Approximations for prime numbers and superposition of integer sequences II

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"But he that sinneth against me wrongeth his own soul: all they that hate me love death" - Proverbs 8:36.

Abstract. I proved two approximations for prime numbers using trigonometric sums.

1. First Approximation

Theorem 1. *Let* $n \in \mathbb{N}$, *for* $1 \leqslant n \leqslant 10$, *then*

$$p^{\natural}(n) = p_{n+1};$$

for $n \geqslant 11$, *then*

$$p^{\natural}(n) < p_{n+1},$$

where p_n *denotes the* pth *number prime and*

$$p^{\natural}(n) = \frac{14n+1}{25}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-1)k}{5}\right) + \frac{14n-3}{25}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-2)k}{5}\right)$$
$$+\frac{16n-13}{25}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-3)k}{5}\right) + \frac{18n-17}{25}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-4)k}{5}\right) + \frac{18n-25}{25}\sum_{k=0}^{4}\cos\left(\frac{2\pi nk}{5}\right),$$

with $\cos(z)$ *denoting the cosine function.*

Proof. The function p_{n+1} have the following integer sequence:

$$\{p_{n+1}\}_{n\geqslant 1} = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, \ldots\}.$$

Now, suppose that the above sequence obeys the following function:

$$p^{\natural}(n) = (\alpha_1 n + \beta_1)p_1(n) + (\alpha_2 n + \beta_2)p_2(n) + (\alpha_3 n + \beta_3)p_3(n) \tag{1}$$
$$+(\alpha_4 n + \beta_4)p_4(n) + (\alpha_5 n + \beta_5)p_5(n),$$

where

$$p_1(n) \overset{\text{def}}{=} \frac{1}{5}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-1)k}{5}\right) = \begin{cases} 1, n \equiv 1 \bmod 5, \\ 0, \text{otherwise}; \end{cases} \tag{2}$$

$$p_2(n) \overset{\text{def}}{=} \frac{1}{5}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-2)k}{5}\right) = \begin{cases} 1, n \equiv 2 \bmod 5, \\ 0, \text{otherwise}; \end{cases} \tag{3}$$

$$p_3(n) \overset{\text{def}}{=} \frac{1}{5}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-3)k}{5}\right) = \begin{cases} 1, n \equiv 3 \bmod 5, \\ 0, \text{otherwise}; \end{cases} \tag{4}$$

$$p_4(n) \overset{\text{def}}{=} \frac{1}{5}\sum_{k=0}^{4}\cos\left(\frac{2\pi(n-4)k}{5}\right) = \begin{cases} 1, n \equiv 4 \bmod 5, \\ 0, \text{otherwise}, \end{cases} \tag{5}$$

$$p_5(n) \overset{\text{def}}{=} \frac{1}{5}\sum_{k=0}^{4}\cos\left(\frac{2\pi nk}{5}\right) = \begin{cases} 1, n \equiv 0 \bmod 5, \\ 0, \text{otherwise}, \end{cases} \tag{6}$$

for $n \geqslant 1$ and $n \in \mathbb{N}$.

So, I can write the system of equations:

$$\begin{cases} \alpha_1+\beta_1=3 \\ 6\alpha_1+\beta_1=17, \end{cases} \tag{7}$$

soon, solving Eq. (7), I encounter

$$\alpha_1=\frac{14}{5}, \beta_1=\frac{1}{5}. \tag{8}$$

Hereafter, I write the system of equations:

$$\begin{cases} 2\alpha_2+\beta_2=5 \\ 7\alpha_2+\beta_2=19, \end{cases} \tag{9}$$

hence, solving Eq. (9), I find

$$\alpha_2=\frac{14}{5}, \beta_2=-\frac{3}{5}. \tag{10}$$

Afterward, I write the system of equations:

$$\begin{cases} 3\alpha_3+\beta_3=7 \\ 8\alpha_3+\beta_3=23, \end{cases} \tag{11}$$

thus, solving Eq.(11), I get

$$\alpha_3=\frac{16}{5}, \beta_3=-\frac{13}{5}. \tag{12}$$

After, I write the system of equations:

$$\begin{cases} 4\alpha_4+\beta_4=11 \\ 9\alpha_4+\beta_4=29, \end{cases} \tag{13}$$

wherefore, solving Eq. (13), I set

$$\alpha_4=\frac{18}{5}, \beta_4=-\frac{17}{5}. \tag{14}$$

Hereafter, I write the system of equations:

$$\begin{cases} 5\alpha_5+\beta_5=13 \\ 10\alpha_5+\beta_5=31, \end{cases} \tag{15}$$

