ESTIMATES FOR VON MANGOLDT, SECOND CHEBYSHEV AND RIEMANN'S J FUNCTIONS

Edigles Guedes

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Abstract. We prove some estimates for von Mangold function, second Chebyshev function and Riemann's J function by elementary methods.[1]

1. Introduction

Using Euler-Maclaurin sum, we demonstrated some estimates for von Mangoldt, second Chebyshev and Riemann's J functions.

Our main goal is to estimate

$$\limsup_{x\to\infty}\frac{\psi(x)}{x\log^2 x}<1,$$

as Theorem 7, by elementary methods.

2. Preliminaries

Theorem 1. *Let γ denotes the Euler's constant. Suppose that N and K are positive integers, then*

$$H_N=\gamma+\log N+\frac{1}{2N}-\sum_{k=1}^{K-1}\frac{B_{2k}}{2kN^{2k}}-\theta\frac{B_{2K}}{2KN^{2K}},$$

for $\theta\in(0,1)$.

Proof. See [1, page 507, Exercise 19]. □

Corollary 2. *Let γ denotes the Euler's constant. Suppose that n is positive integer, then*

$$H_n=\gamma+\log n+\frac{1}{2n}-\frac{B_2}{2n^2}-\theta\frac{B_4}{4n^4},$$

for $\theta\in(0,1)$.

Proof. Set $N=n$ and $K=2$ in Theorem 1. □

3. Theorems

Theorem 3. *For $n\in\mathbb{Z}_{\geqslant 1}$ and $\theta\in(0,1)$, then*

$$\Lambda(n)=\sum_{d|n}\mu(d)H_{n/d}-\frac{1}{2}\sum_{d|n}\mu(d)\frac{d}{n}+\frac{1}{12}\sum_{d|n}\mu(d)\frac{d^2}{n^2}-\frac{\theta}{120}\sum_{d|n}\mu(d)\frac{d^4}{n^4},$$

where $\Lambda(n)$ denotes the Von Mangold function, $\mu(n)$ denotes the Möbius function, H_n denotes the harmonic number and B_n denotes the Bernoulli number.

Proof. We know [1, page 24] that

$$\sum_{d|n}\Lambda(d)=\log n. \qquad (1)$$

By Corollary 2 and (1), we have

$$\sum_{d|n}\Lambda(d)=H_n-\gamma-\frac{1}{2n}+\frac{B_2}{2n^2}+\theta\frac{B_4}{4n^4},$$

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for $\theta \in (0,1)$. Hence, by Möbius inversion formula, we encounter

$$\Lambda(n) = \sum_{d|n} \mu(d) H_{n/d} - \gamma \sum_{d|n} \mu(d) - \frac{1}{2} \sum_{d|n} \mu(d) \frac{d}{n} + \frac{B_2}{2} \sum_{d|n} \mu(d) \frac{d^2}{n^2} + \theta \frac{B_4}{4} \sum_{d|n} \mu(d) \frac{d^4}{n^4},$$

ergo, for $n \geqslant 1$, we obtain

$$\Lambda(n) = \sum_{d|n} \mu(d) H_{n/d} - \frac{1}{2} \sum_{d|n} \mu(d) \frac{d}{n} + \frac{B_2}{2} \sum_{d|n} \mu(d) \frac{d^2}{n^2} + \theta \frac{B_4}{4} \sum_{d|n} \mu(d) \frac{d^4}{n^4}. \tag{2}$$

We put

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}$$

in (2). This completes the proof. □

Theorem 4. *For $n \in \mathbb{Z}_{\geqslant 1}$ and $\theta \in (0,1)$, then*

$$\Lambda(n) < \frac{\sigma_1(n)}{n} + \frac{\sigma_2(n)}{12n^2} - \theta \frac{\sigma_4(n)}{120n^4},$$

where $\Lambda(n)$ denotes the von Mangoldt function and $\sigma_k(n)$ denotes the divisor function.

