Towards A
Quaternionic Spacetime Tensor Calculus

PRELIMINARY DRAFT, Rev.2

Carsten S.P. Spanheimer
carsten.spanheimer (at) student.uni-tuebingen.de

February 1, 2015

Abstract

Introducing a special quaternionic vector calculus on the tangent bundle of a 4-dimensional space, and by forcing a condition of holomorphism, a *Minkowski*-type spacetime emerges, from which gravitation and also the whole *Maxwell* theory of electromagnetic fields arises.

Contents

1 Introduction . . . **2**
- 1.1 Overview . . . 3
- 1.2 Conventions . . . 3

2 Provisions . . . **4**
- 2.1 Scale Invariance . . . 4
- 2.2 Infinitesimal Embedding . . . 5
- 2.3 A Derivative Tensor Picture . . . 6
- 2.4 Analyticity . . . 7
- 2.5 A *Taylor* Series Picture . . . 7
- 2.6 A Vector Form Picture . . . 8
- 2.7 Vector Bases and Flat Metrics . . . 9
- 2.8 Reviewing Quaternions . . . 11
- 2.9 Quaternionic Base and the *Minkowski* Metric . . . 12
- 2.10 A Quaternionic Vector Algebra . . . 12
- 2.11 Quaternionic Forms . . . 14
- 2.12 Exponentiation of the *Jacobi* logarithm . . . 15

3 Decomposing Spacetime **18**
3.1 2nd-Rank Functions and Special Relativity 18
3.2 Summary of the 2nd-Rank Components 21
3.3 3rd-Rank Functions and Gravitation 22
3.4 Purely Spatial Derivatives 25
3.5 Summarizing the Components of 3rd-Rank Derivatives 28
3.6 4th-Rank Functions and Electromagnetism 31
3.7 5th-Rank Functions and *Maxwell*'s Equations 32
3.8 6th-Rank Functions and Electromagnetic Waves 33
3.9 Summarizing Electromagnetism 34
3.10 Complete Derivative Chart 35

4 Conclusion **36**
4.1 What's Been Shown 36
4.2 What's Essential 36
4.3 What's the Gain 37
4.4 Epilog 37

5 Revision History **38**

1 Introduction

Classical field theories are described by scalar/vector/tensor fields, either in an euclidean flat space with a continuous time or in a *Minkowski*-flat spacetime. In both cases, space and time are simplified to forming a flat manifold, which makes it easy to translate from one position to another.

In general relativity, flat spacetime is substituted for a spacetime manifold, of which the gravitational field is located on the tangent bundle, so the manifold is thus distorted by the gravitational field itself. The price to pay is having to integrate along geodesics when constructing even most simple geometric primitives, like lines of translation.

Electromagnetic fields were still separated in a vector bundle on the spacetime manifold. *Einstein*, *Hilbert*, *Weyl*, *Lanczos*, and others sought for a solution to identify electromagnetic field theory on the tangent bundle.

The present text aims to provide a solution to the problem of unifying gravitation with electromagnetic field theory in a context of differential geometry which is related to general relativity. It is shown, that the basic entities of classical (non-quantized) electromagnetic field theory can be identified with components of the tangent bundle of the spacetime manifold.

1.1 Overview

We know from special relativity, that the spacetime manifold should have 3 spatial and one time dimension, and from general relativity, that it needs not be euclidean.

To investigate the structure of a general 4-dimensional tangent bundle, the special viewpoint of a local viewer is taken. Locally, the tangent space is flat and described by an embedding matrix which is locally always unity, but its derivatives need not vanish.

Then the matrix logarithm of that embedding matrix is locally always the zero matrix, but again her derivatives need not vanish. These logarithmic derivative tensors shall be at the center of our investigation.

Now make two reasonable assumptions, which restrict the system,

1. All functions on the tangent bundle shall be harmonic and all components of the derivatives describe holomorphic fields,
2. The logarithmic embedding matrix and all its derivatives are derivatives of a scalar 'master potential'.

This still allows the introduction of imaginary units in the base vectors, and among trivial cases, exactly the choice of a quaternionic vector/tensor algebra induces a space with a behaviour suspectively similar to our physical spacetime.

We can, then without loss of generality, observe the 4-dimensional *Taylor* expansion of a single quaternionic potential, where the mixed-power monomials of some power together determine the derivatives of corresponding tensor rank.

Finally, again without loss of generality, that single potential can be constructed from products of quaternionic differential forms, which allows for encoding quaternionic vectors and thus in a 'generating vector picture' gives rise to well-known geometrical operations like scalar and vector product, gradient, divergence and rotational derivative.

This gives a special-relativistic picture with the *Lorentz* transform and a spatial rotation, and also lets us in a generally-relativistic way identify the fields of gravitation as well as of electromagnetic field theory, that is, the electromagnetic tensor together with vector potential, gauge potential and charge/current density.

1.2 Conventions

Where tensors are written in index notation,
the *Einstein* summation convention is always active, unless noted otherwise.

More unusually[1], contraction indices may be doubled when unambiguous
and named in a suggestive way, like for example in

$$\Gamma^a{}_{\delta\delta}\,\delta^{\delta\delta} \qquad \text{or} \qquad \Gamma_{aaa}\delta^{aa}\,,$$

[1]It should be clear that introducing new conventions or modifying existing ones should be justified by them 1. being unambiguous and 2. actually enhancing readability.

since at least one of the involved tensors is symmetric in the involved index pair(s).

Index instances are printed in bold, for example

$$(T_{\boldsymbol{t}}, T_{\boldsymbol{x}}, T_{\boldsymbol{y}}, T_{\boldsymbol{z}}) \qquad \text{when in } T_a, \qquad a \in \{\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}\,.$$

In matrices, zero elements may be left blank or replaced by a dot $(\cdot)$, the eye may be guided by separation lines, and $+$, $-$ might be used for $+1$, -1 respectively, for example

$$\mathbf{0}_{ab} := \begin{bmatrix} \cdot & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix}, \qquad \delta_{ab} := \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right], \qquad \eta_{ab} := \left[\begin{array}{c|ccc} + & & & \\ \hline & - & & \\ & & - & \\ & & & - \end{array}\right].$$

The *Minkowski* metric will come out with a signature $(+\,-\,-\,-)$, as shown above.

The Nabla operator and the *Laplace* operator are defined as usual in three dimensions,

$$\nabla := \begin{pmatrix} 0 \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}, \qquad \nabla\Phi = \operatorname{grad}\Phi\,,$$

$$\Delta := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right), \qquad \Delta\Phi = \operatorname{div}\operatorname{grad}\Phi\,,$$

and the *d'Alembert* operator in four dimensions,

$$\Box := \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right), \qquad \Box\,\Phi = \frac{\partial^2}{\partial t^2}\Phi - \Delta\Phi\,.$$

2 Provisions

2.1 Scale Invariance

Let us warm up with a one-dimensional toy model. A function

$$g : \mathbb{R} \to \mathbb{R};\ x \mapsto g(x)$$

be differentiable at least once,

$$g'(x) = \tfrac{\partial}{\partial x} g\,.$$

Another function takes the first one to the exponent,

$$f : \mathbb{R} \to \mathbb{R}^+;\ x \mapsto f(x) = e^{g(x)}\,,$$

and the overall derivative is proportional to the latter,

$$f'(x) = g'(x)\, e^{g(x)} = g'(x) f(x)\,,$$

so that the derivative of the inner function is the quotient

$$g'(x) = f'(x)/f(x)\,.$$

Additionally scaling $af(x)$ by a constant factor $a \in \mathbb{R}$, we can run through a cycle of exponentiation, derivation, division and integration,

$$\begin{array}{ccc}
g + c & \xrightarrow{\exp} & af = e^{g+c} = ae^{g} \quad (\text{with } a = e^{c}) \\
{\scriptstyle \int \partial_x + c}\Big\uparrow & & \Big\downarrow{\scriptstyle \partial_x} \\
g' = af'/af & \xleftarrow[1/af]{} & af' = g'\, ae^{g}\,,
\end{array} \tag{1}$$

and observe, that g' does not contain the scale factor $a = e^c$ anymore, which is as arbitrary as the integration constant c. This is the notion of 'scale invariance' which will be employed here.

In another view, $g' = f'/f$ is the derivative of $\log(f)$, and lives in a 'logarithmic space', so multiplication in f can be represented by addition of the logarithms, $f_1 \cdot f_2 = \exp(\,g_1 + g_2)$.

Additionally, f and f' contain f to a power, or 'f–weight', of f^1, but in g' and g the corresponding f–weight is f^0, which is equivalent to saying that the latter expressions are scale-invariant.

2.2 Infinitesimal Embedding

Generalize from the one-dimensional function f to the *Jacobi* matrix of an automorphic embedding of 4-space into 4-space,

$$J^{\mu}{}_{b} : \mathbb{R}^4 \to \mathbb{R}^4\,,$$

where the metric tensor comes out of the *Jacobi* matrix by kind of 'squaring' her through multiplication with her transpose, involving an outer metric, which shall be the flat *Minkowski* metric,

$$g_{ab} \;:=\; \eta_{\eta\eta}\, J^{\eta}{}_{a} J^{\eta}{}_{b}\,. \tag{2}$$

Specify a '*Jacobi* logarithm',

$$\Gamma^{a}{}_{b} \;=\; \log(J^{\mu}{}_{b})\,,$$

so that the *Jacobi* matrix can be seen as the matrix exponential,

$$\exp(\Gamma^{a}{}_{b}) \;=:\; J^{\mu}{}_{b}\,.$$

But to represent matrix multiplication through addition of logarithmic matrices,

$$\overset{1}{J}{}^{\mu}{}_{b} \cdot \overset{2}{J}{}^{\mu}{}_{b} \;=\; \exp(\,\overset{1}{\Gamma}{}^{a}{}_{b} + \overset{2}{\Gamma}{}^{a}{}_{b})\,,$$

matrix multiplication has to commute like addition of the logarithms does, which is true only if the *Jacobi* logarithm is infinitesimally near zero, $\Gamma^{a}{}_{b} \to 0$, so that the *Jacobi* matrix is infinitesimally near identity, $J^a{}_b \to \delta^a{}_b$. This is the notion of an 'infinitesimal embedding' which will be employed here.

The infinitesimal embedding is an embedding of 'my' local tangent spacetime into my infinitesimal neighbourhood, where my tangent spacetime is always orthonormal and my local metric tensor is the flat *Minkowski* metric, $g_{ab} \to \eta_{ab}$.

Though these entities have been fixed at one position (event) in spacetime, their derivatives at that point are still allowed to have any magnitude.

2.3 A Derivative Tensor Picture

Let there be some scalar potential function in 4 dimensions (t, x, y, z),

$$\Gamma(t, x, y, z) : \mathbb{R}^4 \to \mathbb{K}\,,$$

at first leaving open what space $\mathbb{K}$ is, but finally employing $\mathbb{K} = \mathbb{H}$,
the space of quaternionic hypercomplex numbers.

Partial derivation gives a gradient vector,

$$\Gamma_a := \partial_a \, \Gamma(t, x, y, z)\,,$$

and at the second partial derivation a matrix,
which is symmetric from the *Schwarz* rule that partial derivation commutes,

$$\Gamma_{ab} := \partial_b \, \Gamma_a = \partial_a \, \Gamma_b\,,$$

of which the trace is the 4-dimensional *Laplac*ian of the original potential,
which in foresight be defined with a *d'Alembert* operator,

$$\Gamma_{\delta\delta} \, \delta^{\delta\delta} \;=:\; \Box \, \Gamma\,.$$

The third derivation results in a tensor of rank 3, which is again symmetric in all three possible index pairings,

$$\Gamma_{abc} := \partial_c \, \Gamma_{ab}\,,$$

which can be contracted to a vector in three ways, these are

$$\Gamma_{a\delta\delta} \, \delta^{\delta\delta} \;=\; \Gamma_{\delta b\delta} \, \delta^{\delta\delta} \;=\; \Gamma_{\delta\delta c} \, \delta^{\delta\delta} \;=\; \Box \, \Gamma_a\,.$$

And so on deriving n times gives a tensor of rank n, and all derivative tensors are symmetric in all possible index pairings. They can successively be contracted by an even number of ranks, ending with either a scalar (when n is even) or a vector (when n is odd).

2.4 Analyticity

We are interested in holomorphic functions, and a necessary condition is that all contractions of any derivative tensor vanish identically. Stating the contractions explicitly up to the 5th-rank tensors, these are

$$\Box\,\Gamma = \Gamma_{\delta\delta}\,\delta^{\delta\delta} \equiv 0 \qquad \text{(rank 2 to rank 0)}\,,$$
$$\Box\,\Gamma_a = \Gamma_{a\,\delta\delta}\,\delta^{\delta\delta} \equiv 0 \qquad \text{(rank 3 to rank 1)}\,,$$
$$\Box\,\Gamma_{ab} = \Gamma_{ab\,\delta\delta}\,\delta^{\delta\delta} \equiv 0 \qquad \text{(rank 4 to rank 2)}\,,$$
$$\Box\,\Gamma_{abc} = \Gamma_{abc\,\delta\delta}\,\delta^{\delta\delta} \equiv 0 \qquad \text{(rank 5 to rank 3)}\,,$$
$$\Box\Box\,\Gamma_{ab} = \Gamma_{\delta\delta\delta\delta}\,\delta^{\delta\delta}\,\delta^{\delta\delta} \equiv 0 \qquad \text{(rank 4 to rank 0)}\,,$$
$$\Box\Box\,\Gamma_a = \Gamma_{a\delta\delta\delta\delta}\,\delta^{\delta\delta}\delta^{\delta\delta} \equiv 0 \qquad \text{(rank 5 to rank 1)}\,.$$

This also means that all entities are source-free in a 4-dimensional sense, but not necessarily in the 3 spatial subdimensions. Thus the following time-only contractions give corresponding observables, which reveal their respective meanings in the investigations further below,

$$\Gamma_{\boldsymbol{tt}}\,\delta^{\boldsymbol{tt}} \qquad \text{(logarithmic gravitational potential, electromagnetic gauge potential)}\,,$$
$$\Gamma_{a\,\boldsymbol{tt}}\,\delta^{\boldsymbol{tt}} \qquad \text{(electromagnetic vector potential)}\,,$$
$$\Gamma_{ab\,\boldsymbol{tt}}\,\delta^{\boldsymbol{tt}} \qquad \text{(electromagnetic \textit{Faraday} tensor)}\,,$$
$$\Gamma_{\boldsymbol{tttt}}\,\delta^{\boldsymbol{tt}}\delta^{\boldsymbol{tt}} \qquad \text{(logarithmic gravitational source density)}\,,$$
$$\Gamma_{a\,\boldsymbol{tttt}}\,\delta^{\boldsymbol{tt}}\delta^{\boldsymbol{tt}} \qquad \text{(electromagnetic charge/current density)}\,.$$

Forcing all components of the trace vector to zero makes the components of the 1st-rank gradient vector be harmonic functions, and in general, forcing all possible traces of the $k+2$-th rank derivative tensor makes the components of the k-th rank derivative tensor be harmonic functions. Thus all components of all logarithmic tensors have to be harmonic, and the master potential holomorphic.