soon, solving Eq. (15), I obtain

$$\alpha_5=\frac{18}{5}, \beta_5=-5. \tag{16}$$

From (1),(2),(3), (4), (5), (6), (8), (10), (12), (14) and (16), it follows that

$$\begin{aligned} p^{\natural}(n)=&\frac{14n+1}{25}\sum_{k=0}^{4}\cos\Big(\tfrac{2\pi(n-1)k}{5}\Big)+\frac{14n-3}{25}\sum_{k=0}^{4}\cos\Big(\tfrac{2\pi(n-2)k}{5}\Big)\\ &+\frac{16n-13}{25}\sum_{k=0}^{4}\cos\Big(\tfrac{2\pi(n-3)k}{5}\Big)+\frac{18n-17}{25}\sum_{k=0}^{4}\cos\Big(\tfrac{2\pi(n-4)k}{5}\Big)\\ &+\frac{18n-25}{25}\sum_{k=0}^{4}\cos\Big(\tfrac{2\pi nk}{5}\Big). \end{aligned} \tag{17}$$

This leads to the following numerical sequence, which is equal to the original for $1\leqslant n\leqslant 10$; but, for $n\geqslant 11$, their numbers are lower than the original numerical sequence

$$\begin{aligned} \{p^{\natural}(n)\}_{n\geqslant 1}=\{&3,5,7,11,13,17,19,23,29,31,31,33,39,47,49,45,47,55,65,67,\\ &59,61,71,83,85,73,75,87,101,103,87,89,103,119,121,101,103,119,137,139,...\}, \end{aligned} \tag{18}$$

which proves the result. □

2. SECOND APPROXIMATION

Theorem 2. *Let* $n\in\mathbb{N}$, *for* $1\leqslant n\leqslant 20$, *then*

$$p^{\sharp}(n)=p_{n+1};$$

for $n\geqslant 21$, *then*

$$p^{\sharp}(n)<p_{n+1},$$

where p_n *denotes the* pth *number prime and*

$$p^{\sharp}(n)=\frac{17n-2}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-1)k}{10}\Big)+\frac{18n-11}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-2)k}{10}\Big)$$
$$+\frac{18n-19}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-3)k}{10}\Big)+\frac{18n-17}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-4)k}{10}\Big)+\frac{4n-7}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-5)k}{10}\Big)$$
$$+\frac{21n-41}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-6)k}{10}\Big)+\frac{21n-52}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-7)k}{10}\Big)+\frac{22n-61}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-8)k}{10}\Big)$$
$$+\frac{21n-44}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-9)k}{10}\Big)+\frac{21n-55}{50}\sum_{k=0}^{9}\cos\Big(\frac{2\pi nk}{10}\Big),$$

with $\cos(z)$ *denotes the cosine function.*

Proof. The function p_{n+1} have the following integer sequence:

$$\{p_{n+1}\}_{n\geqslant 1}=\{3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,...\}.$$

Now, suppose that the above sequence obeys the following function:

$$p^{\sharp}(n)=(\alpha_1 n+\beta_1)p_1(n)+(\alpha_2 n+\beta_2)p_2(n)+(\alpha_3 n+\beta_3)p_3(n)+(\alpha_4 n+\beta_4)p_4(n) \tag{19}$$
$$+(\alpha_5 n+\beta_5)p_5(n)+(\alpha_6 n+\beta_6)p_6(n)+(\alpha_7 n+\beta_7)p_7(n)+(\alpha_8 n+\beta_8)p_8(n)$$
$$+(\alpha_9 n+\beta_9)p_9(n)+(\alpha_{10} n+\beta_{10})p_{10}(n),$$

where

$$p_1(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-1)k}{10}\Big)=\begin{cases}1, n\equiv 1 \bmod 10,\\ 0, \text{otherwise;}\end{cases} \tag{20}$$

$$p_2(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-2)k}{10}\Big)=\begin{cases}1, n\equiv 2 \bmod 10,\\ 0, \text{otherwise;}\end{cases} \tag{21}$$

$$p_3(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-3)k}{10}\Big)=\begin{cases}1, n\equiv 3 \bmod 10,\\ 0, \text{otherwise;}\end{cases} \tag{22}$$

$$p_4(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-4)k}{10}\Big)=\begin{cases}1, n\equiv 4 \bmod 10,\\ 0, \text{otherwise,}\end{cases} \tag{23}$$

$$p_5(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-5)k}{10}\Big)=\begin{cases}1, n\equiv 5 \bmod 10,\\ 0, \text{otherwise,}\end{cases} \tag{24}$$

$$p_6(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-6)k}{10}\Big)=\begin{cases}1, n\equiv 6 \bmod 10,\\ 0, \text{otherwise,}\end{cases} \tag{25}$$

$$p_7(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-7)k}{10}\Big)=\begin{cases}1, n\equiv 7 \bmod 10,\\ 0, \text{otherwise,}\end{cases} \tag{26}$$

