag{3.31}$$

By replacement $u=\epsilon\eta,$ from (3.31) we obtain

$$2\int_K \left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon \Phi(x)d^3x = 2\int_K \left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon \Phi(x)d^3x =$$

$$\int_{-\frac{2m}{\epsilon}}^{0} (r^2(h_\epsilon^{-\prime\prime})_\epsilon + 2r(h_\epsilon^{-\prime})_\epsilon)\tilde{\Phi}(r)dr =$$

$$= \epsilon \int_{-\frac{2m}{\epsilon}}^{0} \left\{ \frac{\epsilon\eta + 2m}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} - \frac{\epsilon^2\eta^2(\epsilon\eta+2m)}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} \right\} \tilde{\Phi}(\epsilon\eta+2m)d\eta =$$

$$\int_{-\frac{2m}{\epsilon}}^{0} \frac{\epsilon^2\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} + 2m\int_{-\frac{2m}{\epsilon}}^{0} \frac{\epsilon\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} -$$

$$- \int_{-\frac{2m}{\epsilon}}^{0} \frac{\epsilon^4\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} - 2m\int_{-\frac{2m}{\epsilon}}^{0} \frac{\epsilon^3\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} = \tag{3.32}$$

$$\epsilon \int_{-\frac{2m}{\epsilon}}^{0} \frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} - \int_{-\frac{2m}{\epsilon}}^{0} \frac{\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}} +$$

$$+2m\left[\int_{-\frac{2m}{\epsilon}}^{0} \frac{\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} - \int_{-\frac{2m}{\epsilon}}^{0} \frac{\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}} \right].$$

which is calculated to give

$$\mathbf{I}_{\bar{0}}^-(\epsilon) = \mathbf{I}_{\bar{1}}^-(\epsilon) = 2m\frac{\tilde{\Phi}(2m)}{0!}\epsilon^l \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}} \right] d\eta +$$

$$+\frac{\epsilon}{1!}\int_0^{\frac{2m}{\epsilon}} \tilde{\Phi}^{(1)}(\xi)\left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}} \right]\eta d\eta + O(\epsilon^2). \tag{3.33}$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta+2m)$ as

$$\widetilde{\Phi}(\epsilon\eta+2m) = \sum_{l=0}^{n-1} \frac{\Phi^{\alpha\beta(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n\Phi^{\alpha\beta(n)}(\xi) \ ,$$
$$\xi \triangleq \theta\epsilon\eta + 2m \ , \quad 1 > \theta > 0 \ , \quad n = 1 \tag{3.34}$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l\tilde{\Phi}/d\xi^l$.Equation (3.34) gives

$$\lim_{\epsilon\to 0}\mathbf{I}_0^-(\epsilon) = \lim_{\epsilon\to 0}\mathbf{I}_1^-(\epsilon) =$$

$$2m\lim_{\epsilon\to 0}\left\{\frac{\tilde{\Phi}(2m)}{0!}\int_{-\frac{2m}{\epsilon}}^{0}\left[\frac{1}{(\eta^2+1)^{1/2}}-\frac{\eta^2}{(\eta^2+1)^{3/2}}\right]d\eta\right\} = \tag{3.35}$$

$$2m\tilde{\Phi}(2m)\lim_{s\to 0}\left[\int_{-s}^{0}\frac{d\eta}{(\eta^2+1)^{1/2}}-\int_{-s}^{0}\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}\right] =$$

$$= 2m\tilde{\Phi}(2m).$$

where use is made of the relation

$$\lim_{s\to\infty}\left[\int_{-s}^{0}\frac{d\eta}{(u^2+1)^{1/2}}-\int_{-s}^{0}\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}\right] = 1. \tag{3.36}$$

Thus in $\mathcal{S}'_{2m}(B^-(0,2m),k)\subset\mathcal{S}'_{2m}(\mathbb{R}^3,k)$ we obtain

$$w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_1^1 = w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_0^0 = m\tilde{\Phi}(2m). \tag{3.37}$$

Using Egs. (3.12),(3.20),(3.29),(3.37) we obtain

$$\int\left[\left(\mathbf{T}_r^{+r}+\mathbf{T}_\theta^{+\theta}+\mathbf{T}_\phi^{+\phi}+\mathbf{T}_t^{+t}\right)+\left(\mathbf{T}_r^{-r}+\mathbf{T}_\theta^{-\theta}+\mathbf{T}_\phi^{-\phi}+\mathbf{T}_t^{-t}\right)\right]\sqrt{-g}\,d^3x = 0 \tag{3.38}$$

Thus the Tolman formula [3],[4] for the total energy of a static and asymptotically flat spacetime with g the determinant of the four dimensional metric and d^3x the coordinate volume element, gives

$$E_T = \int\left(\mathbf{T}_r^r+\mathbf{T}_\theta^\theta+\mathbf{T}_\phi^\phi+\mathbf{T}_t^t\right)\sqrt{-g}\,d^3x = m, \tag{3.39}$$

IV. Quantum scalar field in curved distributional spacetime

4.1 Canonical quantization in curved distributional spacetime

Much of formalism can be explained with Colombeau generalized scalar field.The basic concepts and methods extend straightforwardly to distributional tensor and distributional spinor fields. To being with let's take a spacetime of arbitrary dimension D, with a metric gμv of signature$(+-\ldots-)$. The action for the Colombeau generalized scalar field $(\varphi_\varepsilon)_\varepsilon\in\mathcal{G}(M)$ is

$$(S_\varepsilon)_\varepsilon = \left(\int_M d^Dx\frac{1}{2}\sqrt{|g_\varepsilon|}\,(g_\varepsilon^{\mu\nu}\partial_\mu\varphi_\varepsilon\partial_\nu\varphi_\varepsilon)-(m^2+R_\varepsilon)\varphi_\varepsilon^2\right)_\varepsilon. \tag{4.1}$$

