Moyal Deformations of Clifford Gauge Theories of Gravity

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Abstract

A Moyal deformation of a Clifford $Cl(3,1)$ Gauge Theory of (Conformal) Gravity is performed for canonical noncommutativity (constant $\Theta^{\mu\nu}$ parameters). In the very special case when one imposes certain constraints on the fields, there are *no* first order contributions in the $\Theta^{\mu\nu}$ parameters to the Moyal deformations of Clifford gauge theories of gravity. However, when one does *not* impose constraints on the fields, there are first order contributions in $\Theta^{\mu\nu}$ to the Moyal deformations in variance with the previous results obtained by other authors and based on different gauge groups. Despite that the generators of $U(2,2), SO(4,2), SO(2,3)$ can be expressed in terms of the Clifford algebra generators this does *not* imply that these algebras are isomorphic to the Clifford algebra. Therefore one should not expect identical results to those obtained by other authors. In particular, there are Moyal deformations of the Einstein-Hilbert gravitational action with a cosmological constant to first order in $\Theta^{\mu\nu}$. Finally, we provide a mechanism which furnishes a plausible cancellation of the huge vacuum energy density.

Keywords: C-space Gravity, Clifford Algebras, Gauge Theories of Gravity, Moyal deformations.

1 Introduction

Many approaches have been taken towards the quantization of gravity. Within the framework of the gauge theory formulations of gravity, a Moyal deformation of gravity has been very popular. Its construction is based, in particular, on the $U(2,2), SO(1,4) \; SO(2,3), SL(2,C)$ groups. Clifford, Division, Exceptional and Jordan algebras are deeply related and essential tools in many aspects in Physics [1], [2], [3], [5]. The aim of this work is to provide a Moyal Deformation of Clifford Gauge Theories of Gravity advanced in [12].

Gravitational theories on canonical noncommutative spacetimes (associated to *constant* noncommutative parameters $\Theta^{\mu\nu}$) was developed earlier [15, 16] in terms of a twisted diffeomorphism algebra. A Noncommutative gravity associated to coordinate-dependent noncommutative parameters $\Theta^{\mu\nu}(X)$ has also been studied by many authors. For instance, on noncommutative spacetimes endowed with a Lie algebraic structure, $[X^\mu, X^\nu] = i\Theta^{\mu\nu}(X) = i f^{\mu\nu}_\rho X^\rho$. An principal example of such noncommutative spacetimes is the κ-Minkowski space associated with DSR (Doubly/Deformed Special Relativity) [17].

An internal gauge theory using a *covariant* star product between two arbitrary Lie algebra valued differential forms on a *symplectic* manifold endowed only with torsion but no curvature was recently developed by [20] . In this case $[X^\mu, X^\nu] = i\Theta^{\mu\nu}(X)$ where $\Theta^{\mu\nu}(X)$ is now a Poisson bivector. If the bivector $\Theta^{\mu\nu}(X)$ has an inverse $\omega_{\mu\nu}(X)$ that is nondegenerate det $\omega_{\mu\nu} \neq 0$ and closed $d\omega = 0$ (so the Jacobi identity is obeyed [20]], then ω is the symplectic two-form of a symplectic manifold $\mathcal{M}$.

The construction of a fully gauge invariant action to all orders in $\Theta^{\mu\nu}$, and the corresponding QFT associated with gauge theories in noncommutative spacetimes based on a Lie-algebraic noncommutativity structure for the $\Theta^{\mu\nu}(X) = i f^{\mu\nu}_\rho X^\rho$, remains a challenging problem due to the fact that the cyclicity property of the integrals is valid up to second order in powers of $\Theta^{\mu\nu}$ [28].

We shall focus on the case when $\Theta^{\mu\nu}$ are constant parameters. The results in this work differ from the results of other authors in several aspects. Mainly in the construction of the generalized action prior to its Moyal deformation. No constraints are imposed by hand on the gauge fields like in [23] and on the scalar fields like in [22].

The outline of this work goes as follows. In section **2** we review the construction of a $Cl(3,1)$ Gauge Theory of (Conformal) Gravity and provide the most generalized gravitational action. In section **3** we perform the Moyal deformation following the Seiberg-Witten map procedure relating the non-Abelian noncommutative gauge fields based on *noncommutative* coordinates and the non-Abelian gauge fields based on *commutative* coordinates.

We find in section **3** that there are *no* first order contributions in the $\Theta^{\mu\nu}$ (constants) parameters to the Moyal deformations of Clifford gauge theories of gravity in the very special case when one *truncates* all the components of the Clifford-valued scalar field $\Phi = \Phi^A \Gamma_A$ to zero *except* $\Phi^{mnpq} \neq 0$, and all the components of the Clifford gauge field $A^A_\mu \Gamma_A$ to zero *except* $A^{ab}_\mu \neq 0$. However, when one does *not* impose such constraints on the fields, there are first order contributions in the $\Theta^{\mu\nu}$ (constants) parameters to the Moyal deformations in variance with the previous results obtained by other authors and based on different gauge groups.

Despite that the generators of $U(2,2), SO(4,2), SO(2,3)$ can be expressed in terms of the Clifford algebra generators this does *not* imply that these algebras are isomorphic to the Clifford algebra. Therefore one should not expect identical results. In particular, there are Moyal deformations of the Einstein-Hilbert gravitational action with a cosmological constant to first order in $\Theta^{\mu\nu}$. Finally, we provide a mechanism which furnishes a plausible cancellation of the huge vacuum energy density.

2 $Cl(3,1)$ Gauge Theory of Conformal Gravity

Let $\eta_{ab} = (-,+,+,+)$, $\epsilon_{0123} = -\epsilon^{0123} = 1$, the real Clifford $Cl(3,1,R)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab}, \ \ \Gamma_5 \ = \ -i\,\Gamma_0\,\Gamma_1\,\Gamma_2\,\Gamma_3, \ \ (\Gamma_5)^2 \ = \ 1; \ \ \ \{\Gamma_5, \Gamma_a\} \ = \ 0; \tag{2.1}$$

$$\Gamma_{abcd} \ = \ \epsilon_{abcd}\,\Gamma_5; \quad \Gamma_{ab} = \frac{1}{2}\,(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)\,. \tag{2.2a}$$

$$\Gamma_{abc} \ = \ \epsilon_{abcd}\,\Gamma_5\,\Gamma^d; \quad \Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5. \tag{2.2b}$$

$$\Gamma_a\,\Gamma_b \ = \ \Gamma_{ab} + \eta_{ab}, \quad \Gamma_{ab}\,\Gamma_5 \ = \ \frac{1}{2}\epsilon_{abcd}\,\Gamma^{cd}, \tag{2.2c}$$

$$\Gamma_{ab}\,\Gamma_c \ = \ \eta_{bc}\,\Gamma_a - \eta_{ac}\,\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \tag{2.2d}$$

$$\Gamma_c\,\Gamma_{ab} \ = \ \eta_{ac}\,\Gamma_b - \eta_{bc}\,\Gamma_a + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \tag{2.2e}$$

$$\Gamma_a\,\Gamma_b\,\Gamma_c \ = \ \eta_{ab}\,\Gamma_c + \eta_{bc}\,\Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \tag{2.2f}$$

$$\Gamma^{ab}\,\Gamma_{cd} \ = \ \epsilon^{ab}{}_{cd}\,\Gamma_5 \ - 4\delta^{[a}_{[c}\,\Gamma^{b]}{}_{d]} \ - 2\delta^{ab}_{cd}. \tag{2.2g}$$

$$\delta^{ab}_{cd} \ = \ \frac{1}{2}\,(\delta^a_c\,\delta^b_d \ - \ \delta^a_d\,\delta^b_c\,). \tag{2.2.h}$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. A representation of the $Cl(3,1)$ algebra exists where the generators

$$\mathbf{1}; \ \Gamma_a \ = \ \Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_4 = -i\Gamma_0; \ \Gamma_5; \quad a = 1,2,3,4 \tag{2.3}$$

are Hermitian; while the generators $\Gamma_a\Gamma_5; \Gamma_{ab}$ for $a, b = 1,2,3,4$ are *anti*-Hermitian. Hence, $i\Gamma_a\Gamma_5; i\Gamma_{ab}$ are Hermitian. Using eqs-(2.1-2.3) allows to write the $Cl(3,1)$ algebra-valued (Hermitian) one-form as

$$\mathbf{A} \ = \ \left(a_\mu\,\mathbf{1} + \ b_\mu\,\Gamma_5 \ + \ e^a_\mu\,\Gamma_a \ + \ i\,f^a_\mu\,\Gamma_a\,\Gamma_5 \ + \ \frac{i}{4}\omega^{ab}_\mu\,\Gamma_{ab} \right)\,dx^\mu. \tag{2.4}$$