2.5 A *Taylor* Series Picture

When the master potential with all its derivatives is holomorphic, then without loss of generality that potential function can be represented by an n-dimensional *Taylor* series expanded at local coordinate zero, where all terms of mixed n-th power arise in exactly the n-th derivative.

In a 2-dimensional example,

$$\Gamma(x,y) := \begin{cases} a_{00} & +\ a_{10}\,x & +\frac{1}{2}\,a_{20}\,x^2 & +\frac{1}{6}\,a_{30}\,x^3 & \dots \\ +\ a_{01}\,iy & +\ a_{11}\,x\,iy & +\frac{1}{2}\,a_{21}\,x^2\,iy & +\frac{1}{6}\,a_{31}\,x^3\,iy & \dots \\ +\frac{1}{2}\,a_{02}\,i^2y^2 & +\frac{1}{2}\,a_{12}\,x\,i^2y^2 & +\frac{1}{4}\,a_{22}\,x^2 i^2y^2 & \ddots & \ddots \\ +\frac{1}{6}\,a_{03}\,i^3y^3 & +\frac{1}{6}\,a_{13}\,x\,i^3y^3 & \ddots & \ddots & \\ \vdots & \vdots & \ddots & & \end{cases} ,$$

where the constant i is not fixed yet.
Defining $i := 1$ gives an $\mathbb{R}^2$ case, and $i^2 := -1$ gives a construction in complex numbers.

Since we are always at point zero, all powers vanish except the one at the tip,

$$\overset{0}{\Gamma}_{|0} = a_{00}\,.$$

Then, of a k-th mixed-order monomial, exactly the k-th derivative is non-vanishing. Terms of power $k < n$ vanish because the higher derivatives are identically zero, and terms of power $k > n$ vanish because we are at point zero, of which all powers vanish.

2.6 A Vector Form Picture

Again without loss of generality, any *Taylor* series can be represented by a linear combination of products of 1st-power functions, for example in 2 dimensions.

A scalar function of **1st order** be defined as

$$\overset{1}{\Gamma} := \overset{1}{\Pi} := \begin{pmatrix} a \\ ib \end{pmatrix} \cdot \begin{pmatrix} \partial x \\ \partial y \end{pmatrix} = \big(a\,\partial x + ib\,\partial y\big)\,,$$

with an at first arbitrary constant i,
and the 1st derivative only unpacks the vector with which we started,

$$\overset{1}{\Gamma}_a = \overset{1}{\Pi}_a := \partial_a \overset{1}{\Pi} = \begin{pmatrix} a \\ ib \end{pmatrix} =: \overset{a}{\Pi}_a\,.$$

Note, that the 2-dimensional 1st-rank tensor has 2 different components out of 2.

From a scalar function of **2nd order**, defined as the product of two vector forms,

$$\overset{2}{\Gamma} = \overset{2}{\Pi} := \left(\begin{pmatrix} a \\ ib \end{pmatrix} \cdot \begin{pmatrix} \partial x \\ \partial y \end{pmatrix}\right)\left(\begin{pmatrix} \alpha \\ i\beta \end{pmatrix} \cdot \begin{pmatrix} \partial x \\ \partial y \end{pmatrix}\right) = a\alpha\, x^2 + i\,(a\beta + b\alpha)\,xy + i^2\, b\beta\, y^2\,,$$

the 2nd derivative gives a matrix,

$$\overset{2}{\Gamma}_{ab} := \partial_b\partial_a \overset{2}{\Gamma} = \begin{bmatrix} 2\,a\alpha & i(a\beta + b\alpha) \\ i(a\beta + b\alpha) & 2\,i^2 b\beta \end{bmatrix},$$

which can be obtained in an easy way from the outer product of the separate first derivatives,

$$\overset{2}{\Pi}_{ab} := \partial_a \overset{a}{\Pi}\, \partial_b \overset{b}{\Pi} = \begin{bmatrix} a\alpha & ia\beta \\ ib\alpha & i^2 b\beta \end{bmatrix}.$$

using the *Leibniz* rule by adding to the outer product matrix its transpose,

$$\overset{2}{\Gamma}_{ab} := \partial_b\partial_a \overset{2}{\Gamma} = \overset{2}{\Pi}_{ab} + \overset{2}{\Pi}_{ba} = \overset{a}{\Pi}_a \overset{b}{\Pi}_b + \overset{a}{\Pi}_b \overset{b}{\Pi}_a = \begin{bmatrix} 2\,a\alpha & i(a\beta + b\alpha) \\ i(a\beta + b\alpha) & 2\,i^2 b\beta \end{bmatrix}.$$

It is symmetric and thus establishes the *Schwarz* rule of commuting partial derivatives. Note, that the 2-dimensional 2nd-rank tensor has only 3 different components out of 4.

From a scalar function of **3rd order**, defined as the product of three vector forms,

$$\overset{3}{\Gamma} \;=\; \overset{3}{\Pi} := \left(\begin{pmatrix} a \\ ib \end{pmatrix}\cdot\begin{pmatrix}\partial x\\ \partial y\end{pmatrix}\right)\left(\begin{pmatrix} \alpha \\ i\beta \end{pmatrix}\cdot\begin{pmatrix}\partial x\\ \partial y\end{pmatrix}\right)\left(\begin{pmatrix} A \\ iB \end{pmatrix}\cdot\begin{pmatrix}\partial x\\ \partial y\end{pmatrix}\right)$$
$$= \; a\alpha A\, x^3 + i\,(b\alpha A + a\beta A + a\alpha B)\, x^2 y + i^2\,(a\beta B + b\alpha B + b\beta A)\, xy^2 + i^3\, b\beta B\, y^3\,,$$

the outer derivative product is

$$\overset{3}{\Pi}_{abc} \;:=\; \partial_a \overset{a}{\Pi}\,\partial_b \overset{c}{\Pi}\,\partial_c \overset{c}{\Pi} \;=\; \left[\begin{bmatrix} a\alpha A & ia\beta A \\ ib\alpha A & i^2 b\beta A \end{bmatrix}\begin{bmatrix} ia\alpha B & i^2 a\beta B \\ i^2 b\alpha B & i^3 b\beta B \end{bmatrix}\right],$$

and from the *Leibniz* rule follows the derivative tensor
as the sum over all permutations of all indices,

$$\overset{3}{\Gamma}_{abc} \;=\; \overset{3}{\Pi}_{abc} + \overset{3}{\Pi}_{bac} + \overset{3}{\Pi}_{bca} + \overset{3}{\Pi}_{cba} + \overset{3}{\Pi}_{cab} + \overset{3}{\Pi}_{acb}$$
$$= \overset{a}{\Pi}_a \overset{b}{\Pi}_b \overset{c}{\Pi}_c + \overset{a}{\Pi}_b \overset{b}{\Pi}_a \overset{c}{\Pi}_c + \overset{a}{\Pi}_b \overset{b}{\Pi}_c \overset{c}{\Pi}_a + \overset{a}{\Pi}_c \overset{b}{\Pi}_b \overset{c}{\Pi}_a + \overset{a}{\Pi}_c \overset{b}{\Pi}_a \overset{c}{\Pi}_b + \overset{a}{\Pi}_a \overset{b}{\Pi}_c \overset{c}{\Pi}_b$$
$$= \left[\begin{bmatrix} 2\,a\alpha A & i(a\beta + b\alpha)A \\ i(a\beta + b\alpha)A & 2\,i^2 b\beta A \end{bmatrix}\begin{bmatrix} 2\,ia\alpha B & i^2(a\beta + b\alpha)B \\ i^2(a\beta + b\alpha)B & 2\,i^3 b\beta B \end{bmatrix}\right]$$
$$+ \left[\begin{bmatrix} 2\,a\alpha A & i(a\beta + b\alpha)A \\ 2\,ia\alpha B & i^2(a\beta + b\alpha)B \end{bmatrix}\begin{bmatrix} i(a\beta + b\alpha)A & 2\,i^2 b\beta A \\ i^2(a\beta + b\alpha)B & 2\,i^3 b\beta B \end{bmatrix}\right]$$
$$+ \left[\begin{bmatrix} 2\,a\alpha A & 2\,ia\alpha B \\ i(a\beta + b\alpha)A & i^2(a\beta + b\alpha)B \end{bmatrix}\begin{bmatrix} i(a\beta + b\alpha)A & i^2(a\beta + b\alpha)B \\ 2\,i^2 b\beta A & 2\,i^3 b\beta B \end{bmatrix}\right]$$
$$= 2\left[\begin{bmatrix} 3\,a\alpha A & i(a\alpha B + a\beta A + b\alpha A) \\ i(a\alpha B + a\beta A + b\alpha A) & i^2(b\beta A + b\alpha B + a\beta B) \end{bmatrix}\begin{bmatrix} i(a\alpha B + a\beta A + b\alpha A) & i^2(b\beta A + b\alpha B + a\beta B) \\ i^2(b\beta A + b\alpha B + a\beta B) & 3\,i^3 b\beta B \end{bmatrix}\right].$$

Note, that the 2-dimensional 3rd-rank tensor has only 4 different components out of 8.

And so on the further derivatives of functions of further order are formed from all permutations over all indices of the outer products of the single derivatives.
They are symmetrical in all possible index pairings, conforming to the *Schwarz* rule.

In the 2-dimensional case, the k-th rank derivative tensor contains at most $k+1$ different components.

2.7 Vector Bases and Flat Metrics

From the previous 2-dimensional example, formulate an orthonormal base vector matrix with an at first arbitrary constant i,

$$\mathfrak{B} \;=\; \mathfrak{B}_{ab} \;=\; \begin{bmatrix} 1 & \\ & i \end{bmatrix},$$

and recognize a metric tensor,

$$\mathfrak{N} \;=\; \mathfrak{N}_{\eta\eta} \;=\; \mathfrak{B}^T\mathfrak{B} \;=\; \mathfrak{B}_{\eta\delta}\,\delta^{\delta\delta}\,\mathfrak{B}_{\delta\eta} \;=\; \begin{bmatrix} 1 & \\ & i^2 \end{bmatrix}.$$

When $i := \pm 1$, then the metric tensor is the identity,

$$\mathfrak{N} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \delta_{\delta\delta} ,$$

and the *Laplace* condition, when contracting a tensor, requires the diagonal components to vanish identically and thus forces the underlying manifold to be flat and the generating vectors to be perpendicular.

When $i^2 := -1$, then the metric tensor is a small *Minkowski* metric,

$$\mathfrak{N} = \begin{bmatrix} +1 & \\ & -1 \end{bmatrix} = \eta_{\eta\eta} ,$$

and the *Laplace* condition, when contracting a tensor, still allows the diagonal components to move and the generating vectors to be arbitrary, with parallel vectors distorting the underlying manifold.

Promote to 4 dimensions.
A purely real vector signature again forces the underlying manifold to be flat and does not result in a *Minkowski* metric,

$$\mathfrak{B}_{ab} = \left[\begin{array}{c|ccc} \pm 1 & & & \\ \hline & \pm 1 & & \\ & & \pm 1 & \\ & & & \pm 1 \end{array}\right] \quad \Rightarrow \quad \mathfrak{N}_{\eta\eta} = \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] = \delta_{\delta\delta} ,$$

so it is not of any use for a description of physical reality.

A complex signature with $i^2 := -1$ does give a *Minkowski* metric in either convention,
but still no possibility to represent spatial rotation,

$$\mathfrak{B}_{ab} = \left[\begin{array}{c|ccc} \pm i & & & \\ \hline & \pm 1 & & \\ & & \pm 1 & \\ & & & \pm 1 \end{array}\right] \quad \Rightarrow \quad \mathfrak{N}_{\eta\eta} = \left[\begin{array}{c|ccc} -1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] = -\eta_{\eta\eta} ,$$

$$\mathfrak{B}_{ab} = \left[\begin{array}{c|ccc} \pm 1 & & & \\ \hline & \pm i & & \\ & & \pm i & \\ & & & \pm i \end{array}\right] \quad \Rightarrow \quad \mathfrak{N}_{\eta\eta} = \left[\begin{array}{c|ccc} 1 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right] = \eta_{\eta\eta} .$$

Finally, a set of quaternionic base vectors gives indeed both a *Minkowski* metric
and a representation of spatial rotations,

$$\mathfrak{B}_{ab} = \left[\begin{array}{c|ccc} \pm 1 & & & \\ \hline & \pm i & & \\ & & \pm j & \\ & & & \pm k \end{array}\right] \quad \Rightarrow \quad \mathfrak{N}_{\eta\eta} = \left[\begin{array}{c|ccc} 1 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right] = \eta_{\eta\eta} ,$$

so the algebra and analytics of that signature shall now be investigated further.

2.8 Reviewing Quaternions

Quaternions, or *Hamilton* numbers, are a generalization of complex numbers to 4 dimensions. Instead of one real and one imaginary part, with the imaginary unit $i^2 := -1$, and the base vectors $\{1, i\}$, quaternions have three imaginary units and a base $\{1, i, j, k\}$,
with definitions

$$i^2 = j^2 = k^2 = -1\,, \quad ij = k\,, \quad jk = i\,, \quad ki = j\,,$$

and corollaries

$$\begin{aligned} &ijk = jki = kij = -1\,, \quad kji = jik = ikj = +1 \\ &iki = +k\,, \quad iik = kii = -k\,. \end{aligned}$$

Multiplication of quaternions is associative, but neither strictly commutative nor strictly anti-commutative, The so defined algebra forms a non-abelian field over $\mathbb{R}^4$.