The corresponding equation of motion is

$$([\Box_\varepsilon + m^2 + \xi R_\varepsilon]\varphi_\varepsilon)_\varepsilon, \varepsilon \in (0,1]. \tag{4.2}$$

Here

$$(\Box_\varepsilon \varphi_\varepsilon)_\varepsilon = \left(|g_\varepsilon|^{-1/2}\partial_\mu |g_\varepsilon|^{1/2} g_\varepsilon^{\mu\nu}\partial_\mu \varphi_\varepsilon\right)_\varepsilon. \tag{4.3}$$

With $\hbar$ explicit, the mass m should be replaced by $m/\hbar$. Separating out a time coordinate x^0, $x^\mu = (x^0, x^i), i = 1,2,3$ we can write the action as

$$(S_\varepsilon)_\varepsilon = \left(\int dx^0 L_\varepsilon\right)_\varepsilon, (L_\varepsilon)_\varepsilon = \left(\int d^{D-1}x \mathcal{L}_\varepsilon\right)_\varepsilon. \tag{4.4}$$

The canonical momentum at a time x^0 is given by

$$(\pi_\varepsilon(\underline{x}))_\varepsilon = (\delta L_\varepsilon/\delta(\partial_0 \varphi_\varepsilon(\underline{x})))_\varepsilon = (|h_\varepsilon|^{1/2} n^\mu \partial_\mu \varphi_\varepsilon(\underline{x}))_\varepsilon, \tag{4.5}$$

where $\underline{x}$ labels a point on a surface of constant x^0, the x^0 argument of $(\varphi_\varepsilon)_\varepsilon$ is suppressed, n^μ is the unit normal to the surface, and $(|h_\varepsilon|)_\varepsilon$ is the determinant of the induced spatial metric $(h_{ij}(\varepsilon))_\varepsilon$. To quantize, the Colombeau generalized field $(\varphi_\varepsilon)_\varepsilon$ and its conjugate momentum $(\pi_\varepsilon(\underline{x}))_\varepsilon$ are now promoted to hermitian operators and required to satisfy the canonical commutation relation,

$$([\varphi_\varepsilon(\underline{x}), \pi_\varepsilon(\underline{y})])_\varepsilon = i\hbar\delta^{D-1}(\underline{x},\underline{y}), \varepsilon \in (0,1]. \tag{4.6}$$

Here $\int d^{D-1}y \delta^{D-1}(\underline{x},\underline{y}) f(\underline{y}) = f(\underline{x})$ for any scalar function $f \in D(\mathbb{R}^3)$, without the use of a metric volume element. We form now a conserved bracket from two complex Colombeau solutions to the scalar wave equation (4.2) by

$$(\langle \varphi_\varepsilon, \phi_\varepsilon\rangle)_\varepsilon = \left(\int_\Sigma d\Sigma_\mu j_\varepsilon^\mu\right)_\varepsilon, \varepsilon \in (0,1]. \tag{4.7}$$

where

$$(j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon = (i/\hbar)\left(|g_\varepsilon|^{1/2} g_\varepsilon^{\mu\nu}(\overline{\varphi}_\varepsilon \partial_\nu \phi_\varepsilon - \varphi_\varepsilon \partial_\nu \overline{\phi}_\varepsilon)\right)_\varepsilon. \tag{4.8}$$

This bracket is called the generalized Klein-Gordon inner product, and $(\langle \phi_\varepsilon, \phi_\varepsilon\rangle)_\varepsilon$ the generalized Klein Gordon norm of $(\varphi_\varepsilon)_\varepsilon$. The generalized current density $(j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon$ is divergenceless,i.e. $(\partial_\mu j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon = 0$ when the Colombeau generalized functions $(\varphi_\varepsilon)_\varepsilon$ and $(\phi_\varepsilon)_\varepsilon$ satisfy the KG equation (4.2), hence the value of the integral in (4.7) is independent of the spacelike surface Σ over which it is evaluated, provided the functions vanish at spatial infinity. The generalized KG inner product satisfies the relations

$$\left(\overline{\langle \varphi_\varepsilon, \phi_\varepsilon\rangle}\right)_\varepsilon = -(\langle \overline{\varphi}_\varepsilon, \overline{\phi}_\varepsilon\rangle)_\varepsilon = (\langle \phi_\varepsilon, \varphi_\varepsilon\rangle)_\varepsilon, \varepsilon \in (0,1]. \tag{4.9}$$

We define now the annihilation operator associated with a complex Colombeau solution $(\phi_\varepsilon)_\varepsilon$ by the bracket of $(\phi_\varepsilon)_\varepsilon$ with the generalized field operator $(\varphi_\varepsilon)_\varepsilon$:

$$(a(\phi_\varepsilon))_\varepsilon = (\langle \phi_\varepsilon, \varphi_\varepsilon\rangle)_\varepsilon. \tag{4.10}$$

It follows from the hermiticity of $(\varphi_\varepsilon)_\varepsilon$ that the hermitian conjugate of $(a(\phi_\varepsilon))_\varepsilon$ is given by

$$(a^{\dagger}(\phi_{\varepsilon}))_{\varepsilon} = -(a(\overline{\phi}_{\varepsilon}))_{\varepsilon}. \tag{4.11}$$

