Approximations for prime numbers and superposition of integer sequences I

By Edigles Guedes

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"God is our refuge and strength, a very present help in trouble. Therefore will not we fear, though the earth be removed, and though the mountains be carried into the midst of the sea; Though the waters thereof roar and be troubled, though the mountains shake with the swelling thereof." - Psalms 46:1-3.

Abstract. I proved some three approximations for prime numbers.

1. First approximation

Theorem 1. *Let* $n\in\mathbb{N}$, *for* $1\leqslant n\leqslant 8$, *then*

$$p^{\flat_{1,1}}(n)=p_{n+1};$$

for $n\geqslant 9$, *then*

$$p^{\flat_{1,1}}(n)<p_{n+1},$$

where p_n *denotes the* pth *prime number and*

$$p^{\flat_{1,1}}(n)=\frac{1}{8}\left\{23n+(n-5)\left[\sin\left(\frac{3\pi n}{2}\right)-\sin\left(\frac{\pi n}{2}\right)\right]+(n-1)\cos(\pi n)-7\right\},$$

with $\sin(z)$ *denotes the sine function and* $\cos(z)$ *denotes the cosine function.*

Proof. The function p_{n+1} have the following integer sequence:

$$\{p_{n+1}\}_{n\geqslant 1}=\{3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,...\}.$$

Now, suppose that the above sequence obeys the following function:

$$p^{\flat}(n)=(\alpha_1 n+\beta_1)p_1(n)+(\alpha_2 n+\beta_2)p_2(n)+(\alpha_3 n+\beta_3)p_3(n) +(\alpha_4 n+\beta_4)p_4(n), \tag{1}$$

where

$$p_1(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi(n-1)k}{4}\right)=\begin{cases}1, n\equiv 1 \bmod 4,\\ 0, \text{otherwise};\end{cases} \tag{2}$$

$$p_2(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi(n-2)k}{4}\right)=\begin{cases}1, n\equiv 2 \bmod 4,\\ 0, \text{otherwise};\end{cases} \tag{3}$$

$$p_3(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi(n-3)k}{4}\right)=\begin{cases}1, n\equiv 3 \bmod 4,\\ 0, \text{otherwise};\end{cases} \tag{4}$$

$$p_4(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi nk}{4}\right)=\begin{cases}1, n\equiv 0 \bmod 4,\\ 0, \text{otherwise},\end{cases} \tag{5}$$

for $n\geqslant 1$ and $n\in\mathbb{N}$.

So, I can write the system of equations:

$$\begin{cases} \alpha_1 + \beta_1 = 3 \\ 5\alpha_1 + \beta_1 = 13, \end{cases} \tag{6}$$

soon, solving Eq. (6), I encounter

$$\alpha_1 = \frac{5}{2}, \beta_1 = \frac{1}{2}. \tag{7}$$

Hereafter, I write the system of equations:

$$\begin{cases} 2\alpha_2 + \beta_2 = 5 \\ 6\alpha_2 + \beta_2 = 17, \end{cases} \tag{8}$$

hence, solving Eq. (8), I find

$$\alpha_2 = 3, \beta_2 = -1. \tag{9}$$

Afterward, I write the system of equations:

$$\begin{cases} 3\alpha_3 + \beta_3 = 7 \\ 7\alpha_3 + \beta_3 = 19, \end{cases} \tag{10}$$

thus, solving Eq.(10), I get

$$\alpha_3 = 3, \beta_3 = -2. \tag{11}$$

After, I write the system of equations:

$$\begin{cases} 4\alpha_4 + \beta_4 = 11 \\ 8\alpha_4 + \beta_4 = 23, \end{cases} \tag{12}$$

wherefore, solving Eq. (12), I set

$$\alpha_4 = 3, \beta_4 = -1. \tag{13}$$

From (1), (2), (3), (4), (5), (7), (9), (11) and (13), it follows that

$$\begin{aligned} p^{\flat_{1,1}}(n) = \frac{5n+1}{8}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi(n-1)k}{4}\right) + \frac{3n-1}{4}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi(n-2)k}{4}\right) \\ + \frac{3n-2}{4}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi(n-3)k}{4}\right) + \frac{3n-1}{4}\sum_{k=0}^{3}\cos\left(\tfrac{2\pi n k}{4}\right). \end{aligned} \tag{14}$$

Simplifying (14), I encounter

$$p^{\flat_{1,1}}(n) = \frac{1}{8}\left\{23n + (n-5)\left[\sin\left(\frac{3\pi n}{2}\right) - \sin\left(\frac{\pi n}{2}\right)\right] + (n-1)\cos(\pi n) - 7\right\}. \tag{15}$$