Quaternions are typically either written as hypercomplex scalars,

$$r + ia + jb + kc\,.$$

or expressed as a matrix algebra, with base

$$1 \mapsto \begin{bmatrix} + & \cdot & \cdot & \cdot \\ \cdot & + & & \\ \cdot & & + & \\ \cdot & & & + \end{bmatrix},\ i \mapsto \begin{bmatrix} \cdot & - & \cdot & \cdot \\ + & \cdot & & \\ \cdot & & \cdot & - \\ \cdot & & + & \cdot \end{bmatrix},\ j \mapsto \begin{bmatrix} \cdot & \cdot & - & \cdot \\ \cdot & \cdot & & + \\ + & & \cdot & \\ \cdot & - & & \cdot \end{bmatrix},\ k \mapsto \begin{bmatrix} \cdot & \cdot & \cdot & - \\ \cdot & \cdot & - & \\ \cdot & + & \cdot & \\ + & & & \cdot \end{bmatrix}.$$

A quaternion can be seen as composed from a real scalar part and an imaginary vector part,

$$\begin{aligned} Q &:= \begin{pmatrix} 1 \\ \begin{pmatrix} i \\ j \\ k \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} d \\ \vec{q} \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix} \cdot \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix} = d + a\,i + b\,j + c\,k \\ R &:= \begin{pmatrix} 1 \\ \begin{pmatrix} i \\ j \\ k \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \delta \\ \vec{r} \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix} \cdot \begin{pmatrix} \delta \\ \alpha \\ \beta \\ \gamma \end{pmatrix} = \delta + \alpha\,i + \beta\,j + \gamma\,k\,, \end{aligned}$$

so that the quaternion product contains both dot product and cross product of the vectors, intermixed with the products from the scalar parts,

$$Q \cdot R = \begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix} \cdot \begin{pmatrix} d\delta + (\vec{q} \cdot \vec{r}) \\ d\vec{r} + \delta\vec{q} + (\vec{q} \times \vec{r}) \end{pmatrix}\,.$$

All in all, this kind of a quaternion algebra is not well suited for representations of 4-vectors. Tensor algebras are more flexible, but the introduction of special multiplication rules like the cross product needs artificial introductions.

In the next sections, the best of both worlds shall be combined, that is, a tensor algebra with built-in quaternionic rules.

2.9 Quaternionic Base and the *Minkowski* Metric

Introducing the convention

$$\bar{\imath} := -i = \tfrac{1}{i}\,, \quad \bar{\jmath} := -j = \tfrac{1}{j}\,, \quad \bar{k} := -k = \tfrac{1}{k}\,,$$

define a quaternionic base vector matrix and its inverse,

$$\mathfrak{B} \;:=\; \mathfrak{B}_{ab} \;=\; \left[\begin{array}{c|ccc} 1 & & & \\ \hline & i & & \\ & & j & \\ & & & k \end{array}\right], \qquad \bar{\mathfrak{B}} \;:=\; \mathfrak{B}^{-1} \;=\; \mathfrak{B}^{ab} \;=\; \left[\begin{array}{c|ccc} 1 & & & \\ \hline & \bar{\imath} & & \\ & & \bar{\jmath} & \\ & & & \bar{k} \end{array}\right],$$

$$\mathfrak{B}\bar{\mathfrak{B}} \;=\; \bar{\mathfrak{B}}\mathfrak{B} \;=\; \mathfrak{B}^{a}{}_{b} \;=\; \mathfrak{B}_{a}{}^{b} \;=\; \mathrm{Id}_4 \;=\; \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right],$$

of which the 2nd power gives a *Minkowski* metric,

$$\mathfrak{N} \;=\; \bar{\mathfrak{N}} \;:=\; \mathfrak{B}\mathfrak{B} \;=\; \bar{\mathfrak{B}}\bar{\mathfrak{B}} \;=\; \left[\begin{array}{c|ccc} 1 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right] \;=\; \eta_{\eta\eta} \;=\; \eta^{\eta\eta}\,,$$

and the 4th power is the identity,

$$\mathfrak{B}^4 \;=\; \bar{\mathfrak{B}}^4 \;=\; \mathfrak{N}^2 \;=\; \bar{\mathfrak{N}}^2 \;=\; \mathrm{Id}_4 \;=\; \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right],$$

$$\mathfrak{B}^3 \;=\; \mathfrak{N}\mathfrak{B} \;=\; \mathfrak{B}\mathfrak{N} \;=\; \bar{\mathfrak{B}}\,, \qquad \bar{\mathfrak{B}}^3 \;=\; \mathfrak{N}\bar{\mathfrak{B}} \;=\; \bar{\mathfrak{B}}\mathfrak{N} \;=\; \mathfrak{B}\,.$$

2.10 A Quaternionic Vector Algebra

In the present context, quaternions will be represented by vectors, but each component 'blessed' with a particular real or imaginary signature,

$$\vec{s} \;:=\; \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \vec{r} \;:=\; \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix},$$

$$S_a \;:=\; \mathfrak{B}\cdot\begin{pmatrix} d \\ \vec{s} \end{pmatrix} \;=\; \begin{pmatrix} d \\ i\,a \\ j\,b \\ k\,c \end{pmatrix}, \qquad R_a \;:=\; \mathfrak{B}\cdot\begin{pmatrix} \delta \\ \vec{r} \end{pmatrix} \;=\; \begin{pmatrix} \delta \\ i\,\alpha \\ j\,\beta \\ k\,\gamma \end{pmatrix},$$

with a common signature

$$\Pi_a \sim \begin{bmatrix} + & i & j & k \end{bmatrix},$$

Then the outer product of two such vectors is a matrix,

$$\Pi_{sr} := S_s\, R_r = \left[\begin{array}{c|ccc} d\delta & i\,d\alpha & j\,d\beta & k\,d\gamma \\ \hline i\,a\delta & -a\alpha & k\,a\beta & \bar{\jmath}\,a\gamma \\ j\,b\delta & \bar{k}\,b\alpha & -b\beta & i\,b\gamma \\ k\,c\delta & j\,c\alpha & \bar{\imath}\,c\beta & -c\gamma \end{array}\right], \tag{3}$$

where each component is blessed by a particular signature,

$$\Pi_{ab} \sim \left[\begin{array}{c|ccc} + & i & j & k \\ \hline i & - & k & \bar{\jmath} \\ j & \bar{k} & - & i \\ k & j & \bar{\imath} & - \end{array}\right].$$

The first column and first row of (3) now contains the quaternion vectors $d\vec{r}$ and $\delta\vec{s}$, the upper left element is the product $d\delta$, and the trace forms a sum of that with the dot product, $d\delta-(\vec{s}\cdot\vec{r})$. The cross product is not yet clear.

But forming the sum of the outer product with its transpose, giving a purely symmetrical (but not self-adjunct) matrix,

$$\Gamma_{ab} := \Pi_{ab} + \Pi_{ba} = \left[\begin{array}{c|ccc} 2d\delta & i\,(d\alpha+\delta a) & j\,(d\beta+\delta b) & k\,(d\gamma+\delta c) \\ \hline i\,(d\alpha+\delta a) & -2a\alpha & k\,(a\beta-b\alpha) & j\,(c\alpha-a\gamma) \\ j\,(d\beta+\delta b) & k\,(a\beta-b\alpha) & -2b\beta & i\,(b\gamma-c\beta) \\ k\,(d\gamma+\delta c) & j\,(c\alpha-a\gamma) & i\,(b\gamma-c\beta) & -2c\gamma \end{array}\right]$$

$$= \left[\begin{array}{c|ccc} \cdot & iv_x & jv_y & kv_z \\ \hline iv_x & \cdot & & \\ jv_y & & \cdot & \\ kv_z & & & \cdot \end{array}\right] + \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & k\,\varrho_{\boldsymbol{z}} & j\,\varrho_{\boldsymbol{y}} \\ \cdot & k\,\varrho_{\boldsymbol{z}} & \cdot & i\,\varrho_{\boldsymbol{x}} \\ \cdot & j\,\varrho_{\boldsymbol{y}} & i\,\varrho_{\boldsymbol{x}} & \cdot \end{array}\right] + 2\left[\begin{array}{c|ccc} d\delta & \cdot & \cdot & \cdot \\ \hline \cdot & -s_x r_x & & \\ \cdot & & -s_y r_y & \\ \cdot & & & -s_z r_z \end{array}\right]. \tag{4}$$

the single vectors are mixed to a cross-wise linear combination,

$$\vec{\nu} := \delta\vec{s} + d\vec{r},$$

the cross product is

$$\vec{\varrho} := \vec{s}\times\vec{r},$$

and the trace is just doubled,

$$\Gamma_{\delta\delta}\,\delta^{\delta\delta} = 2(d\delta - \vec{s}\cdot\vec{r}).$$

The operation of adding the transpose was at first an ad-hoc decision. We should find a justification for that, that is, a situation where this operation occurs in a 'natural' way, and we will.

2.11 Quaternionic Forms

Define an integral operator and a differential operator,

$$\int^a \;=\; \vec{\int} \;:=\; \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}, \qquad \partial_a \;=\; \vec{\partial} \;:=\; \begin{pmatrix} \partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix},$$

to formally compose a hypercomplex scalar function from a quaternionic vector, like

$$\overset{1}{\Gamma} \;:=\; \int^\delta S_\delta \;=\; \begin{pmatrix} d \\ \vec{s} \end{pmatrix} \cdot \mathfrak{B} \cdot \vec{\int} \;=\; \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}t \\ i\,\mathrm{d}x \\ j\,\mathrm{d}y \\ k\,\mathrm{d}z \end{pmatrix} \;=\; d\,\mathrm{d}t + a\,i\,\mathrm{d}x + b\,j\,\mathrm{d}y + c\,k\,\mathrm{d}z\,,$$

(which some may call a 'differential form')
and unpack that quaternionic vector again through derivation,

$$\overset{1}{\Gamma}_a \;:=\; \partial_a \overset{1}{\Gamma} \;=\; \begin{pmatrix} d \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \;=\; S_a\,.$$

Next take the product of two such integrals,

$$\begin{aligned} \overset{2}{\Gamma} \;:&=\; \int^\delta S_\delta \int^\delta R_\delta \;=\; \left(\begin{pmatrix} d \\ \vec{s} \end{pmatrix} \cdot \mathfrak{B} \cdot \vec{\int} \right) \left(\begin{pmatrix} \delta \\ \vec{r} \end{pmatrix} \cdot \mathfrak{B} \cdot \vec{\int} \right) \\ &=\; \left(\begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}t \\ i\,\mathrm{d}x \\ j\,\mathrm{d}y \\ k\,\mathrm{d}z \end{pmatrix} \right) \left(\begin{pmatrix} \delta \\ \alpha \\ \beta \\ \gamma \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}t \\ i\,\mathrm{d}x \\ j\,\mathrm{d}y \\ k\,\mathrm{d}z \end{pmatrix} \right) \\ &=\; (d\,\mathrm{d}t + a\,i\,\mathrm{d}x + b\,j\,\mathrm{d}y + c\,k\,\mathrm{d}z)\,(\delta\,\mathrm{d}t + \alpha\,i\,\mathrm{d}x + \beta\,j\,\mathrm{d}y + \gamma\,k\,\mathrm{d}z)\,, \end{aligned}$$

derive a first time, according to the *Leibniz* product rule,

$$\begin{aligned} \Rightarrow \quad \overset{2}{\Gamma}_a \;&=\; S_a \int^\delta R_\delta + \int^\delta S_\delta\, R_a \\ &=\; \begin{pmatrix} d \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} (\delta\,\mathrm{d}t + i\alpha\,\mathrm{d}x + j\beta\,\mathrm{d}y + k\gamma\,\mathrm{d}z) \;+\; (d\,\mathrm{d}t + ia\,\mathrm{d}x + jb\,\mathrm{d}y + kc\,\mathrm{d}z) \begin{pmatrix} \delta \\ i\,\alpha \\ j\,\beta \\ k\,\gamma \end{pmatrix}, \end{aligned}$$

derive a second time,

$$\Rightarrow \quad \overset{2}{\Gamma}_{ab} = \begin{pmatrix} d \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \begin{pmatrix} \delta & i\alpha & j\beta & k\gamma \end{pmatrix} + \begin{pmatrix} d & ia & jb & kc \end{pmatrix} \begin{pmatrix} \delta \\ i\,\alpha \\ j\,\beta \\ k\,\gamma \end{pmatrix}$$

$$= S_a\, R_b + S_b\, R_a = \left[\begin{array}{c|ccc} d\delta & i\nu_x & j\nu_y & k\nu_z \\ \hline i\nu_x & -a\alpha & k\,\varrho_{\boldsymbol{z}} & j\,\varrho_{\boldsymbol{y}} \\ j\nu_y & k\,\varrho_{\boldsymbol{z}} & -b\beta & i\,\varrho_{\boldsymbol{x}} \\ k\nu_z & j\,\varrho_{\boldsymbol{y}} & i\,\varrho_{\boldsymbol{x}} & -c\gamma \end{array}\right], \tag{5}$$

where $\vec{\nu} := \delta\vec{s} + d\vec{r}, \quad \vec{\varrho} := \vec{s}\times\vec{r}, \quad \Gamma_{\delta\delta}\,\delta^{\delta\delta} = 2(d\delta + \vec{s}\cdot\vec{r})$.

This is enough to arrive at (4). The *Leibniz* product rule introduces the 'add transpose' operation. So differentiation of quaternionic vectors, rather than simple multiplication, gives the known vector cross product by introducing the rotational derivative.

This algebra is commutative with regard to the *Schwarz* rule of partial differentiation, and it is also non-commutative enough to allow for the emergence of a rotational derivative.

Applying the base vector matrix to the quaternionic *Jacobi* logarithm matrix from both sides makes it real-valued,

$$\Gamma_{ab} \sim \left[\begin{array}{c|ccc} + & i & j & k \\ \hline i & - & k & j \\ j & k & - & i \\ k & j & i & - \end{array}\right] \qquad \Rightarrow \qquad \mathfrak{B}\,\Gamma_{ab}\,\mathfrak{B} \sim \left[\begin{array}{c|ccc} + & - & - & - \\ \hline - & + & + & - \\ - & - & + & + \\ - & + & - & + \end{array}\right].$$

2.12 Exponentiation of the *Jacobi* logarithm

Lorentz **Transforms**

As representatives of the *Lorentz* group, assume a hyperbolic rotation with parameter ν_x in the (t,x) plane, together with a circular rotation with parameter ϱ_x in the (y,z) plane.

The corresponding *Jacobi* logarithm matrix can be represented either in a quaternionic or a real-valued form,

$$\overset{\mathbb{H}}{\Gamma}_{ab} = \left[\begin{array}{c|ccc} \cdot & i\,\nu_x & \cdot & \cdot \\ \hline i\,\nu_x & \cdot & & \\ \cdot & & \cdot & i\,\varrho_x \\ \cdot & & i\,\varrho_x & \cdot \end{array}\right] \qquad \Leftrightarrow \qquad \mathfrak{B}\,\overset{\mathbb{H}}{\Gamma}_{ab}\,\mathfrak{B} = \left[\begin{array}{c|ccc} \cdot & -\nu_x & \cdot & \cdot \\ \hline -\nu_x & \cdot & & \\ \cdot & & \cdot & +\varrho_x \\ \cdot & & -\varrho_x & \cdot \end{array}\right] =: \overset{\mathbb{R}}{\Gamma}_{ab},$$

both of which exponentiation gives a valid *Jacobi* matrix, respectively either quaternionic or real-valued. Note, that the quaternionic *Jacobi* logarithm is strictly symmetric (but not self-adjunct), since partial derivatives commute.