From Eq.(4.5) and CCR (4.6) one obtain

$$([a(\varphi_{\varepsilon}), a^{\dagger}(\phi_{\varepsilon})])_{\varepsilon} = (\langle \varphi_{\varepsilon}, \phi_{\varepsilon} \rangle)_{\varepsilon}. \tag{4.12}$$

Note that from Eq.(4.11) follows

$$([a(\varphi_{\varepsilon}), a(\phi_{\varepsilon})])_{\varepsilon} = -(\langle \varphi_{\varepsilon}, \overline{\phi}_{\varepsilon} \rangle)_{\varepsilon}, ([a^{\dagger}(\varphi_{\varepsilon}), a^{\dagger}(\phi_{\varepsilon})])_{\varepsilon} = -(\langle \overline{\varphi}_{\varepsilon}, \phi_{\varepsilon} \rangle) \tag{4.13}$$

Note that if $(\phi_{\varepsilon})_{\varepsilon}$ is a positive norm solution with unit norm hf with, then $(a(\varphi_{\varepsilon}))_{\varepsilon}$ and $a^{\dagger}(\varphi_{\varepsilon})$ satisfy the commutation relation $([a^{\dagger}(\varphi_{\varepsilon}), a(\phi_{\varepsilon})])_{\varepsilon} = 1.$ Suppose now that $|\Psi\rangle$ is a normalized quantum state satisfying $(a(\phi_{\varepsilon})|\Psi\rangle)_{\varepsilon} = 1,$ then for each n, the state $|n, \Psi\rangle = \left((1/\sqrt{n!})(a(\phi_{\varepsilon}))^{n}|\Psi\rangle\right)_{\varepsilon}$ is a normalized eigenstate of the number operator $(N[(\phi_{\varepsilon})])_{\varepsilon} = (a^{\dagger}(\varphi_{\varepsilon})a(\varphi_{\varepsilon}))_{\varepsilon}$ with eigenvalue $n.$ The span of all these states defines a Fock space of the distributional $(\varphi_{\varepsilon})_{\varepsilon}$- wavepacket "$n$-particle excitations" above the state $|\Psi\rangle.$ If we want to construct the full Hilbert space of the field theory in curved distributional spacetime,how can we proceed? We should find a decomposition of the space of complex Colombeau solutions to the wave equation (4.2) $\mathbf{S}$ into a direct sum of a positive norm subspace $\mathbf{S_p}$ and its complex conjugate $\mathbf{S_p},$ such that all brackets between solutions from the two subspaces vanish. That is, we must find a direct sum decomposition:

$$\mathbf{S} = \mathbf{S_p} \oplus \overline{\mathbf{S_p}} \tag{4.14}$$

such that

$$(\langle \varphi_{\varepsilon}, \varphi_{\varepsilon} \rangle)_{\varepsilon} > 0, \forall (\varphi_{\varepsilon})_{\varepsilon} \in \mathbf{S_p} \tag{4.15}$$

and

$$(\langle \varphi_{\varepsilon}, \phi_{\varepsilon} \rangle)_{\varepsilon} > 0, \forall (\varphi_{\varepsilon})_{\varepsilon}, (\phi_{\varepsilon})_{\varepsilon} \in \mathbf{S_p}. \tag{4.16}$$

The condition (4.15) implies that each $(\varphi_{\varepsilon})_{\varepsilon}$ in $\mathbf{S_p}$ can be scaled to define its own harmonic oscillator sub-albegra. The second condition implies, according to (4.13), that the annihilators and creators for $(\varphi_{\varepsilon})_{\varepsilon}$ and $(\phi_{\varepsilon})_{\varepsilon}$ in the subspace $\mathbf{S_p}$ commute amongst themselves:

$$([a(\varphi_{\varepsilon}), a(\phi_{\varepsilon})])_{\varepsilon} = ([a^{\dagger}(\varphi_{\varepsilon}), a^{\dagger}(\phi_{\varepsilon})])_{\varepsilon} = 0. \tag{4.17}$$

Given such a decompostion a total Hilbert space $\mathbf{H}$ for the field theory can be defined as the space of finite norm sums of possibly infinitely many states of the form

$$(a^{\dagger}(\phi_{1,\varepsilon}) \ldots a^{\dagger}(\phi_{n,\varepsilon})|0\rangle)_{\varepsilon}, \tag{4.18}$$

where $|0\rangle$ is a state such that $(a(\phi_{n,\varepsilon})|0\rangle)_{\varepsilon} = 0$ for all $(\phi_{\varepsilon})_{\varepsilon}$ in $\mathbf{S_p}.$ The state $|0\rangle,$ as in classical case, is called a Fock vacuum and Hilbert space $\mathbf{H}$ is called a Fock space. The representation of the field operator on this Fock space is hermitian and satisfies the canonical commutation relations in sense of Colombeau generalized function.