The Clifford-valued gauge field A_μ transforms according to $A'_\mu = U^{-1}\,A_\mu\,U + U^{-1}\partial_\mu U$ under Clifford-valued gauge transformations. The Clifford-valued field strength is $F = dA - i[A, A]$ so that F transforms covariantly $F' = U^{-1}\ F\ U$. Decomposing the field strength in terms of the Clifford algebra generators gives

$$F_{\mu\nu} \ = \ F^1_{\mu\nu}\ \mathbf{1} \ + \ F^5_{\mu\nu}\ \Gamma_5 \ + \ F^a_{\mu\nu}\ \Gamma_a \ + i\ F^{a5}_{\mu\nu}\ \Gamma_a\ \Gamma_5 \ + \ \frac{i}{4} F^{ab}_{\mu\nu}\ \Gamma_{ab}. \tag{2.5}$$

where $F = \frac{1}{2}\ F_{\mu\nu}\ dx^\mu \wedge dx^\nu$. The field-strength (real-valued) components are given by

$$F^1_{\mu\nu} \ = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{2.6a}$$

$$F^5_{\mu\nu} \ = \ \partial_\mu b_\nu - \partial_\nu b_\mu + 2e^a_\mu f_{\nu a} - 2e^a_\nu f_{\mu a} \tag{2.6b}$$

$$F^a_{\mu\nu} \ = \ \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2f^a_\mu b_\nu - 2f^a_\nu b_\mu \tag{2.6c}$$

$$F^{a5}_{\mu\nu} \ = \ \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2e^a_\mu b_\nu - 2e^a_\nu b_\mu \tag{2.6d}$$

$$F^{ab}_{\mu\nu} \ = \ \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{b} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \longleftrightarrow \nu. \tag{2.6e}$$

At this stage we may provide the relation among the $Cl(3,1)$ algebra generators and the the conformal algebra $so(4,2) \sim su(2,2)$ in $4D$. The operators of the Conformal algebra can be written in terms of the Clifford algebra generators as [4]

$$P_a \ = \ \frac{1}{2}\Gamma_a\ (1\ -\ \Gamma_5);\quad K_a \ = \ \frac{1}{2}\Gamma_a\ (1\ +\ \Gamma_5);\quad D \ = \ -\ \frac{1}{2}\ \Gamma_5,\ \ L_{ab} \ = \ \frac{1}{2}\ \Gamma_{ab}. \tag{2.7}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4+4+1+6 = 15$. From the above realization of the conformal algebra generators (2.7), the explicit evaluation of the commutators yields

$$[P_a,\ D] \ = \ P_a;\quad [K_a,\ D] \ = \ -\,K_a;\quad [P_a,\ K_b] \ = \ -\,2g_{ab}\ D\ +\ 2\ L_{ab}$$

$$[P_a,\ P_b] \ = \ 0;\quad [K_a, K_b] \ = \ 0; .... \tag{2.8}$$

which is consistent with the $su(2,2) \sim so(4,2)$ commutation relations. We should notice that the K_a, P_a generators in (2.7) are both comprised of Hermitian Γ_a *and* anti-Hermitian $\pm\Gamma_a\Gamma_5$ generators, respectively. The dilation D operator is Hermitian, while the Lorentz generator L_{ab} is anti-Hermitian. The fact that Hermitian and anti-Hermitian generators are required is consistent with the fact that $U(2,2)$ is a pseudo-unitary group.

Having established this one can infer that the real-valued tetrad V^a_μ field (associated with translations) and its real-valued partner $\tilde{V}^a_\mu$ (associated with conformal boosts) can be defined in terms of the real-valued gauge fields e^a_μ, f^a_μ as follows

$$e^a_\mu\ \Gamma_a \ + \ f^a_\mu\ \Gamma_a\Gamma_5 \ = \ V^a_\mu\ P_a \ + \ \widetilde{V}^a_\mu\ K_a \tag{2.9}$$

From eq-(2.7) one learns that eq-(2.9) leads to

$$e^a_\mu \ - \ f^a_\mu \ = \ V^a_\mu; \quad e^a_\mu \ + \ f^a_\mu \ = \ \tilde V^a_\mu \ \Rightarrow$$

$$e^a_\mu \ = \ \frac{1}{2} \ (V^a_\mu + \tilde V^a_\mu), \quad f^a_\mu \ = \ \frac{1}{2}(\tilde V^a_\mu - V^a_\mu). \tag{2.10}$$

The components of the torsion and conformal-boost curvature of conformal gravity are given respectively by the linear combinations of eqs-(2.6c, 2.6d)

$$F^a_{\mu\nu} \ - \ F^{a5}_{\mu\nu} \ = \ \widetilde F^a_{\mu\nu}[P]; \quad F^a_{\mu\nu} \ + \ F^{a5}_{\mu\nu} \ = \ \widetilde F^a_{\mu\nu}[K] \ \Rightarrow$$

$$F^a_{\mu\nu} \ \Gamma_a \ + \ F^{a5}_{\mu\nu} \ \Gamma_a \ \Gamma_5 \ = \ \widetilde F^a_{\mu\nu}[P] \ P_a \ + \ \widetilde F^a_{\mu\nu}[K] \ K_a. \tag{2.11a}$$

Inserting the expressions for e^a_μ, f^a_μ in terms of the vielbein V^a_μ and $\tilde V^a_\mu$ given by (2.10), yields the standard expressions for the Torsion and conformal-boost curvature, respectively

$$\widetilde F^a_{\mu\nu}[P] \ = \ \partial_{[\mu} \ V^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ V_{\nu]b} \ - \ V^a_{[\mu} \ b_{\nu]}, \tag{2.11b}$$

$$\widetilde F^a_{\mu\nu}[K] \ = \ \partial_{[\mu} \ \tilde V^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ \tilde V_{\nu]b} \ + 2 \ \tilde V^a_{[\mu} \ b_{\nu]}, \tag{2.11c}$$

The Lorentz curvature in eq-(2.6e) can be recast in the standard form as

$$F^{ab}_{\mu\nu} = \mathcal{R}^{ab}_{\mu\nu} \ = \ \partial_{[\mu} \ \omega^{ab}_{\nu]} \ + \ \omega^{ac}_{[\mu} \ \omega^{b}_{\nu]c} \ + \ 2(\ V^a_{[\mu} \ \tilde V^b_{\nu]} \ + \ \tilde V^a_{[\mu} \ V^b_{\nu]} \ ). \tag{2.11d}$$

The components of the curvature corresponding to the Weyl dilation generator given by $F^5_{\mu\nu}$ in eq-(2.6b) can be rewritten as

$$F^5_{\mu\nu} \ = \ \partial_{[\mu} \ b_{\nu]} \ + \ \frac{1}{2} \ (\ V^a_{[\mu} \ \tilde V_{\nu]a} \ - \ \tilde V^a_{[\mu} \ V_{\nu]a} \ ). \tag{2.11e}$$

and the Maxwell curvature is given by $F^1_{\mu\nu}$ in eq-(2.6a). A re-scaling of the vielbein V^a_μ/l and $\tilde V^a_\mu/l$ by a length scale parameter l is necessary in order to endow the curvatures and torsion in eqs-(2.11) with the proper dimensions of $length^{-2}, length^{-1}$, respectively.

To sum up, the real-valued tetrad gauge field V^a_μ (that gauges the translations P_a) and the real-valued conformal boosts gauge field $\widetilde V^a_\mu$ (that gauges the conformal boosts K_a) of conformal gravity are given, respectively, by the linear combination of the gauge fields $e^a_\mu \ \mp \ f^a_\mu$ associated with the $\Gamma_a, \ \Gamma_a \ \Gamma_5$ generators of the Clifford algebra $Cl(3,1)$ of the tangent space of spacetime $\mathcal{M}^4$ after performing a Wick rotation $-i \ \Gamma_0 = \Gamma_4$. A conformal Gravity-Maxwell case is based on the pseudo-unitary algebra $u(2,2) = u(1) \oplus su(2,2) \sim u(1) \oplus so(4,2)$.

Gauge invariant actions involving Yang-Mills terms of the form $\int Tr(F \wedge^* F)$ and theta terms of the form $\int Tr(F \wedge F)$ are straightforwardly constructed. For example, a $SO(4,2)$ gauge-invariant action for conformal gravity is [8]

$$S \ = \ \int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \mathcal{R}^{ab}_{\mu\nu} \ \mathcal{R}^{cd}_{\rho\sigma} \tag{2.12}$$

where the components of the Lorentz curvature 2-form $\mathcal{R}^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu$ are given by eq-(2.11d) after re-scaling the vielbein V^a_μ / l and $\tilde{V}^a_\mu / l$ by a length scale parameter l in order to endow the curvature with the proper dimensions of $length^{-2}$.

The conformal boost symmetry can be fixed by choosing the gauge $b_\mu = 0$ because under infinitesimal conformal boosts transformations the field b_μ transforms as $\delta b_\mu = -2 \ \xi^a \ e_{a\mu} = -2 \ \xi_\mu$; i.e the parameter ξ_μ has the same number of degrees of feedom as b_μ. After fixing the dilational symmetry and setting the torsion to zero which constrains the spin connection $\omega^{ab}_\mu (V^a_\mu)$ to be of the Levi-Civita form given by a function of the vielbein V^a_μ, and eliminating the $\tilde{V}^a_\mu$ field algebraically via its (non-propagating) equations of motion leads to the de Sitter group $SO(4,1)$ invariant Macdowell-Mansouri-Chamseddine-West action [7] (suppressing spacetime indices for convenience)

$$S \ = \ \int \ d^4 x \ (\ R^{ab}(\omega) \ + \frac{1}{l^2} \ V^a \wedge V^b \ ) \ \wedge \ (\ R^{cd}(\omega) \ + \frac{1}{l^2} \ V^c \wedge V^d \ ) \ \epsilon_{abcd}. \quad (2.13)$$

the action (2.20) is comprised of the topological invariant Gauss-Bonnet term $R^{ab}(\omega) \wedge R^{cd}(\omega) \epsilon_{abcd}$; the standard Einstein-Hilbert gravitational action term $\frac{1}{l^2} R^{ab}(\omega) \wedge V^c \wedge V^d \epsilon_{abcd}$, and the cosmological constant term $\frac{1}{l^4} V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd}$. l is the de Sitter throat size; i.e. l^2 is proportional to the square of the Planck scale (the Newtonian coupling constant).