$$p_8(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-8)k}{10}\Big)=\begin{cases}1, n\equiv 8 \bmod 10,\\ 0, \text{otherwise,}\end{cases} \tag{27}$$

$$p_9(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi(n-9)k}{10}\Big)=\begin{cases}1, n\equiv 9 \bmod 10,\\ 0, \text{otherwise,}\end{cases} \tag{28}$$

$$p_{10}(n)\overset{\text{def}}{=}\frac{1}{10}\sum_{k=0}^{9}\cos\Big(\frac{2\pi nk}{10}\Big)=\begin{cases}1, n\equiv 0 \bmod 10,\\ 0, \text{otherwise,}\end{cases} \tag{29}$$

for $n \geqslant 1$ and $n \in \mathbb{N}$.

So, I can write the system of equations:

$$\begin{cases} \alpha_1 + \beta_1 = 3 \\ 11\alpha_1 + \beta_1 = 37, \end{cases} \tag{30}$$

$$\begin{cases} 2\alpha_2 + \beta_2 = 5 \\ 12\alpha_2 + \beta_2 = 41, \end{cases} \tag{31}$$

$$\begin{cases} 3\alpha_3 + \beta_3 = 7 \\ 13\alpha_3 + \beta_3 = 43, \end{cases} \tag{32}$$

$$\begin{cases} 4\alpha_4 + \beta_4 = 11 \\ 14\alpha_4 + \beta_4 = 47, \end{cases} \tag{33}$$

$$\begin{cases} 5\alpha_5 + \beta_5 = 13 \\ 15\alpha_5 + \beta_5 = 53, \end{cases} \tag{34}$$

$$\begin{cases} 6\alpha_6 + \beta_6 = 17 \\ 16\alpha_6 + \beta_6 = 59, \end{cases} \tag{35}$$

$$\begin{cases} 7\alpha_7 + \beta_7 = 19 \\ 17\alpha_7 + \beta_7 = 61, \end{cases} \tag{36}$$

$$\begin{cases} 8\alpha_8 + \beta_8 = 23 \\ 18\alpha_8 + \beta_8 = 67, \end{cases} \tag{37}$$

$$\begin{cases} 9\alpha_9 + \beta_9 = 29 \\ 19\alpha_9 + \beta_9 = 71, \end{cases} \tag{38}$$

$$\begin{cases} 10\alpha_{10} + \beta_{10} = 31 \\ 20\alpha_{10} + \beta_{10} = 73, \end{cases} \tag{39}$$

hence, solving Eq. (30) at Eq. (39), I encounter

$$\begin{gathered} \alpha_1 = \frac{17}{5}, \beta_1 = -\frac{2}{5}; \alpha_2 = \frac{18}{5}, \beta_2 = -\frac{11}{5}; \alpha_3 = \frac{18}{5}, \beta_3 = -\frac{19}{5}; \alpha_4 = \frac{18}{5}, \beta_4 = -\frac{17}{5}; \\ \alpha_5 = 4, \beta_5 = -7; \alpha_6 = \frac{21}{5}, \beta_6 = -\frac{41}{5}; \alpha_7 = \frac{21}{5}, \beta_7 = -\frac{52}{5}; \alpha_8 = \frac{22}{5}, \beta_8 = -\frac{61}{5}; \\ \alpha_9 = \frac{21}{5}, \beta_9 = -\frac{44}{5}; \alpha_{10} = \frac{21}{5}, \beta_{10} = -11. \end{gathered} \tag{40}$$

From (19) at (29) and (40), it follows that

$$\begin{aligned} p^\sharp(n) = & \frac{17n-2}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-1)k}{10}\Big) + \frac{18n-11}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-2)k}{10}\Big) \\ & + \frac{18n-19}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-3)k}{10}\Big) + \frac{18n-17}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-4)k}{10}\Big) \\ & + \frac{4n-7}{10} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-5)k}{10}\Big) + + \frac{21n-41}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-6)k}{10}\Big) \\ & + \frac{21n-52}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-7)k}{10}\Big) + \frac{22n-61}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-8)k}{10}\Big) \\ & + \frac{21n-44}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi(n-9)k}{10}\Big) + \frac{21n-55}{50} \sum_{k=0}^{9} \cos\Big(\tfrac{2\pi nk}{10}\Big). \end{aligned} \tag{41}$$

This leads to the following numerical sequence, which is equal to the original for $1 \leqslant n \leqslant 20$; but, for $n \geqslant 21$, their numbers are lower than the original numerical sequence

$$\{p^\sharp(n)\}_{n\geqslant 1} = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 71, \quad (42)$$
$$77, 79, 83, 93, 101, 103, 111, 113, 115, 105, 113, 115, 119, 133, 143, 145, 155, 155, 157, ...\},$$

which proves the result. □