This leads to the following numerical sequence, which is equal to the original for $1 \leqslant n \leqslant 8$; but, for $n \geqslant 9$, their numbers are lower than the original numerical sequence; namely,

$$\left\{p^{\flat_{1,1}}(n)\right\}_{n\geqslant 1} = \{3, 5, 7, 11, 13, 17, 19, 23, 23, 29, 31, 35, 33, 41, 43, 47, 43, 53, 55, 59, 53, \tag{16}$$

$$65, 67, 71, 63, 77, 79, 83, 73, 89, 91, 95, 83, 73, 89, 91, 95, 83, 101, 103, 107, 93, 113, 115, 119, ...\},$$

which proves the result. □

2. Second approximation

Theorem 2. *Let* $n \in \mathbb{N}$, *for* $1 \leqslant n \leqslant 16$, *then*

$$p^{\flat_{2,2}}(n) = p_{n+1};$$

for $n \geqslant 17$, *then*

$$p^{\flat_{2,2}}(n) < p_{n+1},$$

except in $n=20$ *and* $n=28$*, where* p_n *denotes the* pth *prime number and*

$$p^{\flat_{2,2}}(n)=\frac{1}{8}\left\{23n+(n-5)\left[\sin\left(\frac{3\pi n}{2}\right)-\sin\left(\frac{\pi n}{2}\right)\right]+(n-1)\cos(\pi n)-7\right\},$$

$$+\frac{\theta(n-9)}{8}\left[9(n-8)+4\sin\left(\frac{\pi(n-8)}{2}\right)-4\sin\left(\frac{3\pi(n-8)}{2}\right)+(n-8)\cos\left(\frac{\pi(n-8)}{2}\right)\right.$$

$$\left.+(n-8)\cos\left(\frac{3\pi(n-8)}{2}\right)+(n-24)\cos(\pi(n-8))+16\right],$$

with $\sin(z)$ *denotes the sine function and* $\cos(z)$ *denotes the cosine function and* $\theta(n)$ *denotes the unit step function.*

Proof. The function p_{n+1} have the following integer sequence:

$$\{p_{n+1}\}_{n\geqslant 1}=\{3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,\ldots\}.$$

Now, suppose that the above sequence obeys the following function:

$$p^{\flat_{2,2}}(n)=p^{\flat_{1,1}}(n)+p^{\flat_{1,2}}(n), \tag{17}$$

where

$$p^{\flat_{1,2}}(n)=\theta(n-9)\cdot p^{\flat_{2,1}}(n-8), \tag{18}$$

with $H(n)$ denotes the discrete Heaviside step function or unit step function, defined by

$$\theta(n)\overset{\text{def}}{=}\begin{cases}0, n<0,\\ 1, n\geqslant 0;\end{cases}$$

the function $p^{\flat_{2,1}}(n)$ have the following integer sequence:

$$\left\{p^{\flat_{2,1}}(n)\right\}_{n\geqslant 1}=\{6,2,6,6,10,6,10,12,18,14,16,14,26,18,22,26,38,26,28,26,40,38,40,\ldots\}.$$

Supposing that the above sequence obeys the following function:

$$p^{\flat_{2,1}}(n)=(\alpha_1 n+\beta_1)p_1^{\flat_{2,1}}(n)+(\alpha_2 n+\beta_2)p_2^{\flat_{2,1}}(n)+(\alpha_3 n+\beta_3)p_3^{\flat_{2,1}}(n) \tag{19}$$
$$+(\alpha_4 n+\beta_4)p_4^{\flat_{2,1}}(n),$$

where

$$p_1^{\flat_{2,1}}(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\frac{2\pi(n-1)k}{4}\right)=\begin{cases}1, n\equiv 1 \bmod 4,\\ 0, \text{otherwise};\end{cases} \tag{20}$$

$$p_2^{\flat_{2,1}}(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\frac{2\pi(n-2)k}{4}\right)=\begin{cases}1, n\equiv 2 \bmod 4,\\ 0, \text{otherwise};\end{cases} \tag{21}$$

$$p_3^{\flat_{2,1}}(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\frac{2\pi(n-3)k}{4}\right)=\begin{cases}1, n\equiv 3 \bmod 4,\\ 0, \text{otherwise};\end{cases} \tag{22}$$

$$p_4^{\flat_{2,1}}(n)\overset{\text{def}}{=}\frac{1}{4}\sum_{k=0}^{3}\cos\left(\frac{2\pi nk}{4}\right)=\begin{cases}1, n\equiv 0 \bmod 4,\\ 0, \text{otherwise},\end{cases} \tag{23}$$

for $n\geqslant 1$ and $n\in\mathbb{N}$.