Proof. We easily set the inequalities, for $n \in \mathbb{Z}_{\geqslant 1}$,

$$\sum_{d|n} \mu(d) H_{n/d} \leqslant \sum_{d|n} \mu(d) \sum_{k=1}^{n/d} \frac{1}{k} < \frac{1}{n} \sum_{d|n} \mu(d) d < \frac{1}{n} \sum_{d|n} d = \frac{\sigma_1(n)}{n}, \tag{3}$$

$$\sum_{d|n} \mu(d) \frac{d}{n} \leqslant \frac{1}{n} \sum_{d|n} d = \frac{\sigma_1(n)}{n}, \tag{4}$$

$$\sum_{d|n} \mu(d) \frac{d^2}{n^2} \leqslant \frac{1}{n^2} \sum_{d|n} d^2 = \frac{\sigma_2(n)}{n^2} \tag{5}$$

and

$$\sum_{d|n} \mu(d) \frac{d^4}{n^4} \leqslant \frac{1}{n^4} \sum_{d|n} d^4 = \frac{\sigma_4(n)}{n^4}. \tag{6}$$

From (3), (4),(5), (6) and Theorem 3, we get

$$\Lambda(n) < \frac{\sigma_1(n)}{n} + \frac{\sigma_2(n)}{12n^2} - \theta \frac{\sigma_4(n)}{120n^4},$$

for $n \in \mathbb{Z}_{\geqslant 1}$. This ends the proof. □

Theorem 5. *For $n \in \mathbb{Z}_{\geqslant 1}$ and $\theta \in (0,1)$, then*

$$\psi(n) = \sum_{m \leqslant n} \sum_{d|m} \mu(d) H_{m/d} - \frac{1}{2} \sum_{m \leqslant n} \sum_{d|m} \mu(d) \frac{d}{m} + \frac{1}{12} \sum_{m \leqslant n} \sum_{d|m} \mu(d) \frac{d^2}{m^2} - \frac{\theta}{120} \sum_{m \leqslant n} \sum_{d|m} \mu(d) \frac{d^4}{m^4}.$$

where $\psi(n)$ denotes the second Chebyshev function, $\mu(n)$ denotes the Möbius function, H_n denotes the harmonic number.

Proof. We put the partial summation, for $n \leqslant x$, with respect to n, in both members of the Theorem 3 and find

$$\sum_{n \leqslant x} \Lambda(n) = \sum_{n \leqslant x} \sum_{d|n} \mu(d) H_{n/d} - \frac{1}{2} \sum_{n \leqslant x} \sum_{d|n} \mu(d) \frac{d}{n} + \frac{B_2}{2} \sum_{n \leqslant x} \sum_{d|n} \mu(d) \frac{d^2}{n^2} + \theta \frac{B_4}{4} \sum_{n \leqslant x} \sum_{d|n} \mu(d) \frac{d^4}{n^4}. \tag{7}$$

On the other hand, we well-know that

$$\sum_{n\leqslant x}\Lambda(n)=\psi(x). \tag{8}$$

Replace the rigth hand side of (8) in the left hand side of (7) and $B_2=\frac{1}{6}, B_4=-\frac{1}{30}$, we have

$$\psi(x)=\sum_{n\leqslant x}\sum_{d|n}\mu(d)H_{n/d}-\frac{1}{2}\sum_{n\leqslant x}\sum_{d|n}\mu(d)\frac{d}{n} \tag{9}$$

$$+\frac{1}{12}\sum_{n\leqslant x}\sum_{d|n}\mu(d)\frac{d^2}{n^2}-\frac{\theta}{120}\sum_{n\leqslant x}\sum_{d|n}\mu(d)\frac{d^4}{n^4}.$$

Let $n\to m$ and $x\to n$. This ends the proof. □

Theorem 6. *For $n\in\mathbb{Z}_{\geqslant 1}$, then*

$$\frac{\psi(n)}{n}<(\gamma+\log n)H_n+\frac{1}{2},$$

where $\psi(n)$ denotes the second Chebyshev function, γ denotes the Euler's constant, $\log n$ denotes the natural logarithm function and H_n denotes the harmonic number.