The familiar Einstein-Hilbert gravitational action can also be obtained from a coupling of gravity to a scalar field like it occurs in a Brans-Dicke-Jordan theory of gravity

$$S \ = \ \frac{1}{2} \int \ d^4 x \ \sqrt{g} \ \phi \ \left(\ \frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} \ g^{\mu\nu} \ D^c_\mu \phi) \ + \ b^\mu \ (D^c_\mu \phi) \ + \ \frac{1}{6} \ R \ \phi \ \right). \quad (2.14a)$$

where the conformally covariant derivative acting on a scalar field ϕ of Weyl weight one is

$$D^c_\mu \phi \ = \ (\partial_\mu \ - \ b_\mu) \ \phi \quad (2.14b)$$

Upon fixing the conformal boosts symmetry by setting $b_\mu = 0$ and the dilational symmetry by setting $\phi = constant$ leads to the Einstein-Hilbert action for ordinary gravity.

We proceed next with the introducion of the Clifford-valued scalar field (a hyper-complex valued scalar) defined as

$$\mathbf{\Phi} \ = \ \Phi^A \ \Gamma_A \ = \ \phi \ \mathbf{1} + \phi^a \ \gamma_a + \frac{1}{2!} \ \phi^{ab} \ \gamma_{ab} + \frac{1}{3!} \ \phi^{abc} \ \gamma_{abc} + \frac{1}{4!} \ \phi^{abcd} \ \gamma_{abcd} \quad (2.15)$$

One should mention that the ϕ field appearing in (2.15) must *not* be confused with the scalar field appearing in eqs-(2.14). Now we can propose the most general action as an *extension* of the MMCW action displayed in eq-(2.13) and given by

$$\mathbf{S} \;=\; \int d^4x\, \epsilon^{\mu\nu\rho\sigma} \;<\; \mathbf{F}_{\mu\nu}\, \mathbf{F}_{\rho\sigma}\, \mathbf{\Phi} \;>\; = \; \int d^4x\, \epsilon^{\mu\nu\rho\sigma} \;<\; F^A_{\mu\nu}\, F^B_{\rho\sigma}\, \Phi^C\, \Gamma_A\, \Gamma_B\, \Gamma_C > \tag{2.16}$$

The bracket operation $< ..... >$ denotes extracting the Clifford scalar part of the geometric product of Clifford-valued quantities. It is the analog of taking the trace of a matrix product. The most general action can be decomposed into several pieces $\mathbf{S} = S_1 + S_2 + S_3 + S_4 + S_5$. Defining $\phi^{abcd} \;=\; \epsilon^{abcd}\, \phi^5 = \epsilon^{abcd}\, \varphi$ we have

$$S_5 \;=\; \int d^4x\; \epsilon^{\mu\nu\rho\sigma} \;<\; F^A_{\mu\nu}\; F^B_{\rho\sigma}\; \phi^{abcd}\; \Gamma_A\; \Gamma_B\; \gamma_{abcd} > \;=$$

$$\int d^4x\; \epsilon_{abcd}\; \epsilon^{\mu\nu\rho\sigma}\; \varphi\; \left(\; a_{51} F^{ab}_{\mu\nu}\; F^{cd}_{\rho\sigma} \;+ a_{52}\; F^a_{\mu\nu}\; F^{bcd}_{\rho\sigma} \;+ a_{53}\; F_{\mu\nu}\; F^{abcd}_{\rho\sigma} \;\right) \;+$$

$$\int d^4x\; \epsilon_{abcd}\; \epsilon^{\mu\nu\rho\sigma}\; \varphi\; \left(\; a_{54} F^{ab}_{\mu\nu e}\; F^{ecd}_{\rho\sigma} \;+ a_{55} F^a_{\mu\nu e}\; F^{ebcd}_{\rho\sigma} \;+ a_{56} F^{ab}_{\mu\nu ef}\; F^{efcd}_{\rho\sigma}\right) \tag{2.17}$$

One can rewrite (2.17) in differential form notation as

$$S_5 \;=\; \int \epsilon_{abcd}\; \varphi\; \left(\; a_{51}\; F^{ab} \wedge F^{cd} \;+ a_{52}\; F^a \wedge F^{bcd} \;+ a_{53}\; F \wedge F^{abcd} \;\right) +$$

$$\int \epsilon_{abcd}\; \varphi\; \left(\; a_{54}\; F^{ab}{}_e \wedge F^{ecd} \;+ a_{55}\; F^a{}_e \wedge F^{ebcd} \;+ a_{56}\; F^{ab}{}_{ef} \wedge F^{efcd}\right) \tag{2.18}$$

One can recognize that the MMCW action (2.13) is contained in one *piece* of S_5 and given by

$$S_{MMCW} \;\subset\; \int d^4x\; \epsilon_{abcd}\; \epsilon^{\mu\nu\rho\sigma}\; \varphi\; \left(\; F^{ab}_{\mu\nu}\; F^{cd}_{\rho\sigma}\;\right) \tag{2.19}$$

when $\varphi = 1$ as described by eqs-(2.6e, 2.11). One should notice that when the scalar field φ is *not* constant the expression

$$\int d^4x\; \sqrt{g}\; \varphi\; (\; R_{\mu\nu\rho\sigma}\; R_{\mu\nu\rho\sigma} \;\; - 4\; R_{\mu\nu}\; R^{\mu\nu} \;+\; R^2\;) \tag{2.20}$$

is no longer equal to the Gauss-Bonnet topological invariant due to the key $\varphi(x)$ factor and such terms will now contribute to the equations of motion.

The term $\epsilon_{abcd} F^a \wedge F^{bcd}$ in (2.18) can be rewritten as $F^a \wedge \tilde{F}_a$, while the term $\epsilon_{abcd} F \wedge F^{abcd} = F \wedge \tilde{F}$, etc.... The components $F^{bcd} = F^{bcd}_{\mu\nu} dx^\mu \wedge dx^\nu$, $F^{abcd} = F^{abcd}_{\mu\nu} dx^\mu \wedge dx^\nu$, etc. ... are all given by eqs-(2.4,2.5,2.6) after taking into account the relations among the Clifford algebra generators (gamma matrices) in eqs-(2.1, 2.2). The other terms in the action are

$$S_1 \;=\; \int d^4x\; \epsilon^{\mu\nu\rho\sigma} \;<\; F^A_{\mu\nu}\; F^B_{\rho\sigma}\; \phi\; \Gamma_A\; \Gamma_B\; \mathbf{1} \;>\; =$$

$$\int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; \phi \; \big(\; a_{11} \; F_{\mu\nu} \; F_{\rho\sigma} \; + a_{12} \; F^a_{\mu\nu} \; F_{a \; \rho\sigma} \; + a_{13} \; F^{ab}_{\mu\nu} \; F_{ab \; \rho\sigma} \; \big) \; +$$

$$\int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; \phi \; \big(\; a_{14} \; F^{abc}_{\mu\nu} \; F_{abc \; \rho\sigma} \; + a_{15} \; F^{abcd}_{\mu\nu} \; F_{abcd \; \rho\sigma} \; \big) \tag{2.21}$$

One can rewrite (2.21) in differential form notation as

$$S_1 \; = \; \int \phi \; \big(\; a_{11} \; F \; \wedge \; F \; + a_{12} \; F^a \; \wedge \; F_a \; + a_{13} \; F^{ab} \; \wedge \; F_{ab} \; \big) \; +$$

$$\int \phi \; \big(\; a_{14} \; F^{abc} \; \wedge F_{abc} \; + a_{15} \; F^{abcd} \; \wedge \; F_{abcd} \; \big) \tag{2.22}$$

$$S_3 = \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; F^A_{\mu\nu} \; F^B_{\rho\sigma} \; \phi^{ab} \; \Gamma_A \; \Gamma_B \; \gamma_{ab} \; > \; =$$

$$\int \phi_{ab} \; \big(\; a_{31} \; F^a \wedge F^b \; + a_{32} \; F^{ab} \wedge F \; + a_{33} \; F^a_c \wedge F^{cb} \; \big) \; +$$

$$\int \phi_{ab} \; \big(\; a_{34} \; F^a_{cd} \wedge F^{cdb} \; + a_{35} \; F^a_{cde} \wedge F^{cdeb} \; \big) \tag{2.23}$$

$$S_2 = \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; F^A_{\mu\nu} \; F^B_{\rho\sigma} \; \phi^a \; \Gamma_A \; \Gamma_B \; \gamma_a \; > \; =$$

$$\int \phi_a \; \big(\; a_{21} \; F^a \wedge F \; + a_{22} \; F^a_{\;b} \; \wedge \; F^b \; + a_{23} \; F^a_{bc} \wedge F^{bc} \; + a_{24} \; F^a_{bcd} \wedge F^{bcd} \; \big) \tag{2.24}$$

$$S_4 = \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; F^A_{\mu\nu} \; F^B_{\rho\sigma} \; \phi^{abc} \; \Gamma_A \; \Gamma_B \; \gamma_{abc} \; > \; =$$

$$\int \phi_{abc} \; \big(\; a_{41} \; F^{abc} \wedge F \; + a_{42} \; F^{ab} \wedge F^c \; + a_{43} \; F^{abc}_{\quad d} \wedge F^d \; \big) \; +$$

$$\int \phi_{abc} \; \big(\; a_{44} \; F^{ab}_{\quad d} \wedge F^{dc} \; + a_{45} \; F^{ab}_{\quad de} \wedge F^{dec} \; \big) \tag{2.25}$$

the way to obtain the numerical coefficients a_{ij} is explained in the Appendix.