So, I can write the system of equations:

$$\begin{cases}\alpha_1+\beta_1=6\\ 5\alpha_1+\beta_1=10,\end{cases} \tag{24}$$

soon, solving Eq. (24), I encounter

$$\alpha_1 = 1, \beta_1 = 5. \tag{25}$$

Hereafter, I write the system of equations:

$$\begin{cases} 2\alpha_2 + \beta_2 = 2 \\ 6\alpha_2 + \beta_2 = 6, \end{cases} \tag{26}$$

hence, solving Eq. (26), I find

$$\alpha_2 = 1, \beta_2 = 0. \tag{27}$$

Afterward, I write the system of equations:

$$\begin{cases} 3\alpha_3 + \beta_3 = 6 \\ 7\alpha_3 + \beta_3 = 10, \end{cases} \tag{28}$$

thus, solving Eq.(28), I get

$$\alpha_3 = 1, \beta_3 = 3. \tag{29}$$

After, I write the system of equations:

$$\begin{cases} 4\alpha_4 + \beta_4 = 6 \\ 8\alpha_4 + \beta_4 = 12, \end{cases} \tag{30}$$

wherefore, solving Eq. (30), I set

$$\alpha_4 = \frac{3}{2}, \beta_4 = 0. \tag{31}$$

From (19), (20), (21), (22), (23), (25), (27), (29) and (31), it follows that

$$p^{\flat_{2,1}}(n) = \frac{n+5}{4}\sum_{k=0}^{3} \cos\Big(\frac{2\pi(n-1)k}{4}\Big) + \frac{n}{4}\sum_{k=0}^{3} \cos\Big(\frac{2\pi(n-2)k}{4}\Big) \tag{32}$$

$$+\frac{n+3}{4}\sum_{k=0}^{3} \cos\Big(\frac{2\pi(n-3)k}{4}\Big) + \frac{3n}{8}\sum_{k=0}^{3} \cos\Big(\frac{2\pi nk}{4}\Big),$$

Simplifying (32), I encounter

$$p^{\flat_{2,1}}(n) = \frac{1}{8}\bigg[9n + 4\sin\Big(\frac{\pi n}{2}\Big) - 4\sin\Big(\frac{3\pi n}{2}\Big) + n\cos\Big(\frac{\pi n}{2}\Big) \tag{33}$$

$$+n\cos\Big(\frac{3\pi n}{2}\Big) + (n-16)\cos(\pi n) + 16\bigg].$$

From Theorem 1, (17), (18) and (33), it follows that

$$p^{\flat_{2,2}}(n) = \frac{1}{8}\bigg\{23n + (n-5)\bigg[\sin\Big(\frac{3\pi n}{2}\Big) - \sin\Big(\frac{\pi n}{2}\Big)\bigg] + (n-1)\cos(\pi n) - 7\bigg\} \tag{34}$$

$$+\frac{\theta(n-9)}{8}\bigg[9(n-8) + 4\sin\Big(\frac{\pi(n-8)}{2}\Big) - 4\sin\Big(\frac{3\pi(n-8)}{2}\Big) + (n-8)\cos\Big(\frac{\pi(n-8)}{2}\Big)$$

$$+(n-8)\cos\Big(\frac{3\pi(n-8)}{2}\Big) + (n-24)\cos(\pi(n-8)) + 16\bigg].$$

This leads to the following numerical sequence, which is equal to the original for $1 \leqslant n \leqslant 16$; but, for $n \geqslant 17$, their numbers are lower than the original numerical sequence, except in $n = 20$ and $n = 28$; namely,

$$\big\{p^{\flat_{2,2}}(n)\big\}_{n\geqslant 1} = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 57, 63, 69, 77, \tag{35}$$

$$71, 79, 85, 95, 85, 95, 101, 113, 99, 111, 117, 131, 113, 127, 133, 149, 127, 143, 149, 167, ...\},$$

which proves the result. □

3. Third approximations

Theorem 3. *Let* $n\in\mathbb{N}$, *for* $1\leqslant n\leqslant 24$, *then*

$$p^{\flat_{3,3}}(n)=p_{n+1};$$

for $n\geqslant 25$, *then*

$$p^{\flat_{3,3}}(n)<p_{n+1},$$

except in $n=28,29,32,36,40,44,48,52,56,60,64,68,72,76,80,84,88,92,96,100,104,108,112,116,120,124$, *where* p_n *denotes the* pth *prime number and*