Proof. Now, for $x\geqslant 1$, consider the expressions

$$\sum_{n\leqslant x}\sum_{d|n}\mu(d)\frac{d}{n}=\sum_{d\leqslant x}\mu(d)\frac{d}{x}\left\lfloor\frac{x}{d}\right\rfloor=x\sum_{d\leqslant x}\frac{\mu(d)}{d}\frac{d}{x}+O\left(\sum_{d\leqslant x}\left|\mu(d)\frac{d}{x}\right|\right) \tag{10}$$

$$=O\left(\sum_{d\leqslant x}\frac{d}{x}\right)=O(x);$$

$$\sum_{n\leqslant x}\sum_{d|n}\mu(d)\frac{d^2}{n^2}=\sum_{d\leqslant x}\mu(d)\frac{d^2}{x^2}\left\lfloor\frac{x}{d}\right\rfloor=\sum_{d\leqslant x}\mu(d)\frac{d}{x}+O\left(\sum_{d\leqslant x}\left|\mu(d)\frac{d^2}{x^2}\right|\right) \tag{11}$$

$$=\sum_{d\leqslant x}\mu(d)\frac{d}{x}+O\left(\sum_{d\leqslant x}\frac{d^2}{x^2}\right)=\sum_{d\leqslant x}\mu(d)\frac{d}{x}+O(x);$$

$$\sum_{n\leqslant x}\sum_{d|n}\mu(d)\frac{d^4}{n^4}=\sum_{d\leqslant x}\mu(d)\frac{d^4}{x^4}\left\lfloor\frac{x}{d}\right\rfloor=\sum_{d\leqslant x}\mu(d)\frac{d^3}{x^3}+O\left(\sum_{d\leqslant x}\left|\mu(d)\frac{d^4}{x^4}\right|\right) \tag{12}$$

$$=\sum_{d\leqslant x}\mu(d)\frac{d^3}{x^3}+O\left(\sum_{d\leqslant x}\frac{d^4}{x^4}\right)=\sum_{d\leqslant x}\mu(d)\frac{d^3}{x^3}+O(x);$$

$$\sum_{n\leqslant x}\sum_{d|n}\mu(d)H_{n/d}=\sum_{d\leqslant x}\mu(d)H_{\lfloor x/d\rfloor}\left\lfloor\frac{x}{d}\right\rfloor=x\sum_{d\leqslant x}\mu(d)\frac{H_{\lfloor x/d\rfloor}}{d}+O\left(\sum_{d\leqslant x}|\mu(d)H_{\lfloor x/d\rfloor}|\right) \tag{13}$$

$$=x\sum_{d\leqslant x}\mu(d)\frac{H_{\lfloor x/d\rfloor}}{d}+O\left(\sum_{d\leqslant x}H_{\lfloor x/d\rfloor}\right)=x\sum_{d\leqslant x}\mu(d)\frac{H_{\lfloor x/d\rfloor}}{d}+O(x)$$

and

$$B_2=\frac{1}{6}, B_4=-\frac{1}{30}. \tag{14}$$

From (9), (10), (11), (12), (13) and (14), we get

$$\psi(x)=x\sum_{d\leqslant x}\mu(d)\frac{H_{\lfloor x/d\rfloor}}{d}+O(x)-O\left(\frac{x}{2}\right) \tag{15}$$

$$+\frac{1}{12}\sum_{d\leqslant x}\mu(d)\frac{d}{x}+O\left(\frac{x}{12}\right)-\frac{\theta}{120}\sum_{d\leqslant x}\mu(d)\frac{d^3}{x^3}-O\left(\frac{\theta x}{120}\right)$$

$$=x\sum_{d\leqslant x}\mu(d)\frac{H_{\lfloor x/d\rfloor}}{d}+\frac{1}{12}\sum_{d\leqslant x}\mu(d)\frac{d}{x}-\frac{\theta}{120}\sum_{d\leqslant x}\mu(d)\frac{d^3}{x^3}+O(x).$$

Dividing (15) by x, we take

$$\frac{\psi(x)}{x} = \sum_{d \leqslant x} \mu(d) \frac{H_{\lfloor x/d \rfloor}}{d} + \frac{1}{12} \sum_{d \leqslant x} \mu(d) \frac{d}{x^2} - \frac{\theta}{120} \sum_{d \leqslant x} \mu(d) \frac{d^3}{x^4} + O(1). \tag{16}$$