One may introduce dynamics for the *dimensionless* Clifford-valued scalar field $\mathbf{\Phi}$ otherwise a variation of the action (2.16) with respect to the $\mathbf{\Phi}$ field will trivially constrain the action to zero since in this case $\mathbf{\Phi}$ will act as a Lagrange multiplier. The scalar field contribution to the action for the signature $(-,+,+,+)$ is

$$S[\mathbf{\Phi}] \; = \; \int d^4x \; \sqrt{g} \; < \; - \; \frac{1}{2l^2} \; (D_\mu \; \mathbf{\Phi}^\dagger) \; (D^\mu \; \mathbf{\Phi}) \; - \frac{1}{l^4} \; V(\mathbf{\Phi}) \; > \tag{2.26a}$$

The dagger operation $\mathbf{\Phi}^\dagger$ denotes the reversal operation and is obtained by reversing the order of the Clifford generators. For example, $(\gamma_a \wedge \gamma_b)^\dagger = \gamma_b \wedge \gamma_a$, $(\gamma_a \wedge \gamma_b \wedge \gamma_c)^\dagger = \gamma_c \wedge \gamma_b \wedge \gamma_a$, etc ..... so that

$$< (D_\mu \, \mathbf{\Phi}^\dagger) \, (D^\mu \, \mathbf{\Phi}) \, > \; = \; (D_\mu \phi) \, (D^\mu \phi) + (D_\mu \phi_a) \, (D^\mu \phi^a) + (D_\mu \phi_{ab}) \, (D^\mu \phi^{ab}) +$$

$$(D_\mu \phi_{abc}) \, (D^\mu \phi^{abc}) \; + \; (D_\mu \phi_{abcd}) \, (D^\mu \phi^{abcd}) \qquad (2.26b)$$

where we have omitted combinatorial numerical factors for convenience.

The potential, for example, may be given by a polynomial $V(\mathbf{\Phi}) = \sum_{n=0} a_n \, \mathbf{\Phi}^n$ or a more complicated function. Upon taking the Clifford scalar part of the potential one has $< V(\mathbf{\Phi}) >= \mathcal{V}(\phi, \phi^a, \phi^{ab}, \phi^{abc}, \phi^{abcd})$ which is a complicated (polynomial, for example) expression given in terms of the 16 scalars. For simplicity we shall choose the analog of a quartic Higgs-like potential given by

$$\mathcal{V} \; = \; \frac{1}{l^4} \, \lambda \, (\, |\Phi^A \, \Phi_A| \; - \; \mathbf{v}^2)^2$$

$$\Phi^A \, \Phi_A \; = \; \phi^2 \; + \; \phi^a \phi_a \; + \frac{1}{2!} \, \phi^{ab} \phi_{ab} \; + \frac{1}{3!} \, \phi^{abc} \phi_{abc} \; + \frac{1}{4!} \, \phi^{abcd} \phi_{abcd} \qquad (2.27)$$

the reason one must take the absolute value in $|\Phi^A \Phi_A|$ is because the Clifford scalar norm $\Phi^A \Phi_A$ is *not* positive definite since the 16-dimensional quadratic form has a split $(8, 8)$ signature [**?**] when the tangent space metric η_{ab} is Minkowskian $diag(-1, +1, +1, +1)$.

The gauge covariant derivative acting on the Clifford-valued scalar $\mathbf{\Phi}$ is defined as

$$(D_\mu \Phi^A) \, \Gamma_A \; = \; (\partial_\mu \, \Phi^A) \, \Gamma_A \; - i \, [\; A_\mu^B \, \Gamma_B, \; \Phi^C \, \Gamma_C \;] \; \Rightarrow$$

$$D_\mu \Phi^A \; = \; (\partial_\mu \, \Phi^A) \; -i \, A_\mu^B \, \Phi^C \; < \; [\, \Gamma_B, \, \Gamma_C \,] \, \Gamma^A \; > \; = \; (\partial_\mu \, \Phi^A) \; -i \, A_\mu^B \, \Phi^C \, f_{BC}^A \qquad (2.28a)$$

where we have written the commutator Clifford algebra as $[\Gamma_B, \Gamma_C] = f_{BC}^A \Gamma_A$ and whose structure constants are displayed in the Appendix. Under infinitesimal $Cl(3, 1)$ gauge transformations the Clifford-valued scalar $\mathbf{\Phi}$ field transforms as

$$\delta \Phi^C \; = \; - \, i \, f_{AB}^C \, \xi^A \, \Phi^B, \;\; \xi \; = \; \xi^A \, \Gamma_A \; = \; \tilde{\xi} \, \mathbf{1} \; + \; \xi^a \, \gamma_a \; + \frac{1}{2} \, \xi^{ab} \, \gamma_{ab} \; +$$

$$\frac{1}{3!} \, \xi^{abc} \, \gamma_{abc} \; + \frac{1}{4!} \, \xi^{abcd} \, \gamma_{abcd} \qquad (2.28b)$$

and the gauge covariant derivative transforms as well $\delta(D_\mu \Phi^C) = -i f_{AB}^C \, \xi^A \, D_\mu \Phi^B$.

To sum up, the action $\mathbf{S} + S[\mathbf{\Phi}]$ given by eqs-(2.16-2.26) is comprised of (i) φ times the MMCW Lagrangian (2.13) that contains the Einstein-Hilbert and cosmological constant terms. (ii) Extra terms quadratic in the curvature and torsion. (iii) A coupling of curvature and torsion terms. (iv) kinetic and potential terms for a multiplet of 16 spacetime scalar fields $\phi, \phi^a, \phi^{ab}, \phi^{abc}, \phi^{abcd}$ that from the tangent space point of view behave as a scalar, vector, antisymmetric tensors of rank two and three and a pseudo-scalar field, respectively.

(v) Non-minimal couplings of the scalars and curvature and torsion terms. (vi) terms involving the field strengths associated with conformal boosts, a dilational (Weyl gauge field) and a $U(1)$ Maxwell-like generator as displayed by eqs-(2.6, 2.11). A review of conformal (super) gravity can be found in [8].

The action displayed by eqs-(2.16-2.26) is a more complex generalization of the $f(R,T)$ modified gravity models involving powers of curvature and torsion [9]. It is also a more general extension of the cosmological models based on Brans-Dicke-Jordan gravity [11] and non-minimally coupled Einstein-Electroweak theory [10]. It contains many more terms than a $U(2,2) = SU(2,2) \times U(1)$ gauge theory (conformal gravity and Maxwell theory). In addition it includes the kinetic and potential terms of a multiplet of 16 scalar fields (corresponding to a 4×4 matrix-valued scalar in the 16-dimensional adjoint representation of $U(2,2)$).

3 Moyal Star Product Deformations

The associative and noncommutative Moyal star product when the (inverse) symplectic form $\Omega^{\mu\nu} = -\Omega^{\mu\nu}$ does *not* have an X-dependence is defined as

$$(A_1 \ * \ A_2 \ )(Z) \ = \ exp \left(\ \frac{1}{2} \ \Omega^{\mu\nu} \ \partial_{X^\mu} \ \partial_{Y^\nu} \ \right) \ A_1(X) \ A_2(Y)|_{X=Y=Z} \ =$$

$$\sum_{n=0}^{\infty} \ \frac{(\frac{1}{2})^n}{n!} \ \Omega^{\mu_1\nu_1} \ \Omega^{\mu_2\nu_2} \ .... \ \Omega^{\mu_n\nu_n} \ (\partial^n_{\mu_1\mu_2......\mu_n} \ A_1) \ (\partial^n_{\nu_1\nu_2......\nu_n} \ A_2) \quad (3.1)$$

$$\partial^n_{\mu_1\mu_2......\mu_n} \ A_1(Z) \ \equiv \ \partial_{\mu_1} \ \partial_{\mu_2} \ ...... \ \partial_{\mu_n} \ A_1(Z). \quad (3.2a)$$

$$\partial^n_{\nu_1\nu_2......\nu_n} \ A_2(Z) \ \equiv \ \partial_{\nu_1} \ \partial_{\nu_2} \ ...... \ \partial_{\nu_n} \ A_2(Z). \quad (3.2b)$$

For simplicity we shall take the very special case of canonical noncommutativity $[X^\mu, X^\nu]_* = i\Theta^{\mu\nu} = \Omega^{\mu\nu} = constants$, such that the star product is the standard Moyal one. If the fields and their derivatives vanishing fast enough at infinity, one has the *cyclicity* property of the integral

$$\int \ A \ * B \ = \ \int \ A \ B \ + \ total \ derivative \ = \ \int \ A \ B \ = \ \int \ B \ * \ A \quad (3.3)$$

$$\int \ A \ * \ B \ * \ C \ = \ \int \ A \ (B \ * \ C) \ + \ total \ derivative \ = \ \int \ A \ (B \ * \ C) \ =$$

$$\int \ (B \ * \ C) \ A \ = \ \int \ (B \ * \ C) \ * \ A + total \ derivative \ = \ \int \ B \ *C \ * \ A \quad (3.4)$$

therefore, when the star product is associative and the fields and their derivatives vanishing fast enough at infinity (or there are no boundaries) one has

$$\int A * B * C = \int B * C * A = \int C * A * B . \tag{3.5}$$

The relations (3.3-3.5) are essential in order to construct invariant actions under star gauge transformations of the form $\delta F_{\mu\nu} = i[\xi, F_{\mu\nu}]_*$. The invariance of the actions is due to the associativity property of the star products and the cyclicity property of the integrals and of the Clifford scalar part of the geometric product of the Clifford generators. Taking the scalar part is the analog of the trace of a matrix product.

