Foundations of a mathematical model of physical reality

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Abstract

This paper starts from the idea that physical reality implements a network of a small number of mathematical structures. Only in that way can be explained that observations of physical reality fit so well with mathematical methods.

The mathematical structures do not contain mechanisms that ensure coherence. Thus apart from the network of mathematical structures a model of physical reality must contain mechanisms that manage coherence such that dynamical chaos is prevented.

Reducing complexity appears to be the general strategy. The structures appear in chains that start with a foundation. The strategy asks that especially in the lower levels, the subsequent members of the chain emerge with inescapable self-evidence from the previous member. The chains are interrelated and in this way they enforce mutual restrictions.

As a consequence the lowest levels of a corresponding mathematical model of physical reality are rather simple and can be comprehended by skilled mathematicians.

In order to explain the claimed setup of physical reality, the paper investigates the foundation of the major chain. That foundation is a skeleton relational structure and it was already discovered and introduced in 1936.

The paper does not touch more than the first development levels. The base model that is reached in this way puts already very strong restrictions to more extensive models.

Some of the features of the base model are investigated and compared with results of contemporary physics.

If the model introduces new science, then it has fulfilled its purpose.

Contents

1 Introduction

Physical reality is that what physicists try to model in their theories. It appears that observations of features and phenomena of physical reality can often be explained by mathematical structures and mathematical methods.

This leads to the unorthodox idea that physical reality itself mimics a small set of mathematical structures. In that case physical reality will show the features and phenomena of these structures.

In humanly developed mathematics, mathematical structures appear in chains that start from a foundation and subsequent members of the chain emerge with inescapable self-evidence from the previous member. The chains are often interrelated and impose then mutual restrictions. It is obvious to expect a similar setup for the structures that are maintained by physical reality.

Physical reality is known to show coherence. Its behavior is far from chaotic. The mimicked mathematical structures do not contain mechanisms that ensure coherence. Thus apart from the network of mathematical structures a model of physical reality must contain mechanisms that manage coherence such that dynamical chaos is prevented. In physical reality, reducing complexity appears to be the general strategy.

One chain is expected to play a major role and its foundation can be viewed as the major foundation of the investigated model of physical reality. The discovery of this foundation is essential for explaining how the network of mimicked mathematical structures is configured.

2 The major chain

2.1 The foundation

This paper uses the skeleton relational structure that in 1936 was discovered by Garret Birkhoff and John von Neumann as the major foundation of the model. Birkhoff and von Neumann named it “quantum logic”[1].

The ~25 axioms that define an orthocomplemented weakly modular lattice form the first principles on which the model of physical reality is supposed to be built [2]. Another name for this lattice is ***orthomodular lattice***. Quantum logic has this lattice structure. Classical logic has a slightly different lattice structure. It is an orthocomplemented modular lattice. Due to this resemblance, the discoverers of the orthomodular lattice gave quantum logic its name. The treacherous name “quantum logic” has invited many scientists to deliberate in vain about the significance of the elements of the orthomodular lattice as logical propositions. For our purpose it is better to interpret the elements of the orthomodular lattice as ***construction elements*** rather than as ***logic propositions***.

The selected foundation can be considered as part of a recipe for modular construction. What is missing are the binding mechanism and a way to hide part of the relations that exist inside the modules from the outside of the modules. That functionality is realized in higher levels of the model.

2.2 Extending the major chain

The next level of the major chain of mathematical structures ***emerges with inescapable self-evidence*** from the selected foundation. Not only quantum logic forms an orthomodular lattice, but also the set of closed subspaces of an infinite dimensional separable Hilbert space forms an orthomodular lattice [1].

Where the orthomodular lattice was discovered in the thirties, the Hilbert space was introduced shortly before that time [3]. The Hilbert space is a vector space that features an inner product. The orthomodular lattice that is formed by its set of subspaces makes the Hilbert space a very special vector space.

The Hilbert space adds extra functionality to the orthomodular lattice. This extra functionality concerns the superposition principle and the possibility to store numeric data in eigenspaces of normal operators. In the form of Hilbert vectors the Hilbert space features a finer structure than the orthomodular lattice has.

These features caused that Hilbert spaces were quickly introduced in the development of quantum physics.

Numbers do not exist in the realm of a pure orthomodular lattice. Via the Hilbert space number systems emerge into the model. Number systems do not find their foundation in the major chain. Instead they belong to another chain of mathematical structures. The foundation of that chain concerns mathematical sets.

The Hilbert space can only handle members of a division ring for specifying superposition coefficients, for the eigenvalues of its operators and for the values of its inner products. Only three suitable division rings exist: the real numbers, the complex numbers and the quaternions. These facts were known in the thirties but became a thorough mathematical prove in the second half of the twentieth century [4].

Separable Hilbert spaces act as structured storage media for discrete data that can be stored in real numbers, complex numbers or quaternions. Quaternions enable the storage of 1+3D dynamic geometric data that have an Euclidean geometric structure.

The confinement to division rings puts strong restrictions onto the model. These restrictions reduce the complexity of the whole model.

Thus, selecting a skeleton relational structure that is an orthomodular lattice as the foundation of the model already puts significant restrictions to the model. On the other hand, as can be shown, this choice promotes modular construction. In this way it eases system configuration and the choice significantly reduces the relational complexity of the final model.

3 Consequences of the currently obtained model

The orthomodular lattice can be interpreted as a ***part of a recipe for modular construction***. What is missing are means to ***bind modules*** and means to ***hide relations*** that stay inside the module. This functionality must be supplied by extensions of the model. It is partly supplied by the superposition principle, which is introduced via the separable Hilbert space. For another part

The current model does not yet support coherent dynamics. The selected foundation and its extension to a separable Hilbert space can be interpreted in the following ways:

- Each discrete construct in this model is supposed to expose the skeleton relational structure that is defined as an orthomodular lattice.
- Each discrete construct in this model is either a module or a modular system.
- Every discrete construct in this model can be represented by a closed subspace of a single infinite dimensional separable quaternionic Hilbert space.
- Every module and every modular system in this model can be represented by a closed subspace of a single infinite dimensional separable quaternionic Hilbert space.

However

- Not every closed subspace of the separable Hilbert space represents a module or a modular system.

Modular construction eases system design and system configuration. Modular construction handles its resources in a very economically way. With sufficient resources present it can generate very complicated constructs.

The modular construction recipe is certainly the most influential rule that exists in the generation of physical reality. Even without intelligent design it achieved the construction of intelligent species.

4 Supporting continuums

The separable Hilbert space can only handle discrete numeric data. Physical reality also supports continuums. The eigenspaces of the operators of the separable Hilbert space are countable. Continuums are not countable. Separable Hilbert spaces do not support continuums.

Soon after the introduction of the Hilbert space, scientists tried to extend the separable Hilbert space to a non-separable version that supports operators, which feature continuums as eigenspaces. With his bra-ket notation for Hilbert vectors and operators and by introducing generic functions, such as the Dirac delta function, Paul Dirac introduced ways to handle continuums [5]. This approach became proper mathematical support in the sixties when the Gelfand triple was introduced [6].

Every infinite dimensional separable Hilbert space owns a Gelfand triple. In fact the separable Hilbert space can be seen as ***embedded*** inside this Gelfand triple. How this embedding occurs in mathematical terms is still obscure. It appears that the embedding process allows a certain amount of freedom that is exploited by the mechanisms, which are contained in physical reality and that have the task to ensure coherence.

In the separable Hilbert space the closed subspaces have a well-defined numeric dimension. In contrast, in the non-separable companion the dimension of closed subspaces is in general not defined. The embedding of subspaces of the separable Hilbert space in a subspace of the non-separable Hilbert space that represents an encapsulating composite will at least partly hide the characteristics and interrelations of embedded constituents. This hiding is required for constituents of modular systems.

4.1 Representing continuums and continuous functions

Paul Dirac introduced the bra-ket notation that eases the formulation of Hilbert space habits [5]. By using bra-ket notation, operators that reside in the separable Hilbert space and correspond to continuous functions, can easily be defined starting from an orthogonal base of vectors. This works both in separable Hilbert spaces as well as in non-separable Hilbert spaces.

Let $\{q_i\}$ be the set of rational quaternions and $\{|q_i\rangle\}$ be the set of corresponding base vectors. They are eigenvectors of a normal operator $\mathcal{R} = |q_i\rangle q_i \langle q_i|$. Here we enumerate the base vectors with index i.

$\mathcal{R} = |q_i\rangle q_i \langle q_i|$ is the configuration parameter space operator.

Let $f(q)$ be a quaternionic function.

$f = |q_i\rangle f(q_i) \langle q_i|$ defines a new operator that is based on function $f(q)$.

In a non-separable Hilbert space, such as the Gelfand triple, the continuous function $f(q)$ can be used to define an operator, which features a continuum eigenspace that acts as target space of the function and uses the eigenspace of the reference operator $\mathfrak{R} = |q\rangle q \langle q|$. The eigenspace reference operator $\mathfrak{R} = |q\rangle q \langle q|$ acts as a flat parameter space that is spanned by a quaternionic number system.

Via the continuous quaternionic function $\mathcal{F}(q)$, the operator $\mathcal{F} = |q\rangle \mathcal{F}(q) \langle q|$ defines a curved continuum $\mathcal{F}$. This operator and the continuum reside in the Gelfand triple, which is a non-separable Hilbert space.

Here we no longer enumerate the base vectors with index i. We just use the name of the parameter.

In general the dimension of a subspace loses its significance in the non-separable Hilbert space.

The continuums that appear as eigenspaces in the non-separable Hilbert space can be considered as quaternionic functions that also have a representation in the corresponding infinite dimensional separable Hilbert space. Both representations use a flat parameter space that is spanned by quaternions.

The parameter space operators will be treated as reference operators. The rational quaternionic eigenvalues $\{q_i\}$ that occur as eigenvalues of the reference operator in the separable Hilbert space map onto the rational quaternionic eigenvalues $\{q_i\}$ that occur as subset of the quaternionic eigenvalues $\{q\}$ of the reference operator in the Gelfand triple.

Embedding occurs in a continuum that is defined by a quaternionic function $\mathfrak{C}(q)$***.***

5 Symmetry flavors

Due to the four dimensions of quaternions, quaternionic number systems exist in 16 versions $\{q^x\}$ that differ only in their discrete symmetry set. The quaternionic number systems $\{q^x\}$ correspond to 16 versions $\{q_i^x\}$ of rational quaternions.

Half of these versions are right handed and the other half are left handed.

The index x can be ⓪,①,②,③,④,⑤,⑥,⑦,⑧,⑨,⑩,⑪,⑫,⑬,⑭, or ⑮.

This index represents the ***symmetry flavor*** of the indexed subject.

A reference operator $\mathcal{R}^x = |q_i^x\rangle q_i^x \langle q_i^x|$ in $\mathfrak{H}$ maps into a reference operator $\mathfrak{R}^x = |q^x\rangle q^x \langle q^x|$ in $\mathcal{H}$.

Quaternions can be mapped to Cartesian coordinates along the orthonormal base vectors 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$; with $\boldsymbol{ij} = \boldsymbol{k}$

- If the real part is ignored, then still ***8*** symmetry flavors result
- Symmetry flavors are marked by special indices, for example $\boldsymbol{a}$④
- They are also marked by colors $N, R, G, B, \bar{B}, \bar{G}, \bar{R}, \bar{N}$
- Half of them is right handed, **R**
- The other half is left handed, **L**
- The colored rectangles reflect the directions of the coordinate axes

Symmetry flavors of members of coherent sets:

 $\boldsymbol{a}$⓪ N ***R***

 $\boldsymbol{a}$① R ***L***

 $\boldsymbol{a}$② G ***L***

 $\boldsymbol{a}$③ B ***L***

 $\boldsymbol{a}$④ $\bar{B}$ ***R***

 $\boldsymbol{a}$⑤ $\bar{G}$ ***R***

 $\boldsymbol{a}$⑥ $\bar{R}$ ***R***

 $\boldsymbol{a}$⑦ W ***L***

Per definition, members of coherent sets $\{a_i\}$ of quaternions all feature the same symmetry flavor.