$$p^{\flat_{2,2}}(n)=\frac{1}{8}\left\{23n+(n-5)\left[\sin\left(\frac{3\pi n}{2}\right)-\sin\left(\frac{\pi n}{2}\right)\right]+(n-1)\cos(\pi n)-7\right\},$$

$$+\frac{\theta(n-9)}{8}\left[9(n-8)+4\sin\left(\frac{\pi(n-8)}{2}\right)-4\sin\left(\frac{3\pi(n-8)}{2}\right)+(n-8)\cos\left(\frac{\pi(n-8)}{2}\right)\right.$$

$$\left.+(n-8)\cos\left(\frac{3\pi(n-8)}{2}\right)+(n-24)\cos(\pi(n-8))+16\right]$$

$$+\frac{\theta(n-17)}{8}\left\{6(n-16)+(3n-76)\left[\cos\left(\frac{\pi(n-16)}{2}\right)+\cos\left(\frac{3\pi(n-16)}{2}\right)\right]\right.$$

$$\left.-19\cos(\pi(n-16))+(n-11)\left[\sin\left(\frac{\pi(n-16)}{2}\right)-\sin\left(\frac{3\pi(n-16)}{2}\right)\right]-5\right\},$$

with $\sin(z)$ *denotes the sine function,* $\cos(z)$ *denotes the cosine function and* $\theta(n)$ *denotes the unit step function.*

Proof. The function p_{n+1} have the following integer sequence:

$\{p_{n+1}\}_{n\geqslant 1}=\{3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97,101,...\}.$

Now, suppose that the above sequence obeys the following function:

$$p^{\flat_{3,3}}(n)=p^{\flat_{2,2}}(n)+p^{\flat_{1,3}}(n), \tag{36}$$

where

$$p^{\flat_{1,3}}(n)=\theta(n-17)\cdot p^{\flat_{3,1}}(n-16), \tag{37}$$

with $H(n)$ denotes the discrete Heaviside step function or unit step function, defined by

$$\theta(n)\overset{\text{def}}{=}\begin{cases}0, n<0,\\ 1, n\geqslant 0;\end{cases}$$

the function $p^{\flat_{3,1}}(n)$ have the following integer sequence:

$$\left\{p^{\flat_{3,1}}(n)\right\}_{n\geqslant 1}=\{4,4,2,-4,8,4,4,2,16,8,6,-4,14,16,14,6,26,22,18,8,36,24,24,...\}.$$

Supposing that the above sequence obeys the following function:

$$p^{\flat_{3,1}}(n)=(\alpha_1 n+\beta_1)p_1^{\flat_{3,1}}(n)+(\alpha_2 n+\beta_2)p_2^{\flat_{3,1}}(n)+(\alpha_3 n+\beta_3)p_3^{\flat_{3,1}}(n) \tag{38}$$

$$+(\alpha_4 n+\beta_4)p_4^{\flat_{3,1}}(n),$$

where

$$p_1^{\flat_{3,1}}(n) \overset{\text{def}}{=} \frac{1}{4}\sum_{k=0}^{3} \cos\left(\tfrac{2\pi(n-1)k}{4}\right) = \begin{cases} 1, n \equiv 1 \bmod 4, \\ 0, \text{otherwise}; \end{cases} \tag{39}$$

$$p_2^{\flat_{3,1}}(n) \overset{\text{def}}{=} \frac{1}{4}\sum_{k=0}^{3} \cos\left(\tfrac{2\pi(n-2)k}{4}\right) = \begin{cases} 1, n \equiv 2 \bmod 4, \\ 0, \text{otherwise}; \end{cases} \tag{40}$$

$$p_3^{\flat_{3,1}}(n) \overset{\text{def}}{=} \frac{1}{4}\sum_{k=0}^{3} \cos\left(\tfrac{2\pi(n-3)k}{4}\right) = \begin{cases} 1, n \equiv 3 \bmod 4, \\ 0, \text{otherwise}; \end{cases} \tag{41}$$

$$p_4^{\flat_{3,1}}(n) \overset{\text{def}}{=} \frac{1}{4}\sum_{k=0}^{3} \cos\left(\tfrac{2\pi n k}{4}\right) = \begin{cases} 1, n \equiv 0 \bmod 4, \\ 0, \text{otherwise}, \end{cases} \tag{42}$$

for $n \geqslant 1$ and $n \in \mathbb{N}$.

