Asymptotic expansions for distributions of extremes from generalized Maxwell distribution

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Abstract. In this paper, with optimal normalized constants, the asymptotic expansions of the distribution of the normalized maxima from generalized Maxwell distribution is derived. It shows that the convergence rate of the normalized maxima to the Gumbel extreme value distribution is proportional to $1/\log n$.

Keywords. Expansion; Extreme value distribution; Generalized Maxwell distribution.

1 Introduction

Let $(X_n,\ n \geq 1)$ be a sequence of independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) F_k which obeying the generalized Maxwell distribution (denote by $F_k \sim GMD(k)$). Let $M_n = \max(X_k,\ 1 \leq k \leq n)$ denote the partial maximum of $(X_n,\ n \geq 1)$. The probability density function (pdf) of GMD(k) is given by

$$f_k(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)} x^{2k} \exp\left(-\frac{x^{2k}}{2\sigma^2}\right),\ x > 0,$$

where k, σ is positive and $\Gamma(\cdot)$ represents the Gamma function. For $k = 1$, GMD(1) reduces to the ordinary Maxwell distribution.

Recently, asymptotic properties associated with GMD(k) have been investigated in the literature. Huang et al. (2014) established the Mills inequality, the Mills type ratio and distributional tail representation of GMD(k), and showed that F_k belongs to the domain of attraction Λ of the Gumbel extreme value distribution, i.e., there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$, such that

$$\lim_{n\to\infty} P\left((M_n - b_n)/a_n \leq x\right) = \lim_{n\to\infty} F_k^n(a_n x + b_n) = \Lambda(x),$$

where $\Lambda(x) = \exp\{-e^{-x}\}$. Liu and Liu (2013) established the uniform convergence rate of normalized maxima for GMD(1), i.e., the ordinary Maxwell distribution (MD for short). Huang and Chen (2014) extended the work of Liu and Liu (2013) to the case of $k > 0$.

The aim of this paper is to establish the asymptotic expansion for the distribution of normalized maxima of GMD(k) random variables. The uniform convergence rates and asymptotic expansions of M_n, the maximum of independent and identically distributed random variables for any given cdf F, have been of considerable interest. Hall (1979) derived optimal rates of uniform convergence for the cdf of M_n as F is the standard normal cdf. Nair (1981) obtained asymptotic expansions for the distribution and moments of M_n as F is the standard normal cdf. Peng et al. (2010) established optimal rates of uniform convergence for the cdf of M_n as F is the general error distribution. For other work, see Liao and Peng (2012) for the log-normal distribution, and Liao et al. (2013, 2014), respectively, for logarithmic skew-normal distribution and for skew-normal distribution.

In order to gain the asymptotic expansions of the distribution of normalized maxima from GMD(k), we cite some results from Huang and Chen (2014). They gave the Mills type ratio of GMD(k) as follows: for $k > 0$,

$$\frac{1 - F_k(x)}{f_k(x)} \sim \frac{\sigma^2}{k} x^{1-2k}, \text{ as } x \to \infty. \tag{1.1}$$

It also follows from Huang and Chen (2014) that

$$1 - F_k(x) = c(x) \exp\left(- \int_1^x \frac{g(t)}{f(t)} \,\mathrm{dt} \right) \tag{1.2}$$

for sufficiently large x, where

$$c(x) \to \frac{\exp\left(- 1/(2\sigma^2) \right)}{2^{k/2}\sigma^{1/k}\Gamma(1 + k/2)}, \text{ as } x \to \infty$$

and

$$f(t) = k^{-1}\sigma^2 t^{1-2k}, \; g(t) = 1 - k^{-1}\sigma^2 t^{-2k}. \tag{1.3}$$

Note that $f'(t) \to 0$ and $g(t) \to 1$, and we can choose the norming constants a_n and b_n in such way that b_n is satisfying the equation

$$1 - F_k(b_n) = n^{-1} \tag{1.4}$$

with

$$a_n = f(b_n) \tag{1.5}$$

such that

$$\lim_{n\to\infty} F_k^n(a_n x + b_n) = \Lambda(x).$$

The remainder of this paper are organized as follows. Section 2 gives the main result on asymptotic expansions for the distribution of partial maxima of the GMD(k) with $k > 0$. Some auxiliary lemmas needed to prove the main result and related proof are given in Section 3. In the sequel we shall assume that the parameter $k > 0$.

2 Main result

In this section, we derive an asymptotic expansion for the distribution of normalized maxima from the GMD(k). The distributional expansion could be used to show that the convergence rate of M_n to the Gumbel extreme value distribution is of the order of $O((\log n)^{-1})$.