Quaternionic Exponential

In the quaternionic case, first an index should be raised using the flat *Minkowski* metric $\mathfrak{N} = \mathfrak{B}\mathfrak{B}$, to get a quaternionic *Jacobi* matrix,

$$\overset{\mathbb{H}}{\mathfrak{J}} := \overset{\mathbb{H}}{J}{}^{a}{}_{b} = \begin{cases} \exp\big(\mathfrak{B}\mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab}\big) & = \\ \mathfrak{B}\exp\big(\mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab}\,\mathfrak{B}\big)\bar{\mathfrak{B}} & = \end{cases} \left[\begin{array}{c|ccc} \cosh(\nu_x) & i\,\sinh(\nu_x) & \cdot & \cdot \\ \hline \bar{\imath}\,\sinh(\nu_x) & \cosh(\nu_x) & & \\ \cdot & & \cos(\varrho) & i\,\sin(\varrho) \\ \cdot & & i\,\sin(\varrho) & \cos(\varrho) \end{array}\right],$$

which at a first glance seems to have the wrong symmetries. But notice, that the hyperbolic rotation, though anti-symmetric, is still self-adjunct and the circular transform is, though pro-symmetric, still anti-self-adjunct, so the transforms of a (quaternionic) vector come out correctly in a quaternionic context, for instance,

$$\begin{pmatrix} (\cosh(\varphi)\,d - \sinh(\varphi)\,a) \\ i\,(-\sinh(\varphi)\,d + \cosh(\varphi)\,a) \\ j\,(\cos(\varrho)\,b - \sin(\varrho)\,c) \\ k\,(\sin(\varrho)\,b + \cos(\varrho)\,c) \end{pmatrix} = \left[\begin{array}{c|ccc} \cosh(\varphi) & i\,\sinh(\varphi) & \cdot & \cdot \\ \hline \bar{\imath}\,\sinh(\varphi) & \cosh(\varphi) & & \\ \cdot & & \cos(\varrho) & i\,\sin(\varrho) \\ \cdot & & i\,\sin(\varrho) & \cos(\varrho) \end{array}\right] \begin{pmatrix} d \\ i\,a \\ j\,b \\ k\,c \end{pmatrix}.$$

Real-Valued Exponential

The real-valued *Jacobi* matrix,

$$\overset{\mathbb{R}}{\mathfrak{J}} := \overset{\mathbb{R}}{J}{}^{a}{}_{b} = \begin{cases} \bar{\mathfrak{B}}\exp\big(\mathfrak{B}\mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab}\big)\mathfrak{B} & = \\ \exp\big(\mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab}\,\mathfrak{B}\big) & = \end{cases} \left[\begin{array}{c|ccc} \cosh(\nu_x) & -\sinh(\nu_x) & \cdot & \cdot \\ \hline -\sinh(\nu_x) & \cosh(\nu_x) & & \\ \cdot & & \cos(\varrho) & +\sin(\varrho) \\ \cdot & & -\sin(\varrho) & \cos(\varrho) \end{array}\right],$$

shows the usual symmetries and gives the usual transformation of a (real-valued) vector in a real-valued context, for instance,

$$\begin{pmatrix} \cosh(\varphi)\,d - \sinh(\varphi)\,a \\ -\sinh(\varphi)\,d + \cosh(\varphi)\,a \\ \cos(\varrho)\,b + \sin(\varrho)\,c \\ -\sin(\varrho)\,b + \cos(\varrho)\,c \end{pmatrix} = \left[\begin{array}{c|ccc} \cosh(\varphi) & -\sinh(\varphi) & \cdot & \cdot \\ \hline -\sinh(\varphi) & \cosh(\varphi) & & \\ \cdot & & \cos(\varrho) & +\sin(\varrho) \\ \cdot & & -\sin(\varrho) & \cos(\varrho) \end{array}\right] \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}.$$

Finally,

$$\overset{\mathbb{R}}{\mathfrak{J}} = \bar{\mathfrak{B}}\,\overset{\mathbb{H}}{\mathfrak{J}}\,\mathfrak{B}\,, \qquad \overset{\mathbb{H}}{\mathfrak{J}} = \mathfrak{B}\,\overset{\mathbb{R}}{\mathfrak{J}}\,\bar{\mathfrak{B}}\,.$$

Metric Tensors

The metric tensor from the quaternionic *Jacobi* matrix is formed with twice the base matrix,

$$g_{ab} = (\overset{\mathbb{H}}{\mathfrak{J}}\mathfrak{B})^T(\overset{\mathbb{H}}{\mathfrak{J}}\mathfrak{B}) = \mathfrak{B}\,\overset{\mathbb{H}}{\mathfrak{J}}{}^T\,\overset{\mathbb{H}}{\mathfrak{J}}\,\mathfrak{B} = \mathfrak{B}_{a\alpha}\overset{\mathbb{H}}{J}{}^{\alpha}{}_{\delta}\,\delta^{\delta\delta}\overset{\mathbb{H}}{J}{}^{\beta}{}_{\delta}\,\mathfrak{B}_{\beta b}\,.$$

The corresponding metric tensor from the real-valued *Jacobi* matrix is formed with the *Minkowski* metric,

$$g_{ab} = (\mathfrak{B}\overset{\mathbb{R}}{\mathfrak{J}})^T(\mathfrak{B}\overset{\mathbb{R}}{\mathfrak{J}}) = \overset{\mathbb{R}}{\mathfrak{J}}^T\,\mathfrak{B}\mathfrak{B}\,\overset{\mathbb{R}}{\mathfrak{J}} = \overset{\mathbb{R}}{\mathfrak{J}}^T\mathfrak{N}\overset{\mathbb{R}}{\mathfrak{J}} = \overset{\mathbb{R}}{J}{}^{\eta}{}_{a}\eta_{\eta\eta}\overset{\mathbb{R}}{J}{}^{\eta}{}_{b}\,.$$

The metric tensor of pure *Lorentz* transformations, like those above, is always identically the flat *Minkowski* metric.

Spatial Stretch

A purely diagonal *Jacobi* logarithm is always real-valued,

$$\overset{\mathbb{H}}{\Gamma}_{ab} = \left[\begin{array}{c|ccc} \tau & \cdot & \cdot & \cdot \\ \hline \cdot & -a^2 & & \\ \cdot & & -b^2 & \\ \cdot & & & -c^2 \end{array}\right] \quad\Rightarrow\quad \left\{\begin{array}{l} \mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab}\,\mathfrak{B} = \\ \\ \mathfrak{B}\,\mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab} = \end{array}\right. \left[\begin{array}{c|ccc} \tau & \cdot & \cdot & \cdot \\ \hline \cdot & a^2 & & \\ \cdot & & b^2 & \\ \cdot & & & c^2 \end{array}\right],$$

and constrained by the *d'Alembert* condition, $\tau = a^2 + b^2 + c^2$.

Exponentiation in either way gives an anisotropic stretch,

$$\overset{\mathbb{H}}{\mathfrak{J}} := \overset{\mathbb{H}}{J}{}^{a}{}_{b} = \left\{\begin{array}{l} \exp\big(\mathfrak{B}\mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab}\big) = \\ \\ \exp\big(\mathfrak{B}\overset{\mathbb{H}}{\Gamma}_{ab}\,\mathfrak{B}\big) = \end{array}\right. \left[\begin{array}{c|ccc} \exp(\tau) & \cdot & \cdot & \cdot \\ \hline \cdot & \exp(-a^2) & & \\ \cdot & & \exp(-b^2) & \\ \cdot & & & \exp(-c^2) \end{array}\right],$$

Forming the metric tensor in either way,

$$g_{ab} = \left\{\begin{array}{l} (\overset{\mathbb{H}}{\mathfrak{J}}\mathfrak{B})^T(\overset{\mathbb{H}}{\mathfrak{J}}\mathfrak{B}) = \\ \\ (\mathfrak{B}\overset{\mathbb{H}}{\mathfrak{J}})^T(\mathfrak{B}\overset{\mathbb{H}}{\mathfrak{J}}) = \end{array}\right. \left[\begin{array}{c|ccc} +\exp(2\tau) & \cdot & \cdot & \cdot \\ \hline \cdot & -\exp(-2a^2) & & \\ \cdot & & -\exp(-2b^2) & \\ \cdot & & & -\exp(-2c^2) \end{array}\right],$$

spatial stretch is the only case in which the metric tensor is not flat, but still symmetric and real-valued.

Lemma. The metric tensor from a scalar master potential is always purely diagonal, since the *Jacobi* logarithm is always symmetric,[2] since partial derivation commutes. Thus the metric tensor is also purely real-valued.

Corollary. From vital components of the embedding not showing up in the metric tensor at all, it appears futile to look for the whole picture of spacetime in terms of the metric tensor alone.

Convention. Without explicitly specifying a *Jacobi* logarithm or a *Jacobi* matrix as being quaternionic or real-valued, in the following they shall be understood as quaternionic,

$$\Gamma_{ab} := \overset{\mathbb{H}}{\Gamma}_{ab}\,, \qquad J^a_{\ b} := \overset{\mathbb{H}}{J}{}^{a}{}_{b}\,, \qquad \mathfrak{J} := \overset{\mathbb{H}}{\mathfrak{J}}\,.$$

[2]The metric tensor being always purely diagonal contradicts the current understanding of 'gravitational waves'.

3 Decomposing Spacetime

With the *d'Alembert* operator,

$$\Box \;:=\; {\partial_t}^2 - \operatorname{div}\operatorname{grad}\,,$$

each and every scalar component C of the logarithmic tensors is subject to

$$\Box\, C \;\equiv\; 0 \qquad \Leftrightarrow \qquad {\partial_t}^2 C \;\equiv\; \operatorname{div}\nabla C\,.$$

3.1 2nd-Rank Functions and Special Relativity

From the complete 2nd derivative tensor in (5), identify distinct components and relate them to physical interpretations.

A Vector with Time, Giving Rapidity

Defining

$$\vec{s} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \,,\quad S_a := \mathfrak{B}\begin{pmatrix} \cdot \\ \vec{s} \end{pmatrix} \,,\quad T_a := \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \,,$$

form the master potential as a product of a pure spatial and a pure timely differential form,

$$\vec{\vec{\Gamma}} \;=\; \left(\begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}\right)\left(\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}\right) \;=\; (a\,i\,\mathrm{d}x + b\,j\,\mathrm{d}y + c\,k\,\mathrm{d}z)\cdot \mathrm{d}t\,,$$

derive once,

$$\Rightarrow \quad \vec{\vec{\Gamma}}_a \;=\; \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \mathrm{d}t \;+\; (a\,i\,\mathrm{d}x + b\,j\,\mathrm{d}y + c\,k\,\mathrm{d}z) \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \,,$$

derive a second time,

$$\begin{aligned} \Rightarrow \quad \vec{\vec{\Gamma}}_{ab} \;&=\; \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \big(\,1\,|\,\cdot\,|\,\cdot\,|\,\cdot\,\big) \;+\; \big(\,\cdot\,|\,i\,a\,|\,j\,b\,|\,k\,c\,\big) \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \\ &=\; \left[\begin{array}{c|ccc} \cdot & i\,a & j\,b & k\,c \\ \hline i\,a & \cdot & & \\ j\,b & & \cdot & \\ k\,c & & & \cdot \end{array}\right] \,, \end{aligned} \tag{6}$$

with a **rapidity**, which is logarithmic velocity, Rapidity, $\vec{\nu}$

$$\vec{\nu} \;:=\; \overset{\vec{\nu}}{\Gamma}_{\boldsymbol{st}} \;=\; \begin{pmatrix} a\,i \\ b\,j \\ c\,k \end{pmatrix} ,$$

and is trace-free anyway, thus not constrained by the *Laplace* condition.

Two Perpendicular Vectors

Defining two perpendicular vectors,

$$\vec{s} \;:=\; \begin{pmatrix} a \\ b \\ c \end{pmatrix} , \qquad \vec{r} \;:=\; \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} , \qquad \vec{s} \perp \vec{r} ,$$

$$S_a \;:=\; \mathfrak{B} \cdot \begin{pmatrix} \cdot \\ \vec{s} \end{pmatrix} \;=\; \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} , \qquad R_a \;:=\; \mathfrak{B} \cdot \begin{pmatrix} \cdot \\ \vec{r} \end{pmatrix} \;=\; \begin{pmatrix} \cdot \\ i\,\alpha \\ j\,\beta \\ k\,\gamma \end{pmatrix} ,$$

$$\overset{\vec{\varrho}}{\Gamma} \;:=\; \left(\begin{pmatrix} \cdot \\ \vec{s} \end{pmatrix} \cdot \mathfrak{B} \cdot \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \right) \left(\begin{pmatrix} \cdot \\ \vec{r} \end{pmatrix} \cdot \mathfrak{B} \cdot \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \right) .$$

The derivative is oriented, that is, it swaps sign when the generating vectors $\vec{s}$, $\vec{r}$ are swapped,

$$\overset{\vec{\varrho}}{\Pi}_{sr} \;=\; \partial_r\,\partial_s\,\overset{\vec{\varrho}}{\Gamma} \;=\; S_s\,R_r \;=\; \begin{pmatrix} \cdot \\ a\,i \\ b\,j \\ c\,k \end{pmatrix} \begin{pmatrix} \cdot \\ \alpha\,i \\ \beta\,j \\ \gamma\,k \end{pmatrix} \;=\; \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & k\,a\beta & \bar{\jmath}\,a\gamma \\ \cdot & \bar{k}\,b\alpha & \cdot & i\,b\gamma \\ \cdot & j\,c\alpha & \bar{\imath}\,c\beta & \cdot \end{array} \right] \quad \Rightarrow$$

$$\overset{\vec{\varrho}}{\Gamma}_{ab} \;=\; \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & k\,(a\beta - b\alpha) & j\,(c\alpha - a\gamma) \\ \cdot & k\,(a\beta - b\alpha) & \cdot & i\,(b\gamma - c\beta) \\ \cdot & j\,(c\alpha - a\gamma) & i\,(b\gamma - c\beta) & \cdot \end{array} \right] \;=\; \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & k\,\varrho_{\boldsymbol{z}} & j\,\varrho_{\boldsymbol{y}} \\ \cdot & k\,\varrho_{\boldsymbol{z}} & \cdot & i\,\varrho_{\boldsymbol{x}} \\ \cdot & j\,\varrho_{\boldsymbol{y}} & i\,\varrho_{\boldsymbol{x}} & \cdot \end{array} \right] , \qquad (7)$$

and is again trace-free, with an orientational time-independent **rotation** about an axis Rotation, $\vec{\varrho}$

$$\vec{\varrho} \;:=\; \vec{s} \times \vec{r} \;=\; \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \;=\; \begin{pmatrix} b\gamma - c\beta \\ c\alpha - a\gamma \\ a\beta - b\alpha \end{pmatrix} .$$

Two Parallel Vectors

The outer product of two vectors gives a non-oriented **product tensor**,

$$\Pi_{ab} \;=\; \begin{pmatrix} \cdot \\ a\,i \\ b\,j \\ c\,k \end{pmatrix} \begin{pmatrix} \cdot \\ a\,i \\ b\,j \\ c\,k \end{pmatrix} \;=\; \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & -a^2 & k\,ab & \bar{\jmath}\,ac \\ \cdot & \bar{k}\,ba & -b^2 & i\,bc \\ \cdot & j\,ca & \bar{\imath}\,cb & -c^2 \end{array}\right] ,$$

and the corresponding tensor of all 2nd derivatives is, from the *Leibniz* rule,
the sum of that product matrix with its transpose,

$$\Rightarrow \quad \Gamma_{ab} \;=\; (\Pi_{ab} + \Pi_{ba}) \;\stackrel{\star}{=}\; \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & -2a^2 & & \\ \cdot & & -2b^2 & \\ \cdot & & & -2c^2 \end{array}\right] ,$$

where in the trace, $a^2 + b^2 + c^2$ is the 3D dot product of the generating vectors,

$$\Gamma_{\boldsymbol{ss}}\,\delta^{\boldsymbol{ss}} \;=\; -2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;=\; -2 \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right|^2 , \tag{8}$$

which alone can not fulfil the condition of a vanishing trace, and needs a further compensation.