Continuous quaternionic functions $\psi(q)$ do not switch to other symmetry flavors.

The reference symmetry flavor of function $\psi(q)$ is the symmetry flavor of its parameter space .

Also continuous functions and continuums feature a symmetry flavor. The reference symmetry flavor of a continuous function $\psi(q)$ is the symmetry flavor of the parameter space $\{q\}$.

If a continuous quaternionic function describes the density distribution of a set $\{a_i\}$ of discrete objects a_i, then this set must be attributed with the same symmetry flavor.

6 Modules

Modules are represented by closed subspaces, but not every closed subspace represents a module or modular system. In fact only a small minority of the closed subspaces will act as actual modules. What renders a closed subspace into a module and what combines modules into subsystems or systems? The answers to these questions can only be found by investigating the contents of the closed subspaces.

6.1 Module content

In free translation, the spectral theorem for normal operators that reside in a separable Hilbert space states: "If a normal operator maps a closed subspace onto itself, then the subspace is spanned by an orthonormal base consisting of eigenvectors of the operator." The corresponding eigenvalues characterize this closed subspace.

It is possible to select a quaternionic normal operator for which a subset of the eigenvectors span the closed subspace and the corresponding eigenvalues describe the dynamic geometric data of this module. By ordering the real values of these eigenvalues, the geometric data become functions of what we will call ***progression***.

6.1.1 Progression window

Here we only consider modules for which the content is ***well-ordered***. This means that every progression value is only used once.

For the most primitive modules the closed subspace may be reduced until it covers a generation cycle in which the statistically averaged characteristics of the module mature to fixed values. The resulting closed subspace acts as a ***sliding progression window***.

The sliding window separates a deterministic history from a partly uncertain future. Inside the sliding window ***a dedicated mechanism fills the eigenspace*** of operator $a = |a_j\rangle a_j\langle a_j|$. The mechanism is a function of progression. If it is a cyclic function of progression, then the module is recurrently regenerated.

6.2 Map into a continuum

By imaging the discrete eigenvalues into a reference continuum, the discrete eigenvalues form a ***swarm*** $\{a_j\}$, which is a subset of the rational quaternions $\{q_i\}$. At the same time the discrete eigenvalues form a ***hopping path***. With other words the swarm forms a spatial map of the dynamic hopping of the point-like object.

We use a map $\mathcal{M}_1$ of the swarm into the reference continuum that is the eigenspace of the reference operator, which resides in the Gelfand triple.

In the model, two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are relevant. The first map $\mathcal{M}_1$ has the flat reference continuum $\mathfrak{R} = \{q\}$ as target. ***This reference continuum is not affected by the imaging***. The second map $\mathcal{M}_2$ has the curved continuum $\mathfrak{C}(q)$ as target. $\mathfrak{C}(q)$ ***is affected by the embedding process***.

If the image of the hopping path is closed in the reference continuum $\mathfrak{R} = \{q\}$, then the swarm stays at the same location in the map $\mathcal{M}_1$ onto the reference continuum. This does not need to be the case for the map $\mathcal{M}_2$ into the embedding continuum $\mathfrak{C}(q)$. The two target continuums $\mathfrak{R}$ and $\mathfrak{C}$ reside as eigenspaces in the Gelfand triple.

We will interpret the two maps to work in succession. The second map $\mathcal{M}_2$ maps the reference continuum $\{q\}$ that resides in the Gelfand triple into the embedding continuum $\mathfrak{C}(q)$, which also resides in the Gelfand triple.

6.3 Coherent elementary modules

In the map to the reference continuum, coherent elementary modules feature a closed hopping path. Further, for coherent elementary modules, the map of the location swarm into the reference continuum corresponds to an operator that is defined by a continuous function. That continuous function is a normalized location density distribution and it has a Fourier transform.

Coherence is ensured by a mechanism that selects the eigenvalues such that a coherent swarm is generated.

Coherent elementary modules are characterized by a symmetry flavor. When mapped into a reference continuum that is eigenspace of reference operator $\mathfrak{R}^x = |q^x\rangle q^x\langle q^x|$ the module is characterized by a ***symmetry related charge***, which is *located at the center of symmetry*.

The size and the sign of the symmetry related charge depends on the difference of the symmetry flavor of the coherent swarm $\{a_j^x\}$ with respect to the symmetry flavor of the reference continuum $\mathcal{R}^{⓪}$.

6.4 The function of coherence

Embedding of point-like objects into the affected embedding continuum spreads the reach of the separate embedding locations and offers the possibility to bind modules. The spread of the embedded point-like object is defined by the Green's function of the non-homogeneous wave equation. However, spurious embedding locations have not enough strength and not enough reach to implement an efficient binding effect. In contrast, coherent location swarms offer enough locality and enough strength in order to bind two coherent swarms that are sufficiently close.

Imaging of the location swarm into the reference continuum is only used to define coherence and to indicate the influence of the symmetry related charges. The embedding into the affected continuum is used to exploit the corresponding potential binding effect of the swarm.

7 Fields

In this paper, fields are continuums that are target spaces of quaternionic functions that define eigenspaces of operators, which reside in the Gelfand triple.

Quaternionic functions and their differentials can be split in a real scalar functions and imaginary vector functions.

7.1 Maps

The orthomodular base model consist of two related Hilbert spaces.

- A separable Hilbert space $\mathfrak{H}$ that acts as a descriptor of the properties of all discrete objects.
- A non-separable Hilbert space $\mathcal{H}$ that acts as a descriptor of the properties of all continuums.

Controlling mechanisms fill the module related subspaces with data and embed Hilbert space $\mathfrak{H}$ into Hilbert space $\mathcal{H}$.

A closed subspace in $\mathfrak{H}$ maps into a subspace of $\mathcal{H}$.

7.2 Parameter spaces

The reference operators $\mathfrak{R}^x$ that reside in the Gelfand triple deliver simple fields that can act as flat parameter spaces. These fields are not affected by the maps. However, $\mathfrak{R}^{⓪}$ acts as the playground of the symmetry related charges.

7.3 Embedding field

In $\mathcal{H}$ the operator $\mathfrak{C} = |q^{⓪}\rangle\mathfrak{C}(q^{⓪})\langle q^{⓪}|$ is defined by function $\mathfrak{C}(q^{⓪})$ and represents an embedding continuum $\mathfrak{C}$. This continuum gets affected by the embedding process and thus deforms dynamically.

The embedding continuum is always and everywhere present. It is deformed and vibrated by discrete artifacts that are embedded in this field.

7.4 Symmetry related fields

Due to their four dimensions, quaternionic number systems exist in sixteen versions that only differ in their symmetry flavor. The elements of coherent sets of quaternions belong to the same symmetry flavor.

Symmetry related charges are located at centers of such symmetries. The size and the sign of the symmetry related charge depends on the difference of the symmetry flavor of the coherent set with respect to the symmetry flavor of the embedding continuum. The coherent sets that belong to different symmetry related charges try to compensate the symmetry differences. Equally signed charges repel and differently signed charges attract.

The symmetry related charges ***generate a field*** that differs from the embedding continuum. This symmetry related field also plays a role in the binding of modules, but that role differs from the role of the embedding continuum.

The symmetry related field can affect the locations of the symmetry related charges in the first map $\mathcal{M}_1$. *This means that with the centers of symmetry also the corresponding coherent swarms are relocated.*

The symmetry related charges do not directly affect the embedding continuum $\mathfrak{C}(q)$. Their effects are confined to map $\mathcal{M}_1$. However, with their action the symmetry related charges ***relocate*** the centers of the corresponding coherent swarms.

The symmetry related charges are point charges. As a consequence the range of the field that is generated by a single charge is rather limited. The corresponding Green's function diminishes as 1/r with distance r from the charge.

Fields of point charges superpose. A wide spread distribution of symmetry related point charges can generate a corresponding wide spread symmetry related field.

Since the coherent swarms that underlay the symmetry related charges are recurrently regenerated, this also holds for the locations of the corresponding charges.

7.5 Free space

In the separable Hilbert space, the eigenvectors that do not belong to a module subspace together span free space. Thus module eigenvalues form open subsets of the set of rational quaternions.

Between any two elements of $\{a_j\}$ exist infinite many elements of $\{q_i\}$ that are not eigenvalues of location operator $a = |a_j\rangle a_j\langle a_j|$.

$\{q_i\}$ is eigenspace of reference operator $\mathcal{R} = |q_i\rangle q_i\langle q_i|$.

The subspace is a closed set of Hilbert vectors, but it is not a closed set of eigenvectors of $\mathcal{R}$. Thus, despite the fact that the module subspace is closed, free space is mixed with the playground of the module's swarm. The swarm is not a closed set of eigenvalues.

7.6 Information transfer

Information travels in the embedding continuum $\mathfrak{C}(q)$. It is transported by wave fronts that occur in this continuum. These wave fronts are solutions of the homogeneous wave equation. For not too violent conditions, the embedding process is described by the non-homogeneous wave equation.

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\chi = \frac{\partial^2\chi}{\partial\tau^2} + \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2} = \rho \quad (1)$$

The homogeneous wave equation is a restriction of the non-homogeneous wave equation that results after zeroing the triggers ρ.

8 The orthomodular base model

Now we have achieved a level in which the major chain of mathematical structures does no longer offer an inescapable self-evident extension. The model uses separable and non-separable Hilbert spaces in order to store numeric data that can describe a series of discrete objects that are embedded in a continuum. The real parts of the parameters can be used to order the parameters and the target values of functions. If properly ordered these descriptions can represent a sequence of static status quos. However, this model contains no means to control the ***coherence*** between the subsequent members of the sequence.

We will call this stage of the model development "***The orthomodular base model***". Any further development of the model involves the insertion of mechanisms that ensure the coherence between the subsequent members of the sequence of static status quos.

The orthomodular base model describes the relational structure of modular systems. Via the management mechanisms it can add characteristics to the modules. These characteristics are based on eigenvalues of normal operators that reside in the separable Hilbert space and have eigenvectors in the closed subspace that represents the module. The Hilbert spaces only support storage and description. The management mechanisms represent the actual drivers of the model.

The numeric data that occur in the orthonormal base model must be taken from division rings. The most elaborate choice for these data are quaternions. The peculiarities of these quaternions influence the features and the behavior of the discrete objects and the fields that occur in the orthonormal model.

Many of these peculiarities are hardly known by scientists. As far as they apply to this paper these subjects are treated in the Appendix.

9 Embedding

In the form of eigenvalues of reference operator $\mathcal{R}^x$ the set $\{a_i^x\}$ correspond to sets of eigenvectors $\{|a_i^x\rangle\}$ that span a corresponding closed subspace. This restricts operator $\mathcal{R}^x$ to operator $\sigma^x = |a_j^x\rangle a_j^x \langle a_j^x|$. The selection may change dynamically in a stepwise way. In that case the closed subspace acts as a sliding window within a larger subspace that covers all progression values, including the history of the sliding window. The sliding window covers the recurrent regeneration of the set $\{a_i^x\}$. During this period the statistical properties of the set stabilize.

9.1 The mapper

The mapper $\wp^x$ maps subsets$\{a_i^x\}$ of $\{q_i^x\}$ onto the continuum $\mathfrak{C}$ defined by function $\mathfrak{C}(q)$.

The action of the mapper can be split into three steps.

The two first steps form a map from a subspace of the eigenspace of $\mathcal{R}^x$ to the corresponding eigenspace of $\mathfrak{R}^{⓪}$.

The first step converts $\mathcal{R}^x$ into $\mathcal{R}^{⓪}$. It only switches the symmetry flavor of the reference operator.