Theorem 2.1. *Let $F_k(x)$ represent the cdf of GMD(k). For normalizing constants a_n and b_n given, respectively, by (1.4) and (1.5), we have*

$$b_n^{2k}\left[b_n^{2k}\left(F_k^n(a_n x + b_n) - \Lambda(x) \right) - l_k(x)\Lambda(x) \right] \to \left(w_k(x) + \frac{l_k^2(x)}{2} \right)\Lambda(x)$$

as $n \to \infty$, where $l_k(x)$ and $w_k(x)$ are, respectively, given by

$$l_k(x) = k^{-1}\sigma^2\left[(2k-1)x^2 - 2x\right]e^{-x}/2$$

and

$$w_k(x) = -k^{-2}\sigma^4\left[3(2k-1)^2x^4 - 4(2k+1)(2k-1)x^3 + 24x^2 - 48kx\right]e^{-x}/24.$$

Remark 2.1. *By the definition of b_n, it is easy to check that $b_n^{2k} = O(1/\log n)$. Hence, Theorem 2.1 shows that the convergence rate of $F_k^n(a_nx+b_n)$ tending to its extreme value limit is proportional to $1/\log n$. Further, the convergence rate of $b_n^{2k}(F_k^n(a_nx+b_n)-\Lambda(x))$ tending to its limit is also proportional to $1/\log n$.*

Remark 2.2. *As the mentioned in the Introduction, we get ordinary Maxwell pdf when $k=1$, and then Theorem 2.1 shows also that the asymptotic expansion of the distribution of normalized maximum from ordinary Maxwell distribution is*

$$\bar{b}_n^2\left[\bar{b}_n^2\left(F_1^n(\bar{a}_nx+\bar{b}_n) - \Lambda(x)\right) - l_1(x)\Lambda(x)\right] \to \left(w_1(x) + \frac{l_1^2(x)}{2}\right)\Lambda(x)$$

as $n \to \infty$, with norming constants determined by

$$1 - F_1(\bar{b}_n) = n^{-1} \text{ and } \bar{a}_n = \sigma^2\bar{b}_n^{-1},$$

where $l_1(x)$ and $w_1(x)$ are, respectively, given by

$$l_1(x) = \frac{1}{2}\sigma^2(x^2-2x)e^{-x}$$

and

$$w_1(x) = -\frac{1}{8}\sigma^4\left(x^4 - 4x^3 + 8x^2 - 16x\right)e^{-x}.$$

3 Auxiliary results and proofs

In order to obtain expansions for the distribution of the normalized maximum of GMD(k) random varibles, we provide the following tail decomposition of GMD(k).

Lemma 3.1. *Let $F_k(x)$ represent the cdf of GMD(k). For large x, we have*

$$\begin{aligned} 1 - F_k(x) =& f_k(x)\frac{\sigma^2}{k}x^{1-2k}\left(1 + k^{-1}\sigma^2x^{-2k} + k^{-2}(1-2k)\sigma^4x^{-4k} + O(x^{-6k})\right) \\ =& \frac{\exp(-1/(2\sigma^2))}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\Big(1 + k^{-1}\sigma^2x^{-2k} + k^{-2}(1-2k)\sigma^4x^{-4k} \\ &+ O(x^{-6k})\Big)\exp\left(-\int_1^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right) \end{aligned} \tag{3.1}$$

with $f(t)$ and $g(t)$ given by (1.3).

Proof. By integration by parts, we have

$$\begin{aligned}1-F_k(x) =&f_k(x)\frac{\sigma^2}{k}x^{1-2k}\Bigg(1+\frac{\sigma^2}{k}x^{-2k}+\frac{(1-2k)\sigma^4}{k^2}x^{-4k}+\frac{(1-2k)(1-4k)\sigma^6}{k^3}x^{-6k}\Bigg)\\&+\frac{(1-2k)(1-4k)(1-6k)}{2^{k/2}k^3\sigma^{1/k-6}\Gamma(1+k/2)}\int_x^\infty t^{-6k}\exp\Bigg(-\frac{t^{2k}}{2\sigma^2}\Bigg)\,\mathrm{dt}.\end{aligned}\tag{3.2}$$

It is easy to show by utilizing L'Hospital's rule that

$$\lim_{x\to\infty}\frac{\int_x^\infty t^{-6k}\exp(-\frac{t^{2k}}{2\sigma^2})\,\mathrm{dt}}{x^{1-6k}\exp(-\frac{x^{2k}}{2\sigma^2})}=0.\tag{3.3}$$