4.2 Defining distributional outgoing modes

For illustration we consider the non-rotating,uncharged d-dimensional SAdS BH with a distributional line element

$$(ds^2_\varepsilon)_\varepsilon = \left(-f_\varepsilon dt^2 + f_\varepsilon^{-1} dr^2\right)_\varepsilon + r^2 d\Omega^2_{d-2}, \varepsilon \in (0,1], \tag{4.19}$$

where

$$\begin{cases} f_\varepsilon \neq 0, \varepsilon \in (0,1], \\ f_{\varepsilon=0} = 1 + \dfrac{r^2}{L^2} - \dfrac{r_0^{d-3}}{r^{d-3}}, \end{cases} \tag{4.20}$$

where $d\Omega^2_{d-2}$ is the metric of the $(d-2)$-sphere, and the AdS curvature radius squared L^2 is related to the cosmological constant by $L^2 = -(d-2)(d-1)/2\Lambda.$ The parameter r_0 is proportional to the mass M of the spacetime: $M = (d-2)A_{d-2} r_0^{d-3}/16\pi,$ where $A_{d-2} = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2].$ The distributional Schwarzschild geometry corresponds to $L \to \infty.$The corresponding equation of motion (4.2) for massless case are

$$\begin{gathered} (\nabla_\mu \nabla^\mu \varphi_\varepsilon)_\varepsilon = \frac{(d-2)\gamma}{4(d-1)} (R_\varepsilon)_\varepsilon, \\ (G_{\varepsilon,\mu\nu})_\varepsilon + \Lambda (g_{\varepsilon,\mu\nu})_\varepsilon = 8\pi G (T_{\varepsilon,\mu\nu})_\varepsilon, \\ (T_{\varepsilon,\mu\nu})_\varepsilon \sim \delta(x). \end{gathered} \tag{4.21}$$

The time-independence and the spherical symmetry of the metric imply the canonical decomposition

$$(\varphi_\varepsilon(t,r,\theta))_\varepsilon = e^{-i\omega t}\left(\sum \frac{\Psi_{lm,\varepsilon}(r) Y_{lm}(\theta)}{r^{(d-2)/2} + \varepsilon}\right)_\varepsilon, \tag{4.22}$$

where $Y_{lm}(\theta)$ denotes the d-dimensional scalar spherical harmonics, satisfying

$$\Delta\Omega_{d-2} Y_{lm}(\theta) = -l(l+d-3) Y_{lm}(\theta), \tag{4.23}$$

the Laplace-Beltrami operator.Substituting the decomposition (4.22) into Eq. (6) one get a radial wave equation

$$\left(f_\varepsilon^2 \frac{d^2\Psi_{lm,\varepsilon}(r)}{dr^2} + f_\varepsilon f_\varepsilon' \frac{d\Psi_{lm,\varepsilon}(r)}{dr} + (\omega^2 - V_{lm,\varepsilon}(r))\Psi_{lm,\varepsilon}(r)\right)_\varepsilon = 0. \tag{4.24}$$

We define now a “tortoise” distributional coordinate $(r^*_\varepsilon)_\varepsilon = (r^*_\varepsilon(r))_\varepsilon$ by the relation

$$\left(\frac{dr^*_\varepsilon}{dr}\right)_\varepsilon = \left(f_\varepsilon^{-1}(r)\right)_\varepsilon. \tag{4.25}$$

By using a “tortoise” distributional coordinate the Eq.(4.24) can be written in the form of a Schrödinger equation with the potential $V_{lm,\varepsilon}(r)$

$$\left(\frac{d\Psi_\varepsilon(r^*_\varepsilon)}{dr^*_\varepsilon}\right)_\varepsilon + ((\omega^2 - V_\varepsilon(r^*_\varepsilon))\Psi_\varepsilon(r^*_\varepsilon))_\varepsilon = 0. \tag{4.26}$$

Note that the tortoise distributional coordinate $(r^*_\varepsilon(r))_\varepsilon$ becomes to infinite Colombeau constant $[(r^*_\varepsilon(r_+))_\varepsilon] = [(\ln\varepsilon)_\varepsilon]$ at the horizon, i.e. as $r \to r_+,$ but its behavior at infinity is strongly dependent on the cosmological constant:

$[(r^*_\varepsilon(r_+))_\varepsilon] = +\infty$ for asymptotically-flat spacetimes, and $[(r^*_\varepsilon(r_+))_\varepsilon]$ = finite Colombeau constant for the SAdS$_d$ geometry.

4.2.1. Boundary conditions at the horizon of the distributional SAdS BH geometry.

For most spacetimes of interest the potential $(V_\varepsilon(r^*_\varepsilon(r)))_\varepsilon = 0$ as $r = r_+$, i.e. $(|r^*_\varepsilon(r)|)_\varepsilon = +\infty$, and in this limit solutions to the wave equation (4.26) behave as

$$(\Psi_\varepsilon(t, r^*_\varepsilon))_\varepsilon \sim (\exp[-i\omega(t \pm r^*_\varepsilon(r))])_\varepsilon, \text{ as } r \sim r_+. \tag{4.27}$$

Note that classically nothing should leave the horizon and thus classically only ingoing modes (corresponding to a plus sign) should be present,i.e.

$$(\Psi_\varepsilon(t, r^*_\varepsilon))_\varepsilon \sim (\exp[-i\omega(t + r^*_\varepsilon(r))])_\varepsilon, \text{ as } r \sim r_+. \tag{4.28}$$

Note that for non-extremal spacetimes, the tortoise coordinate tends to

$$(r^*_\varepsilon(r))_\varepsilon = \left(\int f^{-1}_\varepsilon(r)dr\right)_\varepsilon \sim \left[\left(f^{'}_\varepsilon(r_+)\right)_\varepsilon\right]^{-1} (\ln(|r - r_+| + \varepsilon))_\varepsilon \text{ as } r \sim r_+, \tag{4.29}$$

where $\left(f^{'}_\varepsilon(r_+)\right)_\varepsilon > 0$. Therefore near the horizon, outgoing modes behave as

$$\begin{cases} (\exp[-i\omega(t - r^*_\varepsilon(r))])_\varepsilon = \{(\exp[-i\omega v^*_\varepsilon(t,r)])_\varepsilon\}(\exp[-2i\omega r^*_\varepsilon(r)])_\varepsilon = \\ \quad = \{(\exp[-i\omega v^*_\varepsilon(t,r)])_\varepsilon\}\left\{\left([|r - r_+| + \varepsilon]^{2i\omega/f^{'}_\varepsilon(r_+)}\right)_\varepsilon\right\}, \end{cases} \tag{4.30}$$

where $(v^*_\varepsilon(t,r))_\varepsilon = t + (r^*_\varepsilon(r))_\varepsilon$. Now Eq. (4.30) shows that outgoing modes is Colombeau generalized function of class $\mathcal{G}(\mathbb{R})$.