Time Squared

Be a pure logarithmic **Time dilation**,

$$\overset{\chi}{\Pi}_{ab} \;=\; \begin{pmatrix} d \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} d \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \;=\; \left[\begin{array}{c|ccc} d^2 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array}\right] \quad \Rightarrow \quad \overset{\chi}{\Gamma}_{ab} \;=\; 2 \left[\begin{array}{c|ccc} d^2 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array}\right] ,$$

with

$$\chi \;:=\; 2d^2 \;=\; \Gamma_{\boldsymbol{tt}} ,$$

which alone does not fulfil the trace condition.

Combining time squared with two parallel vectors,

$$\overset{\chi}{\Pi}_{ab} \;=\; \begin{pmatrix} \cdot \\ a\,i \\ b\,j \\ c\,k \end{pmatrix} \begin{pmatrix} \cdot \\ a\,i \\ b\,j \\ c\,k \end{pmatrix} \;+\; \begin{pmatrix} d \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} d \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad \Rightarrow \quad \overset{\chi}{\Gamma}_{ab} \;=\; 2 \left[\begin{array}{c|ccc} d^2 & \cdot & \cdot & \cdot \\ \hline \cdot & -a^2 & & \\ \cdot & & -b^2 & \\ \cdot & & & -c^2 \end{array}\right] , \tag{9}$$

the *d'Alembert* trace condition is met by

$$d^2 \;=\; \vec{s} \cdot \vec{s} ,$$

and we get

$$\chi \;:=\; \overset{\chi}{\Gamma}_{\boldsymbol{tt}} \;=\; 2\,d^2 \;=\; 2\,\vec{s}\cdot\vec{s}\,,$$

which is the logarithmic gravitational potential, and also a logarithmic measure of time-shift, where $\chi > 0$ means a redshift, and $\chi < 0$ a blueshift.

The popular redshift number z, with λ_1 the wavelength at the emitter (there) and λ_0 the wavelength at the receiver (here), or the respective frequencies f_1, f_0,

$$z \;=\; \frac{\lambda_0}{\lambda_1} - 1 \;=\; \frac{f_1}{f_0} - 1\,,$$

can in this context be expressed

$$z \;=\; \exp\left(\,\chi - 1\,\right), \qquad \chi \;=\; \ln(\,z + 1\,)\,.$$

Below (16), the logarithmic time shift will also be identified with an electromagnetic gauge potential, of which the 4-gradient gives the electromagnetic vector potential,

$$\begin{pmatrix} \partial_t \\ \nabla \end{pmatrix} \chi \;=\; \begin{pmatrix} \Phi_{\mathrm{el}} \\ \vec{A}_{\mathrm{mag}} \end{pmatrix}.$$

3.2 Summary of the 2nd-Rank Components

From the trace condition follows a continuity equation for the logarithmic time and space coordinates, for which Γ is a gauge potential,

$$\Gamma_{\delta\delta}\,\delta^{\delta\delta} \;\equiv\; 0 \;=\; \Box\,\Gamma \;=\; \partial_t\tau - \mathrm{div}\,\vec{\xi} \qquad \Leftrightarrow \qquad \partial_t\tau \;\equiv\; \mathrm{div}\,\vec{\xi}\,,$$

with the slice

$$\chi \;:=\; \partial_t\tau \;=\; \Gamma_{\boldsymbol{tt}}\,\delta^{\boldsymbol{tt}} \quad \sim \left[\begin{array}{c|ccc} + & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \quad \left(\begin{array}{l} \text{logarithmic time shift} \\ \text{logarithmic gravitational potential} \\ \text{electromagnetic gauge potential} \end{array}\right),$$

which compensates a non-euclidean

$$\mathrm{div}\,\vec{\xi} \;=\; -\chi \;=\; \Gamma_{\boldsymbol{ss}}\,\delta^{\boldsymbol{ss}} \quad \sim \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & - & & \\ & & - & \\ & & & - \end{array}\right] \quad \text{(logarithmic gravitational space squeeze)}\,.$$

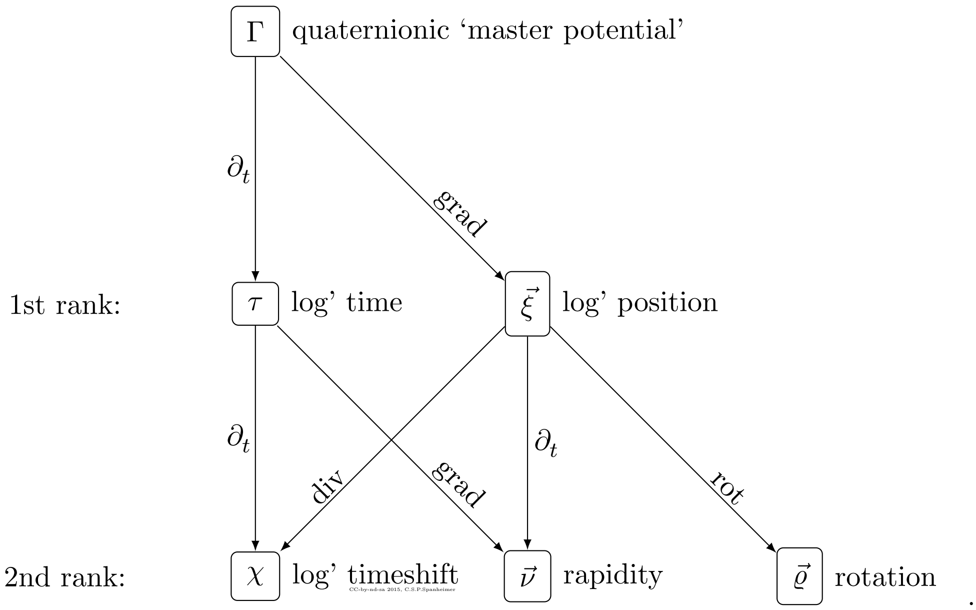
Further the euclidean entities,

$$\vec{\nu} \;=\; \mathrm{grad}\,\tau \;=\; \partial_t\,\vec{\xi} \;=\; \Gamma_{\boldsymbol{ts}} \quad \sim \left[\begin{array}{c|ccc} \cdot & i & j & k \\ \hline i & \cdot & & \\ j & & \cdot & \\ k & & & \cdot \end{array}\right] \quad \text{(rapidity, logarithmic velocity)}\,,$$

$$\vec{\varrho} \;=\; \mathrm{rot}\,\vec{\xi} \quad \sim \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & k & j \\ & k & \cdot & i \\ & j & i & \cdot \end{array}\right] \quad \text{(time-independent static rotation)}\,.$$

Derivative Chart

Summarizing the derivatives up to the 2nd rank in an overwiew chart,



3.3 3rd-Rank Functions and Gravitation

Time Derivatives of the 2nd-Rank Components

Taking the former 2nd-rank derivatives and just deriving over/multiplying by time gives the easier part of the 3rd-rank derivative components.

One Vector, Twice Time

Defining

$$\vec{s} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} , \quad S_a := \mathfrak{B} \begin{pmatrix} \cdot \\ \vec{s} \end{pmatrix} , \quad T_a := \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} ,$$

assume a master potential

$$\overset{\vec{\alpha}}{\Gamma} = \left(\begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \right) \left(\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \right)^2 = (a\,i\,\mathrm{d}x + b\,j\,\mathrm{d}y + c\,k\,\mathrm{d}z) \cdot \mathrm{d}t^2 ,$$

and from the outer product

$$\overset{\vec{\alpha}}{\Pi}_{abc} = S_a\,T_b\,T_c = \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

find the third derivative,

$$\Rightarrow \quad \overset{\vec{a}}{\Gamma}_{abc} = \overset{\vec{\alpha}}{\Pi}_{abc} + \overset{\vec{\alpha}}{\Pi}_{bac} + \overset{\vec{\alpha}}{\Pi}_{bca} + \overset{\vec{\alpha}}{\Pi}_{cba} + \overset{\vec{\alpha}}{\Pi}_{cab} + \overset{\vec{\alpha}}{\Pi}_{acb}$$
$$= 2 \left[\left[\begin{array}{c|ccc} \cdot & i\,a & j\,b & k\,c \\ \hline i\,a & \cdot & & \\ j\,b & & \cdot & \\ k\,c & & & \cdot \end{array} \right] \left[\begin{array}{c|ccc} i\,a & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array} \right] \left[\begin{array}{c|ccc} j\,b & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array} \right] \left[\begin{array}{c|ccc} k\,c & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array} \right] \right] , \qquad (10)$$

containing a **celerity field** $\vec{\alpha}$, which is a logarithmic acceleration,

Celerity, $\vec{\alpha}$

$$\overset{\vec{a}}{\Gamma}_{a\,\boldsymbol{tt}} = t^2\,\vec{s} = 2 \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} =: 2\,\vec{\alpha} .$$

which can be seen as either the time derivative of rapidity,
or the gradient of the logarithmic gravitational potential, $\vec{\alpha} = \partial_t\,\vec{\nu} = \nabla\chi$.

Two Perpendicular Vectors with Time

Defining two perpendicular generating vectors,

$$\vec{s} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} , \quad \vec{r} := \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} , \quad \vec{s} \perp \vec{r}, \quad S_a := \mathfrak{B} \begin{pmatrix} \cdot \\ \vec{s} \end{pmatrix} , \quad R_a := \mathfrak{B} \begin{pmatrix} \cdot \\ \vec{r} \end{pmatrix} , \quad T_a := \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} ,$$

the outer product and the third derivative are

$$\overset{\vec{\omega}}{\Pi}_{abc} \;=\; S_a\, R_b\, T_c \;=\; \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \begin{pmatrix} \cdot \\ i\,\alpha \\ j\,\beta \\ k\,\gamma \end{pmatrix} \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad\Rightarrow\quad \overset{\vec{\omega}}{\Gamma}_{abc} \;=$$

$$\left[\left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & k\,\omega_{\boldsymbol{z}} & j\,\omega_{\boldsymbol{y}} \\ \cdot & k\,\omega_{\boldsymbol{z}} & \cdot & i\,\omega_{\boldsymbol{x}} \\ \cdot & j\,\omega_{\boldsymbol{y}} & i\,\omega_{\boldsymbol{x}} & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & k\,\omega_{\boldsymbol{z}} & j\,\omega_{\boldsymbol{y}} \\ \hline \cdot & \cdot & & \\ k\,\omega_{\boldsymbol{z}} & & \cdot & \\ j\,\omega_{\boldsymbol{y}} & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & k\,\omega_{\boldsymbol{z}} & \cdot & i\,\omega_{\boldsymbol{x}} \\ \hline k\,\omega_{\boldsymbol{z}} & \cdot & & \\ \cdot & & \cdot & \\ i\,\omega_{\boldsymbol{x}} & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & j\,\omega_{\boldsymbol{y}} & i\,\omega_{\boldsymbol{x}} & \cdot \\ \hline j\,\omega_{\boldsymbol{y}} & \cdot & & \\ i\,\omega_{\boldsymbol{x}} & & \cdot & \\ \cdot & & & \cdot \end{array}\right]\right],$$

with an **angular velocity** about an axis

Angular velocity, $\vec{\omega}$

$$\vec{\omega} \;:=\; (\vec{s}\times\vec{r})\,t \;=\; \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} t \;=\; \begin{pmatrix} b\gamma - c\beta \\ c\alpha - a\gamma \\ a\beta - b\alpha \end{pmatrix} t\,,$$

that lives in logarithmic space, rather than in the exponential.

Triple Time

With $\quad T := (d|\cdot|\cdot|\cdot),\quad$ we get a pure time dilation,

$$\overset{\Phi_{\rm el}}{\Pi}_{TTT} \;=\; \begin{pmatrix} d \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} d \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} d \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \;=\; \left[\begin{array}{c|ccc} d^3 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array}\right] \quad\Rightarrow\quad \overset{\Phi_{\rm el}}{\Gamma}_{abc} \;=\; 6 \left[\begin{array}{c|ccc} d^3 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array}\right] \;=\; 6 \left[\begin{array}{c|ccc} \Phi_{\rm el} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array}\right],$$

which will below be identified with electric potential,

$$\Phi_{\rm el} \;=\; \tfrac{1}{6}\Gamma_{\boldsymbol{ttt}}\,,$$

and is observable in the trace, but alone does not compensate according to the *d'Alembert* condition, unless it vanishes identically.

Two Parallel Vectors with Time

Again with

$$\vec{s} := \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad S_a := \mathfrak{B}\begin{pmatrix} \cdot \\ \vec{s} \end{pmatrix}, \quad T_a := \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix},$$

we get an outer product and a third derivative,

$$\Pi_{abc} \;=\; S_a\, S_b\, T_c \;=\; \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad \Rightarrow$$

$$\Gamma_{abc} \;=\; -2 \left[\left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & a^2 & & \\ \cdot & & b^2 & \\ \cdot & & & c^2 \end{array} \right] \left[\begin{array}{c|ccc} \cdot & a^2 & \cdot & \cdot \\ \hline a^2 & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array} \right] \left[\begin{array}{c|ccc} \cdot & \cdot & b^2 & \cdot \\ \hline \cdot & \cdot & & \\ b^2 & & \cdot & \\ \cdot & & & \cdot \end{array} \right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & c^2 \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ c^2 & & & \cdot \end{array} \right] \right] ,$$

of which contraction yields a time-dependent squeeze of space,

$$\Gamma_{aaa}\delta^{aa} \;=\; -2\,(\vec{s}\cdot\vec{s})\,t \;=\; -2 \begin{pmatrix} |\vec{s}|^2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \;=\; -2 \begin{pmatrix} 3\,\Phi_{\mathrm{el}} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} ,$$

which compensates with electric potential.