Thus, by (1.1), (1.2), (3.2) and (3.3), for large x, we can have

$$\begin{aligned}1-F_k(x)&=f_k(x)\frac{\sigma^2}{k}x^{1-2k}\Bigg(1+\frac{\sigma^2}{k}x^{-2k}+\frac{(1-2k)\sigma^4}{k^2}x^{-4k}+O(x^{-6k})\Bigg)\\&=\frac{\exp(-\frac{1}{2\sigma^2})}{2^{\frac{k}{2}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}\Bigg(1+\frac{\sigma^2}{k}x^{-2k}+\frac{(1-2k)\sigma^4}{k^2}x^{-4k}\\&\quad+O(x^{-6k})\Bigg)\exp\Bigg(-\int_1^x\frac{g(t)}{f(t)}\,\mathrm{dt}\Bigg).\end{aligned}$$

The desired result follows. □

In order to prove Theorem 2.1, the following auxiliary result is needed.

Lemma 3.2. *Let $v_k(b_n;x)=n\log F_k(a_nx+b_n)+e^{-x}$. For normalizing constants a_n and b_n given, respectively, by (1.4) and (1.5), we have*

$$\lim_{n\to\infty}b_n^{2k}\Big[b_n^{2k}v_k(b_n;x)-l_k(x)\Big]=w_k(x),\tag{3.4}$$

where $l_k(x)$ and $w_k(x)$ are given by Theorem 2.1.

Proof. Obviously, $b_n\to\infty$ if and only if $n\to\infty$ since $1-F_k(b_n)=n^{-1}$. The following facts can be gained by (1.1)

$$\lim_{n\to\infty}\frac{1-F_k(a_nx+b_n)}{n^{-1}}=e^{-x}\tag{3.5}$$

and

$$\lim_{n\to\infty}\frac{1-F_k(a_nx+b_n)}{b_n^{-2j}}=0,\quad j=2,4.\tag{3.6}$$

Set

$$B_k(n,x)=\frac{1+\frac{\sigma^2}{k}b_n^{-2k}+\frac{(1-2k)\sigma^4}{k^2}b_n^{-4k}+O(b_n^{-6k})}{1+\frac{\sigma^2}{k}(a_nx+b_n)^{-2k}+\frac{(1-2k)\sigma^4}{k^2}(a_nx+b_n)^{-4k}+O\Big((a_nx+b_n)^{-6k}\Big)}.$$

It is easy to check that $\lim_{n\to\infty}B_k(n,x)=1$ and

$$B_k(n,x)-1=\Bigg(\frac{2\sigma^4}{k}b_n^{-4k}x+\frac{4(1-2k)\sigma^6}{k^2}b_n^{-6k}x+O(b_n^{-6k})\Bigg)(1+o(1)).$$

Then

$$\lim_{n\to\infty} \frac{B_k(n,x)-1}{b_n^{-2k}} = 0 \tag{3.7}$$

and

$$\lim_{n\to\infty} \frac{B_k(n,x)-1}{b_n^{-4k}} = \frac{2\sigma^4}{k}x. \tag{3.8}$$

By (3.1), we have

$$\begin{aligned}\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x} =& B_k(n,x)\exp\left(\int_0^x \left(\frac{k}{\sigma^2}a_n(a_ns+b_n)^{2k-1} - \frac{a_n}{a_ns+b_n} - 1\right)\mathrm{ds}\right)\\ =& B_k(n,x)\Big\{1+\int_0^x \left(\frac{k}{\sigma^2}a_n(a_ns+b_n)^{2k-1} - \frac{a_n}{a_ns+b_n} - 1\right)\mathrm{ds}\\ &+\frac{1}{2}\left(\int_0^x \left(\frac{k}{\sigma^2}a_n(a_ns+b_n)^{2k-1} - \frac{a_n}{a_ns+b_n} - 1\right)\mathrm{ds}\right)^2(1+o(1))\Big\}. \end{aligned}\tag{3.9}$$

Combining with (3.5), (3.6), (3.7), (3.8) and (3.9), we have

$$\begin{aligned}&\lim_{n\to\infty} b_n^{2k}v_k(b_n;x)\\ &= \lim_{n\to\infty}\frac{\log F_k(a_nx+b_n)+(1-F_k(b_n))e^{-x}}{n^{-1}b_n^{-2k}}\\ &= \lim_{n\to\infty}\frac{-(1-F_k(a_nx+b_n))-\frac{1}{2}(1-F_k(a_nx+b_n))^2(1+o(1))}{n^{-1}b_n^{-2k}} + \lim_{n\to\infty}\frac{(1-F_k(b_n))e^{-x}}{n^{-1}b_n^{-2k}}\\ &= \lim_{n\to\infty}\frac{1-F_k(a_nx+b_n)}{n^{-1}}\frac{\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x}-1}{b_n^{-2k}}\\ &= e^{-x}\lim_{n\to\infty}\left\{B_k(n,x)b_n^{2k}\left(\int_0^x \left(\frac{k}{\sigma^2}a_n(a_ns+b_n)^{2k-1} - \frac{a_n}{a_ns+b_n} - 1\right)\mathrm{ds}\right)(1+o(1)) + \frac{B_k(n,x)-1}{b_n^{-2k}}\right\}\\ &= e^{-x}\lim_{n\to\infty}\int_0^x b_n^{2k}\left(\frac{k}{\sigma^2}a_n(a_ns+b_n)^{2k-1} - \frac{a_n}{a_ns+b_n} - 1\right)\mathrm{ds}\\ &= \frac{\sigma^2}{2k}\Big((2k-1)x^2-2x\Big)e^{-x} = l_k(x), \end{aligned}\tag{3.10}$$