V. Distributional SAdS BH spacetime-induced vacuum dominance

We shall assume now any distributional spacetime which is conformally static in both the asymptotic past and future. We will be considered distributional spacetime which is conformally flat in the asymptotic past,i.e.

$$\begin{cases} ds^2_\varepsilon \sim (f^2_{\varepsilon,in}(-dt^2 + d\vec{x}^2))_\varepsilon & \text{asymp. past} \\ ds^2_\varepsilon \sim (f^2_{\varepsilon,out}(-dt^2 + h_{\varepsilon,ij}dx^i dx^j))_\varepsilon, & \text{asymp. future} \end{cases} \tag{5.1}$$

where $\varepsilon \in (0,1]$ $(f_{\varepsilon,\mathbf{J}})_\varepsilon = (f_{\varepsilon,\mathbf{J}}(t,\vec{x}))_\varepsilon > 0, \mathbf{J} \in \{in, out\}$, are smooth functions and $h_{\varepsilon,ij} = h_{\varepsilon,ij}(\vec{x}), i,j = 1,2,3$, are the components of an arbitrary distributional spatial metric. Note that we use the same labels t and $\vec{x} = (x^1, x^2, x^3)$ for coordinates in the asymptotic past and future only for simplicity; they are obviously defined on non-intersecting regions of the spacetime.) In each of these asymptotic regions the distributional field $(\Phi_\varepsilon)_\varepsilon$ can be written as $(\Phi_\varepsilon)_\varepsilon = (\tilde{\Phi}_\varepsilon / f_{\varepsilon,\mathbf{J}})_\varepsilon$, where $(\tilde{\Phi}_\varepsilon)_\varepsilon$ satisfies

$$-\left(\frac{\partial^2}{\partial t^2}\tilde{\Phi}\right)_\varepsilon = -(\Delta_{\varepsilon,\mathbf{J}}\tilde{\Phi}_\varepsilon)_\varepsilon + (V_{\varepsilon,\mathbf{J}}\tilde{\Phi}_\varepsilon)_\varepsilon, \tag{5.2}$$

where $(\Delta_{\varepsilon,in})_\varepsilon$ is the flat Laplace operator, $(\Delta_{\varepsilon,out})_\varepsilon$ is the Laplace operator associated with the spatial metric $(h_{\varepsilon ij})_\varepsilon$, and the effective potential V_J is given by

$$\begin{cases} (V_{\varepsilon,\mathbf{J}})_\varepsilon = \left(\frac{\Delta_\varepsilon \mathbf{J} f_{\varepsilon,\mathbf{J}}}{f_{\varepsilon,\mathbf{J}}}\right)_\varepsilon + \left(f^2_{\varepsilon,\mathbf{J}}(m^2+\xi R_\varepsilon)\right)_\varepsilon = \\ (1-6\xi)\left(\frac{\Delta_\varepsilon \mathbf{J} f_{\varepsilon,\mathbf{J}}}{f_{\varepsilon,\mathbf{J}}}\right)_\varepsilon + m^2\left(f^2_{\varepsilon,\mathbf{J}}\right)_\varepsilon + \xi(K_{\varepsilon,\mathbf{J}})_\varepsilon, \end{cases} \tag{5.3}$$

with $(K_{\varepsilon,in})_\varepsilon = 0$, $K_{\varepsilon,out} = K_{\varepsilon,out}(\vec{x})$ the scalar curvature associated with the spatial distributional metric $h_{\varepsilon,ij}$,

We assume now this condition: (i) the massless ($m = 0$) field with arbitrary coupling ξ in spacetimes which are asymptotically flat in the past and asymptotically static in the future,i.e. $f_{in} = 1$ and $f_{\varepsilon,out} = f_{\varepsilon,out}(\vec{x})$, as those describing the formation of a static BH from matter initially scattered throughout space, and (ii) the massless, conformally coupled field ($m = 0$ and $\xi = 1/6$). With this assumptions for the potential, two different sets of positive-norm distributional modes, $\left(u^{(+)}_{\varepsilon,\vec{k}}\right)_\varepsilon$ and $\left(v^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon$, can be naturally defined by the requirement that they are the solutions of Eq.(5.2) which satisfy the asymptotic conditions:

$$\left(u^{(+)}_{\varepsilon,\vec{k}}\right)_\varepsilon \sim^{\text{past}} (16\pi^3\omega_{\vec{k}})^{-1/2}\ e^{-i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{x})}\left(f^{-1}_{\varepsilon,in}\right)_\varepsilon \tag{5.4}$$

and

$$\left(v^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon \sim^{\text{future}} (2\varpi_\alpha)^{-1/2} e^{-i\varpi_\alpha t}\left(f^{-1}_{\varepsilon,out}\ F_{\varepsilon,\alpha}(\vec{x})\right)_\varepsilon, \tag{5.5}$$