The second step embeds $\mathfrak{H}$ into $\mathcal{H}$ by mapping $\mathcal{R}^{⓪}$ to $\mathfrak{R}^{⓪}$.It is a map between quaternions with rational valued components and a continuum consisting of quaternions that have real valued components. The discrete set and the continuum have the same symmetry flavor, which is the reference symmetry flavor.

The mapper $\wp^x$ is affected by the movements of the symmetry related charges that are initiated by the symmetry related field. It means that the centers of the coherent swarms are ***relocated*** due to the effects of the symmetry related fields on the locations of the symmetry related charges.

The third step is performed completely inside $\mathcal{H}$ by operator $\mathfrak{C}$.

The symmetry flavor switch occurs in $\mathfrak{H}$ and the curvature of the continuum occurs in $\mathcal{H}$.

9.2 Coherence

Closed subspaces of a separable Hilbert space are characterized by a countable set of eigenvalues of a normal operator. Dedicated mechanisms ensure the coherence of the set of eigenvalues.

Due to the four dimensions of quaternions, quaternionic number systems exist in 16 versions that only differ in their discrete symmetry set. For example right handed quaternions exist and left handed quaternions exist.

A ***coherent set*** of discrete quaternions is defined by two criteria:

1. All members of the set belong to the same symmetry flavor.
2. The set can be described by a continuous density distribution.

The second requirement involves a ***map*** $\wp^x(\{a_i^x\})$ onto a continuum that ***embeds*** the elements $\{a_i^x\}$ of the coherent set. The continuum is defined by the ***quaternionic function*** $\mathfrak{C}(q^{⓪})$, which has a flat parameter space that is spanned by a quaternionic number system $\{q^{⓪}\}$. The real valued continuous location density distribution $\rho_0(q^{⓪})$ describes the density distribution of set $\{a_j^x\}$ within set $\{q_i^x\}$.

An ***ordered coherent set*** is ordered with respect to the real parts of its members.

In a ***well-ordered coherent set*** all members have different real parts.

A well-ordered coherent set contains a well-defined hopping path. Also the hops form a discrete distribution. The landing locations form a well ordered swarm and the hops are also well-ordered. However, the subsequent hops have quite stochastic directions and sizes. Still the continuous location density distribution $\rho_0(q^{⓪})$ that describes the set of locations also characterizes the density distribution of the hops. Both are functions of the progression that is stored in the real parts of the eigenvalues.

The hops are eigenvalues of a hop operator. The hop operator and the landing location operator share the corresponding eigenvectors.

It is possible to define an imaginary function $\boldsymbol{\rho}(q^{⓪})$ that defines the ***average local displacement***. Together with the location density distribution $\rho_0(q^{⓪})$ it forms a quaternionic function:

$$\rho(q^{⓪}) = \rho_0(q^{⓪}) + \boldsymbol{\rho}(q^{⓪}).$$

We will call this function a ***density function***. The well-ordered coherent set $\{a_j^x\}$, which can be described by a dynamic continuous density distribution $\rho(q^{⓪})$ may also have a Fourier transform $\tilde{\rho}(p)$. In that case we call the set a ***coherent swarm***. The coherent swarm owns a ***displacement generator***. This means that at first approximation the swarm $\{a_j^x\}$ ***moves as one unit***. Having a Fourier transform is a higher level coherence requirement.

Defined in this way, the density function has lost its relation with the symmetry flavor of the discrete set $\{a_j^x\}$. However, it is possible to restore that relation by defining:

$$\rho^x(q^{⓪}) = \rho_0(q^{⓪}) + \boldsymbol{\rho}^x(q^{⓪})$$

The directions of the hops are stochastically distributed. This would mean that $\boldsymbol{\rho}^x(q^{⓪}) = \mathbf{0}$. However, the embedding causes an extra curvature of the continuum. This means that the curvature of the embedding continuum $\mathfrak{C}$ may change and that a corresponding flow is generated in this continuum. This produces a relative flow of the map of density distribution ρ^x with respect to $\mathfrak{C}$.

9.3 Embedding set elements

Embedding a single element a_j^x of the subset $\{a_j^x\}$ of the eigenspace of $\mathcal{R}^x$ in continuum $\mathfrak{C}$ involves first the conversion to the reference symmetry flavor. Next this element is mapped from the eigenspace of $\mathcal{R}^{⓪}$ in $\mathfrak{H}$ into to the eigenspace of $\mathfrak{R}^{⓪}$ in $\mathcal{H}$. The symmetry related fields may cause a relocation of the center of symmetry. Finally this discrete quaternion is embedded in the continuum $\mathfrak{C}$.

Locally the curved continuum $\mathfrak{C}$ is represented by ψ, which is nearly flat. For that reason for ψ we can use the quaternionic nabla ∇.

$$\nabla = \left\{\frac{\partial}{\partial\tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \quad (1)$$

ψ is considered to cover the images of all elements of $\{a_j^x\}$. This makes ψ a normalizable function.

The duration of the embedding is very short and is quickly released. Current mathematics lacks a proper description of the full embedding process, but it already contains equations that properly describe the situation before, after and during the embedding.

What happens can be described by the non-homogeneous wave equation.

$$\nabla\nabla^*\psi = \nabla_0\nabla_0\psi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi = \rho_j \tag{2}$$

Before and after the embedding ρ_j equals zero. In this condition any solution of the homogeneous wave equation will proceed as it did before.

During the embedding ρ_j represents the embedded discrete quaternion. The embedding results in the emission of a spherical wave front, which is a solution of the homogeneous wave equation. The non-homogeneous wave equation may be limited by special conditions:

$$\nabla_0\nabla_0\psi = -\lambda^2\,\psi \tag{3}$$

This reduces the non-homogeneous wave equation to a screened Poisson equation:

$$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi - \lambda^2\psi = \rho_j \tag{4}$$

The 3D solution of this equation is determined by the screened Green's function $G(r)$.

Green functions represent solutions for point sources.

$$G(r) = \frac{\exp(-\lambda\, r)}{r} \tag{5}$$

$$\psi = \iiint G(\boldsymbol{r}-\boldsymbol{r}')\,\rho(\boldsymbol{r}')\,d^3\boldsymbol{r}' \tag{6}$$

The continuum is touched and as a reaction it gets curved. The embedded particle will vanish, but traces in the continuum stay and represent the curvature. However, also these traces fade away.

Solutions of the wave equation can be found via the continuity equations:

$$\nabla\psi = \phi\,;\ \nabla^*\phi = \rho_j \tag{7}$$

And

$$\nabla^{*}\psi = \varphi \,; \nabla\, \varphi = \rho_j \tag{8}$$

Solutions of the homogeneous wave equation that cover an odd number of dimensions are known to represent wave fronts or combinations of wave fronts. These wave fronts proceed with fixed speed c. However, due to their diminishing amplitude, the spherical wave fronts fade away.

Embedding a single element of $\{a_j^x\}$ may cause the emission of a single spherical wave front. The amplitude of spherical wave fronts diminishes as 1/r with distance r from the source. This is also the form of the Green's function of the Poisson equation for the three dimensional isotropic case. This fact forms the origin of the curvature of the embedding continuum ψ.

Embedding a single hop may cause the emission of a single one dimensional wave front. The amplitude of one dimensional wave fronts keeps constant. The direction of the one dimensional wave front relates to the direction of the hop. This phenomenon may represent quanta that leave or enter the object that is represented by the swarm $\{a_j^x\}$.

9.4 Embedding the full set

If the full set is considered, then this means that the view integrates over the full cycle of progression steps that represent the generation of the swarm $\{a_j^x\}$.

If embedding of the full set $\{a_j^x\}$ is considered, then ρ represents the density distribution of the full set. In that case the continuity equations: $\nabla\,\varphi = \rho$ and $\nabla^{*}\phi = \rho$ determine what happens to the embedding continuum ψ, which locally represents $\mathfrak{C}$. As already indicated, due to the extra curvature the map of ρ may flow relative to ψ. The set $\{a_j^x\}$ is well-ordered. It means that each of its elements only exists during a small interval. Before that interval the element did not exist. It is ***generated*** by a stochastic mechanism. After the embedding this element of $\{a_j^x\}$ vanishes into history. Only its value is stored in an eigenvalue of operator $\mathcal{o}^x = |a_j^x\rangle a_j^x \langle a_j^x|$ that maps the subspace spanned by $\{|a_j^x\}$ onto itself. The operator $\mathcal{o}^x$ and the corresponding subspace have a dynamic definition. That definition covers a certain period, which represents a progression window.

In the embedding continuum $\mathfrak{C}$, the traces of what happened are the emitted vibrations and wave fronts that independent of the progression window keep proceeding. The spherical wave fronts do not vanish, but they fade away. With them the curvature also fades away. However, the recurrent embedding process keeps this curvature alive in a dynamical fashion. It drags the curvature with the subspace that represents the corresponding module.

The averaged Green's functions now indicate the averaged effects of the recurrent embedding on the curvature of ψ. The result is that the corresponding potential no longer represents a singularity.

9.5 Subspace dimension

In $\mathfrak{H}$ the dimension of the subspace that represents the set $\{a_j^x\}$ has a clear significance. In order to comprehend what this dimension and the spread of the set do to the function ψ we use the Green's function. The Green's function represents the influence of the embedding of a single point-like artifact into ψ. That artifact can be a landing point or a hop. If we do this for the three dimensional case, then the shape of the Green's function is $\mathcal{g}_j = 1/r$.

We replace ρ_j by ρ/N, multiply by the Green's function $\mathcal{g}_j$ and integrate over the space covered by ψ. Here N represents the number of elements in the set. ρ_j represents the effect of the single element a_j^x. For example, in case of an isotropic Gaussian distribution ρ/N the contributions to the integral will equal $\mathfrak{G}(r) = \mathrm{ERF}(r)/r$. In total N of those contributions [7] will be added. $N\,\mathfrak{G}(r)$ represents the gravitation potential.

This indicates that N directly relates to mass, which determines the strength of deformation of ψ.

If $\|\rho\| = N$, then $\nabla\,\varphi = \rho$ means $\|\nabla\,\varphi\| = N$.

This is a version of the coupling equation, which holds for all quaternionic normalizable functions φ and ρ, where φ is differentiable. If there are N landing locations, then there are also N hops.

10 Attaching characteristics to a module

10.1 Module subspace

We take one closed subspace as an example.

In free translation, the spectral theorem for normal operators that reside in a separable Hilbert space states: "If a normal operator maps a closed subspace onto itself, then the subspace is spanned by an orthonormal base consisting of eigenvectors of the operator."

The corresponding eigenvalues characterize this closed subspace.

The normal operator $\mathcal{O}^x = |a_i^x\rangle a_i^x \langle a_i^x|$ that maps the closed subspace onto itself ***may*** correspond to a ***companion operator*** $|\wp^x(a_i^x)\rangle \wp^x(a_i^x) \langle \wp^x(a_i^x)|$ that resides in the non-separable companion of the Hilbert space. The target of the mapper $\wp^x$ is a curved continuum that is characterized by the reference symmetry flavor. The index ^{x} indicates the symmetry flavor of the set $\{a_i^x\}$ of eigenvalues of operator $\mathcal{O}^x$.

The Hilbert spaces are structured storage places and in that way they can describe things. They possess no means that enable them to control what happens. That is the task of management mechanisms. However, the mechanism is restricted by the properties of the Hilbert spaces.

Here we take the position that the eigenvalues of operator $\mathcal{O}^x = |a_i^x\rangle a_i^x \langle a_i^x|$ are generated by a mechanism that implements a stochastic process. This process does not reside in the Hilbert spaces, but part of its behavior can be described by a series of operators. Some of these operators reside in the separable Hilbert space $\mathfrak{H}$. Other participating operators reside in the non-separable Hilbert space $\mathcal{H}$.