One should notice, for example, that when one has a Lie-algebraic type of noncommutativity, the $\Theta' s$ are now X-dependent $[X^\mu, X^\nu]_* = i\Theta^{\mu\nu}(X) = if^{\mu\nu}_\rho X^\rho$ so that the cyclicity property *no longer holds* since the star product is X-dependent. For a detailed study of how to remedy this problem see [25].

Due to the noncommutativity of the spacetime coordinates, the components of the Clifford-algebra valued field strength are now *modified* as follows

$$\mathcal{F}_{\mu\nu} = \mathcal{F}^C_{\mu\nu} \Gamma_C = (\partial_\mu \mathcal{A}^C_\nu - \partial_\nu \mathcal{A}^C_\mu) \Gamma_C -$$

$$\frac{i}{2} (\mathcal{A}^A_\mu * \mathcal{A}^B_\nu - \mathcal{A}^B_\nu * \mathcal{A}^A_\mu) \{ \Gamma_A, \Gamma_B \} - \frac{i}{2} (\mathcal{A}^A_\mu * \mathcal{A}^B_\nu + \mathcal{A}^B_\nu * \mathcal{A}^A_\mu) [\Gamma_A, \Gamma_B]. \tag{3.6}$$

The commutators $[\Gamma_A, \Gamma_B]$ and anti-commutators $\{ \Gamma_A, \Gamma_B \}$ in eq-(3.6), where A, B are polyvector-valued indices, can be obtained from all the relations provided in the Appendix. Notice that both the standard commutators *and* anticommutators of the gammas appear in eq-(3.6) and which now define the Clifford-algebra valued field strength in noncommutative spacetimes; i.e. if the products of fields were to commute one would have had only the Lie algebra commutator $\mathcal{A}^A_M \mathcal{A}^J_B [\Gamma_A, \Gamma_B]$ pieces without the anti-commutator $\{\Gamma_A, \Gamma_B\}$ contributions in the r.h.s of eq-(3.6).

We should remark that one is *not* deforming the Clifford algebra involving $[\Gamma_A, \Gamma_B]$ and $\{ \Gamma_A, \Gamma_B \}$ in eq-(3.6) but it is the "point" product algebra $\mathcal{A}^A_M * \mathcal{A}^B_N$ of the fields which is being deformed. (Quantum) q-Clifford algebras have been studied extensively by [29].

The symmetrized star product in terms of $\Theta^{\mu\nu} = constants$ is

$$\mathcal{A}^A_\mu *_s \mathcal{A}^B_\nu \equiv \frac{1}{2} \left(\mathcal{A}^A_\mu * \mathcal{A}^B_\nu + \mathcal{A}^B_\nu * \mathcal{A}^A_\mu \right) = \mathcal{A}^A_\mu \mathcal{A}^B_\nu +$$

$$\frac{i^2}{2!} \Theta^{\alpha\beta} \Theta^{\kappa\lambda} (\partial_\alpha \partial_\kappa \mathcal{A}^A_\mu) (\partial_\beta \partial_\lambda \mathcal{A}^B_\nu) + \ldots\ldots \tag{3.7}$$

the antisymmetrized (Moyal bracket) star product is

$$\mathcal{A}^A_\mu *_a \mathcal{A}^B_\nu \equiv \frac{1}{2} \left(\mathcal{A}^A_\mu * \mathcal{A}^B_\nu - \mathcal{A}^B_\nu * \mathcal{A}^A_\mu \right) = i \, \Theta^{\alpha\beta} (\partial_\alpha \mathcal{A}^A_\mu) (\partial_\beta \mathcal{A}^B_\nu) + \ldots\ldots \tag{3.8}$$

Early works on Moyal deformations of gravity can be found in [21],[18],[15]. Examples of an X-dependent $\Theta^{\mu\nu}(x)$ occurs in κ-deformed Minkowski spacetimes [17]. An extension of the Seiberg – Witten (SW) map for X-dependent $\Theta^{\mu\nu}(x)$ was provided by [25], [19], [20], [26], [27], among others, relating the non-Abelian noncommutative gauge fields based on *noncommutative* coordinates and the non-Abelian gauge fields based on *commutative* coordinates. It is then when one may construct the proper expressions for the $deformed$ field strengths, associated with the *noncommutative* coordinates, in terms of the undeformed field strengths. Since the former involve the universal enveloping algebra that is $infinite$ dimensional one must find a criteria to reduce the number of the degrees of freedom to a finite one; this is attained via the Seiberg-Witten map.

The main advantage of recurring to a Clifford algebraic formulation described in this work, is that both the commutator and anticommutator algebra in eq-(3.6) *closes* and this will simplify the laborious and cumbersome Seiberg-Witten procedure, involving the universal enveloping algebra. One may now proceed to perform the Moyal deformations of the field strengths and the action in a straightforward fashion.

The Moyal deformation of the terms S_5 encoding the MMCW gravitational action with a cosmological constant is given by

$$S_{(5)*} \;=\; \int d^4x\; \epsilon^{\mu\nu\rho\sigma}\; <\; \mathcal{F}^A_{\mu\nu}\; * \mathcal{F}^B_{\rho\sigma}\; * \phi^{abcd}\; \Gamma_A\; \Gamma_B\; \gamma_{abcd} > \;=$$

$$\int d^4x\; \epsilon_{abcd}\; \epsilon^{\mu\nu\rho\sigma}\; \varphi * \big(\; a_{51}\mathcal{F}^{ab}_{\mu\nu}\; * \mathcal{F}^{cd}_{\rho\sigma}\; + a_{52}\; \mathcal{F}^a_{\mu\nu}\; * \mathcal{F}^{bcd}_{\rho\sigma}\; + a_{53}\; \mathcal{F}_{\mu\nu}\; * \mathcal{F}^{abcd}_{\rho\sigma}\; \big) \;+$$

$$\int d^4x\; \epsilon_{abcd}\; \epsilon^{\mu\nu\rho\sigma}\; \varphi * \big(\; a_{54}\mathcal{F}^{ab}_{\mu\nu e}\; * \mathcal{F}^{ecd}_{\rho\sigma}\; + a_{55}\mathcal{F}^a_{\mu\nu e}\; * \mathcal{F}^{ebcd}_{\rho\sigma}\; + a_{56}\mathcal{F}^{ab}_{\mu\nu ef}\; * \mathcal{F}^{efcd}_{\rho\sigma}\big) \tag{3.9}$$

Before studying the Moyal deformations given by the action (3.9) one needs to establish the dictionary among the different Clifford $Cl(3,1)$ gauge field components and the fields of conformal gravity. From eqs-(2.2,2.4) one can infer the following correspondence

$$\mathcal{A}^{ab}_\mu \leftrightarrow \omega^{ab}_\mu,\;\; \mathcal{A}^a_\mu \leftrightarrow e^a_\mu,\;\; \mathcal{A}^{abc}_\mu \leftrightarrow f^a_\mu,\;\; \mathcal{A}^{abcd}_\mu \leftrightarrow b_\mu,\;\; \mathcal{A}_\mu \leftrightarrow a_\mu \tag{3.10}$$

Let us look at the $first$ order Θ-corrections to the components of $F^{ab}_{\mu\nu}$ given by eq-(2.6e) upon using eq-(3.6) and the equations in the Appendix

$${}^{(1)}\mathcal{F}^{ab}_{\mu\nu} \;=\; \mathcal{F}^{ab}_{\mu\nu} \;+\; \Theta^{\alpha\beta}\; \partial_\alpha \mathcal{A}^{abe}_\mu\; \partial_\beta \mathcal{A}_{\nu e} \;-\; \Theta^{\alpha\beta}\; \partial_\alpha \mathcal{A}^{abef}_\mu\; \partial_\beta \mathcal{A}_{\nu ef} \tag{3.11}$$

Repeating this procedure with the other field strength components in eqs-(2.6a-2.6d) yields the $first$ order Θ-corrections

$${}^{(1)}\mathcal{F}_{\mu\nu} \;=\; \mathcal{F}_{\mu\nu} \;+ 2\; \Theta^{\alpha\beta}\; \partial_\alpha \mathcal{A}^e_\mu\; \partial_\beta \mathcal{A}_{\nu e} \;-$$

$$2\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{ef}\ \partial_\beta \mathcal{A}_{\nu ef} - 2\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{efg}\ \partial_\beta \mathcal{A}_{\nu efg}\ +$$
$$2\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{efgh}\ \partial_\beta \mathcal{A}_{\nu efgh} \tag{3.12}$$

$$^{(1)}\mathcal{F}^a_{\mu\nu}\ =\ \mathcal{F}^a_{\mu\nu}\ - 2\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{aef}\ \partial_\beta \mathcal{A}_{\nu ef} \tag{3.13}$$

$$^{(1)}\mathcal{F}^{abc}_{\mu\nu}\ =\ \mathcal{F}^{abc}_{\mu\nu}\ + 2\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{ab}\ \partial_\beta \mathcal{A}^c_\nu\ - \frac{1}{2}\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{abef}\ \partial_\beta \mathcal{A}^c_{\nu ef} \tag{3.14}$$

$$^{(1)}\mathcal{F}^{abcd}_{\mu\nu}\ =\ \mathcal{F}^{abcd}_{\mu\nu} + 2\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{ab}\ \partial_\beta \mathcal{A}^{cd}_\nu\ + \frac{1}{2}\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{abe}\ \partial_\beta \mathcal{A}^{cd}_{\nu e}\ -$$
$$\frac{1}{4}\ \Theta^{\alpha\beta}\ \partial_\alpha \mathcal{A}_\mu^{abef}\ \partial_\beta \mathcal{A}^{cd}_{\nu ef} \tag{3.15}$$

We have indicated in the previous equations (3.11-3.15) that one has a first order correction by attaching explicitly a superscript $^{(1)}$ to the field strength expressions in the left hand side. The expressions for the components of $\mathcal{F}^A_{\mu\nu}$ in the right hand side are obtained explicitly from eqs-(2.6a-2.6e) by *replacing* the *commutative* gauge fields A^A_μ for the *noncommutative* ones $\mathcal{A}^A_\mu$.

