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Exact solution of the problem of random walks on 2-and 3-dimensional simple cubic grids in the form of combinatorial expressions.

Abstract

The obtained combinatorial formulas describing random walks on a simple cubic grid.

For the case of 2 dimensions - accurate and simple.

For the case of 3 dimensions - accurate, but, unfortunately, not compact.

A simple cubic grid is defined as a set of node locations of the end of the N - dimensional vector

$$\mathbf{k}=n_1 *\mathbf{k_1} + n_2 *\mathbf{k_2} + \ldots + n_N *\mathbf{k_N} ,$$

where k_i - guides mutually perpendicular unit vector, and n_i - runs over all possible integer values: 0, 1, 2, 3,... The model grid in the 1-dimensional case is the X-axis with units in their integer values in the 2-dimensional case it is a sheet with a drawing of a square grid, in 3-d is a cubic grid, etc.

The task of random walks on grid is as follows: let at the moment L= 0 "walking point" is placed at the origin (0,0,0,...). What is the probability P_L (X,Y,Z,...) at step L, the point will be in the node (X,Y,Z,...) taking into account the fact that:
- all values of X,Y,Z,... and L are arbitrary integers 0,
- point one-step randomly moved to one of the nodes adjacent to the current,
- the probability of this transition is the grid constant and inversely proportional to the number of adjacent nodes. (Thus, this probability is equal to ½ for direct, ¼ for the plane, 1/6 in the case of 3 dimensions and, in General, $\boldsymbol{P} = \frac{1}{2N}$, where N = 1, 2, 3,... is the dimension of the space in the "conventional" sense.)

It is known [1] the solution to this problem in General for the case of any dimension:

$$\boldsymbol{P_L} (\mathbf{k}) = \boldsymbol{P_L} (k_1 ,k_2 ,\ldots, k_N) = \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} [\frac{1}{N} (c_1 +c_2 +\ldots+c_N)]^L *\exp(-i\varphi \mathbf{k})\, d^N\varphi ,$$

where $c_j = \cos \varphi_j = \frac{1}{2} [\exp(i\varphi_j)+ \exp(-i\varphi_j)]$; $\varphi * \mathbf{k} = \varphi_1 k_1 +\ldots+ \varphi_N k_N$; a $d^N\varphi = d\varphi_1 *\ldots* d\varphi_N$.

However, for specific calculations and evaluations, it is always desirable to have a simple expression.
For one-dimensional case such a simple solution is known and is given by the following combinatorial formula. The probability that the L-th step, the point will be in the node's X coordinate is:

$$\boldsymbol{P_L} (X) = \left(1/2^L \right) * \binom{L}{\frac{L+X}{2}}$$

where L is the number of "step", the X - coordinate of the node. The symbol $\binom{a}{b}$ in the right part means $\frac{a!}{b!(a-b)!}$, where b! - factorial.

We obtained similar solutions in the form of combinatorial formulas: accurate and easy for the case of 2 dimensions and exact, but "heavy" for 3 measurements.

The basis for the conclusion was based on the obvious connection: the probability of getting at step L to the node in the grid is proportional to the number of paths leading to this node (i.e., ending in him). Obviously, the probability P_L (X,Y,Z,...) falling into a given node is calculated by dividing the (normalized) number of trajectories K_L (X,Y,Z,...) , resulting in the same, the sum of the number of ALL possible trajectories in the whole grid at the same step.
Because, at each step, the point can make a single leap for 2N different directions to any of the 2N neighboring nodes, it is obvious that the sum of the number of ALL possible trajectories in the whole grid at step L is equal to $\mathbf{(2\mathit{N})^{\mathit{L}}}$.
Note that ALL paths in step L have the same length equal to L, because in one step, the point can move only to an adjacent node, i.e., the "1". Check that under the length of the trajectory is not Cartesian distance from the origin to the given node, namely the gained sum of individual displacements for L steps, and so on., the length of the trajectory is equal to 1*L.

Counting the number of paths leading to a particular node is sufficiently transparent iterative process based on the observation that the access to the node A on step L can only be from one of the adjacent nodes. Therefore, the number of paths leading to the node A in step with the number (L), is the sum of the number of trajectories, leading to ALL ADJOINING nodes, but in the PREVIOUS step with the number (L-1).