$\{a_i^x\}$ forms a well-ordered coherent set. All elements belong to different progression values. They belong to the same symmetry flavor and with respect to the quaternionic number system $\{q^x\}$ they own a continuous density distribution $\rho_0(q^x)$.

The stochastic process can be considered as a combination of a stochastic selector, such as a Poisson process and a binomial process, which is implemented by a 3D spread function. The binomial process locally attenuates the Poisson process. The ***stochastic spread function*** $\mathcal{S}$ produces a distribution of discrete locations that can be described by a density distribution ρ.

The involved operators and mechanisms are:

- In the separable Hilbert space a ***reference operator*** $\mathcal{R}^x = |q_i^x\rangle q_i^x \langle q_i^x|$ provides the parameter space of involved functions. The set of eigenvalues $\{q_i^x\}$ of this operator represent all rational members of a quaternionic number system $\{q^x\}$ that features a symmetry flavor, which is indicated with index ^{x}.

- In the non-separable Hilbert space a ***reference operator*** $\mathfrak{R}^x = |q^x\rangle q^x \langle q^x|$ provides the parameter space of involved functions. The set of eigenvalues $\{q^x\}$ of this operator represent all members of a quaternionic number system $\{q^x\}$ that features a symmetry flavor, which is indicated with index ^{x}.
- The ***density operator*** $|a_j^x\rangle\rho(a_j^x)\langle a_j^x|$, resides in separable Hilbert space $\mathfrak{H}$ and represents the density $\rho(q_i^x)$ of the discrete distribution $\{a_j^x\}$ that is generated by the stochastic spread function $\mathcal{S}$ during a period of progression that covers the progression values of the set $\{a_j^x\}$.
- The ***stochastic selection mechanism*** selects parameter values a_j^x according to the density operator $|a_j^x\rangle\rho(a_j^x)\langle a_j^x|$ that represents the density $\rho(q_i^x)$ of the discrete distribution $\{a_j^x\}$ within the set $\{q_i^x\}$ that is generated by the stochastic spread function $\mathcal{S}$.
- The eigenvectors$\{|a_j^x\rangle\}$ that belong to the eigenvalues $\{a_j^x\}$ of operator $\sigma^x = |a_j^x\rangle a_j^x \langle a_j^x|$ span the considered closed subspace and characterize the module that is represented by this subspace.
- Due to the action of the symmetry related fields, the mapper $\wp^x$ ***reallocates*** the center of the swarm $\{a_j^x\}$.
- The ***target space operator*** $|q^{⓪}\rangle\mathfrak{C}^{⓪}(q^{⓪})\langle q^{⓪}|$ resides in the non-separable Hilbert space $\mathcal{H}$ and is implemented by a continuous mapping function $\mathfrak{C}^{⓪}(q^{⓪})$.
- The ***density operator*** $|q^x\rangle\wp^x\big((\rho(q^x))\big)\langle q^x|$ resides in the non-separable Hilbert space $\mathcal{H}$ and represents the density $\wp^x((\rho(q^x)))$ of the discrete distribution $\{\wp^x(a_j^x)\}$ that is generated by the stochastic spread function $\mathcal{S}$ via the convolution $\mathcal{P} = \wp \circ \mathcal{S}$ of the map $\wp$ and the spread function $\mathcal{S}$.

Thus the selection mechanism and the combination of the operators that reside in the separable Hilbert space produce a sequence of eigenvalues $\{a_j^x\}$ of operator $\sigma^x = |a_i^x\rangle a_i^x \langle a_i^x|$ that map onto the closed target set in the continuum that is formed by the density operator $|q^x\rangle\wp^x\big((\rho(q^x))\big)\langle q^x|$ that represents the convolution $\mathcal{P} = \wp^x \circ \mathcal{S}$.

$\{a_j^x\}$ is a coherent subset of $\{q_i^x\}$, which form the eigenvalues of $\mathcal{R}^x = |q_i^x\rangle q_i^x \langle q_i^x|$.

$\mathfrak{C}^{⓪}(q^{⓪})$ represents the continuum eigenspace of the target space operator $|q^{⓪}\rangle\mathfrak{C}^{⓪}(q^{⓪})\langle q^{⓪}|$.

Since $\mathcal{P}(q)$ is a continuous function, $\{\mathcal{P}(a_j^x)\}$ is a discrete coherent subset of the continuous target space $\{\mathfrak{C}^{⓪}(q^{⓪})\}$.

The target subset $\{\mathcal{P}(a_j^x)\}$ represents the freedom that is left by the embedding of the separable Hilbert space into the non-separable Hilbert space. This imaging process is described by the convolution:

$$\mathcal{P} = \wp^x \circ \mathcal{S}_j \tag{1}$$

$\mathcal{S}_j$ is a stochastic spatial spread function and varies with each subsequent progression step.

The mapper $\wp^x$ produces an exact map that is influenced by the symmetry related charges.

The exact target location $\mathcal{P}(a_j^x)$ is not known beforehand, but after selection of the source eigenvalue a_j^x the image $\wp^x(a_j^x)$ is exactly known and is stored in the eigenspaces of the respective operators.

Averaged over all selections, $\mathcal{P}$ produces a blurred image.

The average $\boldsymbol{a}^x$ of the imaginary parts of all $\{a_j^x\}$ is the center location of the set. The combination of all involved operators and the selection mechanism produces a blurred image of $\boldsymbol{a}^x$.

The blur only concerns the imaginary part of the quaternion(s).

10.2 History, presence and future

In the orthomodular base model, the eigenvalues of the reference operators are not touched by management mechanisms or by the embedding process.

Presence is marked by a progression value that occurs in the real part of quaternionic eigenvalues. History is marked by lower real parts of quaternionic eigenvalues. ***Progression sensitive operators*** are characterized by the fact that they have known and fixed eigenvalues if the real part of the eigenvalue is lower than the present progression value. At the same time the current eigenvalues of these operators are influenced by the controlling mechanisms. ***Future*** eigenvalues of these operators are considered to be unknown.

In the orthomodular base model ***presence, history and future are artificial concepts***. History is defined with respect to the current real value of the eigenvalues of the reference operators.

The eigenspaces of ***progression sensitive operators*** exactly describe the history. The history is fixed. Thus also the historic eigenvalues are not touched by management mechanisms or by the embedding process. However, these operators do not yet describe the ***future***. The future is constructed by the management mechanisms and the embedding process. This means that these mechanisms depend on the progression parameter. The mechanisms only affect ***the current eigenvalues***. These eigenvalues describe the presence.

Progression sensitive operators are related to functions that use a flat parameter space which is defined using the reference operator.

The subspace that represents a module covers a sliding part of the last history. The dimension N of the subspace determines the number of covered progression instances. Inside the subspace progression rules the cyclic regeneration process. The subspace covers one cycle of that regeneration process. This period is governed by a controlling mechanism.

Thus, the progression window covers a recycling period in which the statistical properties of the set $\{a_j^x\}_N$ stabilize. This period is a property of the stochastic generation mechanism. The stochastic generation mechanisms exist in a series of types that each have their own characteristics.

10.3 Map of well-ordered coherent set

Since the source eigenvalues $\{a_j^x\}$ are all quaternions, they can be ordered with respect to their real value. All source eigenvalues have different real parts. That real value contains the sequence number. The set of source eigenvalues forms a ***well-ordered coherent set***. As a consequence, the image of the map of the source eigenvalues onto the continuum eigenspace can be described by a dynamic continuous location density distribution in which the sequence number acts as the progression parameter. This also means that $\{a_j^x\}$ describes a ***hopping path***.

10.4 Coherent swarm

The well-ordered coherent set $\{a_j^x\}$, which can be described by a dynamic continuous location density distribution $\rho(q^x)$ may also have a Fourier transform. In that case we call the set a ***coherent swarm***. The coherent swarm owns a displacement generator. This means that at first approximation the swarm $\{a_j^x\}$ ***moves as one unit***. Having a Fourier transform is a higher level coherence requirement.

Having a Fourier transform means that the swarm can be represented by a wave package. On movement, wave packages tend to disperse. Since the dynamic continuous location density distribution only describes the swarm, it is continuously regenerated. As a consequence, movement does not disperse the swarm. Thus due to recurrent regeneration, no danger of dispersion exists.

On the other hand the representation by a wave package indicates that the swarm $\{a_j^x\}$ may take the form of an interference pattern. That interference pattern is still a location swarm. It is not constructed by interfering waves!

10.5 The coherent map

Thus in the ***special case*** that a companion operator$|\wp^x(a_i^x)\rangle\wp^x(a_i^x)\langle\wp^x(a_i^x)|$ of the normal operator $|a_i^x\rangle a_i^x\langle a_i^x|$ that maps the subspace onto itself exists and the source eigenvalues $\{a_i^x\}$ form a well ordered coherent set, then the embedding of the module can be described by a progression dependent continuous mapping function $\wp$, which produces a blurred image $\mathcal{P}(\boldsymbol{a})$ of the average of the source eigenvalues. $\wp$ uses a flat parameter space that is spanned by a quaternionic number system. This ***mapper*** includes the actions of the symmetry related fields. The coherent set of source eigenvalues can be considered to be generated by a mechanism that can be characterized by a source location spread function $\mathcal{S}$. This function has fixed statistical characteristics, uses quaternions as its target values and progression as its parameter value. The progression parameter is taken from the parameter space of $\wp$. Now the blurred image $\mathcal{P}$ is the convolution of the mapping function $\wp$ and the source location spread function $\mathcal{S}$.

$$\mathcal{P} = \wp \circ \mathcal{S} \tag{1}$$

The coherent set of source eigenvalues can be described by a discrete source location density distribution $\{a_i^x\}$. If these eigenvalues are generated in a sequence, then for each member of this sequence the represented object can be considered to occupy a single source location. In this way the object can be considered to hop between the elements of the coherent swarm of eigenvalues. Each landing location corresponds with a hop. The sequence number can act as the progression parameter. The progression parameter is stored in the real part of the landing location eigenvalue.

We will call this special case “the coherent map”.

10.6 Stochastic generation

A Poisson process or equivalent stochastic generator may produce germs that are used by the spread function.

The spread function produces the locations. The combination of the sequence number and location is stored in the eigenvalue. The spread function is spherical symmetric and is best treated in spherical coordinates. The location is specified in the independent variables radius r, polar angle φ and azimuth θ. The order of these specifications may vary between mechanism types. This order and the direction in which the angles run influence the hopping path.

The generation process takes place in the realm of the separable Hilbert space. The interpretation of the sequence number as progression value occurs as an aftermath. During swarm generation, the notion of speed is meaningless.

The embedding process occurs in the realm of the Gelfand triple. With respect to hopping nothing moves in the embedding continuum. This means that hopping speed is irrelevant. However, the embedding process generates deformations and or vibrations of the embedding continuum. These movements are independent of the hop size. They may depend on hop direction.

10.7 Generation cycle

The generation by the stochastic spatial spread function $\mathcal{S}$ is done before the map $\wp$. This means that it occurs in the realm of the separable Hilbert space and this generation process is not (yet) affected by the embedding in the non-separable Hilbert space.

The stochastic generation process determines the short term cyclic part of the dynamical behavior of the object. The corresponding cycle period lasts until the spatial statistical characteristics of the generation result stabilize. Thus, the stochastic generation process is characterized by spatial statistical characteristics that are obtained after averaging over complete cycles of the generation process. These characteristics are the statistical characteristics of the coherent swarm.

The collection $\{\mathcal{P}(a_i^x)\}$ taken over the full generation cycle represents a spatial map of the cyclic dynamic behavior of the object.