3.4 Purely Spatial Derivatives

Deriving the 2nd-rank space derivatives again over space gives new entities with their own rules.

Three Arbitrary Vectors

Given three arbitrary generating vectors,

$$\vec{s} := \begin{pmatrix} \cdot \\ a \\ b \\ c \end{pmatrix} , \; \vec{r} := \begin{pmatrix} \cdot \\ \alpha \\ \beta \\ \gamma \end{pmatrix} , \; \vec{q} := \begin{pmatrix} \cdot \\ A \\ B \\ C \end{pmatrix} \quad \Rightarrow \quad S_a := \begin{pmatrix} \cdot \\ i\,a \\ j\,b \\ k\,c \end{pmatrix} , \; R_a := \begin{pmatrix} \cdot \\ i\,\alpha \\ j\,\beta \\ k\,\gamma \end{pmatrix} , \; Q_a := \begin{pmatrix} \cdot \\ i\,A \\ j\,B \\ k\,C \end{pmatrix} ,$$

calculate their outer product,

$$\Pi_{srq} \;=\; S_a\, R_b\, Q_c \;=\;$$

$$\left[[\cdot] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \bar{\imath}\,a\alpha A & j\,a\beta A & \bar{k}\,a\gamma A \\ \cdot & \bar{\jmath}\,b\alpha A & \bar{\imath}\,b\beta A & -b\gamma A \\ \cdot & k\,c\alpha A & +c\beta A & \bar{\imath}\,c\gamma A \end{array} \right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \bar{\jmath}\,a\alpha B & \bar{\imath}\,a\beta B & +a\gamma B \\ \cdot & i\,b\alpha B & \bar{\jmath}\,b\beta B & k\,b\gamma B \\ \cdot & -c\alpha B & \bar{k}\,c\beta B & \bar{\jmath}\,c\gamma B \end{array} \right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \bar{k}\,a\alpha C & -a\beta C & \bar{\imath}\,a\gamma C \\ \cdot & +b\alpha C & \bar{k}\,b\beta C & \bar{\jmath}\,b\gamma C \\ \cdot & i\,c\alpha C & j\,c\beta C & \bar{k}\,c\gamma C \end{array} \right] \right] ,$$

introducing 2-symmetry in the first index pair,

$$
\begin{aligned}
&\Pi_{srq} + \Pi_{rsq} \;= \\
= &\left[[\cdot] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & j\,A\varrho_{\boldsymbol z} & \bar k\,A\varrho_{\boldsymbol y} \\ \cdot & j\,A\varrho_{\boldsymbol z} & \cdot & -A\varrho_{\boldsymbol x} \\ \cdot & \bar k\,A\varrho_{\boldsymbol y} & -A\varrho_{\boldsymbol x} & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \bar\imath\,B\varrho_{\boldsymbol z} & -B\varrho_{\boldsymbol y} \\ \cdot & \bar\imath\,B\varrho_{\boldsymbol z} & \cdot & k\,B\varrho_{\boldsymbol x} \\ \cdot & -B\varrho_{\boldsymbol y} & k\,B\varrho_{\boldsymbol x} & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & -C\varrho_{\boldsymbol z} & i\,C\varrho_{\boldsymbol y} \\ \cdot & -C\varrho_{\boldsymbol z} & \cdot & \bar\jmath\,C\varrho_{\boldsymbol x} \\ \cdot & i\,C\varrho_{\boldsymbol y} & \bar\jmath\,C\varrho_{\boldsymbol x} & \cdot \end{array}\right] \right] \\
-\,2 &\left[[\cdot] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & i\,a\alpha A & & \\ \cdot & & i\,b\beta A & \\ \cdot & & & i\,c\gamma A \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & j\,a\alpha B & & \\ \cdot & & j\,b\beta B & \\ \cdot & & & j\,c\gamma B \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & k\,a\alpha C & & \\ \cdot & & k\,b\beta C & \\ \cdot & & & k\,c\gamma C \end{array}\right] \right],
\end{aligned}
$$

using

$$
\vec\varrho \;:=\; \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \;=\; \begin{pmatrix} b\gamma - c\beta \\ c\alpha - a\gamma \\ a\beta - b\alpha \end{pmatrix} .
$$

The full 6-symmetry,

$$
\begin{aligned}
\Gamma_{abc} \;=\; &\Pi_{srq} + \Pi_{rsq} + \Pi_{qsr} + \Pi_{qrs} + \Pi_{sqr} + \Pi_{rqs} \;= \\
-\,2 &\left[\left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \bar\jmath\,A\varrho_{\boldsymbol z} & k\,A\varrho_{\boldsymbol y} \\ \cdot & \bar\jmath\,A\varrho_{\boldsymbol z} & \bar\imath\,B\varrho_{\boldsymbol z} & V \\ \cdot & k\,A\varrho_{\boldsymbol y} & V & i\,C\varrho_{\boldsymbol y} \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & j\,A\varrho_{\boldsymbol z} & i\,B\varrho_{\boldsymbol z} & V \\ \cdot & i\,B\varrho_{\boldsymbol z} & \cdot & \bar k\,B\varrho_{\boldsymbol x} \\ \cdot & V & \bar k\,B\varrho_{\boldsymbol x} & \bar\jmath\,C\varrho_{\boldsymbol x} \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \bar k\,A\varrho_{\boldsymbol y} & V & \bar\imath\,C\varrho_{\boldsymbol y} \\ \cdot & V & k\,B\varrho_{\boldsymbol x} & j\,C\varrho_{\boldsymbol x} \\ \cdot & \bar\imath\,C\varrho_{\boldsymbol y} & j\,C\varrho_{\boldsymbol x} & \cdot \end{array}\right] \right] \qquad (11) \\
-\,2 &\left[[\cdot] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & V \\ \cdot & & V & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & V \\ \cdot & & \cdot & \\ \cdot & V & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & V & \\ \cdot & V & \cdot & \\ \cdot & & & \cdot \end{array}\right] \right] \qquad (12) \\
-\,2 &\left[[\cdot] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & i\,a\alpha A & j\,a\alpha B & k\,a\alpha C \\ \cdot & j\,a\alpha B & i\,b\beta A & \\ \cdot & k\,a\alpha C & & i\,c\gamma A \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & j\,a\alpha B & i\,b\beta A & \\ \cdot & i\,b\beta A & j\,b\beta B & k\,b\beta C \\ \cdot & & k\,b\beta C & j\,c\gamma B \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & k\,a\alpha C & & i\,c\gamma A \\ \cdot & & k\,b\beta C & j\,c\gamma B \\ \cdot & i\,c\gamma A & j\,c\gamma B & k\,c\gamma C \end{array}\right] \right] \qquad (13) \\
-\,4 &\left[[\cdot] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & i\,a\alpha A & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & j\,b\beta B & \\ \cdot & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & k\,c\gamma C \end{array}\right] \right], \qquad (14)
\end{aligned}
$$

with (11) containing, up to the sign, 6 different components and each possible contraction gives the same **vector triple product**,

$$\Gamma_{aaa}\delta^{aa} + = \; 2\begin{pmatrix} \cdot \\ B\varrho_{\boldsymbol{z}} - C\varrho_{\boldsymbol{y}} \\ C\varrho_{\boldsymbol{x}} - A\varrho_{\boldsymbol{z}} \\ A\varrho_{\boldsymbol{y}} - B\varrho_{\boldsymbol{x}} \end{pmatrix} \; = \; 2\left(\begin{pmatrix} \cdot \\ a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} \cdot \\ \alpha \\ \beta \\ \gamma \end{pmatrix}\right) \times \begin{pmatrix} \cdot \\ A \\ B \\ C \end{pmatrix}$$
$$= \; 2\,(\vec{s}\times\vec{r}) \times \vec{q} \; = \; 2\,(\vec{q}\cdot\vec{s})\,\vec{r} \; - \; 2\,(\vec{q}\cdot\vec{r})\,\vec{s},$$

then (12) contains, in 6 identical instances, the **scalar triple product** of the generating vectors,

$$V \; := \; \left(\begin{pmatrix} \cdot \\ a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} \cdot \\ \alpha \\ \beta \\ \gamma \end{pmatrix}\right) \cdot \begin{pmatrix} \cdot \\ A \\ B \\ C \end{pmatrix} \; = \; \big(\varrho_{\boldsymbol{x}}A + \varrho_{\boldsymbol{y}}B + \varrho_{\boldsymbol{z}}C\big) \; = \; \vec{\varrho}\cdot\vec{q}$$
$$= \; (\vec{s}\times\vec{r})\cdot\vec{q} \; = \; (\vec{r}\times\vec{q})\cdot\vec{s} \; = \; (\vec{q}\times\vec{s})\cdot\vec{r}\,.$$

Then (13) contains 6 different components and each possible contraction gives the same

$$\Gamma_{aaa}\delta^{aa} + = \; -2\left(\begin{pmatrix} \cdot \\ a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \cdot \\ \alpha \\ \beta \\ \gamma \end{pmatrix}\right) \begin{pmatrix} \cdot \\ A\,i \\ B\,j \\ C\,k \end{pmatrix} \; = \; -2\,(\vec{s}\cdot\vec{r})\,\vec{q}\,.$$

Finally, the contractions of (14) are anisotropic and not rotationally invariant,

$$\Gamma_{aaa}\delta^{aa} + = \; -\,4\begin{pmatrix} \cdot \\ i\,a\alpha A \\ j\,b\beta B \\ k\,c\gamma C \end{pmatrix}.$$

So the total contraction is

$$\Gamma_{aaa}\delta^{aa} \; = \; -2\,\vec{S}(\vec{r}\cdot\vec{q}) \; + 2\,\vec{R}(\vec{q}\cdot\vec{s}) \; - 2\,\vec{Q}(\vec{s}\cdot\vec{r}) \; - 4\begin{pmatrix} \cdot \\ i\,a\alpha A \\ j\,b\beta B \\ k\,c\gamma C \end{pmatrix}.$$

Three Perpendicular Vectors

With three perpendicular vectors, $\vec{s}\perp\vec{r}\perp\vec{q}\perp\vec{s}$,
the **scalar triple product** is at maximum,

$$|V| \; = \; |\vec{s}|\,|\vec{r}|\,|\vec{q}|\;,$$

and the total contraction vanishes,

$$\Gamma_{aaa}\delta^{aa} \; = \; \vec{0}\,.$$

Two Parallel Vectors, One Perpendicular

With any two of the three generating vectors being parallel, the **scalar triple product** vanishes,

$$|V| = \vec{0},$$

and the contraction depends on which two of the vectors being parallel.
With the outer two, $\vec{s}$ and $\vec{q}$, being parallel,

$$\Gamma_{aaa}\delta^{aa} = -2\left(2\begin{pmatrix} \cdot \\ i\,\alpha a^2 \\ j\,\beta b^2 \\ k\,\gamma c^2 \end{pmatrix} - \vec{R}\,|\vec{s}|^2\right) = -2\begin{pmatrix} \cdot \\ i\,\alpha\,(+a^2-b^2-c^2) \\ j\,\beta\,(-a^2+b^2-c^2) \\ k\,\gamma\,(-a^2-b^2+c^2) \end{pmatrix}.$$

With an outer and the middle, $\vec{s}$ and $\vec{r}$, being parallel,

$$\Gamma_{rrr}\delta^{rr} = -2\left(2\begin{pmatrix} \cdot \\ i\,Aa^2 \\ j\,Bb^2 \\ k\,Cc^2 \end{pmatrix} + \vec{Q}\,|\vec{r}|^2\right) = -2\begin{pmatrix} \cdot \\ i\,A\,(3a^2+b^2+c^2) \\ j\,B\,(a^2+3b^2+c^2) \\ k\,C\,(a^2+b^2+3c^2) \end{pmatrix}.$$

Three Parallel Vectors

With three parallel vectors, $\vec{s} = \vec{r} = \vec{q}$, again the **scalar triple product** vanishes,

$$|V| = \vec{0},$$

and the contraction is

$$\Gamma_{rrr}\delta^{rr} = -2\left(2\begin{pmatrix} \cdot \\ i\,a^3 \\ j\,b^3 \\ k\,c^3 \end{pmatrix} + \vec{R}\,|\vec{r}|^2\right) = -2\begin{pmatrix} \cdot \\ i\,a\,(3a^2+b^2+c^2) \\ j\,b\,(a^2+3b^2+c^2) \\ k\,c\,(a^2+b^2+3c^2) \end{pmatrix}.$$

3.5 Summarizing the Components of 3rd-Rank Derivatives

NOTE: From here on, not everything is clear yet.
All 3 possible contractions of the 3rd-rank derivative tensor,

$$\Gamma_{a\,\delta\delta}\,\delta^{\delta\delta} = \Gamma_{\delta\,b\,\delta}\,\delta^{\delta\delta} = \Gamma_{\delta\delta\,c}\,\delta^{\delta\delta} \stackrel{!}{=} 0 \tag{15}$$

induce a 4-vector which contains the electromagnetic vector potential,

$$\begin{aligned}
\Gamma_{a\,\delta\delta}\,\delta^{\delta\delta} \;=\; &-\;2\,\mathfrak{B}\,t^2 \begin{pmatrix} t \\ \vec{s}+\vec{r}+\vec{q} \end{pmatrix} && \begin{matrix}\text{(triple time)} \\ \text{(celerity)}\end{matrix} \\
&+\;2\,\mathfrak{B} \begin{pmatrix} t\,(\vec{s}\cdot\vec{r}+\vec{r}\cdot\vec{q}+\vec{q}\cdot\vec{s}) \\ -\vec{s}\,(\vec{r}\cdot\vec{q})+\vec{r}\,(\vec{q}\cdot\vec{s})-\vec{q}\,(\vec{s}\cdot\vec{r}) \end{pmatrix} && \begin{matrix}\text{(squeeze over time)} \\ \text{(isotropic squeeze over space)}\end{matrix} \\
&-\;4\,\mathfrak{B} \begin{pmatrix} \cdot \\ a\alpha A \\ b\beta B \\ c\gamma C \end{pmatrix} && \text{(anisotropic squeeze over space, gravity)} \quad .
\end{aligned}$$