Having written the above expressions (3.11-3.15) for the noncommutative field strengths in terms of the noncommutative gauge fields $\mathcal{A}^A_\mu$ it remains to write the latter noncommutative fields in terms of the commutative fields A^A_μ via the Seiberg-Witten map procedure. A lengthy procedure (see [22], [23]) yields the following expression for the noncommutative field strengths $\mathcal{F}_{\mu\nu}$ in terms of the commutative fields, after ommiting the Clifford-valued internal indices for simplicity since $\mathcal{F}_{\mu\nu} \equiv \mathcal{F}^A_{\mu\nu}\Gamma_A$, $\mathbf{F}_{\mu\nu} \equiv F^A_{\mu\nu}\Gamma_A$, $\mathbf{A}_\mu \equiv A^A_\mu \Gamma_A$,

$$\mathcal{F}_{\mu\nu}\ =\ \mathbf{F}_{\mu\nu}\ +\ \frac{1}{2}\Theta^{\alpha\beta}\{\mathbf{F}_{\mu\alpha},\ \mathbf{F}_{\nu\beta}\}\ -\ \frac{1}{4}\Theta^{\alpha\beta}\{\mathbf{A}_\alpha,\ (\partial_\beta + \mathcal{D}_\beta)\mathbf{F}_{\mu\nu}\} + \cdots \tag{3.16}$$

where the covariant derivative is defined in the adjoint representation

$$\mathcal{D}_\sigma \mathbf{F}_{\mu\nu}\ =\ \partial_\sigma \mathbf{F}_{\mu\nu}\ - i\ [\mathbf{A}_\sigma,\ \mathbf{F}_{\mu\nu}]. \tag{3.17}$$

Similarily, the Seiberg-Witten map allows to express the noncommutative scalar fields components present in the Clifford-valued field $\hat{\mathbf{\Phi}}$ in terms of the commutative scalar fields components present in the Clifford-valued field $\mathbf{\Phi}$

$$\hat{\mathbf{\Phi}}\ =\ \mathbf{\Phi}\ - \frac{1}{4}\ \Theta^{\alpha\beta}\ \{A_\alpha,\ (\partial_\beta + D_\beta)\mathbf{\Phi}\}\ +\ \ldots \tag{3.18}$$

see [22] for the case of a $SO(2,3)$-valued scalar field.

All that rests now is to evaluate the individual components of $\mathcal{F}_{\mu\nu} \equiv \mathcal{F}^A_{\mu\nu}\Gamma_A$ in the left hand side of (3.16) after performing the geometric products of the Clifford algebra generators appearing in the right hand side of (3.16) due to the decomposition of $\mathbf{F}_{\mu\nu} \equiv F^A_{\mu\nu}\Gamma_A$, $\mathbf{A}_\mu \equiv A^A_\mu \Gamma_A$. A similar procedure is performed in eq-(3.18).

We shall focus for now on the contribution up to first order in the Θ-terms to the Clifford bivector components $\mathcal{F}^{ab}_{\mu\nu}\gamma_{ab}$

$$
{}^{(1)}\mathcal{F}^{ab}_{\mu\nu} \; = \; F^{ab}_{\mu\nu} \; + \frac{1}{2} \; \Theta^{\alpha\beta} \; \left(\; F^{abc}_{\mu\alpha} \; F_{\nu\beta c} \; - \; F^{abcd}_{\mu\alpha} \; F_{\nu\beta cd} \; \right) +
$$

$$
\frac{1}{2} \; \Theta^{\alpha\beta} \; \left(\; F_{\mu\alpha c} \; F^{cab}_{\nu\beta} \; - \; F_{\mu\alpha cd} \; F^{cdab}_{\nu\beta} \; \right) \; + \; \ldots \tag{3.19}
$$

The extra terms in (3.19) are of the form $\Theta(A\partial F + AAF)$. For example

$$
-\frac{1}{4} \; \Theta^{\alpha\beta} \; \left(\; A^{abc}_\alpha \; \partial_\beta F_{\mu\nu c} \; - \; A^{abcd}_\alpha \; \partial_\beta F_{\mu\nu cd} \; \right) \; - \; \frac{1}{4} \; \Theta^{\alpha\beta} \; A^{abc}_\alpha \; A_{\beta cd} \; F^d_{\mu\nu} \tag{3.20}
$$

A similar procedure yields the expression for the noncommutative scalar field $\hat{\phi}^{abcd} = \epsilon^{abcd}\hat{\varphi}$ in terms of the commutative scalar and gauge fields.

The higher order corrections in Θ are obtained from the higher order terms in the definition of the Moyal star products and in those terms generated by the Seiberg-Witten map. Comparing our results, based on the Moyal deformations provided by eq-(3.9), with the results of others we should emphasize that the authors [23] had for their starting $U(2,2)$ invariant Lagrangian only the two terms (omitting numerical factors)

$$
L \; = \; \epsilon_{abcd} \; \left(\; F^{ab} \wedge F^{cd} \; + \; F \wedge F^{abcd} \; \right) \tag{3.21}
$$

instead of the *six* terms present in eq-(2.18). Secondly, they imposed by hand several *constraints* on the fields such that $F_{\mu\nu} = F^a_{\mu\nu} = F^{abc}_{\mu\nu} = F^{abcd}_{\mu\nu} = 0$. And thirdly, they set $\varphi = constant$.

Whereas the authors [22] used the Seiberg-Witten map procedure to construct a model of noncommutative gravity based on the gauge theory of $SO(2,3)$ defined over a noncommutative spacetime characterized by $\Theta^{\mu\nu}$ = constants. The starting Lagrangian in [22] was chosen to be

$$
L \; = \; \epsilon_{abcd} \; \varphi \; F^{ab} \wedge F^{cd} \tag{3.22}
$$

They found a cancellation of the Θ-terms to first order and which agrees with the results obtained by the authors [23] (for the group $U(2,2)$) when one has a canonical noncommutativity. It appears that the cancellation of the first order terms in $\Theta^{\mu\nu}$ might be model-independent.

Let us examine carefully the Moyal deformation of the eq-(2.16) after one inserts the explicit expressions for the *noncommutative* fields inside the integral

$$
\int \; d^4x \; \epsilon^{\mu\nu\rho\sigma} \; < \; \hat{\mathbf{\Phi}} \; * \; \mathcal{F}_{\mu\nu} \; * \mathcal{F}_{\rho\sigma} \; > \tag{3.23}
$$

the Θ-terms up to $first$ order in the integrand will be

$$
\mathbf{\Phi} \; (\mathcal{F}_{\mu\nu} \; * \mathcal{F}_{\rho\sigma})^{(1)} \; + \; \hat{\mathbf{\Phi}}^{(1)} \; (F_{\mu\nu} \; F_{\rho\sigma}) \; +
$$

$$\frac{i}{2}\ \Theta^{\alpha\beta}\ \partial_\alpha \Phi\ \partial_\beta (F_{\mu\nu}\ F_{\rho\sigma}) \tag{3.24}$$

The last term is a total derivative after an integration by parts due to the condition $\Theta^{\alpha\beta}\partial_\alpha\partial_\beta(...) = 0$. Hence the last term decouples (it can be dropped if the fields vanish fast enough at infinity or there are no boundaries). This is to be expected if one does not wish to introduce imaginary terms to the Moyal deformed action. The hats represent the noncommuative scalars and $\hat{\mathbf{\Phi}}^{(1)}$ is the first order contribution in Θ to the noncommutative scalar field. $\mathbf{\Phi}$ is the Clifford-valued scalar field with *commutative* components.