10.8 Model wide progression steps and cycles

Each closed subspace that represents a coherent swarm is governed by a mechanism that ensures dynamic and spatial coherence. In fact many different types of such mechanisms exist. They correspond to elementary particle types. If these modules combine into composites, then the generation cycles must synchronize. This asks for a model wide progression step that is much shorter than any swarm generation cycle. A RTOS-like management mechanism must schedule the generation of composites from completed modules.

10.9 Swarm behavior

The coherent swarm moves as one unit. This means that the represented object features two kinds of movement. The first kind stays internal to the swarm. During the corresponding generation process, the hopping speed has no significance. The second kind concerns the swarm as a whole. The speed of the swarm makes physical sense.

Inside the swarm, the represented object hops from swarm element to swarm element. The hopping path is folded and if the swarm is at rest, then the hopping path is closed. Adding extra hops causes movement of the swarm. Adding a closed string of hops in a cyclic fashion causes an oscillation of the swarm. From observations it follows that in composites, such as atoms only certain oscillation modes are tolerated. Adding an arbitrary open string of hops opens the hopping path. In that case the sum of all hops is no longer zero. As a consequence the swarm will move. This motion gets its origin in the separable Hilbert space. And is mapped onto the continuum. This movement is recognizable relative to the parameter space.

A dynamic local change of the mapping function $\wp$ may move the swarm relative to other swarms. Such changes may occur when discrete objects curve the embedding continuum. This kind of movement gets its origin in the non-separable Hilbert space. Relative to the parameter space the effect of this movement is not recognizable.

10.10 Swarm characteristics

The swarm has a central location, which in separable Hilbert space is defined as the average $\boldsymbol{a}$ of the imaginary parts of the coherent set of source eigenvalues $\{a_i^x\}$ and in the non-separable Hilbert space it is defined by the image $\wp(\boldsymbol{a})$. This target value corresponds to an object source location $\boldsymbol{a}$ in the flat parameter space of $\wp$. The source location may move as a function of progression.

In the continuum the observed image of the swarm cannot move faster than the speed with which information can be transported.

The speed of transfer of information is set by the speed of information carriers. These information carriers are one-dimensional wave fronts. The quaternionic wave equation describes the way in which these wave fronts proceed.

The statistical characteristics of the swarm and the symmetry flavor of the swarm are sources for the properties that characterize the types of the objects that are represented by a coherent swarm.

10.11 Swarm diversity

The mechanism that generates the swarm determines the characteristics of the swarm. Apart from the number of elements of the swarm, the properties of the swarm appear to depend on its symmetry flavor. Due to the four dimensions of quaternions will quaternionic number systems, coherent swarms, quaternionic continuums and continuous quaternionic functions exist In 16 versions that only differ in their symmetry flavor.

Here we use the diversity that is represented by the standard model of contemporary physics as reference for naming elementary object types.

Elementary particle types have different masses. In the orthomodular base model this means that the corresponding closed subspaces have different dimensions and that correspondingly the swarms have different numbers of elements. It takes a type dependent number of progression steps for generating the corresponding swarm.

10.11.1 Fermions

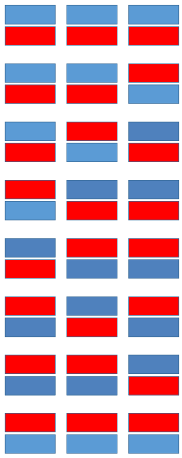
Embedding couples coherent swarms that possess symmetry flavor ψ^x to an embedding continuum that has symmetry flavor $\varphi^{⓪}$. If this symmetry flavor of the embedding continuum is fixed, then varying the symmetry flavor of the coherent swarm creates sixteen different elementary object types. Half of these types concern anti-particles. Again half of these sub-types concern left-handed quaternions and the other half are right-handed. Isotropic types represent another category. Anisotropic types occur in three versions that are deviated by the dimension in which the anisotropy occurs.

The difference in the symmetry flavors between the members of the pair $\{\psi^x, \varphi^y\}$ can be related to the electric charge, the color charge and the spin of the corresponding elementary particle.

Fermions are known to have half integer spin. In contemporary physics, their "color" structure becomes noticeable when composites are formed.

- Symmetry flavors are marked by special indices, for example $\boldsymbol{\psi}^{④}$
- They are also marked by colors $N, R, G, B, \bar{B}, \bar{G}, \bar{R}, \bar{N}$
- Half of them is right handed, **R**
- The other half is left handed, **L**
- $\boldsymbol{\psi}^{⓪}$is the reference symmetry flavor
- The colored rectangles reflect the directions of the axes

Result of coupling ψ^x to $\varphi^{⓪}$



$\psi^{⓪}$	*neutrino*	*0*	***R***
$\psi^{①}$	*R upquark*	*⅔*	***L***
$\psi^{②}$	*G upquark*	*⅔*	***L***
$\psi^{③}$	*B upquark*	*⅔*	***L***
$\psi^{④}$	$\bar{B}$ *downquark*	*-⅓*	***R***
$\psi^{⑤}$	$\bar{G}$ *downquark*	*-⅓*	***R***
$\psi^{⑥}$	$\bar{R}$ *downquark*	*-⅓*	***R***
$\psi^{⑦}$	*electron*	*-1*	***L***

The value of electric charge relates to the number of dimensions in which symmetry flavors differ. The sign of the electric charge relates to the direction in which the difference occurs. It also relates to the difference in handedness of the involved quaternions.

Color charge appears to relate to the index of the dimension in which the difference occurs. Isotropic differences correspond to “neutral” colors.

Quarks have “partial” electric charge. Up-quarks have electric charge + ⅔e. Down-quarks have electric charge - ⅓ e.

10.11.2 Massive Bosons

Massive bosons couple to an embedding continuum in a similar way as fermions do. Fermions and bosons appear to contribute to a common gravitation potential. This means that bosons embed in the same embedding field as fermions do. Boson swarms feature color-neutral symmetry flavors. Bosons are known to feature integer spin.

This can be explained when fermions are generated in a polar angle first and azimuthal angle second way, while bosons are generated in an azimuthal angle first and polar angle second fashion. The polar angle takes 2π radians and the azimuthal angle takes π radians.

Massive bosons are observable as W_{-}, W_{+} and Z particles. W_{+} is the antiparticle of W_{-}. Until now, there is no indication of the existence of quark-like bosons. At least their “color” structure cannot be observed.

10.11.3 Spin axis

Fermion swarms and boson swarms contain a hopping path that can be walked into two directions. That hopping path may implement spin.

If the swarm is at rest (does not move), then the hopping path is closed.

For bosons the spin axis may be coupled to the polar axis. The polar angle runs from 0 through 2π. For fermions the spin axis may be coupled to the azimuth axis. The azimuthal angle runs from 0 through π.

Nothing is said yet about the fact and the corresponding influence that the number of hops can be even or odd. And nothing is said yet about whether the opening hop and the closing hop are coupled in a symmetric or asymmetric sense.

10.12 Mass and energy

10.12.1 Having mass

Having mass can be interpreted as the capability to curve the continuum that embeds the concerned object. More mass corresponds to more curvature.

The dimension of the closed subspace, which represents a discrete object has a physical significance. Any eigenvector that contributes to spanning the closed subspace increases the dimension of the subspace. If all elements of the swarm contribute separately to the curvature of the embedding continuum, then the total curvature is proportional to the dimension of the subspace. In that case, this dimension relates to the mass of the object that corresponds to the swarm. If extra hops are added that cause movements or oscillations, then this adds to the mass in the form of kinetic energy. The extra hops may enter or leave in strings. Inside the swarm the hops that cause oscillation are stored as closed strings. Outside of the swarm the strings are open and appear as information messengers.

The fact that fermions and massive bosons contribute to a common gravitation potential means that they curve the same embedding continuum.

10.12.2 Information messengers

Information messengers represent open strings of hops. At the same time they are solutions of the homogeneous wave equation. This means that they can be viewed as strings of one dimensional wave fronts. One dimensional wave fronts do not diminish their amplitude as function of the distance to their emission point. In an otherwise flat continuum the one dimensional wave fronts and thus the information messengers proceed with the speed of information transfer. The energy carried by information messengers is proportional to the number of one-dimensional wave fronts that they contain. If the duration of emission, absorption and passage is fixed, then the apparent frequency of information messengers is proportional to their energy.

In contemporary physics the information messengers are known as ***photons***. From experiments we know that the energy of photons is proportional to their frequency. Thus if photons are information messengers then this suggests that at least locally, the emission, the absorption and the passage of information messengers takes a fixed number of progression cycles.

10.12.3 Mass energy equivalence

Creation and annihilation of elementary particles shows the equivalence of mass and energy.

10.12.3.1 Suggested creation process

Creation of elementary particles starts with the combination of two photons that came from opposite directions into an intermediate object. The intermediate object is a very short lived massive object that consists of as many paired elements as wave fronts are contained in the constituting photons. The wave fronts will convert into hops. The long chain of paired hops will then rip apart into two folded hopping strings that each form a coherent location swarm. Next the two swarms will split and move in opposite directions.

10.12.3.2 Suggested annihilation process

Annihilation of elementary particles starts with the combination of an elementary particle and its anti-particle that come from opposite directions into an intermediate object. The intermediate object is a very short lived massive object that consists of as many paired elements as elements are contained in the constituting coherent location swarms. The hops will convert into wave fronts. The long chain of paired wave fronts will then rip apart into two separate chains of wave fronts. Next these photons leave in opposite directions.

10.13 Relation to the wave function

The concept of wave function is used by contemporary physics in order to represent the state of a quantum physical object. The wave function is a complex amplitude probability distribution. Its squared modulus is a normalized density distribution of locations where the owner of the wave function can be detected. The value of this continuous distribution equals the probability of finding the owner at the location that is defined by the value of the parameter of the distribution.

If the detection is actually performed, then the object will be converted into something else. By the adherents of the Copenhagen interpretation, this fact is known as "the collapse of the wave function".

The normalized density distribution of locations where the owner of the wave function can be detected corresponds to the map of a coherent swarm on a flat continuum eigenspace of the companion operator in the orthomodular base model.

Thus the concept of the coherent map of a well-ordered coherent set on a flat continuum eigenspace of the companion operator in the orthonormal base model leads directly to an equivalent of the concept of the wave function in contemporary physics. Both concepts cannot be verified by experiments. The equivalence indicates that the suggested coherent map extension of the orthomodular base model runs in a sensible direction.

11 Traces of embedding

The actual embedding of a discrete eigenvalue in the continuum does not last longer than a single progression step. For each object, the embedding occurs only once at every used progression step. The source eigenvalue a_j is stored in the eigenspace of the location operator that resides in the separable Hilbert space. Immediately afterwards the embedding is released and is replaced by another embedding at a slightly different location a_{j+1} in the target continuum. This recurrent embedding process generates the map of the well-ordered coherent set of source eigenvalues $\{a_j\}$.

In the non-separable Hilbert space the map $\{\wp(a_j)\}$ affects the target subspace of the continuum eigenspace. This is done in a special way. Locally, the effect is determined by the non-homogeneous ***wave equation***. The homogeneous wave equation and the Poisson equation are restrictions of the non-homogeneous wave equation. The homogeneous wave equation controls the situation just before and after the actual embedding action. The Poisson equation determines the situation during the actual embedding action. The embedding results in the emission of a 3D ***wave front***. The solution of the Poisson equation folds and thus ***curves*** the target subspace of the continuum. After release of the embedding, the 3D wave front keeps proceeding, but then it will quickly diminish its amplitude as function of the distance to the emission location. The effects of the solutions of the non-homogeneous wave equation for all participating elements of the swarm combine and form an ***embedding potential***.

In general can be said that the embedding of discrete artifacts trigger vibrations and deformations of the embedding continuum. The vibrations can be wave fronts and oscillations and are solutions of the homogeneous wave equation. These solutions are restricted by local conditions and by the configuration of the triggers. For free particles these solutions are isotropic in one, two or three dimensions. In atoms the embedding of the electrons determine the configuration of triggers that cause spherical harmonics as solutions of the homogeneous wave equation.