In accordance with the described algorithm for L in the range from 1 to 20-30 steps were counted number table for the number of paths leading to relevant nodes (for the first quadrant in the case of 2 dimensions, and for the first octant in 3 dimensions).
It was found that in these tables, all numbers, as lying on the straight, parallel axes and diagonal lines, can be represented, as members of the same progressions.
This progression was similar progression, describing the binomial coefficients.
Defining progression explicitly and analyzed factors were able, then, to build from these coefficients arithmetic progression of order M.
Recall that an arithmetic progression of order M, this is called an arithmetic progression that its M-th difference dM is constant (see, in particular, the number 29 in [2]).
As M, and the difference between built progressions were individual values for each sequence.
Further, the process of analysis and construction of new sequences were repeated, as far as this was possible. (New progression was constructed from the coefficients of the previous progressions). This recurrence has led to increasingly dense bundle already received progressions in progression following levels of aggregation.

To explain how the source table are highlighted in the above progression, consider the simplest case for 2 dimensions.
At each step, the trajectory length L do not end in any, and only at certain nodes. These nodes, of course, surround the origin. For example, in step L, the node with coordinate (x= L, y= 0) has already been achieved, although only one path, which is straight along the X-axis length (L). Node (x= (L+1), y= 0) in this step is not yet achieved.
Thus, for step L numbers in the table are equal: for node (x= L, y= 0) the number of paths ending at node is 1, and for node (x= (L+1), y= 0) the number is 0.
At each current step will be called nodes, which terminates in at least one path, populated, and the nodes for which the number ending in these trajectories is 0 - empty.
Filled part of the table visually looks like extending from the origin with each step L, square, or rhombus isosceles. We will assume that it is a square.

This square is limited to its "external" parties. "Outer side" are straight, inclined at an angle of 45 degrees to the axes X and Y.
These are straight out of them, "towards infinity", are only empty nodes, and these straight lines formed by the "external" (about the origin) is filled with nodes.
Returning to the example. It turned out in particular that the numbers that make up these external parties are members of the well-known progression. This progression is painted binomial coefficients of degree L.
More specifically: for L=3 "outer" side of the square formed by the numbers 1 - 3 - 3 - 1, for L=4: 1 - 4 - 6 - 4 - 1, for L=5: 1 - 5 - 10 - 10 - 5 - 1 etc.

Further, as already mentioned, numerous parcels of private formulas describing the various detected progression, this process (package) managed to finish in the case of 2-dimensional grid. It was obtained the following expression for the NUMBER of PATHS leading to the node (X,Y) in step L:

$$\boldsymbol{K_L(X,Y)} = \begin{pmatrix} L \\ \frac{L+X+Y}{2} \end{pmatrix} * \begin{pmatrix} L \\ \frac{L+X-Y}{2} \end{pmatrix}$$

And, so acting, for the case of 2-dimensional grid final solution of the problem of random walks is very simple:

$$\boldsymbol{P_L}\,(X,Y) = \left(1/4^L \right) * \begin{pmatrix} L \\ \frac{L+X+Y}{2} \end{pmatrix} * \begin{pmatrix} L \\ \frac{L+X-Y}{2} \end{pmatrix}$$

where P_L (X,Y) is the probability that a point on the step number L will be in the node with coordinates (X, Y), and the symbol $\begin{pmatrix} a \\ b \end{pmatrix}$ at the right side, as before, means $\frac{a!}{b!(a-b)!}$.

For the case of 3-dimensional grid convolution managed to bring only the penultimate stage, namely:

$$\boldsymbol{P_L}\,(X,Y,Z) = \left(1/6^L \right) *$$

$$\sum_{j=0}^{\frac{L-(X+Y+Z)}{2}} \{ \begin{pmatrix} L \\ X+j \end{pmatrix} * \begin{pmatrix} L-(X+j) \\ j \end{pmatrix} * \begin{pmatrix} L-X-2j \\ \frac{(L-X-2j)-(Y+Z)}{2} \end{pmatrix} * \begin{pmatrix} L-X-2j \\ \frac{(L-X-2j)-(Y-Z)}{2} \end{pmatrix} \},$$

where the values P, L, X, Y, Z - similar to the previous cases, and j is the technical index of the current summation.

Note that due to the complete symmetry axis, it is obvious that the result does not change when replacing X with Y (or Z) and, respectively, Y (or Z) on the X. The author believes that this observation may help to complete convolution and obtaining a compact and simple formula.

On the other hand, it is known that for N=1 and 2, the movement of a wandering point back (i.e. wandering point with probability= 1 returns to the origin), and for any N>=3, this is not the case - the movement of non-returnable.
For example, for our case, a simple dimensional grid, the probability of return to the origin is set to p= 0,3405... [1]. Perhaps, this fact did not allow the author to obtain a simpler expression for N= 3.

So it is, for the case of 2 dimensions, a SIMPLE combinatorial formula describing random walks on the grid. For the case of same 3 dimensions the EXACT formula, but unfortunately, not compact.

Bibliography

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2. Dwight H.B. Tables of integrals, 1961