where $\vec{k} \in \mathbb{R}^3$, $\omega_{\vec{k}} := \|\vec{k}\|$, $\varpi_\alpha > 0$, and $(F_{\varepsilon,\alpha}(\vec{x}))_\varepsilon$ are Colombeau solutions of

$$([-\Delta_{\varepsilon,out} + V_{\varepsilon,out}(\vec{x})]F_{\varepsilon,\alpha}(\vec{x}))_\varepsilon = \varpi^2_\alpha (F_{\varepsilon,\alpha}(\vec{x}))_\varepsilon \tag{5.6}$$

satisfying the normalization

$$\left(\int_{\Sigma_{out}} d^3x\sqrt{h}\ F_{\varepsilon,\alpha}(\vec{x})^* F_{\varepsilon,\beta}(\vec{x})\right)_\varepsilon = \delta_{\alpha,\beta} \tag{5.7}$$

on a Cauchy surface Σ_{out} in the asymptotic future. Note that each $F_{\varepsilon,\alpha}, \varepsilon \in (0,1]$ can be chosen to be real without loss of generality. There are reasonable situations where the distributional modes $\left(v^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon$, given in Eq. (5.5), together with distributional modes $\left(v^{(-)}_{\varepsilon,\alpha}\right)_\varepsilon$ fail to form a complete set of normal modes. This happens whenever the operator $([-\Delta_{\varepsilon,out} + V_{\varepsilon,out}(\vec{x})])_\varepsilon$ in Eq. (5.2) happens to possess normalizable i.e., satisfying Eq. (5.7) eigenfunctions with negative eigenvalues, $\varpi^2_\alpha = -\Omega^2_\alpha < 0$. In this case, additional positive-norm modes $\left(w^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon$ with the asymptotic behavior

$$\left(w^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon \sim^{\text{future}} (e^{\Omega_\alpha t - i\pi/12} + e^{-\Omega_\alpha t + i\pi/12})\left(\frac{F_{\varepsilon,\alpha}(\vec{x})}{\sqrt{2\Omega_\alpha}\ f_{\varepsilon,out}(\vec{x})}\right)_\varepsilon \tag{5.8}$$

and their complex conjugates $\left(w^{(-)}_{\varepsilon,\alpha}\right)_\varepsilon$ are necessary in order to expand an arbitrary Colombeau solution of Eq.(5.1) As a direct consequence, at least some of the in-modes $\left(u^{(\pm)}_{\varepsilon,\vec{k}}\right)_\varepsilon$ (typically those with low $\omega_{\vec{k}}$) eventually undergo an exponential

growth.This asymptotic divergence is reflected on the unbounded increase of the vacuum fluctuations,

$$(\langle\Phi^2_\varepsilon(\vec{x})\rangle)_\varepsilon \quad \sim^{\text{future}} \quad \frac{\kappa\, e^{2\bar{\Omega}t}}{2\bar{\Omega}}\left[\left(\frac{\bar{F}_\varepsilon(\vec{x})}{f_{\varepsilon,out}(\vec{x})}\right)_\varepsilon\right]^2[1+\mathcal{O}(e^{-\epsilon t})], \tag{5.9}$$

where $\bar{F}(\vec{x})$ is the eigenfunction of Eq. (5.6) associated with the lowest negative eigenvalue allowed, $\varpi^2_\alpha = -\bar{\Omega}^2$, ϵ is some positive constant, and κ is a dimensionless constant (typically of order unity) whose exact value depends globally on the spacetime structure (since it crucially depends on the projection of each $\left(u^{(\pm)}_{\varepsilon,\vec{k}}\right)_\varepsilon$ on the mode $\left(w^{(\pm)}_{\varepsilon,\alpha}\right)_\varepsilon$ whose $\varpi^2_\alpha = -\bar{\Omega}^2$; κ also depends on the initial state, here assumed to be the vacuum $|0\rangle_{in}$). As one would expect, these wild quantum fluctuations give an important contribution to the vacuum energy stored in the field. In fact, the expectation value of its distributional energy-momentum tensor, $(\langle T_{\varepsilon,\mu\nu}(\vec{x})\rangle)_\varepsilon, \varepsilon \in (0,1]$, in the asymptotic future is found to be dominated by this exponential growth:

$$\begin{cases} (\langle T_{\varepsilon,00}(\vec{x})\rangle)_\varepsilon \sim^{\text{future}} \{(\langle\Phi^2_\varepsilon(\vec{x})\rangle)_\varepsilon\}\left\{\frac{(1-4\xi)}{2}\left(\bar{\Omega}^2+\frac{(D\bar{F}_\varepsilon)^2}{\bar{F}^2}+m^2f^2_\varepsilon+\xi K_\varepsilon\right)_\varepsilon\right. \\ \left. +(1-6\xi)\left(\frac{2\xi D^2f_\varepsilon}{f_\varepsilon}+\frac{(Df_\varepsilon)^2}{2f^2_\varepsilon}-\frac{D_if_\varepsilon D^i\bar{F}_\varepsilon}{f_\varepsilon\bar{F}_\varepsilon}\right)_\varepsilon+\mathcal{O}(e^{-\epsilon t})\right\}, \end{cases} \tag{5.10}$$