We evaluate easily the inequality, for $x \geqslant 1$,

$$\sum_{d \leqslant x} \mu(d) \frac{H_{\lfloor x/d \rfloor}}{d} = \sum_{d \leqslant x} \frac{\mu(d)}{d} \left(\gamma + \log \lfloor x/d \rfloor + \frac{1}{2\lfloor x/d \rfloor} - \frac{B_2}{2 \lfloor x/d \rfloor^2} - \theta \frac{B_4}{4 \lfloor x/d \rfloor^4} \right) \tag{17}$$

$$= \gamma \sum_{d \leqslant x} \frac{\mu(d)}{d} + \sum_{d \leqslant x} \frac{\mu(d)}{d} \log \lfloor x/d \rfloor + \frac{1}{2} \sum_{d \leqslant x} \frac{\mu(d)}{d\lfloor x/d \rfloor} - \frac{B_2}{2} \sum_{d \leqslant x} \frac{\mu(d)}{d\lfloor x/d \rfloor^2} - \theta \frac{B_4}{4} \sum_{d \leqslant x} \frac{\mu(d)}{d\lfloor x/d \rfloor^4}$$

$$\leqslant \gamma H_x + H_x \log x + \frac{1}{2} - \frac{B_2}{2} \frac{1+x}{2x} - \theta \frac{B_4}{4} \frac{(1+x)^2}{4x^2},$$

where

$$\sum_{d \leqslant x} \frac{\mu(d)}{d} \leqslant \sum_{d \leqslant x} \frac{1}{d} = H_x,$$

$$\sum_{d \leqslant x} \frac{\mu(d)}{d} \log \lfloor x/d \rfloor \leqslant \sum_{d \leqslant x} \frac{\log x}{d} = H_x \log x,$$

$$\sum_{d \leqslant x} \frac{\mu(d)}{d\lfloor x/d \rfloor} \leqslant \sum_{d \leqslant x} \frac{\mu(d)d}{dx} = \frac{1}{x} \sum_{d \leqslant x} \mu(d) \leqslant \frac{1}{x} \sum_{d \leqslant x} 1 = 1,$$

$$\sum_{d \leqslant x} \frac{\mu(d)}{d\lfloor x/d \rfloor^2} \leqslant \sum_{d \leqslant x} \frac{\mu(d)d^2}{dx^2} = \sum_{d \leqslant x} \frac{\mu(d)d}{x^2} \leqslant \frac{1}{x^2} \sum_{d \leqslant x} d = \frac{1+x}{2x},$$

$$\sum_{d \leqslant x} \frac{\mu(d)}{d\lfloor x/d \rfloor^4} \leqslant \sum_{d \leqslant x} \frac{\mu(d)d^4}{dx^4} = \sum_{d \leqslant x} \frac{\mu(d)d^3}{x^4} \leqslant \frac{1}{x^4} \sum_{d \leqslant x} d^3 = \frac{(1+x)^2}{4x^2}.$$

On the other hand, we evaluate

$$\sum_{d \leqslant x} \mu(d) \frac{d}{x^2} \leqslant \frac{1}{x^2} \sum_{d \leqslant x} d = \frac{1+x}{2x}, \tag{18}$$

$$\sum_{d \leqslant x} \mu(d) \frac{d^3}{x^4} \leqslant \frac{1}{x^4} \sum_{d \leqslant x} d^3 = \frac{(1+x)^2}{4x^2} \tag{19}$$

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}. \tag{20}$$

From (16), (17), (18), (19) and (20), we find

$$\frac{\psi(x)}{x} \leqslant \gamma H_x + H_x \log x + \frac{1}{2} - \frac{1+x}{24x} + \theta \frac{(1+x)^2}{480x^2}$$

$$+ \frac{1+x}{24x} - \theta \frac{(1+x)^2}{480x^2} + O(1),$$

accordingly, for $x > 1$, we obtain

$$\frac{\psi(x)}{x} < (\gamma + \log x) H_x + \frac{1}{2}.$$

Let $x \to n$. This ends the proof. □

Theorem 7. *We have*

$$\limsup_{x \to \infty} \frac{\psi(x)}{x \log^2 x} < 1.$$

Proof. We have the following estimate for harmonic number

$$H_x \sim \gamma + \log x + \frac{1}{2x}. \tag{21}$$

From Theorem 5 and (21), we obtain

$$\frac{\psi(x)}{x} < \gamma^2 + 2\gamma \log x + \frac{\gamma}{2x} + \log^2 x + \frac{\log x}{2x} + \frac{1}{2}. \tag{22}$$