Defining

$$\begin{aligned}
\vec{\alpha} \;&:=\; \operatorname{grad}\chi \;=\; \partial_t\,\vec{\nu} \;=\; t^2\,(\vec{s}+\vec{r}+\vec{q}) \qquad \text{(celerity)}\,, \\
\vec{g} \;&:=\; 2 \begin{pmatrix} \cdot \\ a\alpha A \\ b\beta B \\ c\gamma C \end{pmatrix} \qquad \text{(gravity)}\,,
\end{aligned}$$

from the trace condition follow two continuity equations,

$$\begin{aligned}
\Gamma_{\boldsymbol{t}\,\delta\delta}\,\delta^{\delta\delta} \;\equiv\; 0 \;=\; 2\left(\partial_t\chi-\operatorname{div}\vec{\nu}\right) \qquad &\Leftrightarrow \qquad \partial_t\chi \;\equiv\; \operatorname{div}\vec{\nu}\,, \\
\Gamma_{\boldsymbol{s}\,\delta\delta}\,\delta^{\delta\delta} \;\equiv\; 0 \;=\; 2\left(\vec{\alpha}-\vec{g}-\vec{A}_{\mathrm{mag}}\right) \qquad &\Leftrightarrow \qquad \vec{A}_{\mathrm{mag}} \;=\; \vec{\alpha}-\vec{g} \quad (?)\,,
\end{aligned}$$

with the slices

$$\begin{aligned}
\Phi_{\mathrm{el}} \;&:=\; \Gamma_{\boldsymbol{t}\,\boldsymbol{tt}}\,\delta^{\boldsymbol{tt}} \;=\; \partial_t\chi \;=\; \operatorname{div}\vec{\nu}\,, \\
\vec{A}_{\mathrm{mag}} \;&:=\; \Gamma_{\boldsymbol{s}\,\boldsymbol{tt}}\,\delta^{\boldsymbol{tt}} \;=\; \vec{\alpha}-\vec{g} \quad (?)\,.
\end{aligned}$$

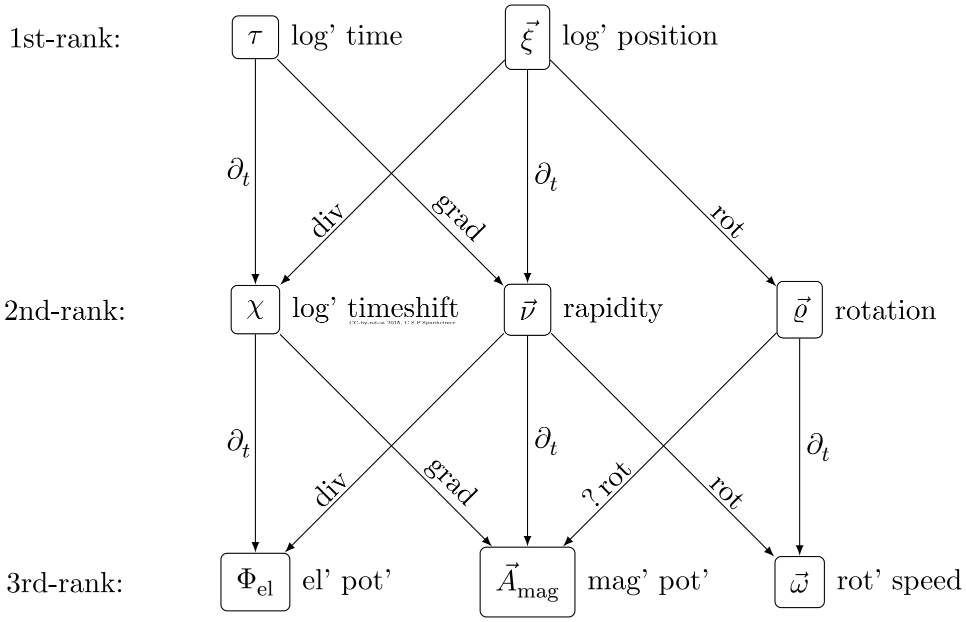
In electromagnetic gauge theory, the electromagnetic vector potential is the 4-gradient of a gauge potential, which can be identified here with logarithmic gravitational potential, χ, Φ_{el}, $\vec{A}_{\mathrm{mag}}$

$$\begin{pmatrix} \Phi_{\mathrm{el}} \\ \vec{A}_{\mathrm{mag}} \end{pmatrix} \;=\; \begin{pmatrix} \partial_t \\ \nabla \end{pmatrix} \chi\,. \tag{16}$$

Components hidden from the trace are

$$\begin{aligned}
\vec{\omega} \;&:=\; \partial_t\vec{\varrho} \;=\; (\vec{s}\times\vec{r} \;+\; \vec{r}\times\vec{q} \;+\; \vec{q}\times\vec{s})\,t \qquad \text{(rotation over time)}\,, \\
V \;&:=\; (\vec{s}\times\vec{r})\cdot\vec{q} \;=\; (\vec{r}\times\vec{q})\cdot\vec{s} \;=\; (\vec{q}\times\vec{s})\cdot\vec{r} \qquad \text{(volume span)}\,.
\end{aligned}$$

Derivative Chart



Decomposing the 3rd-rank derivative tensor into purely 3-symmetric parts,

(euclidean, occuring in no trace)

$$\overset{\vec{\omega}}{\Gamma}_{abc} \sim \left[\left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & k & j \\ & k & \cdot & i \\ & j & i & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & \cdot & k & j \\ \hline \cdot & \cdot & & \\ k & & \cdot & \\ j & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & k & \cdot & i \\ \hline k & \cdot & & \\ \cdot & & \cdot & \\ i & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & j & i & \cdot \\ \hline j & \cdot & & \\ i & & \cdot & \\ \cdot & & & \cdot \end{array}\right]\right] \qquad \text{(rotation over time, } \vec{\omega}\text{)}\,,$$

(non-euclidean, occuring in the time trace)

$$\overset{\Phi_{\text{el}}}{\Gamma}_{abc} \sim \left[\left[\begin{array}{c|ccc} + & & & \\ \hline & - & & \\ & & - & \\ & & & - \end{array}\right] \left[\begin{array}{c|ccc} \cdot & - & & \\ \hline - & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & - & \\ \hline & \cdot & & \\ - & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & - \\ \hline & \cdot & & \\ & & \cdot & \\ - & & & \cdot \end{array}\right]\right] \qquad \text{(electric potential, } \Phi_{\text{el}}\text{)}\,,$$

(non-euclidean, occuring in the space traces)

$$\overset{\vec{\alpha}}{\Gamma}_{abc} \sim \left[\left[\begin{array}{c|ccc} \cdot & i & j & k \\ \hline i & \cdot & & \\ j & & \cdot & \\ k & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} i & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} j & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} k & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right]\right] \qquad \text{(celerity, } \vec{\alpha}\text{)}\,,$$

$$\overset{\vec{g}}{\Gamma}_{abc} \sim \left[\left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \bar{\imath} & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & & \\ & & \bar{\jmath} & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \bar{k} \end{array}\right]\right] \qquad \text{(gravitation)}\,,$$

$$\Gamma_{abc} \sim \left[\left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & & \bar{k} \\ & & \bar{\imath} & \\ & \bar{k} & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & \bar{\imath} & \\ & \bar{\imath} & \cdot & \\ & & & \bar{\jmath} \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \bar{k} & & \\ & & \cdot & \bar{\jmath} \\ & & \bar{\jmath} & \cdot \end{array}\right]\right]$$

$$+ \left[\left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & \bar{\jmath} & \\ & \bar{\jmath} & \cdot & \\ & & & \bar{\imath} \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \bar{\jmath} & & \\ & & \cdot & \bar{k} \\ & & \bar{k} & \cdot \end{array}\right] \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & & \bar{\imath} \\ & & \bar{k} & \\ & \bar{\imath} & & \cdot \end{array}\right]\right] \quad (\mathrm{rot}\,\vec{\varrho})\,.$$

3.6 4th-Rank Functions and Electromagnetism

NOTE: From here on, not everything is clear yet.

Of the 4th-rank tensor, we investigate contractions with no more than rank 2.
The total contraction condition,

$$\Gamma_{\delta\delta\delta\delta}\delta^{\delta\delta}\delta^{\delta\delta} \;\equiv\; 0 \;=\; \Box\,\chi \;=\; \Box\,\vec{\nu} \;=\; \Box\,\vec{\varrho}\,,$$

is equivalent to the *Lorenz* gauge, *Lorenz* gauge

$$\Box\,\chi \;\equiv\; 0 \;=\; \partial_t\Phi_{\mathrm{el}} - \mathrm{div}\,\vec{A}_{\mathrm{mag}} \qquad\Rightarrow\qquad \partial_t\Phi_{\mathrm{el}} \;\equiv\; \mathrm{div}\,\vec{A}_{\mathrm{mag}}\,.$$

The non-euclidean part of the total contraction induces
the source of logarithmic timeshift and thus a logarithmic **mass/energy density**, ϱ_{grav}

$$\Gamma_{\boldsymbol{tttt}}\delta^{\boldsymbol{tt}}\delta^{\boldsymbol{tt}} \;=\; \rho_{\mathrm{gr}} \;=\; \partial_t{}^2\chi \;=$$
$$-\,\Gamma_{\boldsymbol{ttss}}\delta^{\boldsymbol{tt}}\delta^{\boldsymbol{ss}} \;=\; -8\,(\vec{s}\cdot\vec{r})\,(\vec{q}\cdot\vec{p}) \;-8\,(\vec{s}\cdot\vec{q})\,(\vec{r}\cdot\vec{p}) \;-8\,(\vec{s}\cdot\vec{p})\,(\vec{r}\cdot\vec{q})\,.$$

The lesser condition $\Gamma_{ab\,\delta\delta}\,\delta^{\delta\delta} \overset{!}{=} 0$ induces a matrix
which is equivalent to the **electromagnetic *Faraday* tensor**, F_{ab}

$$F_{ab} \;:=\; \Gamma_{ab\,\boldsymbol{tt}}\delta^{\boldsymbol{tt}} \;=\; \left[\begin{array}{c|ccc} \cdot & iE_x & jE_y & kE_z \\ \hline iE_x & \cdot & k\,B_z & j\,B_y \\ jE_y & k\,B_z & \cdot & iB_x \\ kE_z & j\,B_y & iB_x & \cdot \end{array}\right],$$

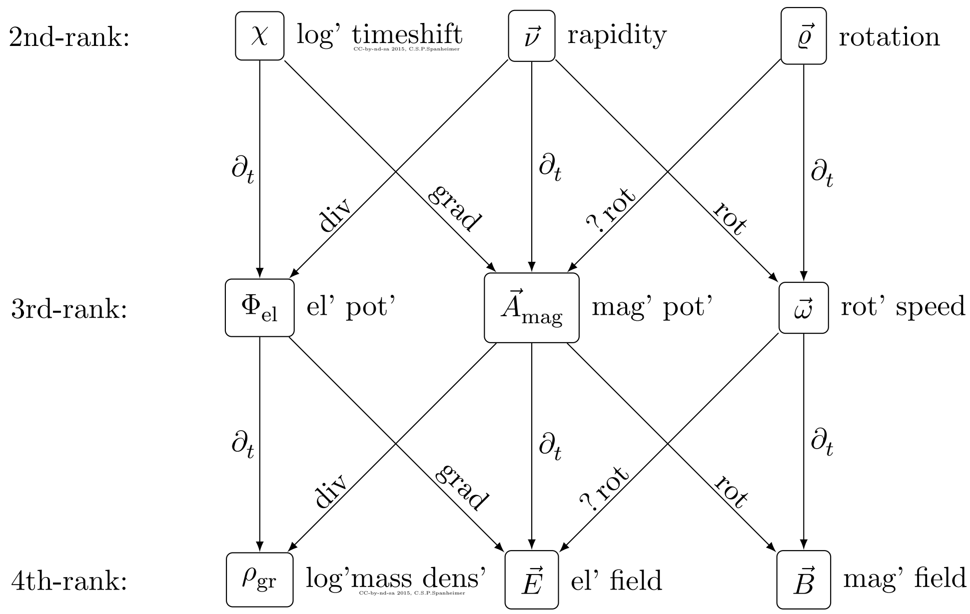
with an **electric field**, $\vec{E}$

$$\vec{E} \;=\; \nabla\Phi_{\mathrm{el}} \;=\; \nabla\partial_t\chi$$
$$\;=\; \partial_t\vec{A}_{\mathrm{mag}} \;=\; \partial_t\nabla\chi\,,$$

and a **magnetic field**, $\vec{B}$

$$\vec{B} \;=\; \mathrm{rot}\,\vec{A}_{\mathrm{mag}} \;=\; \mathrm{rot}\,\nabla\chi\,.$$

Derivative Chart



3.7 5th-Rank Functions and *Maxwell*'s Equations

Maxwell's homogeneous '**rot**$\vec{E}$' equation follows immediately, up to a sign, $\mathrm{rot}\,\vec{E} \equiv \partial_t \vec{B}$

$$\begin{aligned} \mathrm{rot}\,\vec{E} &= \mathrm{rot}\,\nabla\Phi_{\mathrm{el}} \;=\; \mathrm{rot}\,\nabla\partial_t\chi \\ &= \mathrm{rot}\,\partial_t\vec{A}_{\mathrm{mag}} \;=\; \mathrm{rot}\,\partial_t\nabla\chi \\ &= \partial_t\,\mathrm{rot}\,\nabla\chi \;=\; \partial_t\,\mathrm{rot}\,\vec{A}_{\mathrm{mag}} \\ &= \partial_t\vec{B}\,, \end{aligned}$$

and *Maxwell*'s 'div $\vec{B}$' equation, $\mathrm{div}\,\vec{B}$

$$\mathrm{div}\,\vec{B} \;=\; \mathrm{div}\,\mathrm{rot}\,\vec{A}_{\mathrm{mag}} \;\overset{?}{\equiv}\; 0\,,$$

TODO: is there a proof that div $\vec{B}$ vanishes identically ? From div rot $\equiv 0$?