So, I can write the system of equations:

$$\begin{cases} \alpha_1 + \beta_1 = 4 \\ 5\alpha_1 + \beta_1 = 8, \end{cases} \tag{43}$$

soon, solving Eq. (43), I encounter

$$\alpha_1 = 1, \beta_1 = 3. \tag{44}$$

Hereafter, I write the system of equations:

$$\begin{cases} 2\alpha_2 + \beta_2 = 4 \\ 6\alpha_2 + \beta_2 = 4, \end{cases} \tag{45}$$

hence, solving Eq. (45), I find

$$\alpha_2 = 0, \beta_2 = 4. \tag{46}$$

Afterward, I write the system of equations:

$$\begin{cases} 3\alpha_3 + \beta_3 = 2 \\ 7\alpha_3 + \beta_3 = 4, \end{cases} \tag{47}$$

thus, solving Eq.(47), I get

$$\alpha_3 = \frac{1}{2}, \beta_3 = \frac{1}{2}. \tag{48}$$

After, I write the system of equations:

$$\begin{cases} 4\alpha_4 + \beta_4 = -4 \\ 8\alpha_4 + \beta_4 = 2, \end{cases} \tag{49}$$

wherefore, solving Eq. (49), I set

$$\alpha_4 = \frac{3}{2}, \beta_4 = -10. \tag{50}$$

From (38), (39), (40), (41), (42), (44), (46), (48) and (50), it follows that

$$p^{\flat_{3,1}}(n) = \frac{n+3}{4}\sum_{k=0}^{3} \cos\left(\tfrac{2\pi(n-1)k}{4}\right) + \sum_{k=0}^{3} \cos\left(\tfrac{2\pi(n-2)k}{4}\right) \tag{51}$$

$$+\frac{n+1}{8}\sum_{k=0}^{3} \cos\left(\tfrac{2\pi(n-3)k}{4}\right) + \frac{3n-20}{8}\sum_{k=0}^{3} \cos\left(\tfrac{2\pi n k}{4}\right),$$

Simplifying (51), I encounter

$$p^{\flat_{3,1}}(n) = \frac{1}{8}\left\{6n + (3n-28)\left[\cos\left(\frac{\pi n}{2}\right) + \cos\left(\frac{3\pi n}{2}\right)\right]\right. \tag{52}$$

$$\left. -19\cos(\pi n) + (n+5)\left[\sin\left(\frac{\pi n}{2}\right) - \sin\left(\frac{3\pi n}{2}\right)\right] - 5\right\}.$$

From Theorem 2, (36), (37) and (52), it follows that

$$p^{\flat_{3,3}}(n) = \frac{1}{8}\left\{23n + (n-5)\left[\sin\left(\frac{3\pi n}{2}\right) - \sin\left(\frac{\pi n}{2}\right)\right] + (n-1)\cos(\pi n) - 7\right\} \tag{53}$$

$$+\frac{\theta(n-9)}{8}\left[9(n-8) + 4\sin\left(\frac{\pi(n-8)}{2}\right) - 4\sin\left(\frac{3\pi(n-8)}{2}\right) + (n-8)\cos\left(\frac{\pi(n-8)}{2}\right)\right.$$

$$\left. + (n-8)\cos\left(\frac{3\pi(n-8)}{2}\right) + (n-24)\cos(\pi(n-8)) + 16\right]$$

$$+\frac{\theta(n-17)}{8}\left\{6(n-16) + (3n-76)\left[\cos\left(\frac{\pi(n-16)}{2}\right) + \cos\left(\frac{3\pi(n-16)}{2}\right)\right]\right.$$

$$\left. -19\cos(\pi(n-16)) + (n-11)\left[\sin\left(\frac{\pi(n-16)}{2}\right) - \sin\left(\frac{3\pi(n-16)}{2}\right)\right] - 5\right\}.$$

This leads to the following numerical sequence, which is equal to the original for $1 \leqslant n \leqslant 24$; but, for $n \geqslant 25$, their numbers are lower than the original numerical sequence, except in $n = 28$, $29, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100, 104, 108, 112, 116, 120, 124$; namely,

$$\left\{p^{\flat_{3,3}}(n)\right\}_{n \geqslant 1} = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, \tag{54}$$

$$89, 97, 97, 99, 107, 121, 115, 115, 125, 145, 133, 131, 143, 169, 151, 147, 161, 193, ...\},$$

which proves the result. □