Divide (22) by $\log^2 x$ and let $x \to \infty$, as follows

$$\limsup_{x\to\infty} \frac{\psi(x)}{x \log^2 x} < \lim_{x\to\infty} \left(\frac{\gamma^2}{\log^2 x} + \frac{2\gamma}{\log x} + \frac{\gamma}{2x \log x} + 1 + \frac{1}{2x \log x} + \frac{1}{2 \log x} \right) = 1.$$

This completes the proof. □

Theorem 8. *For $\theta \in (0,1)$ and $n \in \mathbb{Z}_{\geqslant 1}$, then*

$$\psi(n) = \sum_{m=1}^{n} \sum_{d=1}^{m} \frac{\mu(d) H_{\lfloor m/d \rfloor}}{d} \sum_{r=0}^{d-1} e^{\frac{2\pi i m r}{d}} - \frac{1}{2} \sum_{m=1}^{n} \frac{1}{m} \sum_{d=1}^{m} \mu(d) \sum_{r=0}^{d-1} e^{\frac{2\pi i m r}{d}} \tag{23}$$

$$+ \frac{1}{12} \sum_{m=1}^{n} \frac{1}{m^2} \sum_{d=1}^{m} \mu(d) d \sum_{r=0}^{d-1} e^{\frac{2\pi i m r}{d}} - \frac{\theta}{120} \sum_{m=1}^{n} \frac{1}{m^4} \sum_{d=1}^{m} \mu(d) d^3 \sum_{r=0}^{d-1} e^{\frac{2\pi i m r}{d}}$$

or

$$\psi(n) = \sum_{m=1}^{n} \sum_{d=1}^{m} \frac{\mu(d) H_{\lfloor m/d \rfloor}}{d} \sum_{r=0}^{d-1} \cos\left(\frac{2\pi m r}{d}\right) - \frac{1}{2} \sum_{m=1}^{n} \frac{1}{m} \sum_{d=1}^{m} \mu(d) \sum_{r=0}^{d-1} \cos\left(\frac{2\pi m r}{d}\right)$$

$$+ \frac{1}{12} \sum_{m=1}^{n} \frac{1}{m^2} \sum_{d=1}^{m} \mu(d) d \sum_{r=0}^{d-1} \cos\left(\frac{2\pi m r}{d}\right) - \frac{\theta}{120} \sum_{m=1}^{n} \frac{1}{m^4} \sum_{d=1}^{m} \mu(d) d^3 \sum_{r=0}^{d-1} \cos\left(\frac{2\pi m r}{d}\right),$$

where $\psi(n)$ denotes the second Chebyshev function, $\mu(n)$ denotes the Möbius functin, H_n denotes the Harmonic number and B_n denotes the Bernoulli number.

Proof. Set the partial summation, for $n \leqslant x$, with respect to n, in both members of the Theorem 3:

$$\sum_{n \leqslant x} \Lambda(n) = \sum_{n \leqslant x} \sum_{d|n} \mu(d) H_{n/d} - \frac{1}{2} \sum_{n \leqslant x} \frac{1}{n} \sum_{d|n} \mu(d) d \tag{24}$$

$$+ \frac{B_2}{2} \sum_{n \leqslant x} \frac{1}{n^2} \sum_{d|n} \mu(d) d^2 + \theta \frac{B_4}{4} \sum_{n \leqslant x} \frac{1}{n^4} \sum_{d|n} \mu(d) d^4.$$

On the other hand, we know that

$$\sum_{n \leqslant x} \Lambda(n) = \psi(x). \tag{25}$$

Replace the rigth hand side of the Eq. (25) in the left hand side of the Eq. (24)

$$\psi(x) = \sum_{n \leqslant x} \sum_{d|n} \mu(d) H_{n/d} - \frac{1}{2} \sum_{n \leqslant x} \frac{1}{n} \sum_{d|n} \mu(d) d \tag{26}$$

$$+ \frac{B_2}{2} \sum_{n \leqslant x} \frac{1}{n^2} \sum_{d|n} \mu(d) d^2 + \theta \frac{B_4}{4} \sum_{n \leqslant x} \frac{1}{n^4} \sum_{d|n} \mu(d) d^4.$$