Electric charges appear to gather in centers of symmetry. In spherical symmetric conditions this coincides with the local center of gravitation symmetry.

11.1 Embedding potentials

In this model, embedding potentials form the averages over a small period of progression and over a region of space of the effects of deformations of the embedding continuum and emissions of wave fronts that occur during the embedding of particles. The deformation is described by solutions of the Poisson equation, which is a restriction of the non-homogeneous wave equation. Mathematically these potentials are described by Green's functions or by weighted averages of these Green's functions. The Green's function of a single embedding is spherical symmetric. It describes how the embedding continuum is deformed by a single embedding occurrence.

The shape of the Green's function $G(r)$ of a single embedding corresponds with the shape of the amplitude of the spherical wave front that is emitted at the embedding instant. The wave fronts that are emitted during the embedding of the members of the location swarm are isotropic 3D wave fronts. Their spreading is controlled by the 3D version of the Huygens principle. This means that their amplitude decreases with the distance r from the source as $1/r$.

The spherical wave fronts quickly fade away and their effect is smoothed by the averaging.

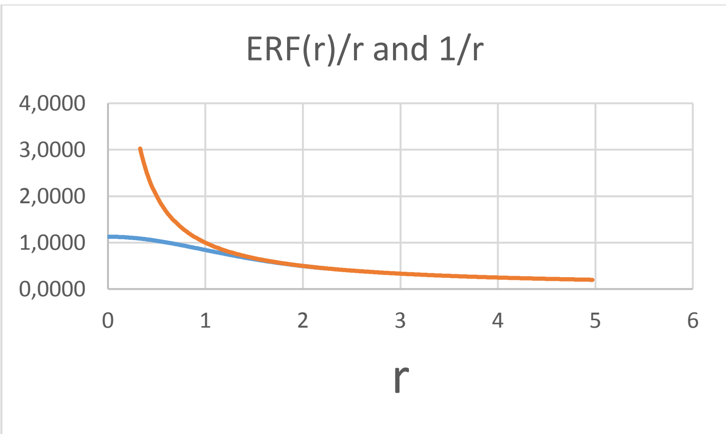
Here we consider a simplified situation. With an isotropic density distribution $\rho_0(r)$ in the swarm the scalar potential $\varphi_0(R)$ can be estimated as:

$$\varphi_0(R) = \iiint_0^R \rho_0(\boldsymbol{r}')\, G(\boldsymbol{r} - \boldsymbol{r}')d^3\boldsymbol{r}' \qquad (1)$$

R is the distance to the center of the swarm.

If the density distribution $\rho_0(\boldsymbol{r})$ approaches a 3D Gaussian distribution, then this integral equals [10]:

$$\varphi_0(R) = \mathrm{ERF}(R)/R \qquad (2)$$



We suppose that this distribution is a good estimate for the structure of the swarm of a free electron. It is remarkable that this potential (the blue curve) has no singularity at $R = 0$. At the same time, already at a short distance of the center the function very closely approaches $1/R$ (the orange curve).

The term $\mathrm{ERF}(R)$ indicates the influence of the spread of the embedding locations. This view can be used to determine the spatially averaged effect of the single embeddings. The set $\{a_j^x\}_N$ corresponds to N instances of such spatially averaged contributions. This approach shows that curvature and thus mass is directly related to the size of the set and to dimension of the subspace that represents the module.

In contemporary physics the embedding potential $\varphi_0(R)$ is known as the gravitation potential. It describes the curvature of the embedding continuum.

11.1.1 Inertia

If the swarm moves with uniform speed $\boldsymbol{v}$, then this goes together with a vector potential $\boldsymbol{\varphi}$.

$$\boldsymbol{\varphi}(R) = \iiint_0^R \boldsymbol{v}\, \rho_0(\boldsymbol{r}')\, G(\boldsymbol{r} - \boldsymbol{r}') d^3\boldsymbol{r}' = \boldsymbol{v}\, \varphi_0(R) \tag{1}$$

If the swarm accelerates, then this goes together with a field $\boldsymbol{\epsilon}$ that counteracts the acceleration.

$$\boldsymbol{\epsilon} = \dot{\boldsymbol{\varphi}}(R) = \dot{\boldsymbol{v}}\, \varphi_0(R) \tag{2}$$

11.2 Symmetry related potential

All elements of the coherent swarm have the same symmetry flavor. The effects of symmetry flavor coupling work over the whole reach of the coherent swarm. The source of this influence is located at the target value of the mapping function $\wp(a)$. The charge at this location depends on the difference between the symmetry flavor of the coherent swarm and the symmetry flavor of the embedding continuum.

Also here the quaternionic wave equation describes what happens, but the charge stays at its center location. If the swarm stays at rest, then the charge stays static as well and the governing equation is:

$$\nabla^*\nabla\varphi = \nabla_0\nabla_0\varphi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi = \rho$$

Here φ represents the quaternionic electric potential and ρ represents the distribution of electric charges.

For the static electric potential this reduces to

$$\langle \nabla, \nabla \rangle \varphi = \rho$$

11.2.1 Difference with gravitation potential

The electrostatic potential deviates in many aspects from the gravitation potential. Where every element of the swarm contributes separately to the gravitation potential, will the electrostatic potential only depend on the symmetry flavor of the complete swarm. It is generated by the complete swarm and not by the separate elements. The virtual location of the electrostatic charge coincides with the location of the center of symmetry of the swarm. For elementary particles, the strength of the symmetry related potential does not depend on the number of involved swarm elements.

12 Composites

Closed subspaces can combine into wider subspaces. If in the disjunction no eigenvectors of the location operator are shared between the constituents, then the constituents stay independent and keep their characteristics. Still superposition coefficients may rule the relative contribution of these properties. The properties are added per property type and these sums are not affected by the superposition.

12.1 Closed strings

Elementary particles are represented by coherent location swarms that also implement a folded hopping path. At rest this hopping path is closed. Adding extra hops may open the hopping path. This means that the sum of all hops may no longer equal zero. As a consequence the swarm moves. If a closed string of hops is added, then on average the swarm still stays at the same location, but at the same time the swarm oscillates. Such oscillations occur inside atoms.

The added hops act for the whole swarm as displacement generators. The corresponding quaternions act as superposition coefficients.

Quaternionic superposition coefficients may act as rotators. Special rotators can switch the color charge of quarks. They do not affect color-neutral swarms.

12.2 Open strings

The closed strings of superposition coefficients enter and leave the composite as open strings.

Messengers are open strings that relate to particular swarm oscillations. They are known as ***photons***.

Messengers are also represented by strings of one-dimensional wave fronts.

Gluons are open strings that relate to swarm rotations. They can switch the color charge of quarks

Color confinement stimulates that in composites the combined color charge is neutralized.

12.3 Binding

The potentials are a means to bind constituents of composites. Embedding potentials form pitches. If the particles move or oscillate the pitches become ditches.

12.3.1 Orthomodular model

The orthomodular base model suggests that at every progression step in every participating elementary particle only one swarm element is influenced by the currently existing potentials.

12.3.2 Gravitation

In the orthomodular base model, this is obvious for the gravitation potential which describes the deformation of the embedding continuum that is caused by these constituents. All embedding events contribute separately to the deformation of the embedding continuum. The constituents produce pitches into the embedding continuum and when they oscillate these pitches transform into ditches. The strength of the gravitation potential depends on the number of involved swarm elements.

12.3.3 Symmetry related potential

The origin of the symmetry related potential can also take a role in the binding of constituents, but this is questionable. The source of the symmetry related potential is probably located at the center of mass of the composite and is not located at the centers of mass of the constituents. If the sources of this potential would be located on the centers of mass of the constituents, then in case of oscillating constituents, this would result in ongoing emission of electromagnetic radiation.

12.4 Contemporary physics

Here we compare with results of contemporary physics.

12.4.1 Atoms

For stable composites, such as atoms, an ongoing emission of electromagnetic radiation is obviously not the case. Still the behavior of atoms with respect to absorption and emission of photons indicate that the electrons oscillate in concordance with the patterns of spherical harmonics. However, this oscillation occurs in the embedding continuum and does not concern the "location" of the electron.

For atoms and its composites, the strength of the symmetry related potential does not depend on the number of involved swarm elements.

The shell of atoms is described by spherical harmonics that are solutions of the homogeneous wave equation. This equation describes vibrations of the embedding continuum. These vibrations are caused by (non-isotropic) recurrent embedding of the electrons. The restriction of the wave equation to the conditions that define the spherical harmonics is the Helmholtz equation.

12.4.2 Hadrons

In hadrons the situation is different. There the binding is regulated by gluons. Gluons are capable of rotating quarks such that their color charge switches to another value. Gluons can join in strings. As rotators they act in pairs. Gluons do not affect isotropic swarms.

12.4.3 Standard model

In the standard model of contemporary physics the symmetry related potential that governs the binding of electrons in atoms is considered to be the electromagnetic potential.

The standard model suggests the existence of other potentials that implement weak and strong forces. Gluons play a role in the strong force. Massive bosons play a role in the weak force. Introducing strong and weak forces suggests that the potentials act on the full swarm and not on the individual swarm elements.

13 The space-progression model

The embedding continuum $\mathfrak{C}$ is defined by an almost continuous quaternionic function $\mathfrak{C}(q^{\circledcirc})$, which in the non-separable Hilbert space $\mathcal{H}$ is specified by operator $\mathfrak{C} = |q^{\circledcirc}\rangle\mathfrak{C}(q^{\circledcirc})\langle q^{\circledcirc}|$. It has a flat parameter space that is spanned by the eigenvalues of reference operator $\mathfrak{R}^{\circledcirc} = |q^{\circledcirc}\rangle q^{\circledcirc}\langle q^{\circledcirc}|$. The continuum $\mathfrak{C}$ corresponds to the map $\wp$, which images eigenvalues of reference operator

$\mathcal{R}^{⓪} = \left| q_i^{⓪} \right\rangle q_i^{⓪} \left\langle q_i^{⓪} \right|$ in $\mathfrak{H}$ onto continuum $\mathfrak{C}$. The real parts of the parameters store progression. Progression steps in $\mathfrak{H}$ and flows in $\mathcal{H}$.

The length of a path between two points in $\mathfrak{C}$ will generally be different from the distance between the corresponding parameter locations. Similar reasoning's hold for the image of a square and the image of a cube. These differences may change dynamically. Apart from these displacements and distortions, also rotations (vortices) may occur. At every progression instant the map $\wp$ is uniquely defined. Its inverse is in general not defined. A map of a volume onto the surface of a space cavity will not be excluded.

Point-like artifacts are recurrently selected from $\{q_i^x\}$ by a type specific stochastic mechanism and then embedded in continuum $\mathfrak{C}$. Their target location is determined by map $\wp$. The selections form a coherent spatial swarm $\{a_j^x\}$, which represents the point-like object in $\mathfrak{H}$. The spatial centers of the swarms move with respect to $\{q_i^x\}$ and thus their image in $\mathcal{H}$ moves with respect to $\mathfrak{C}$.

13.1 Metric

$\mathfrak{C}$ has a flat parameter space that is spanned by a quaternionic number system.

In almost flat space, the quaternionic nabla can rely on the fact that displacements in the embedding continuum depend on displacements in parameter space in a simple way:

$$ds_{flat} = c_0^\tau\, dq_\tau + c_0^x\, \boldsymbol{i}\, dq_x + c_0^y\, \boldsymbol{j}\, dq_y + c_0^z\, \boldsymbol{k}\, dq_z = c^\mu(q) dq_\mu \tag{1}$$

Here the coefficients $c_0^\mu(q)$ are real functions. dq_μ are real numbers.