The first two terms of eq-(3.24) gives

$$-\frac{\Theta^{\alpha\beta}}{4}\ <\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{F}_{\mu\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ -$$
$$-\frac{\Theta^{\alpha\beta}}{4}\ <\ \mathbf{F}_{\mu\nu}\ \mathbf{F}_{\rho\sigma}\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ +$$
$$\frac{\Theta^{\alpha\beta}}{2}\ <\ \{\mathbf{F}_{\alpha\mu},\ \mathbf{F}_{\beta\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ +\ \ldots \tag{3.25}$$

The terms that one must extract the Clifford scalar part $< ... >$ are of the form

$$\Theta^{\alpha\beta}\ <\ \{\mathbf{F}_{\alpha\mu},\ \mathbf{F}_{\beta\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{3.26}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \ \{\ \mathbf{F}_{\alpha\rho},\ \mathbf{F}_{\beta\sigma}\ \}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{3.27}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \mathbf{F}_{\rho\sigma}\ \{\ \mathbf{F}_{\alpha\beta},\ \mathbf{\Phi}\ \}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{3.28}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \mathbf{F}_{\rho\sigma}\ \{\ \mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{\Phi}\ \}\ \ >\ \epsilon^{\mu\nu\rho\sigma} \tag{3.29}$$

$$\Theta^{\alpha\beta}\ <\ \mathbf{F}_{\mu\nu}\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{F}_{\rho\sigma}\}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma}\ +$$
$$\Theta^{\alpha\beta}\ <\ \{\mathbf{A}_\alpha,\ (\partial_\beta + D_\beta)\ \mathbf{F}_{\mu\nu}\}\ \mathbf{F}_{\rho\sigma}\ \mathbf{\Phi}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{3.30}$$

$$\frac{i}{2}\ \Theta^{\alpha\beta}\ <\ (\partial_\alpha \mathbf{F}_{\mu\nu})\ (\partial_\beta \mathbf{F}_{\rho\sigma})\ \mathbf{\Phi}\ > \tag{3.31}$$

To simplify the calculations let us *truncate* all the components of the field $\Phi = \Phi^A \Gamma_A$ to zero *except* $\Phi^{mnpq} \neq 0$, and all the components of $A^A_\mu \Gamma_A$ to zero *except* $A^{ab}_\mu \neq 0$. In this case one will have in explicit components form for the term in eq-(3.28) the following

$$\Theta^{\alpha\beta}\ <\ F^{ab}_{\mu\nu}\ \gamma_{ab}\ F^{cd}_{\rho\sigma}\ \gamma_{cd}\ \{F^{rs}_{\alpha\beta}\ \gamma_{rs},\ \phi_{mnpq}\ \gamma^{mnpq}\}\ >\ \epsilon^{\mu\nu\rho\sigma} \tag{3.32}$$

Recurring to the expressions displayed in the Appendix allow us to extract the Clifford scalar part $< ... >$ of the geometric products of the Clifford $Cl(3,1)$

algebra generators in eq-(3.32). After some straightforward but lengthy algebra it yields (up to a numerical factor)

$$\Theta^{\alpha\beta} \; \eta_{ac} \; F^{ap}_{\mu\nu} \; F^{cq}_{\rho\sigma} \; F^{mn}_{\alpha\beta} \; \phi_{mnpq} \; \epsilon^{\mu\nu\rho\sigma} \; = \; 0 \tag{3.33}$$

The reason this last expression eq-(3.33) is *vanishing* is due to the contraction structure of the tangent space indices and the *antisymmetry* of *all* the terms of eq-(3.33) under the exchange of indices with the *exception* of the (flat) tangent space metric $\eta_{ac} = \eta_{ca}$.

Following the same procedure with eq-(3.27) and using the same symmetry (antisymmetry) argument in the contraction of indices gives for the Clifford scalar part

$$\Theta^{\alpha\beta} \; \eta_{ac} \; F^{ap}_{\alpha\rho} \; F^{cq}_{\beta\sigma} \; F^{mn}_{\mu\nu} \; \phi_{mnpq} \; \epsilon^{\mu\nu\rho\sigma} \; = \; 0 \tag{3.34}$$

identical vanishing results occur with eq-(3.29)

$$\Theta^{\alpha\beta} \; \eta_{ac} \; F^{ap}_{\mu\nu} \; F^{cq}_{\rho\sigma} \; A^{mn}_{\alpha} \; (\partial_\beta + D_\beta)\phi_{mnpq} \; \epsilon^{\mu\nu\rho\sigma} \; = \; 0 \tag{3.35}$$

and with eq-(3.26).

The explicitly gauge noncovariant eq-(3.30) yields

$$\Theta^{\alpha\beta} \; \eta_{ac} \; \phi_{mnpq} \; F^{am}_{\mu\nu} \; A^{cn}_{\alpha} \; (\partial_\beta + D_\beta) F^{pq}_{\rho\sigma} \; \epsilon^{\mu\nu\rho\sigma} \; -$$

$$\Theta^{\alpha\beta} \; \eta_{ca} \; \phi_{mnpq} \; F^{cq}_{\rho\sigma} \; A^{an}_{\alpha} \; (\partial_\beta + D_\beta) F^{mp}_{\mu\nu} \; \epsilon^{\mu\nu\rho\sigma} \; = \; 0 \tag{3.36}$$

A way to see why eq-(3.36) is zero can be obtained by relabeling the indices $\mu\nu \leftrightarrow \rho\sigma, q \leftrightarrow m, a \leftrightarrow c$ in the second line of eq-(3.36) so that it becomes identical to the first line and leading to an exact cancellation due to the key minus sign in eq-(3.36) and antisymmetry $F^{pq}_{\rho\sigma} = -F^{qp}_{\rho\sigma}$.

Finally we examine eq-(3.31) giving

$$\frac{i}{2} \; \Theta^{\alpha\beta} \; (\partial_\alpha F^{mn}_{\mu\nu}) \; (\partial_\beta F^{pq}_{\rho\sigma}) \; \phi_{mnpq} \; \epsilon^{\mu\nu\rho\sigma} = \; 0 \tag{3.37}$$

The reason eq-(3.37) is zero is due to an overall antisymmetry. Relabeling the indices in eq-(3.37) $\mu\nu \leftrightarrow \rho\sigma, \alpha \leftrightarrow \beta, mn \leftrightarrow pq$ and due to the antisymmetry of $\Theta^{\alpha\beta} = -\Theta^{\beta\alpha}$ it leads to

$$\frac{i}{2} \, \Theta^{\alpha\beta} \, (\partial_\alpha F^{mn}_{\mu\nu}) \, (\partial_\beta F^{pq}_{\rho\sigma}) \, \phi_{mnpq} \, \epsilon^{\mu\nu\rho\sigma} \; = \; - \; \frac{i}{2} \, \Theta^{\alpha\beta} \, (\partial_\beta F^{pq}_{\rho\sigma}) \, (\partial_\alpha F^{mn}_{\mu\nu}) \; \phi_{pqmn} \, \epsilon^{\rho\sigma\mu\nu} =$$

$$- \; \frac{i}{2} \; \Theta^{\alpha\beta} \; (\partial_\alpha F^{mn}_{\mu\nu}) \; (\partial_\beta F^{pq}_{\rho\sigma}) \; \phi_{mnpq} \; \epsilon^{\mu\nu\rho\sigma} \tag{3.38}$$

therefore, if $X = -X \Rightarrow X = 0$.

Therefore, the Clifford scalar part of the *first* order contributions in the $\Theta^{\alpha\beta}$ terms of the Moyal-deformed action is *vanishing* when one *truncates* all the components of $\Phi = \Phi^A \Gamma_A$ to zero *except* $\Phi^{mnpq} \neq 0$, and all the components of $A^A_\mu \Gamma_A$ to zero *except* $A^{ab}_\mu \neq 0$. If one does *not* impose such truncation, one will have to consider the Moyal deformations of all other expressions in eqs-(2.21-2.25). It is unlikely that there is a cancellation of the Θ-terms up to first order in this most general case.

For example, let us examine the first order contribution in $\Theta^{\alpha\beta}$ of

$$\int \ < \ \left(F_{\mu\nu} * F^{abcd}_{\rho\sigma} * \phi_{abcd} \right)^{(1)} \ > \ \epsilon^{\mu\nu\rho\sigma} \tag{3.39}$$

One of the terms is

$$\frac{i}{2} \ \Theta^{\alpha\beta} \ (\partial_\alpha F_{\mu\nu}) \ (\partial_\beta F^{abcd}_{\rho\sigma}) \ \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \neq 0 \tag{3.40}$$

which is clearly nonvanishing and furnishes an imaginary contribution to the Moyal deformed action. The other imaginary contribution can be dropped because it yields a total derivative term

$$\int \ \frac{i}{2} \ \Theta^{\alpha\beta} \ \partial_\alpha (F_{\mu\nu} \ F^{abcd}_{\rho\sigma}) \ \partial_\beta \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ =$$

$$\int \ \frac{i}{2} \ \Theta^{\alpha\beta} \ \partial_\alpha \left(\ F_{\mu\nu} \ F^{abcd}_{\rho\sigma} \ \partial_\beta \phi_{abcd} \ \right) \ \epsilon^{\mu\nu\rho\sigma} \tag{3.41}$$

after an integration by parts.

One may cancel the contribution in eq-(3.40) by adding to eq-(3.39) the term

$$\int \ < \ \left(F^{abcd}_{\rho\sigma} * F_{\mu\nu} * \phi_{abcd} \right)^{(1)} \ > \ \epsilon^{\mu\nu\rho\sigma} \tag{3.42}$$

which amounts to a trivial *symmetrization* of the ordering in the products of the field strengths. Not surprisingly, due to this trivial *symmetrization*, there is cancellation due to the antisymmetry of $\Theta^{\alpha\beta}$.

Eq-(3.40) is gauge covariant because $\partial_\alpha F_{\mu\nu} = D_\alpha F_{\mu\nu}$ and $\partial_\beta F^{abcd}_{\rho\sigma} = D_\beta F^{abcd}_{\rho\sigma}$ after writing $F^{abcd}_{\rho\sigma} = \epsilon^{abcd} G_{\rho\sigma}$. Because there are a lot of gauge noncovariant terms in the expansion in powers of Θ, the authors [24] used the method of composite fields which enables to write the final results in a manifestly gauge covariant way. Therefore, the final results are manifestly gauge covariant as they should be.