We well-know that

$$\sum_{r=0}^{d-1} e^{\frac{2\pi i n r}{d}} = \begin{cases} d, \text{if } d|n; \\ 0, \text{otherwise.} \end{cases} \tag{27}$$

From (26) and (27), it follows that

$$\psi(x)=\sum_{n=1}^{x}\sum_{d=1}^{n}\frac{\mu(d)H_{\lfloor n/d\rfloor}}{d}\sum_{r=0}^{d-1}e^{\frac{2\pi i n r}{d}}-\frac{1}{2}\sum_{n=12}^{x}\frac{1}{n}\sum_{d=1}^{n}\mu(d)\sum_{r=0}^{d-1}e^{\frac{2\pi i n r}{d}} \tag{28}$$

$$+\frac{B_2}{2}\sum_{n=1}^{x}\frac{1}{n^2}\sum_{d=1}^{n}\mu(d)d\sum_{r=0}^{d-1}e^{\frac{2\pi i n r}{d}}+\theta\frac{B_4}{4}\sum_{n=1}^{x}\frac{1}{n^4}\sum_{d=1}^{n}\mu(d)d^3\sum_{r=0}^{d-1}e^{\frac{2\pi i n r}{d}}.$$

Set $x\to n$ and $n\to m$ in (28); and, again, put (14) in (28)

$$\psi(n)=\sum_{m=1}^{n}\sum_{d=1}^{m}\frac{\mu(d)H_{\lfloor m/d\rfloor}}{d}\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}-\frac{1}{2}\sum_{m=1}^{n}\frac{1}{m}\sum_{d=1}^{m}\mu(d)\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}} \tag{29}$$

$$+\frac{1}{12}\sum_{m=1}^{n}\frac{1}{m^2}\sum_{d=1}^{m}\mu(d)d\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}-\frac{\theta}{120}\sum_{m=1}^{n}\frac{1}{m^4}\sum_{d=1}^{m}\mu(d)d^3\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}.$$

We well-know that

$$\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)=\begin{cases} d, \text{if } d|m;\\ 0, \text{otherwise.}\end{cases} \tag{30}$$

From (14), (26) and (30), it follows that

$$\psi(n)=\sum_{m=1}^{n}\sum_{d=1}^{m}\frac{\mu(d)H_{\lfloor m/d\rfloor}}{d}\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)-\frac{1}{2}\sum_{m=1}^{n}\frac{1}{m}\sum_{d=1}^{m}\mu(d)\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)$$

$$+\frac{1}{12}\sum_{m=1}^{n}\frac{1}{m^2}\sum_{d=1}^{m}\mu(d)d\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)-\frac{\theta}{120}\sum_{m=1}^{n}\frac{1}{m^4}\sum_{d=1}^{m}\mu(d)d^3\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right).$$

This ends the proof. □

Theorem 9. *For $\theta\in(0,1)$ and $n\in\mathbb{Z}_{\geqslant 2}$, then*

$$J(n)=\sum_{m\leqslant n}\frac{1}{\log m}\sum_{d|m}\mu(d)H_{m/d}-\frac{1}{2}\sum_{m\leqslant n}\frac{1}{m\log m}\sum_{d|m}\mu(d)d$$

$$+\frac{1}{12}\sum_{m\leqslant n}\frac{1}{m^2\log m}\sum_{d|m}\mu(d)d^2-\frac{\theta}{120}\sum_{m\leqslant n}\frac{1}{m^4\log m}\sum_{d|m}\mu(d)d^4,$$

where $J(n)$ denotes the Riemann's J function, $\mu(n)$ denotes the Möbius functin, H_n denotes the Harmonic number and B_n denotes the Bernoulli number.

Proof. In [2, page 63], we know

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{\log n}=J(x), \tag{31}$$

where $J(x)$ denotes the Riemann's J function.