However, more generally holds in curved space:

$$ds_{\mathfrak{C}} = c^\tau\, dq_\tau + c^x\, dq_x + c^y\, dq_y + c^z\, dq_z = c^\mu(q)\, dq_\mu \tag{2}$$

Here the coefficients $c^\mu(q)$ are full quaternionic functions.

$$ds(q) = ds_\nu(q) e^\nu = d\mathfrak{C} = \sum_{\mu=0\ldots3} \frac{\partial \mathfrak{C}}{\partial q_\mu} dq^\mu = c_\mu(q) dq^\mu \tag{3}$$

$d\mathfrak{C}$ is a quaternionic differential.

q is the quaternionic location.

c^μ is a quaternionic function.

At very small scale where $\mathfrak{C}$ is nearly flat holds Pythagoras:

$$c^2 dt^2 = ds\, ds^* = dq_\tau^2 + dq_x^2 + dq_y^2 + dq_z^2 \tag{4}$$

and Minkowski:

$$dq_\tau^2 = d\tau^2 = c^2\, dt^2 - dq_x^2 - dq_y^2 - dq_z^2 \tag{5}$$

In strongly curved space the differential of the embedding continuum function $\mathfrak{C}$ defines a kind of quaternionic metric.

$$ds\, ds^* = c_\mu(q) dq^\mu\, (c_\nu(q) dq^\nu)^* = c_\mu(q)\, c_\nu^*(q)\, dq^\mu dq^\nu = g_{\mu\nu}(q)\, dq^\mu dq^\nu \tag{6}$$

14 Restricting the orthomodular base model

Not all closed subspaces of the separable Hilbert space will represent modules that act as construction elements. Only closed subspaces for which a location generating mechanism governs, will act as modular construction elements. The management mechanisms that ensure spatial coherence will enforce this rule. The mechanisms appear to work in a step-wise fashion. This introduces a model-wide notion of progression in the model. Progression steps in the separable Hilbert space and it flows in the non-separable Hilbert space. The restriction converts the static model into a dynamic model in which special mechanisms ensure spatial and dynamical coherence. These are coupled due to the fact that the well-ordered coherent set of source eigenvalues represents a spatial map of the dynamic behavior of the source eigenvalue. At the same time the continuity of the mapping function $\wp$ ensures that the coherence is preserved in the image $\mathcal{P}$ of the set.

15 Role of the incoherent subspaces

Incoherent subspaces correspond to closed subspaces that do not correspond to a well-ordered coherent set of eigenvalues. If the subspace still is spanned by eigenvectors of the reference operator, then these eigenvalues may still produce an image in the continuum eigenspace of the companion location operator in the non-separable Hilbert space. Those images may produce spurious traces of embedding.

16 Conclusion

It appears sensible to suggest that physical reality mimics a network of mathematical structures that is controlled by a set of coherence ensuring management mechanisms. This setup aims at reducing relational complexity and it prevents dynamical chaos. The network consists of chains of structures that each start with a rather simple foundation. The major chain starts with an orthomodular lattice.

In this way an orthomodular base model emerges with inescapable evidence. This model treats all discrete objects as modules or modular systems that are embedded in continuums. This is supported by an infinite dimensional separable Hilbert space and a companion non-separable Hilbert space. Both Hilbert spaces act as structured storage media. The management mechanisms ensure the dynamic and spatial coherence. This leads to a model in which progression steps in the discrete part and flows in the continuous part of the model.

The embedding process creates the triggers that deforms the continuum in a dynamical way and causes vibrations in that continuum. Without these triggers the continuum is not affected.

A symmetry related field overlays the embedding continuum and features electric charges that are concentrated on local symmetry centers. In spherical symmetric conditions this coincides with the local center of gravitation symmetry.

The habits and diversity of quaternions play an essential role in the extension of the orthomodular base model. These habits cause a large variety of module types that differ in their properties and in their behavior. The generation of the modules is controlled both by these habits and by stochastic management mechanisms. The behavior of the modules and of the continuums is restricted by the embedding process.

History is precisely determined and stored in the Hilbert spaces. The controlling mechanisms act in a short period around the current progression value. Each mechanism acts in a sliding window that is represented by a closed subspace of the separable Hilbert space. The future is unknown, but it is restricted by the capabilities of the controlling mechanisms. These capabilities are restricted by the properties of the Hilbert spaces.

The development of mathematical tools that are used by physicist did not always occur in sync with the sometimes violent development of physical theories. Sometimes choices were made that would not have been taken when the proper mathematical tools were developed in an earlier phase. The paper shows that when looking back on this development , some leading physicists did not always provide the most sensible choice. They cannot be blamed for that choice, but as a consequence, the models of contemporary physics are more complicated than is necessary and do not reach as deep as is possible. It will be difficult to repair that situation.

If the target is to investigate the foundations of physical reality, then it is sensible to apply the most advanced mathematical tools and obey the restrictions that are set by these tools.

Appendix

1 Quaternionic calculus

Quaternions have features and capabilities that are hardly known [8]. Some of them are treated here.

Quaternions are hyper-complex numbers that consist of a real scalar and a three dimensional real vector [8]. The vector plays the role of the imaginary part. Quaternions keep these parts in one compact unit. This has the advantage that it is immediately clear that these parts belong together.

It is not necessary to treat quaternions as one unit. Contemporary physics has selected for the option to treat the real part and the imaginary part separately. This has generated unhappy far reaching consequences.

1.1 Quaternions

We indicate the real part of quaternion a by the suffix a_0.

We indicate the imaginary part of quaternion a by bold face $\boldsymbol{a}$.

$$a = a_0 + \boldsymbol{a} \tag{1}$$

The product of two quaternions does not commute and exists in two versions:

$$f = f_0 + \boldsymbol{f} = d\, e$$

$$f_0 = d_0 e_0 - \langle \boldsymbol{d}, \boldsymbol{e} \rangle \tag{2}$$

$$\boldsymbol{f} = d_0 \boldsymbol{e} + e_0 \boldsymbol{d} \pm \boldsymbol{d} \times \boldsymbol{e} \tag{3}$$

The $\pm$ sign indicates the influence of right or left handedness of the number system.

$\langle \boldsymbol{d}, \boldsymbol{e} \rangle$ is the inner product of $\boldsymbol{d}$ and $\boldsymbol{e}$.

$\boldsymbol{d} \times \boldsymbol{e}$ is the outer product of $\boldsymbol{d}$ and $\boldsymbol{e}$.

1.2 Symmetry flavors

Due to their four dimensions, quaternionic number systems exist in 16 versions that differ in their discrete symmetry sets. Half of these versions are right handed and the other half are left handed.

Quaternions can be mapped to Cartesian coordinates along the orthonormal base vectors 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$; with $\boldsymbol{ij} = \boldsymbol{k}$

- If the real part is ignored, then still ***8*** symmetry flavors result
- Symmetry flavors are marked by special indices, for example $\boldsymbol{a}^{④}$
- They are also marked by colors $N, R, G, B, \bar{B}, \bar{G}, \bar{R}, \bar{N}$
- Half of them is right handed, **R**
- The other half is left handed, **L**
- The colored rectangles reflect the directions of the coordinate axes

Symmetry flavors of members of coherent sets:

 $\boldsymbol{a}^{⓪} N$ ***R***

 $\boldsymbol{a}^{①} R$ ***L***

 $\boldsymbol{a}^{②} G$ ***L***

 $\boldsymbol{a}^{③} B$ ***L***

 $\boldsymbol{a}^{④} \bar{B}$ ***R***

 $\boldsymbol{a}^{⑤} \bar{G}$ ***R***

 $\boldsymbol{a}^{⑥} \bar{R}$ ***R***

 $\boldsymbol{a}^{⑦} W$ ***L***

Members of coherent sets $\{a_i\}$ of quaternions all feature the same symmetry flavor.

Continuous quaternionic functions $\psi(q)$ do not switch to other symmetry flavors.

The reference symmetry flavor of function $\psi(q)$ is the symmetry flavor of its parameter space .

Also continuous functions and continuums feature a symmetry flavor. The reference symmetry flavor of a continuous function $\psi(q)$ is the symmetry flavor of the parameter space $\{q\}$.

If the continuous quaternionic function describes the density distribution of a set $\{a_i\}$ of discrete objects a_i, then this set must be attributed with the same symmetry flavor.

1.3 Symmetry flavor conversion tools

1.3.1 Conjugation

Quaternionic conjugation

$$(\psi^{x})^{*} = \psi^{(7-x)}; x = ⓪, ①, ②, ③, ④, ⑤, ⑥, ⑦ \tag{1}$$

1.3.2 Rotation

Quaternions are often used to represent rotations.

$$c = ab/a \tag{1}$$

rotates the imaginary part of b that is perpendicular to the imaginary part of a over an angle 2θ, where $a = |a|\, exp(2\pi \boldsymbol{i}\theta)$.

Via quaternionic rotation, the following normalized quaternions ϱ^{x} can shift the indices of symmetry flavors of coordinate mapped quaternions and for quaternionic functions:

$$\varrho^{①} = \frac{1+\boldsymbol{i}}{\sqrt{2}};\ \varrho^{②} = \frac{1+\boldsymbol{j}}{\sqrt{2}}; \varrho^{③} = \frac{1+\boldsymbol{k}}{\sqrt{2}};\ \varrho^{④} = \frac{1-\boldsymbol{k}}{\sqrt{2}};\ \varrho^{⑤} = \frac{1-\boldsymbol{j}}{\sqrt{2}};\ \varrho^{⑥} = \frac{1-\boldsymbol{i}}{\sqrt{2}} \tag{2}$$

$$\boldsymbol{ij} = \boldsymbol{k};\ \ \boldsymbol{jk} = \boldsymbol{i};\ \ \boldsymbol{ki} = \boldsymbol{j} \tag{2}$$

$$\varrho^{⑥} = \left(\varrho^{①}\right)^{*} \tag{3}$$

For example

$$\psi^{③} = \varrho^{①}\psi^{②}/\varrho^{①} \tag{4}$$

$$\psi^{③}\varrho^{①} = \varrho^{①}\psi^{②} \tag{5}$$

$$\psi^{⓪} = \varrho^{x}\psi^{⓪}/\varrho^{x};\ \psi^{⑦} = \varrho^{x}\psi^{⑦}/\varrho^{x} \tag{6}$$

Also strings of symmetry flavor convertors may change the index of symmetry flavor of the multiplied quaternion or quaternionic function. The convertors can act on each other.

For example:

$$\varrho^{①}\varrho^{②} = \varrho^{②}\varrho^{③} = \varrho^{③}\varrho^{①} = \frac{1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{2} \tag{7}$$

The result is an isotropic quaternion. This means:

$$\varrho^{①}\psi^{②}/\varrho^{x} = \varrho^{②}\psi^{③}/\varrho^{x} = \psi^{(x+1)} \tag{8}$$

Here $(x+1)$ means $\boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to \boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k}$, or ①→②→③→①→②→③ and so on.

1.4 Differential calculus

In a flat continuum we can use the quaternionic nabla

$$\nabla = \left\{\frac{\partial}{\partial\tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \frac{\partial}{\partial\tau} + \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} = \nabla_0 + \boldsymbol{\nabla} \tag{1}$$

$$\Phi = \Phi_0 + \boldsymbol{\Phi} = \nabla\psi \tag{2}$$

$$\Phi_0 = \nabla_0\psi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\psi}\rangle \tag{3}$$

$$\boldsymbol{\Phi} = \nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{4}$$

In Maxwell equations the equivalent terms have been given separate names. Maxwell equations use coordinate time t rather than proper time τ. See section on space-progression models.

1.4.1 The coupling equation

The coupling equation represents a peculiar property of the quaternionic differential equation.

We start with two normalized functions ψ and φ and a normalizable function $\Phi = m\,\varphi$.