where the last step follows by the dominated convergence theorem and

$$\lim_{n\to\infty} b_n^{2k}\left(\frac{k}{\sigma^2}a_n(a_ns+b_n)^{2k-1}-1\right) = \frac{2k-1}{2k}\sigma^2 s$$

and

$$\lim_{n\to\infty}\frac{a_nb_n^{2k}}{a_ns+b_n} = \frac{\sigma^2}{k}.$$

By arguments similar to (3.10), we have

$$\begin{aligned}&\lim_{n\to\infty} b_n^{2k}\left[b_n^{2k}v_k(b_n;x)-l_k(x)\right]\\ &= \lim_{n\to\infty}\frac{\log F_k(a_nx+b_n)+n^{-1}e^{-x}-n^{-1}b_n^{-2k}l_k(x)}{n^{-1}b_n^{-4k}}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{\log F_k(a_n x + b_n) + (1 - F_k(b_n))e^{-x}(1 - l_k(x)e^x b_n^{-2k})}{n^{-1} b_n^{-4k}} \\
&= \lim_{n\to\infty} \frac{-(1 - F_k(a_n x + b_n)) + (1 - F_k(b_n))e^{-x}(1 - l_k(x)e^x b_n^{-2k})}{n^{-1} b_n^{-4k}} \\
&= \lim_{n\to\infty} \frac{1 - F_k(a_n x + b_n)}{n^{-1}} \frac{\frac{1-F_k(b_n)}{1-F_k(a_n x+b_n)} e^{-x}(1 - l_k(x)e^x b_n^{-2k}) - 1}{b_n^{-4k}} \\
&= e^{-x} \lim_{n\to\infty} \Bigg\{ B_k(n,x) b_n^{4k} \Bigg[\int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1 \right) \mathrm{ds} - l_k(x) e^x b_n^{-2k} \Bigg] \\
&\quad + \frac{1}{2} B_k(n,x) b_n^{4k} \Bigg[\int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1 \right) \mathrm{ds} \Bigg]^2 \\
&\quad - B_k(n,x) l_k(x) e^x b_n^{2k} \int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1 \right) \mathrm{ds} + \frac{B_k(n,x) - 1}{b_n^{-4k}} \Bigg\} \\
&= -\frac{\sigma^4}{24k^2} \Big[3(2k-1)^2 x^4 - 4(2k+1)(2k-1)x^3 + 24x^2 - 48kx \Big] e^{-x} = w_k(x).
\end{aligned}
$$

The proof is complete. □

Proof of Theorem 2.1. By (3.10), we have $v_k(b_n; x) \to 0$ and

$$
\left| \sum_{i=3}^{\infty} \frac{v_k^{i-3}(b_n; x)}{i!} \right| < \exp\Big(|v_k(b_n; x)| \Big) \to 1
$$

as $n \to \infty$. By applying Lemma 3.2, we have

$$
\begin{aligned}
& b_n^{2k} \Big[b_n^{2k} \Big(F_k^n(a_n x + b_n) - \Lambda(x) \Big) - l_k(x)\Lambda(x) \Big] \\
=& b_n^{2k} \Big[b_n^{2k} \Big(\exp\Big(v_k(b_n; x) \Big) - 1 \Big) - l_k(x) \Big] \Lambda(x) \\
=& \Bigg[b_n^{2k} \Big(b_n^{2k} v_k(b_n; x) - l_k(x) \Big) + b_n^{4k} v_k^2(b_n; x) \Bigg(\frac{1}{2} + v_k(b_n; x) \sum_{i=3}^{\infty} \frac{v_k^{i-3}(b_n; x)}{i!} \Bigg) \Bigg] \Lambda(x) \\
\to& \left(w_k(x) + \frac{l_k^2(x)}{2} \right) \Lambda(x),
\end{aligned}
$$

as $n \to \infty$.

We obtain the desired result. □

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