$$\begin{cases} (\langle T_{\varepsilon,0i}(\vec{x})\rangle)_\varepsilon \sim^{\text{future}} \\ \{(\langle\Phi^2_\varepsilon(\vec{x})\rangle)_\varepsilon\}\left\{(1-4\xi)\left(\frac{\bar{\Omega}D_i\bar{F}_\varepsilon}{\bar{F}_\varepsilon}\right)_\varepsilon-(1-6\xi)\left(\frac{\bar{\Omega}D_if_\varepsilon}{f_\varepsilon}\right)_\varepsilon+\mathcal{O}(e^{-\epsilon t})\right\}, \end{cases} \tag{5.11}$$

$$\begin{aligned} &(\langle T_{\varepsilon,ij}(\vec{x})\rangle)_\varepsilon \sim^{\text{future}} \\ &\{(\langle\Phi^2_\varepsilon(\vec{x})\rangle)_\varepsilon\}\left\{(1-2\xi)\left(\frac{D_i\bar{F}_\varepsilon D_j\bar{F}_\varepsilon}{\bar{F}^2_\varepsilon}\right)_\varepsilon-2\xi\left(\frac{D_iD_j\bar{F}_\varepsilon}{F_\varepsilon}\right)_\varepsilon+\xi\left(\widetilde{R}_{\varepsilon,ij}\right)_\varepsilon\right. \\ &+\frac{(1-4\xi)h_{ij}}{2}\left(\bar{\Omega}^2-\left(\frac{(D\bar{F}_\varepsilon)^2}{\bar{F}^2_\varepsilon}-m^2f^2_\varepsilon-\xi K_\varepsilon\right)_\varepsilon\right) \\ &+(1-6\xi)\left[\left(\frac{D_if_\varepsilon D_jf_\varepsilon}{f^2_\varepsilon}-\frac{D_ifD_j\bar{F}}{f_\varepsilon\bar{F}_\varepsilon}-\frac{D_jf_\varepsilon D_i\bar{F}_\varepsilon}{f_\varepsilon\bar{F}_\varepsilon}\right)_\varepsilon+\right. \\ &\left.\left.+(h_{\varepsilon,ij})_\varepsilon\left(\frac{2\xi D^2f_\varepsilon}{f_\varepsilon}-\frac{(Df_\varepsilon)^2}{2f^2_\varepsilon}+\frac{D_kf_\varepsilon D^k\bar{F}_\varepsilon}{f_\varepsilon\bar{F}_\varepsilon}\right)_\varepsilon\right]+\mathcal{O}(e^{-\epsilon t}),\right\} \end{aligned} \tag{5.12}$$

where D_i is the derivative operator compatible with the distributional metric $(h_{\varepsilon,ij})_\varepsilon$ (so that $\Delta_{out} = D^2$), $\left(\widetilde{R}_{\varepsilon,ij}\right)_\varepsilon$ is the associated distributional Ricci tensor so that $(K_{\varepsilon,out})_\varepsilon = \left(h^{ij}_\varepsilon\widetilde{R}_{\varepsilon,ij}\right)_\varepsilon$, and we have omitted the subscript *out* in $(f_{\varepsilon,out})_\varepsilon$ and $(K_{\varepsilon,out})_\varepsilon$ for simplicity. The Eqs. (5.10-5.12), together with Eq.(5.9), imply that on time scales determined by $\bar{\Omega}^{-1}$, the vacuum fluctuations of the field should overcome any other classical source of energy, therefore taking control over the evolution of the

background geometry through the semiclassical Einstein equations (in which $(\langle T_{\varepsilon,\mu\nu}\rangle)_\varepsilon$ is included as a source term for the distributional Einstein tensor). We are then confronted with a startling situation where the quantum fluctuations of a field, whose energy is usually negligible in comparison with classical energy components, are forced by the distributional background spacetime to play a dominant role. We are still left with the task of showing that there exist indeed well-behaved distributional background spacetimes in which the operator $[(-\Delta_{\varepsilon,out} + V_{\varepsilon,out}(\vec{x}))_\varepsilon]$ possesses negative eigenvalues $\varpi_\alpha^2 < 0$, condition on which depends Eq(5.9). Experience from quantum mechanics tells us that this typically occurs when $(V_{\varepsilon,out})_\varepsilon$ gets sufficiently negative over a sufficiently large region. It is easy to see from Eq. (5.3) that, except for very special geometries (as the flat one), one can generally find appropriate values of $\xi \in \mathbb{R}$ which make $(V_{\varepsilon,out})_\varepsilon$ as negative as would be necessary in order to guarantee the existence of negative eigenvalues. For distributional BH spacetime using Eq.(5.9)-Eq.(5.12) one obtain

$$(\langle\Phi_\varepsilon^2(r)\rangle)_\varepsilon \quad \sim^{\text{future}} \quad \frac{\kappa\ e^{2\bar{\Omega}t}\left[f_\varepsilon^{'}(r_+)\right]^2}{2\bar{\Omega}}\left[\left(\frac{\bar{F}_\varepsilon(r)}{(|r-r_+|+\varepsilon)}\right)_\varepsilon\right]^2, r \to r_+ \tag{5.13}$$