Here m is a fixed quaternion. Function φ can be adapted such that m becomes a real number.

$$\|\psi\| = \|\varphi\| = 1 \tag{1}$$

These normalized functions are supposed to be related by:

$$\Phi = \nabla\psi = m\,\varphi \tag{2}$$

$\Phi = \nabla\psi$ defines the ***differential equation***. (3)

$\nabla\psi = \Phi$ formulates a differential ***continuity equation***. (4)

$\nabla\psi = m\,\varphi$ formulates the ***coupling equation***. (5)

1.4.1.1 Special forms of the coupling equation

The existence of symmetry flavors of quaternionic functions gives rise to special forms of the coupling equation for symmetry flavors $\{\psi^x, \psi^y\}$ of the shared base function $\psi^{⓪}$.

For two cases the situation is uncovered by the Dirac equation. It represents an equation for the free electron and an equation for the free positron.

For example, the Dirac equation for the free electron in quaternionic format runs:

$$\nabla\psi = m_e\,\psi^* \qquad (1)$$

ψ^* and ψ are symmetry flavors of the same base function.

The Dirac equation for the free positron runs:

$$\nabla^*\psi^* = m_e\,\psi \qquad (2)$$

These equations differ in the sign of a curl term. Together they constitute a special non-homogeneous wave equation:

$$\nabla^*\nabla\psi = \nabla^*(m_e\,\psi^*) = m_e^2\,\psi \qquad (3)$$

1.4.2 Transformations

The value of ϕ in

$$\phi = \nabla\psi \qquad (1)$$

does not change after the transformation

$$\psi \to \psi + \xi = \psi + \nabla^*\chi \qquad (2)$$

where

$$\nabla\xi = \nabla\nabla^*\chi = 0 \tag{3}$$

The rotations

$$\psi \to \vartheta\psi\vartheta^{-1};\ \varphi \to \vartheta\varphi\vartheta^{-1} \tag{4}$$

do not affect the validity of

$$\phi = \nabla\psi \tag{5}$$

if

$$\nabla(\vartheta\psi\vartheta^{-1}) = \vartheta(\nabla\psi)\vartheta^{-1} \tag{6}$$

while

$$\vartheta\phi\vartheta^{-1} = \vartheta(\nabla\psi)\vartheta^{-1} = \nabla(\vartheta\psi\vartheta^{-1}) \tag{7}$$

1.4.3 The non-homogeneous wave equation

Locally, the wave function is considered to act in a rather flat continuum χ.

The quaternionic wave equation exists in a homogeneous ($\rho = 0$) and in non-homogeneous ($\rho \neq 0$) form.

$$\nabla^*\nabla\chi \equiv \nabla_0\nabla_0\chi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\chi = \rho \tag{1}$$

1.4.4 Diversity

The non-homogeneous wave equation corresponds to two continuity equations.

With $\nabla\chi = \varphi$ follows $\nabla^*\varphi = \rho$ (2)

ψ involves the embedding continuum. ρ involves the description of the coherent swarm of triggers.

The non-homogeneous wave equation couples the sign flavor of the embedding continuum to the sign flavor of the coherent location swarm. Whether this occurs in the first or in the second continuity equation is not yet clear.

We can apply the Dirac equation for the electron and the positron as a template. There we used the equation for the elementary particle as the first continuity equation and the equation of the antiparticle as the second continuity equation. For example:

$$\nabla\psi^{⓪} = m_x\psi^{\otimes} \tag{3}$$

$$\nabla^*\psi^{\otimes} = m_x\psi^{⓪} \tag{4}$$

$$\nabla^*\left(\nabla\psi^{⓪}\right) = m_x\nabla^*\psi^{\otimes} = m_x^2\psi^{⓪} \tag{5}$$

Superscript ⊗ represents the sign flavor of the swarm of the elementary particle before it gets embedded.

This template represents the diversity of the fermions. $\psi^{⑦}$ represents the swarm of the electron.

1.4.5 Restrictions of the non-homogeneous wave equation

Depending on local and temporal conditions the non-homogeneous wave equation can be restricted to different versions. Examples are the homogeneous wave equation, the Poisson equation, the screened Poisson equation and the Helmholtz equation.

1.4.5.1 The non-homogeneous wave equation

The homogeneous wave equation is taken as:

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\chi = \frac{\partial^2\psi}{\partial\tau^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{3}$$

The function ρ represents the temporary presence of one or more discrepant discrete objects.

Depending of local conditions the (non-)homogeneous wave equations has several groups of solutions. ρ may trigger vibrations that are solutions of the homogeneous wave equation.

Near the embedding location the homogeneous wave equation applies between two embedding occurrences and the non-homogeneous wave equation applies during the embedding.

$$\nabla^*\nabla\chi_0 = 0 \tag{4}$$

Equation (2) has 3D isotropic wave fronts as one group of its solutions. χ_0 is a scalar function. By changing to polar coordinates it can be deduced that a general solution is given by:

$$\chi_0(r,\tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} \tag{5}$$

Where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

$$\nabla^*\nabla\boldsymbol{\chi} = 0 \tag{6}$$

Here $\boldsymbol{\chi}$ is a vector function.

Equation (4) has one dimensional wave fronts as one group of its solutions:

$$\boldsymbol{\chi}(z,\tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{7}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function.

The imaginary $\boldsymbol{i}$ represents the base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the one dimensional wave front.

1.4.5.2 Poisson equation

The Poisson equation is a special condition of the non-homogeneous wave equation in which some terms are zero or have a special value.

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\chi = \rho \tag{1}$$

$$\nabla_0\nabla_0\chi = -\lambda^2\,\chi \tag{2}$$

$$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\chi - \lambda^2\chi = \rho \tag{3}$$

The 3D solution of this equation is determined by the screened Green's function $G(r)$.

Green functions represent solutions for point sources.

$$G(r) = \frac{\exp(-\lambda\, r)}{r} \tag{4}$$

$$\chi = \iiint G(\boldsymbol{r} - \boldsymbol{r}')\,\rho(\boldsymbol{r}')\,d^3\boldsymbol{r}' \tag{5}$$

G(r) has the shape of the Yukawa potential [14]

In case of $\lambda = 0$ it is the Coulomb or gravitation potential of a point source.

1.5 Space-progression models

Different versions of the wave equation exist. One is based on Maxwell's equations. Another is based on quaternionic differential calculus.

These wave equations correspond to two different space-progression models.

1.5.1 The Maxwell-Huygens wave equation

In Maxell format the wave equation uses coordinate time t. It runs as:

$$\partial^2\psi/\partial t^2 - \partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2 - \partial^2\psi/\partial z^2 = 0 \tag{1}$$

Papers on Huygens principle work with this formula or it uses the version with polar coordinates.

For isotropic conditions in three participating dimensions the general solution runs:

$$\psi = f(r - ct)/r, \text{ where } c = \pm 1; f \text{ is real} \tag{2}$$

In a single participating dimension the general solution runs:

$$\psi = f(x - ct), \text{ where } c = \pm 1; f \text{ is real} \tag{3}$$

1.5.2 The homogeneous quaternionic wave equation

Locally, the wave function is considered to act in a rather flat continuum χ.

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\chi = 0 \tag{1}$$

First we look at:

$$\nabla^*\nabla\chi_0 = 0 \tag{2}$$

χ_0 is a scalar function. For isotropic conditions in three participating dimensions equation (2) has three dimensional spherical wave fronts as one group of its solutions.

By changing to polar coordinates it can be deduced that a general solution is given by:

$$\chi_0(r,\tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} \tag{3}$$

where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

Next we consider the vector function $\boldsymbol{\chi}$

$$\nabla^*\nabla\boldsymbol{\chi} = 0 \tag{4}$$

Equation (4) has one dimensional wave fronts as one group of its solutions:

$$\boldsymbol{\chi}(z,\tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{5}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function.

The imaginary $\boldsymbol{i}$ represents the base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the one dimensional wave front.

1.5.3 Relativity

The orthomodular base model applies the homogeneous quaternionic wave equation for establishing the model's speed of information transfer. This equation offers 1D wave fronts as one of its possible solutions.

These wave fronts can act as information carriers. They proceed with constant speed $c = \pm 1$. Their amplitude does not diminish with distance from the source. Thus these carriers can travel huge distances and still keep their integrity. In contrast 3D wave fronts proceed with the same speed, but their amplitude diminishes as $1/r$ with distance r from the source.

In his introduction of special relativity in 1905, Einstein used the Maxwell based wave equation [10] in order to derive the speed of information transfer in his models. This resulted in a spacetime model that features a Minkowski signature.

The Maxwell based wave equation uses coordinate time t. The quaternionic wave equation uses progression τ. Comparing these two parameters becomes difficult when space is curved, but for infinitesimal steps, space can be considered to be flat and the progression step becomes a proper time step. In that situation holds:

$$\text{Coordinate time step vector} = \text{proper time step vector} + \text{spatial step vector} \tag{1}$$

Or in Pythagoras format:

$$(\Delta t)^2 = (\Delta\tau)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{2}$$

The formula indicates that the coordinate time step corresponds to the step of a full quaternion, which is a superposition of a proper time step and a spatial step.

An infinitesimal spacetime step Δs is usually presented as an infinitesimal proper time step $\Delta\tau$.

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{3}$$

The signs on the right side form the Minkowski signature $(+,-,-,-)$.

Quaternions offer a Euclidean signature $(+,+,+,+)$ as is shown in formula (2).

The Lorentz transform uses a speed parameter that is compared with the maximum speed of information transfer. Einstein and most contemporary physics models use coordinate time based speed for this purpose.

The orthomodular base model will use progression based speed for that purpose. As a consequence it supports a space-progression model that features an Euclidean signature.

2 Related historic discoveries

Physical models use mathematical tools. The development of mathematical tools did not evolve in sync with the development of the physical models that use these tools. Complicated mathematical tools may take several decades before they mature.

[1] Quantum logic was introduced by Garret Birkhoff and John von Neumann in their 1936 paper. G. Birkhoff and J. von Neumann, *The Logic of Quantum Mechanics*, Annals of Mathematics, Vol. 37, pp. 823–843

[2] The lattices of quantum logic and classical logic are treated in detail in: http://vixra.org/abs/1411.0175 .

[3] The Hilbert space was discovered in the first decades of the 20-th century by David Hilbert and others. http://en.wikipedia.org/wiki/Hilbert_space.

[4] In the sixties Constantin Piron and Maria Pia Solèr proved that the number systems that a separable Hilbert space can use must be division rings. See: "Division algebras and quantum theory" by John Baez. http://arxiv.org/abs/1101.5690 and http://www.ams.org/journals/bull/1995-32-02/S0273-0979-1995-00593-8/

[5] Paul Dirac introduced the bra-ket notation, which popularized the usage of Hilbert spaces. Dirac also introduced its delta function, which is a generalized function. Spaces of generalized functions offered continuums before the Gelfand triple arrived.

[6] In the sixties Israel Gelfand and Georgyi Shilov introduced a way to model continuums via an extension of the separable Hilbert space into a so called Gelfand triple. The Gelfand triple often gets the name rigged Hilbert space. It is a non-separable Hilbert space. http://www.encyclopediaofmath.org/index.php?title=Rigged_Hilbert_space .

[7] Potential of a Gaussian charge density: http://en.wikipedia.org/wiki/Poisson%27s_equation#Potential_of_a_Gaussian_charge_density .

[8] Quaternionic function theory and quaternionic Hilbert spaces are treated in: http://vixra.org/abs/1411.0178 .

[9] In 1843 quaternions were discovered by Rowan Hamilton. http://en.wikipedia.org/wiki/History_of_quaternions

Later in the twentieth century quaternions fell in oblivion.

[10] http://en.wikipedia.org/wiki/Wave_equation#Derivation_of_the_wave_equation

[11] http://en.wikipedia.org/wiki/Yukawa_potential

[12] These discoveries are also used as foundations by the author's e-book "The Hilbert Book Model Game". http://vixra.org/abs/1405.0340 .