There are many other terms in eq-(3.39) whose contribution is nonvanishing and real to first order in Θ, for example

$$\Theta^{\alpha\beta} \ F^{rs}_{\alpha\rho} \ F_{\beta\sigma rs} \ F^{abcd}_{\mu\nu} \ \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \neq \ 0 \tag{3.43}$$

$$\Theta^{\alpha\beta} \ F_{\mu\nu} \ F^{ab}_{\alpha\rho} \ F^{cd}_{\beta\sigma} \ \phi_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \neq \ 0 \tag{3.44}$$

due to the fact that now $F_{\mu\nu}$ and $F^{abcd}_{\mu\nu}$ are no longer zero. In particular, the terms of eq-(3.44) clearly form part of the deformed action $S_{(5)*}$ in eq-(3.9) and encoding the Moyal deformations of the MMCW gravitational action with a cosmological constant given by eq-(2.13) to first order in $\Theta^{\mu\nu}$. By setting

$\phi_{abcd} = \epsilon_{abcd}\varphi$ and recurring to the decomposition of $F^{ab}_{\alpha\rho}, F^{cd}_{\beta\sigma}$ provided in eqs-(2.11d, 2.13) one will have that eq-(3.44) yields the following Θ corrections to the vacuum energy density (in the modified action)

$$\frac{\varphi}{l^4}\ \Theta^{\alpha\beta}\ F_{\mu\nu}\ V^a_\alpha\ V^b_\rho\ V^c_\beta\ V^d_\sigma\ \epsilon_{abcd}\ \epsilon^{\mu\nu\rho\sigma} \tag{3.45}$$

where V^a_α is the vielbein field. If one identifies $\frac{\varphi}{l^2} \sim \frac{1}{G} = \frac{1}{L_P^2}$ and $\frac{\varphi}{l^4} = \rho_{vacuum}$ one can cancel the enormous ρ_{vacuum} energy density (when $\varphi = 1$) if the terms in eq-(3.45) are of the same order of magnitude, which implies that

$$\frac{\varphi}{l^4}\ \left(V^a_\mu\ V^b_\nu\ V^c_\rho\ V^d_\sigma\ +\ \Theta^{\alpha\beta}\ F_{\mu\nu}\ V^a_\alpha\ V^b_\rho\ V^c_\beta\ V^d_\sigma \right)\ \epsilon_{abcd}\ \epsilon^{\mu\nu\rho\sigma}\ =\ 0 \tag{3.46}$$

Setting the magnitude of the constant $\Theta^{\alpha\beta}$ parameters to be of the order of the Planck scale squared L_P^2 will fix the values of $F_{\mu\nu}$ in eq-(3-46) that furnish a cancellation of the huge vacuum energy density. Hence, the second terms in eq-(3.46) provide in general the x-dependent corrections to the vacuum energy density (cosmological constant). This result should be contrasted with those in [22].

One should notice that despite the generators of $U(2,2), SO(4,2), SO(2,3)$ can be expressed in terms of the Clifford algebra generators this does *not* imply that these algebras are *isomorphic* to the Clifford algebra. Hence one should *not* expect identical results as those obtained by other authors.

To sum up, when one does *not* impose constraints on the fields, there are first order contributions in the $\Theta^{\mu\nu}$ (constants) parameters in the Moyal deformations of a Clifford gauge theory formulation of gravity in variance with the previous results obtained by other authors and based on different gauge groups. This could provide a plausible cancellation mechanism of the huge vacuum energy density $1/L_P^4$. The *first* order contributions in the $\Theta^{\alpha\beta}$ terms of the Moyal-deformed action is *vanishing* in the special case when one *truncates* all the components of $\Phi = \Phi^A \Gamma_A$ to zero *except* $\Phi^{mnpq} \neq 0$, and all the components of $A^A_\mu \Gamma_A$ to zero *except* $A^{ab}_\mu \neq 0$.

Similarly, one obtains the Moyal deformations of the action $S[\mathbf{\Phi}]$ corresponding to the Clifford-valued scalar field $\mathbf{\Phi}$. Firstly, there is a modification of the gauge covariant derivative term (2.28a) due to the noncommutativity of the pointwise product of fields. Both commutators and anticommutators will appear in the Moyal deformations of eq-(2.28a) as they did in eq-(3.6). This will lead to corrections in powers of Θ of the gauge covariant derivative terms. Secondly, one performs the Moyal star products among all the terms present in the Clifford-valued scalar field action as it was done in eq-(3.9) after recurring to eq-(3.18).

APPENDIX

In this Appendix we shall write the (anti) commutator relations for the Clifford algebra generators and explain how to obtain the numerical coefficients a_{ij} in eqs-(2.16-2.25).

$$\frac{1}{2}\{\,\gamma_a,\,\gamma_b\,\} \;=\; g_{ab}\,\mathbf{1};\;\; \frac{1}{2}[\,\gamma_a,\,\gamma_b\,] \;=\; \gamma_{ab} \;=\; -\,\gamma_{ba},\;\; a,b=1,2,3,\cdots,m \quad (A.1)$$

$$[\,\gamma_a,\,\gamma_{bc}\,] \;=\; 2\;g_{ab}\;\gamma_c\;\;-2\;g_{ac}\;\gamma_b,\;\;\{\,\gamma_a,\,\gamma_{bc}\,\} \;=\; 2\;\gamma_{abc} \qquad (A.2)$$

$$[\,\gamma_{ab},\,\gamma_{cd}\,] \;=\; -2\;g_{ac}\;\gamma_{bd}\;\;+2\;g_{ad}\;\gamma_{bc}\;\;-2\;g_{bd}\;\gamma_{ac}\;\;+2\;g_{bc}\;\gamma_{ad} \qquad (A.3)$$

In general one has [30]

$$pq=\mathbf{odd},\;\;[\gamma_{m_1m_2....m_p},\,\gamma^{n_1n_2....n_q}] \;=\; 2\,\gamma^{n_1n_2....n_q}_{m_1m_2....m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\,\delta^{[n_1n_2}_{[m_1m_2}\;\gamma^{n_3....n_q]}_{m_3......m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\;\delta^{[n_1....n_4}_{[m_1....m_4}\;\;\gamma^{n_5....n_q]}_{m_5......m_p]} \;-\; .... \qquad (A.4)$$

$$pq=\mathbf{even},\;\;\{\,\gamma_{m_1m_2....m_p},\,\gamma^{n_1n_2....n_q}\,\} \;=\; 2\,\gamma^{n_1n_2....n_q}_{m_1m_2....m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\,\delta^{[n_1n_2}_{[m_1m_2}\;\gamma^{n_3....n_q]}_{m_3......m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\;\delta^{[n_1....n_4}_{[m_1....m_4}\;\;\gamma^{n_5....n_q]}_{m_5......m_p]} \;-\; .... \qquad (A.5)$$

$$pq=\mathbf{even},\;\;[\gamma_{m_1m_2....m_p},\,\gamma^{n_1n_2....n_q}] \;=\; \frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\delta^{[n_1}_{[m_1}\;\gamma^{n_2....n_q]}_{m_2....m_p]} \;-$$

$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\;\delta^{[n_1n_2n_3}_{[m_1m_2m_3}\;\;\gamma^{n_4....n_q]}_{m_4......m_p]} \;+\; .... \qquad (A.6)$$

$$pq=\mathbf{odd},\;\;\{\,\gamma_{m_1m_2....m_p},\,\gamma^{n_1n_2....n_q}\,\} \;=\; \frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\delta^{[n_1}_{[m_1}\;\gamma^{n_2....n_q]}_{m_2....m_p]} \;-$$

$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\;\delta^{[n_1n_2n_3}_{[m_1m_2m_3}\;\;\gamma^{n_4....n_q]}_{m_4......m_p]} \;+\; .... \qquad (A.7)$$

The generalized Kronecker delta is defined as the determinant

$$\delta^{a_1a_2.....a_k}_{b_1b_2.....b_k} \;\equiv\; det\begin{pmatrix} \delta^{a_1}_{b_1} & \ldots & \ldots & \delta^{a_1}_{b_k} \\ \delta^{a_2}_{b_1} & \ldots & \ldots & \delta^{a_2}_{b_k} \\ ------------ & & -------------- & \\ \delta^{a_k}_{b_1} & \ldots & \ldots & \delta^{a_k}_{b_k} \end{pmatrix} \qquad (A.8)$$

These equations are all that is need to evaluate the numerical coefficients of the action provided by eqs-(2.16-2.26). For instance if one wishes to extract the scalar part of the Clifford geometric product of $< \gamma_{mnp}\gamma_{rst}\gamma_{uv} >$, all one needs is to extract the *bivector* part of the product

$$\gamma_{mnp}\ \gamma^{rst} \ = \ \frac{1}{2}\ [\gamma_{mnp},\ \gamma^{rst}]\ +\frac{1}{2}\ \{\gamma_{mnp},\gamma^{rst}\}\ =$$

$$\frac{1}{2}\ (\ 2\ \gamma_{mnp}{}^{rst}\ - 36\ \delta_{[mn}^{[rs}\ \gamma_{p}{}^{t]}\ )\ +\frac{1}{2}\ (\ 18\ \delta_{[m}^{[r}\ \gamma_{np]}{}^{st]}\ - 12\ \delta_{[mnp]}^{rst}\ ) \qquad (A.9)$$

From eq-(A.9) one learns that its *bivector* piece is

$$-\ \frac{1}{2}\ 36\ \delta_{[mn}^{[rs}\ \gamma_{p}{}^{t]} \qquad (A.10)$$

and whose contraction with γ_{uv} will bring up the scalar part as follows

$$<\ \gamma^{pt}\ \gamma_{uv} > \ = \ -4\ \delta_{[uv]}^{pt} \qquad (A.11)$$

In this fashion one extracts the scalar part of the Clifford triple geometric product of generators and obtains the numerical coefficients a_{ij} in the action displayed by eqs-(2.16-2.26).

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