The maximum contraction gives a 4-vector of electrical charge/current density,

$$\Gamma_{a\,\boldsymbol{tttt}}\,\delta^{\boldsymbol{tt}}\delta^{\boldsymbol{tt}} \;=\; \tfrac{1}{\varepsilon_0}\begin{pmatrix}\rho_{\mathrm{el}}\\ \vec{\jmath}\end{pmatrix} \;=\; \begin{pmatrix}\partial_t\\ \nabla\end{pmatrix}\left(\partial_t{}^2 \equiv \Delta\right)\chi\,,$$

of which the inhomogeneous equations follow,
first *Maxwell*'s 'div $\vec{E}$' equation (with a different sign),
which actually defines an electric source density, $\mathrm{div}\,\vec{E} =: \rho_{\mathrm{el}}/\varepsilon_0$

$$\mathrm{div}\,\vec{E} \;=\; \Gamma_{\boldsymbol{t}\,\boldsymbol{tttt}}\,\delta^{\boldsymbol{tt}}\delta^{\boldsymbol{tt}} \;=:\; \rho_{\mathrm{el}}/\varepsilon_0\,,$$

and then *Maxwell*'s 'rot $\vec{B}$' equation (with a different sign),
which actually defines an electric current density, $\vec{\jmath}/\varepsilon_0$

$$\begin{aligned}\Gamma_{\boldsymbol{s\,tttt}}\,\delta^{\boldsymbol{tt}}\delta^{\boldsymbol{tt}} &= \partial_t\vec{E} \;\stackrel{?}{=}\; \operatorname{rot}\vec{B} \\ &= \nabla\Delta\chi \;=\; \partial_t{}^2\nabla\chi \;=:\; \vec{\jmath}/\varepsilon_0\,,\end{aligned}$$

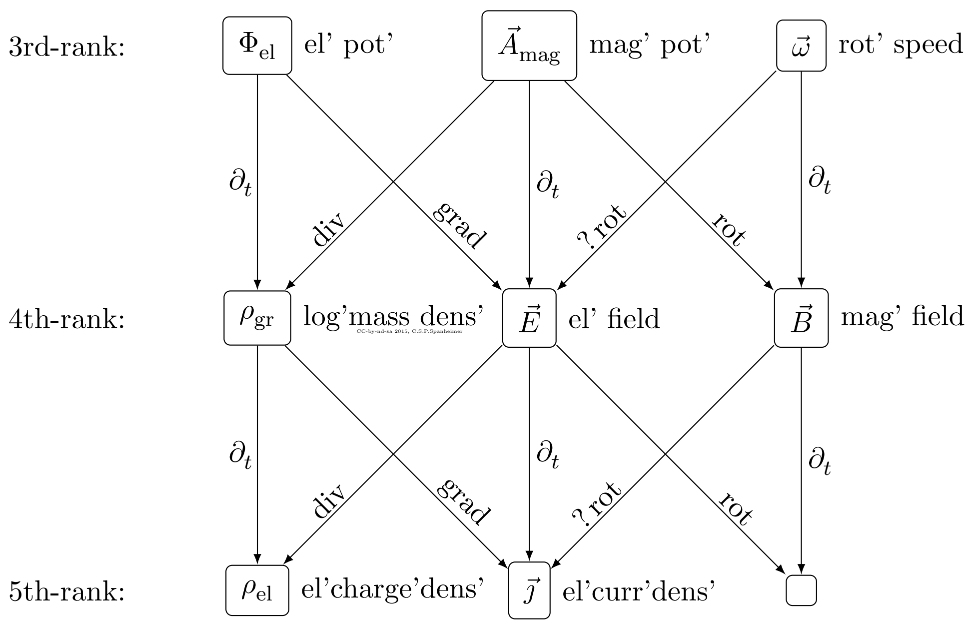
with $\partial_t\vec{E}$

$$\begin{aligned}\partial_t\vec{E} &= \partial_t\nabla\Phi_{\text{el}} \;=\; \partial_t\nabla\partial_t\chi \\ &= \partial_t{}^2\vec{A} \;=\; \partial_t{}^2\nabla\chi\,,\end{aligned}$$

and (TODO: not yet clear) $\operatorname{rot}\vec{B}$

$$\begin{aligned}\operatorname{rot}\vec{B} &= \operatorname{rot}\operatorname{rot}\vec{A}_{\text{mag}} \;=\; \nabla\operatorname{div}\vec{A}_{\text{mag}} - \Delta\vec{A}_{\text{mag}} \quad (?) \\ &= \operatorname{rot}\operatorname{rot}\nabla\chi \;=\; \nabla\Delta\chi - \Delta\nabla\chi \quad (?)\,.\end{aligned}$$

Derivative Chart



3.8 6th-Rank Functions and Electromagnetic Waves

From

$$\rho_{\text{el}} \;=\; \operatorname{div}\vec{E}\,, \qquad \vec{\jmath} \;=\; \partial_t\vec{E}\,,$$

and since

$$\partial_t\,\rho_{\text{el}} \;=\; \partial_t\operatorname{div}\vec{E} \quad\equiv\quad \operatorname{div}\partial_t\vec{E} \;=\; \operatorname{div}\vec{\jmath},$$

follows continuity of electric charge density,

$$\Box\,(\partial_t{}^2 \equiv \Delta)\,\chi \;\equiv\; 0 \qquad\Rightarrow\qquad \partial_t\,\rho_{\text{el}} \;\equiv\; \operatorname{div}\vec{\jmath}.$$

Finally the electromagnetic 'wave equations'

$$\begin{aligned} \Box \vec{E} &\equiv \vec{0} \quad &\Leftrightarrow \quad &(\partial_t{}^2 \equiv \Delta)\, \vec{E}\,, \\ \Box \vec{B} &\equiv \vec{0} \quad &\Leftrightarrow \quad &(\partial_t{}^2 \equiv \Delta)\, \vec{B}\,, \end{aligned}$$

summarized by

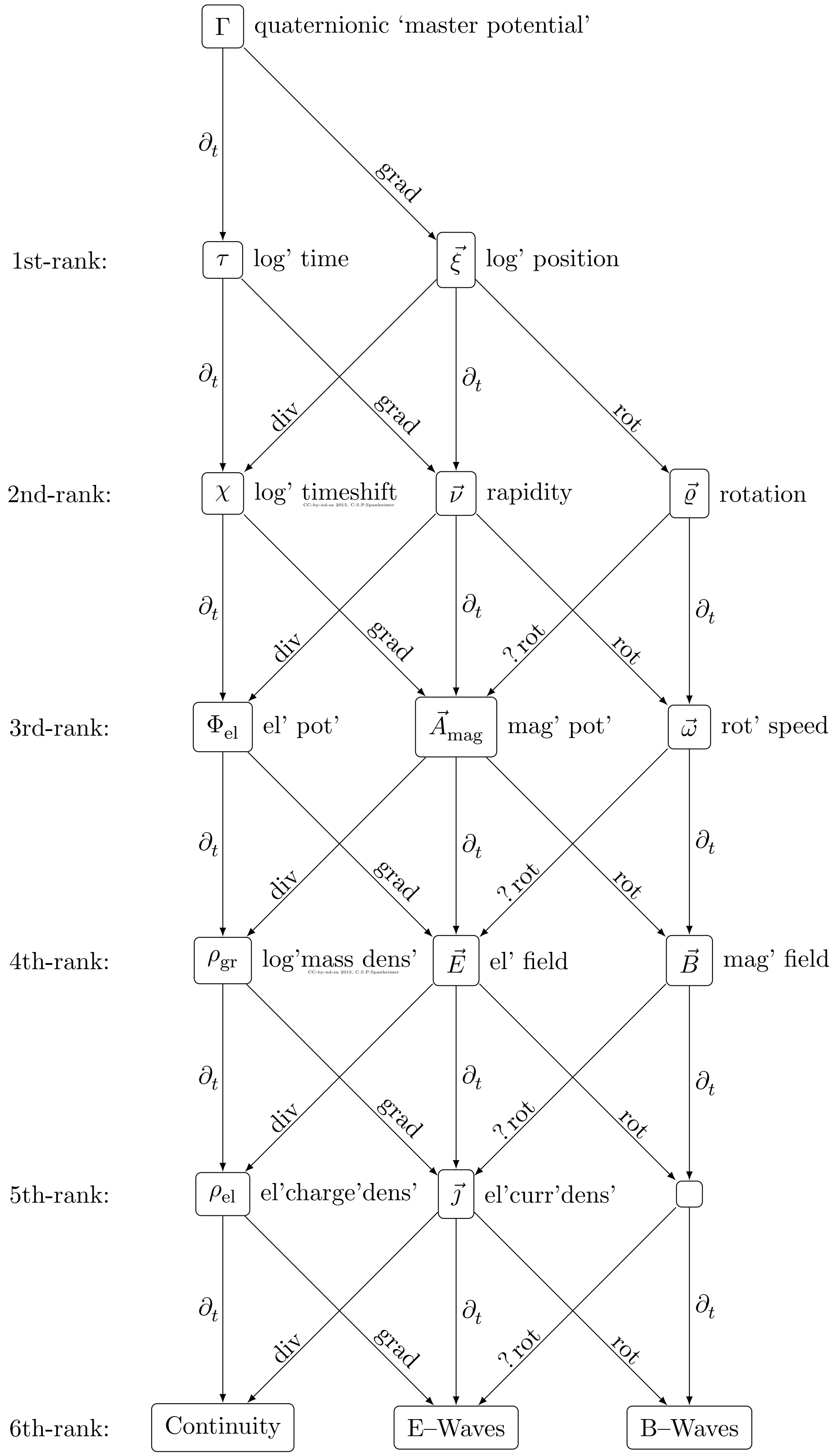
$$\Box \begin{pmatrix} \partial_t \\ \mathrm{rot} \end{pmatrix} \vec{A}_{\mathrm{mag}} \equiv \vec{0}\,,$$

are no more than the *d'Alembert* condition which holds true for each component of each derivative tensor and tells nothing more than a dispersion relation.

3.9 Summarizing Electromagnetism

Mathematical Definition Here	Usual Physical Definition
Gauged Potentials	
$\Phi := \partial_t \chi = \mathrm{div}\, \vec{\nu}$	$\Phi = \Phi_0 + \partial_t \chi$
$\vec{A} := \nabla \chi = \partial_t \vec{\nu}$	$\vec{A} = \vec{A}_0 + \nabla \chi$
Lorenz Gauge	
$\Box \chi \equiv 0$	$\partial_t \Phi - \mathrm{div}\, \vec{A} \equiv 0$
Fields	
$\vec{E} := \partial_t \vec{A} = \nabla \Phi$	$\vec{E} := \partial_t \vec{A} - \nabla \Phi$
$\vec{B} := \mathrm{rot}\, \vec{A}$	$\vec{B} := \mathrm{rot}\, \vec{A}$
Maxwell's Equations	
$\mathrm{rot}\, \vec{E} \equiv \partial_t \vec{B}$	$\mathrm{rot}\, \vec{E} + \partial_t \vec{B} \equiv \vec{0}$
$\mathrm{div}\, \vec{B} \equiv 0$	$\mathrm{div}\, \vec{B} \equiv 0$
$\partial_t \vec{E} \stackrel{?}{=} \mathrm{rot}\, \vec{B} =: \vec{\jmath}/\varepsilon_0$	$\mathrm{rot}\, \vec{B} - \partial_t \vec{E} =: \vec{\jmath}/\varepsilon_0$
$\mathrm{div}\, \vec{E} =: \rho_{\mathrm{el}}/\varepsilon_0$	$\mathrm{div}\, \vec{E} =: \rho_{\mathrm{el}}/\varepsilon_0$
Continuity of Charge	
$\Box \Delta \chi \equiv 0$	$\partial_t \rho_{\mathrm{el}} \equiv \mathrm{div}\, \vec{\jmath}$
Wave Equations	
$\Box \vec{E} \equiv 0$	$\Box \vec{E} = 0$
$\Box \vec{B} \equiv 0$	$\Box \vec{B} = 0$

3.10 Complete Derivative Chart



4 Conclusion

4.1 What's Been Shown

In the logarithmic derivative tensors, parallels to the most fundamental physical fields have been identified,

in the 2nd rank,
a logarithmic *Lorentz* boost from rapidity, together with a static rotation,
and a logarithmic gravitational potential,
which is equivalent to a possible electromagnetic gauge potential,

in the 3rd rank,
a logarithmic celerity and a logarithmic gravitational field, a rotation over time,
possibly electric and magnetic vector potentials,

in the 4th rank,
a logarithmic gravitational source density,
a possible electromagnetic field tensor and the *Lorenz* gauge,

in the 5th rank,
possibly *Maxwell*'s equations,
together with an electromagnetic charge and current density.

Further, an infinite series of gauge potentials occurs, of which the topmost one is the 'master potential' function.

4.2 What's Essential

- Working on the tangent bundle of a spacetime manifold
  similar to *Einstein*'s general relativity.
- More fundamental than the metric tensor, $g_{\mu\nu}$,
  is the *Jacobi* matrix, $J^{\mu}{}_{b}$, from which the metric results.
- For an infinitesimal embedding, the logarithm of the *Jacobi* matrix, Γ_{ab},
  is yet more fundamental and introduces a notion of 'scale invariance',
  which was an objective of *Weyl*'s.
- Requiring all possible contractions of all logarithmic derivative tensors to vanish identically enforces holomorphism and also results in the functional determinant being at unity everywhere, as *Einstein* required.
- Assuming a 'master potential', all derivative tensors are symmetrical in all index pairs.
  and the metric tensor is always purely diagonal.
- 'Blessing' the space coordinates with a signature of quaternionic units
  introduces an algebra with torsion, which is essential to yielding rotation and magnetism,
  but is different from *Cartan*'s torsion.

- Most essential is a philosophy of detection instead of engineering,
  of investigation rather than invention.

4.3 What's the Gain

From such a quaternionic derivative tensor framework, the concepts of analytic geometry and 3D vector calculus emerge in a natural way, that is dot product, cross product, *Grassmann*'s identity, triple scalar and vector product, as well as divergence, gradient, rotational derivative, time derivative, and the operators of *d'Alembert* and *Laplace*.

It would not work as a vector calculus pasted on a flat euclidean or any arbitrary space, since some of the entities are not rotationally invariant or *Lorentz*-covariant with respect to a flat euclidean or *Minkowski* metric.

But on the tangent space of its own manifold, *Lorentz* covariance is fulfilled by definition, on the other hand distorting spacetime and giving rise to topological questions.

4.4 Epilog

Mathematics has often been used to deductively engineer models of physical observations.

Contrary to that, in the present context mathematics herself is asked to reveal her own nature and the most reasonable behavior to create in an inductive way a mathematical reality which might, at deeper inspection, actually resemble our physical universe.

This introduces a philosophy of detection instead of engineering, of investigation rather than invention.

Any addition which adds to the freedom of a given model will spoil further investigation to 'hear, what mathematics itself speaks'. If at all, then additions which restrict a model are acceptable.

Mathematicians have investigated plenty of mathematically insane situations, like functions being not continuous and thus not differentiable, spaces being incomplete or not compact.

From that, mathematics can tell us in reverse, what is a sane environment to possibly induce a viable reality: Functions being differentiable infinitely often, being holomorphic on a complete compact space, which might even be a general manifold with a tangent bundle.

Additional restricting assumptions made here are that any fields shall be living on the tangent bundle alone and that the tangent bundle is governed by a single 'master potential'.

In 3+1 dimensions, mandating the possibility of spatial rotations induces a quaternonic algebra and allows exactly for a gravitational, an electric, and a magnetic field, and some electromagnetic waves.

Still open is the freedom of a manifold to form different topologies, where supposedly phenomena of stable particles (or 'topological centers') as well as of topological adhesive ('nuclear') forces might be found.

5 Revision History

Rev.2

Fixed the order of first occurences:

- added '...arbitrary constant *i*' in 2.6
- 'Introducing the convention...' moved from 2.10 to 2.9
- 'Applying the base vector matrix...' moved from 2.9 to the end of 2.11

Changed to '$\Gamma_{\boldsymbol{ss}}\,\delta^{\boldsymbol{ss}}$' in (8),
minor changes of wording in 2.12, 3.1, 3.3, 3.7, 4.4,
minor changes in the derivative charts.

Rev.1

In 2.5 added missing factors to the *Taylor* series.
Added contact info to the title.