$$\begin{gathered}(\langle T_{\varepsilon,00}(r\rangle)_\varepsilon \sim^{\text{future}} \\ \{(\langle\Phi_\varepsilon^2(r)\rangle)_\varepsilon\}\left\{\frac{(1-4\xi)}{2}\left(\bar{\Omega}^2 + \frac{(D\bar{F}_\varepsilon(r_+))^2}{\bar{F}^2(r_+)} + m^2(|r-r_+|+\varepsilon)^2 + \xi K_\varepsilon\right)_\varepsilon\right. \\ \left.+(1-6\xi)\left(\frac{2\xi f_\varepsilon^{'}(r_+)D^2 f_\varepsilon(r_+)}{(|r-r_+|+\varepsilon)} + \frac{\left[f_\varepsilon^{'}(r_+)\right]^2(Df_\varepsilon(r_+))^2}{2(|r-r_+|+\varepsilon)^2} - \frac{f_\varepsilon^{'}(r_+)D_i f_\varepsilon(r_+)D^i\bar{F}_\varepsilon}{(|r-r_+|+\varepsilon)\bar{F}_\varepsilon}\right)_\varepsilon\right\}, \\ r \to r_+,\end{gathered} \tag{5.14}$$

$$\begin{cases} (\langle T_{\varepsilon,0i}(r)\rangle)_\varepsilon \sim^{\text{future}} \\ \{(\langle\Phi_\varepsilon^2(r)\rangle)_\varepsilon\}\left\{(1-4\xi)\left(\frac{\bar{\Omega}D_i\bar{F}_\varepsilon(r_+)}{\bar{F}_\varepsilon(r_+)}\right)_\varepsilon - (1-6\xi)\left(\frac{\bar{\Omega}f_\varepsilon^{'}(r_+)D_i f_\varepsilon(r_+)}{(|r-r_+|+\varepsilon)}\right)_\varepsilon\right\}, \\ r \to r_+, \end{cases} \tag{5.15}$$

$$
\begin{aligned}
&(\langle T_{\varepsilon,ij}(r)\rangle)_\varepsilon \overset{\text{future}}{\sim} \\
&\{(\langle \Phi^2_\varepsilon(r)\rangle)_\varepsilon\}\Bigg\{(1-2\xi)\left(\frac{D_i\bar{F}_\varepsilon D_j\bar{F}_\varepsilon}{\bar{F}^2_\varepsilon}\right)_\varepsilon - 2\xi\left(\frac{D_iD_j\bar{F}_\varepsilon}{F_\varepsilon}\right)_\varepsilon + \xi\left(\widetilde{R}_{\varepsilon,ij}\right)_\varepsilon \\
&+\frac{(1-4\xi)h_{ij}}{2}\left(\bar{\Omega}^2 - \left(\frac{(D\bar{F}_\varepsilon)^2}{\bar{F}^2_\varepsilon} - m^2(|r-r_+|+\varepsilon)^2 - \xi K_\varepsilon\right)_\varepsilon\right) \\
&+(1-6\xi)\Bigg[\left(\frac{\left[f'_\varepsilon(r_+)\right]^2 D_if_\varepsilon D_jf_\varepsilon}{(|r-r_+|+\varepsilon)^2} - \frac{f'_\varepsilon(r_+)D_if D_j\bar{F}}{(|r-r_+|+\varepsilon)\bar{F}_\varepsilon} - \frac{f'_\varepsilon(r_+)D_jf_\varepsilon D_i\bar{F}_\varepsilon}{(|r-r_+|+\varepsilon)\bar{F}_\varepsilon}\right)_\varepsilon + \\
&+(h_{\varepsilon,ij})_\varepsilon\left(\frac{2f'_\varepsilon(r_+)\xi D^2f_\varepsilon}{(|r-r_+|+\varepsilon)} - \frac{\left[f'_\varepsilon(r_+)\right]^2(Df_\varepsilon)^2}{2(|r-r_+|+\varepsilon)^2} + \frac{f'_\varepsilon(r_+)D_kf_\varepsilon D^k\bar{F}_\varepsilon}{(|r-r_+|+\varepsilon)\bar{F}_\varepsilon}\right)_\varepsilon\Bigg], \\
&r \to r_+.
\end{aligned}
\tag{5.16}
$$

Remark 5.1.Note that in spite of the unbounded growth at $r \to r_+$ in Eq.(5.13)-Eq.(5.16), $(\langle T_{\varepsilon,\mu\nu}\rangle)_\varepsilon$ is covariantly conserved: $(\nabla_\mu\langle T^\mu_{\varepsilon,\nu}\rangle)_\varepsilon = 0$. In the static case $(f_{\varepsilon,out})_\varepsilon = (f_{\varepsilon,out}(\vec{x}))_\varepsilon$, for instance for distributional BH geometry, this implies that the total vacuum energy is kept constant, although it continuously flows from spatial regions where its density is negative to spatial regions where it is positive.

Remark 5.2. Note that the singular behavior at $r \to r_+$ appearing in Eq.(5.13)-Eq.(5.16) leads only to asymptotic divergences, i.e. all the quantities remain finite everywhere except horizon.

VI.Conclusions and remarks.

We have shown that a succesfull approach for dealing with curvature tensor valued distribution is to first impose admisible the nondegeneracy conditions on the metric tensor, and then take its derivatives in the sense of classical distributions in space $\mathcal{S}'_{2m}(\mathbb{R}^3,k), k \geqslant 2$.

The distributional meaning is then equivalent to the junction condition formalism. Afterwards, through appropiate limiting procedures, it is then possible to obtain well behaved distributional tensors with support on submanifolds of $d \leq 3,$ as we have shown for the energy-momentum tensors associated with the Schwarzschild spacetimes. The above procedure provides us with what is expected on physical grounds. However, it should be mentioned that the use of new supergeneralized functions (supergeneralized Colombeau algebras $\widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$). in order to obtain superdistributional curvatures, may renders a more rigorous setting for discussing situations like the ones considered in this paper.

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