Dividing the Theorem 3 by $\log n$ and summating for $n\leqslant x$, with respect to n, we find

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{\log n}=\sum_{n\leqslant x}\frac{1}{\log n}\sum_{d|n}\mu(d)H_{n/d}-\frac{1}{2}\sum_{n\leqslant x}\frac{1}{n\log n}\sum_{d|n}\mu(d)d \tag{32}$$

$$+\frac{B_2}{2}\sum_{n\leqslant x}\frac{1}{n^2\log n}\sum_{d|n}\mu(d)d^2+\theta\frac{B_4}{4}\sum_{n\leqslant x}\frac{1}{n^4\log n}\sum_{d|n}\mu(d)d^4.$$

From (14), (31) and (32), it follows that

$$J(x)=\sum_{n\leqslant x}\frac{1}{\log n}\sum_{d|n}\mu(d)H_{n/d}-\frac{1}{2}\sum_{n\leqslant x}\frac{1}{n\log n}\sum_{d|n}\mu(d)d \tag{33}$$

$$+\frac{1}{12}\sum_{n\leqslant x}\frac{1}{n^2\log n}\sum_{d|n}\mu(d)d^2-\frac{\theta}{120}\sum_{n\leqslant x}\frac{1}{n^4\log n}\sum_{d|n}\mu(d)d^4.$$

Let $n\rightarrow m$ and $x\rightarrow n$. This completes the proof. □

Theorem 10. *For $\theta\in(0,1)$ and $n\in\mathbb{Z}_{\geqslant 2}$, then*

$$J(n)=\sum_{m=2}^{n}\frac{1}{\log m}\sum_{d=1}^{m}\frac{\mu(d)H_{\lfloor m/d\rfloor}}{d}\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}-\frac{1}{2}\sum_{m=2}^{n}\frac{1}{m\log m}\sum_{d=1}^{m}\mu(d)\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}$$

$$+\frac{1}{12}\sum_{m=2}^{n}\frac{1}{m^2\log m}\sum_{d=1}^{m}\mu(d)d\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}-\frac{\theta}{120}\sum_{m=2}^{n}\frac{1}{m^4\log m}\sum_{d=1}^{m}\mu(d)d^3\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}},$$

and

$$J(n)=\sum_{m=2}^{n}\frac{1}{\log m}\sum_{d=1}^{m}\frac{\mu(d)H_{\lfloor m/d\rfloor}}{d}\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)-\frac{1}{2}\sum_{m=2}^{n}\frac{1}{m\log m}\sum_{d=1}^{m}\mu(d)\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)$$

$$+\frac{1}{12}\sum_{m=2}^{n}\frac{1}{m^2\log m}\sum_{d=1}^{m}\mu(d)d\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)-\frac{\theta}{120}\sum_{m=2}^{n}\frac{1}{m^4\log m}\sum_{d=1}^{m}\mu(d)d^3\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right),$$

where $J(n)$ denotes the Riemann's J function, $\mu(n)$ denotes the Möbius functin, H_n denotes the Harmonic number and B_n denotes the Bernoulli number.

Proof. From (27), (30) and Theorem 9, we encounter

$$J(n)=\sum_{m=2}^{n}\frac{1}{\log m}\sum_{d=1}^{m}\frac{\mu(d)H_{\lfloor m/d\rfloor}}{d}\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}-\frac{1}{2}\sum_{m=2}^{n}\frac{1}{m\log m}\sum_{d=1}^{m}\mu(d)\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}$$

$$+\frac{1}{12}\sum_{m=2}^{n}\frac{1}{m^2\log m}\sum_{d=1}^{m}\mu(d)d\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}}-\frac{\theta}{120}\sum_{m=2}^{n}\frac{1}{m^4\log m}\sum_{d=1}^{m}\mu(d)d^3\sum_{r=0}^{d-1}e^{\frac{2\pi i m r}{d}},$$

and

$$J(n)=\sum_{m=2}^{n}\frac{1}{\log m}\sum_{d=1}^{m}\frac{\mu(d)H_{\lfloor m/d\rfloor}}{d}\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)-\frac{1}{2}\sum_{m=2}^{n}\frac{1}{m\log m}\sum_{d=1}^{m}\mu(d)\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)$$

$$+\frac{1}{12}\sum_{m=2}^{n}\frac{1}{m^2\log m}\sum_{d=1}^{m}\mu(d)d\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right)-\frac{\theta}{120}\sum_{m=2}^{n}\frac{1}{m^4\log m}\sum_{d=1}^{m}\mu(d)d^3\sum_{r=0}^{d-1}\cos\left(\frac{2\pi m r}{d}\right),$$

This completes the proof. □

References

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E-mail adress: edigles.guedes@gmail.com